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Existence of solutions for turbulence shell models

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There once lived a man
who learned how to slay dragons
and gave all he possessed
to mastering the art.

After three years
he was fully prepared but,
alas, he found no opportunity
to practise his skills.

(Zhuāngzǐ)

As a result he began
to teach how to slay dragons.

(René Thom)

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Introduction

The aim of the thesis is to study the existence of solutions for turbulence shell models. The thesis collects the two major works done by the author, both mainly devoted to existence results. The first one, [26], improves a standard existence result in the mixed turbulence shell model, extending from finite energy initial conditions to almost all initial conditions, where almost all is respect to a Gaussian measure on the infinite dimensional space of initial conditions. The second one, a theorem from [10], is an existence result for high general turbulence tree model with a force acting on the first component.
The work is divided into four chapter, in the first two chapters we give an introduction to the concept of shell model and a taste of some existence results on the standard dyadic model, discussing also the problem of uniqueness of solutions. The third chapter basically shows the work done in [26]. In the fourth chapter we introduce tree models with the aim of proving the existence theorem from [10]. Finally, in chapter 4, we prove also a result on the tree model similar to the one on the mixed model obtained in [26].

The classical dyadic model

It is accepted that the dynamic of a fluid is well described by the Navier-Stokes equation (NSE). The full Navier-Stokes equation is the following

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i + f_i$$

combined with the continuity equation $\partial_i v_i = 0$. The Navier-Stokes equation has generated lot of problems that are open nowadays and it is natural to introduce simplified models of turbulence from NSE.
If one takes the Fourier series of the spectral version of the NSE obtains, as we will show in the first chapter, the following equation

$$\partial_t v_i(n) = -i n_j \left(\frac{2\pi}{L}\right) \sum_{n'} \left(\delta_{il} - \frac{n_i n'_l}{n_j^2}\right) v_y(n') v_l(n-n') - \nu n_j^2 v_i(n) + f_i(n),$$

with wave vectors $y(n) = \frac{2\pi n}{L}$. So in place of the NSE we have an infinite dimensional dynamic system where nonlinear terms are quadratic in the velocities. Shell models are built to mimic the dynamic system obtained from the Fourier series of the spectral NSE. They are simplified version but they have an energy cascade similar to the one of the Kolmogorov theory. The first building block in the shell models approach to turbulence consists on studying the direct cascade dyadic model models. The classical dyadic model (inviscid and unforced), extensively studied in [20], [22] and others , is the following

$$\begin{cases} \frac{d}{dt}X_n = k_{n-1}X_{n-1}^2 - k_n X_n X_{n+1} \\ X_n(0) = x_n \end{cases} \tag{1}$$

where $k_n = 2^{\beta n}$ with $\beta > 0$, $X = (X_n)_{n\in\mathbb{N}}$ is a sequence of real functions $X_n = X_n(t)$ with $X_0(t) = 0$ and $x = (x_n)_{n\in\mathbb{N}}$ is the initial condition.
We can note that if one looks only for positive x_n components, the first equation of the system means that the n-th component has derivative that grows taking from the $(n-1)$-th component and giving to the $(n+1)$-th. In this sense we think to this model as a "direct cascade" model, the energy flows from large to small scale, and this is typical of the dynamic of a 3D-fluid. An "inverse cascade" model is better interpreted by a model with different sign in the non-linearity, and this mimics a 2D-fluid dynamic. Of course this heuristic is valid as long as one has positive initial conditions, but we will see that if one looks for uniqueness of a solution cannot go so much far from all positive initial conditions.

A quantity, $E(t) = \sum_{n=1}^{\infty} X_n^2(t)$, called energy, is formally (but not rigorously) preserved along the trajectories of the dyadic shell model, in the sense that $\frac{d}{dt}\sum_{n=1}^{\infty} X_n^2(t) = 0$. Hence, for the initial conditions that have finite energy, i.e. $\sum_{n+1}^{\infty} x_n^2 < \infty$, we can prove the existence of solutions in the following way:

- We build a truncated version of the infinite dimensional dynamic system, a N-dimensional dynamic system that approximates for $N \to \infty$ the dyadic model in the Galerkin sense.
- As we have built the truncated version in a way that the energy is now rigorously preserved along the trajectories, we have the existence and uniqueness of solutions for all initial condition for each N-dimensional system.
- Using Ascoli-Arzelà theorem we extract a converging subsequence of solutions from the approximating ones and showing that the limit solves the equation (1) in the integral form we have the proof of existence.

It is possible to extend this heuristic to other turbulence shell models, as long as we have the energy conservation, that guarantees both the well-posedness for the truncated system and the use of Ascoli-Arzelà theorem, and thanks to the quadratic non-linearity we usually have easy estimates to prove that the limit obtained via Ascoli-Arzelà theorem really solves the infinite dimensional system.
The first result we can prove (after the existence for finite energy initial conditions that follows the sketch above) is that a weak solution, i.e. a sequence of functions that formally solve the equations of the system, once it gets positive it remains positive. Moreover, if any component X_n satisfies $X_n(t_0) > 0$ $(\geq)$ for a certain t_0, then for any $t > 0$ we have $X_n(t) > 0$ $(\geq)$. So, if we take an initial condition with all positive entries, all weak solutions with said initial condition would be positive for any $t \geq 0$. Introducing $H = H^0$ as the class of finite energy

sequences, i.e. ℓ^2 sequences, and more generally, given k_n as the coefficients of a shell model, the Hilbert space H^s with $s \in \mathbb{R}$ is the set of $x \in \mathbb{R}^{\mathbb{N}}$ such that the H^s norm $\|x\|_{H^s} \overset{\text{def}}{=} \sqrt{\sum_{i=1}^{\infty} k_i^{2s} x_i^2}$ is finite, we can generalize saying that if an initial condition x belongs to H, then there exists at least one finite solution bounded in H which becomes positive in finite time.

Hence the positiveness requirement doesn't concern with existence of solutions, as finite energy does, however it helps for uniqueness. A big part of the second chapter is dedicated to prove that there are some initial conditions in H, with all negative components, such that they admit infinite-many finite energy solutions. To get the result we will introduce the concept of stationary and self-similar solution. A stationary solution is a sequence of real numbers $x = (x_n)_{n \geq 1}$ such that $x \in H$ and, with a given $f \in \mathbb{R}$

$$\begin{cases} 0 = f - k_1 x_1 x_2 \\ 0 = k_{n-1} x_{n-1}^2 - k_n x_n x_{n+1}, \quad n \geq 2. \end{cases}$$

A self-similar solution is a finite energy solution X such that there exists a function φ and a sequence of real numbers $(y_n)_{n \geq 1}$ and satisfies for all $n \geq 1$ and $t \geq 0$, $X_n(t) = y_n \varphi(t)$.
The heuristic to find infinite-many solutions for the same initial condition is the following, we first show that there exists a bounded non-decreasing sequence of positive real numbers $(r_n)_{n \geq 1}$ such that

$$X_n(t) = \frac{k_n^{-\frac{1}{3}} r_n}{t - t_0}$$

are the only self-similar solution with $X_n > 0$ and where t_0 is a free parameter. The set of self-similar solution is given by the above ones and their forward shift and their modifications in the following way

$$X_n(t) = \begin{cases} 0 & n = 0, 1, \dots, m \\ \pm k_1^{-\frac{1}{3}} r_1 \frac{k_m^{-1}}{t - t_0} & n = m + 1 \\ k_{n-m}^{-\frac{1}{3}} r_{n-m} \frac{k_m^{-1}}{t - t_0} & n \geq m + 2 \end{cases}$$

where $m \geq 1$ and $t_0 < 0$ can be chosen freely. The energy of these self-similar solutions is strictly a decreasing function that tends to zero, hence we have in this case a phenomenon of "anomalous dissipation", more precisely the conservation of the energy (that formally holds) is violated.
After this step we know that there exists a self-similar solution X whose total energy is strictly decreasing. With a time inversion, that consists of a transformation that send weak solutions for initial condition x into weak solution with initial condition $-X(T)$, we consider an associate weak solution

$$Y(t) \overset{\text{def}}{=} -X(T - t).$$

From this for every $s \in [0, T]$ we can build a different solution, we consider Y^s as the solution obtained attaching Y on $[0, s]$ to a Leray-Hopf solution (that is a weak solution with non-increasing energy) on $[s, \infty)$ with initial condition

$Y(s) = -X(T-s)$.
With this trick we have build infinite-many different solutions with starting negative initial condition $-X(T)$, where the difference between Y^{s_1} and Y^{s_2} with $s_1 < s_2$ is guarantee by the behaviour of the energy, in Y^{s_1} the energy strictly increase on $[0, s_1]$ and it is not increasing (by construction) on $[s_1, \infty)$, so Y^{s_2} is a proper different solution from Y^{s_1} since differently on $[s_1, s_2]$ we have that the energy of Y^{s_2} is strictly increasing.

As we already anticipated, differently from non-positive ones, for positive initial conditions with finite energy we have the uniqueness of solutions.
The first step in this direction is to prove that if for an initial condition x it holds that any weak solution X with x as initial condition has the property

$$\lim_{n\to\infty} 2^{-n} k_n \int_0^t X_n^3(s)ds = 0,$$

for $t \geq 0$, so there exists only a unique weak solution with initial condition x.
The second step is to prove that, for any weak solution X with a finite energy initial condition x with all non-negative components there exists a constant a constant c depending only on β such that the following inequality holds for all $n \geq 1$ and $M > 0$

$$\mathcal{L}(X_n > M) = \mathcal{L}\{t \geq 0 : X_n(t) > M\} \leq \frac{c\|x\|^2}{k_n M^3}$$

where $\mathcal{L}$ stands for the Lebesgue measure.
In the third step we use the estimates done in the second step to prove that for an initial condition x with finite energy and all positive components holds the property made in the first step, so from this we get uniqueness.

A stationary solution for the mixed model

Hence for the classic dyadic model existence holds for any finite energy initial condition x, moreover the solution is unique if x has also non-negative components. Now, passing from the results of the second chapter to the results of the third chapter, we consider a mixed shell model. For a generic mixed shell model

$$X_n' = \alpha(k_n X_{n-1}^2 - k_{n+1} X_n X_{n+1}) - \beta(k_n X_{n+1}^2 - k_{n-1} X_{n-1} X_n),$$

the existence of solutions holds, as for the classic dyadic model, for finite energy initial conditions. The reader can note that we still have a sort of energy conservation in the mixed model, the quantity $\sum_{i=1}^{\infty} X_i^2(t)$ is formally preserved along the trajectories of any solution, hence we can apply the sketch of the existence proof made for the classic dyadic model: we first construct an approximating finite dimensional system (in the Galerkin sense) where the conservation of energy rigorously holds. Then, thanks to the energy conservation, we have the existence and uniqueness of solutions for any finite dimensional system of the Galerkin sequence, and then via Ascoli-Arzelà theorem we extract a limit that turns out to be a solution for the mixed model.

The works done in [26] starts from this point and has the aim to improve the existence of solutions, passing from finite energy initial conditions to a larger class. Following the techniques introduced by Albeverio in his works [1], [2], [3] and later perfectioned by Flandoli [16], [17], we look for a choice of coefficients of the mixed shell models to let to a certain class of Gaussian measures to be invariant for the system. More precisely we have also to construct the Galerkin approximating system such that for each system of the sequence, if N is the dimension of the system the projection of the Gaussian measure on the first N component is invariant.
The candidate Gaussian measure will be $\mu_r = \bigotimes_{i=1}^{\infty} \mathcal{N}(0, r^2)$, where $r \in \mathbb{R}$ is a free parameter. To let to the projection on the first N components of μ_r, μ_r^N, to be invariant for the N-dimensional system of the Galerkin sequence, it musts hold $\mathrm{div}(b(x)f(x)) = 0$, where $b(x)$ is the vector field and $f(x)$ the density function of μ_r^N with respect to the Lebesgue measure. To satisfy this condition the coefficients of the mixed model must be $\alpha = \beta$ that we set equal to 1 without loss of generality. The approximating Galerkin sequence in this way is, for $1 \leq n \leq N$

$$\frac{d}{dt}X_n(t) = k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t) - k_n X_{n+1}^2(t) + k_{n-1}X_{n-1}(t)X_n(t),$$

with $k_n = 2^{\beta n}$, $\beta > 0$ for $1 < n < N$ and $k_0 = k_1 = k_N = k_{N+1} = 0$. It also holds that the energy $\sum_{i=1}^{N} X_i^2(t)$ is preserved along the trajectory of any solution of this system, hence we still have the existence and uniqueness of solutions for any initial conditions of the N-dimensional approximating system.

The existence of an invariant measure opens the door to the introduction of the concept of random solution, that is a solution for a random initial condition, with respect to the probability measure μ_r^N. More precisely, let $(\Omega, \mathcal{F}, P)$ be an abstract probability space, for fixed N let Y_r^N be a random variable

$$Y_r^N : (\Omega, \mathcal{F}, P) \to (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^{\mathbb{N}})),$$

with law μ_r^N. We call $(\Omega, \mathcal{F}, P, U_r^N)$ a N-finite random solution if U_r^N is defined on the abstract probability space $(\Omega, \mathcal{F}, P) \times [0, T]$ to $\mathbb{R}^\infty$, all k-coordinates of U_r^N are almost surely for each time $t \in [0, T]$ equal to 0 if $k > N$ and for $k \leq N$ almost surely

$$U_r^N{}_{(k)}(\omega, t) = F_r^N{}_{(k)}(\omega, t),$$

where, for $\omega \in \Omega$, the function

$$F_r^N(\omega) : [0, T] \to \mathbb{R}^N$$

is the unique solution of the N-dimensional shell model with initial conditions

$$X(0) = Y_r^N(\omega).$$

Since we have taken as law for the random initial condition an invariant measure for the system, by construction the law of $U_r^N(t)$ is, for any $t \in [0, T]$,

$$\tilde{\mu}_r^N = \mu_r^N \otimes \bigotimes_{N+1}^{\infty} \delta_0.$$

A finite random solution almost surely belongs to H^s for any $s < 0$, moreover, thanks to the invariance and some technical details, we have uniform in N estimates on the $L^p(0,T;H^s)$ and $W^{1,p}(0,T;H^s)$ norms of the random variable:

- For every $s < 0$, $r > 0$, $p > 1$, $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, not depending on N, such that
$$P(\|U_r^N\|_{L^p(0,T;H^s)} \leq C_\epsilon) > 1 - \epsilon,$$
for each $N \in \mathbb{N}$.

- For every $s < -1$, $r > 0$, $p > 1$ $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that
$$P(\|U_r^N\|_{W^{1,p}(0,T;H^s)} \leq C_\epsilon) > 1 - \epsilon,$$
for each $N \in \mathbb{N}$.

Having this estimates we can go for a compactness result on the laws of the N-finite random solution. If we consider a sequence of N-finite random solutions we claim that there exists a converging subsequence of these laws that converge in the topology of $L^p(0,T;H^s)$ and in the topology of $C(0,T;H^s)$ for any $s < 0$. This is true thanks to a combination of Aubin-Lions lemma and Prohorov theorem.
For Aubin-Lions lemma, given $s_0 < 0$ and $s_1 < -1$ the set

$$K_{R_1,R_2} = \{X | \|X\|_{L^p(0,T;H^{s_0})} \leq R_1, \|X\|_{W^{1,p}(0,T;H^{s_1})} \leq R_2\}$$

is relatively compact in $L^p(0,T;H^s)$ for any $s < 0$ such that $s_0 > s > s_1$. Moreover, thanks to the estimates done before, we have a uniform bound on the measure of such a relative compact set, since it holds for any $p_0, r_1 > 1$ that

$$P(\|U_r^N\|_{L^{p_0}(0,T;H^s)} \leq c_\varepsilon) \geq 1 - \varepsilon$$

for all $N \in \mathbb{N}$ and

$$P(\|U_r^N\|_{W^{1,r_1}(0,T;H^{s_1})} \leq c_\varepsilon) \geq 1 - \varepsilon$$

for all $N \in \mathbb{N}$. So we have that given $\varepsilon > 0$ there exist $R_1(\varepsilon)$, $R_2(\varepsilon)$ such that the family of laws satisfies

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(L^p(0,T;H^s)) | \mu(K^c_{R_1,R_2}) \leq \varepsilon\},$$

hence, since $\bar{K}_{R_1,R_2} \supseteq K_{R_1,R_2}$ we have $\bar{K}^c_{R_1,R_2} \subseteq K^c_{R_1,R_2}$, and we remark that $\bar{K}_{R_1,R_2}$ is compact. Hence

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(L^p(0,T;H^s)) | \mu(\bar{K}^c_{R_1,R_2}) \leq \varepsilon\},$$

so the family of laws $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ is tight in the topology of $L^p(0,T;H^s)$ and we can now apply Prohorov theorem to have a converging subsequence. Moreover, since the estimates of the L^p-norms hold for any $p > 1$, we can apply a result from Simon to get the same convergence in law also in the topology of $C(0,T;H^s)$.

From what done above we get a convergence of a sequence of laws to a certain measure. We can then apply Skorokhod theorem to have a sequence of random variables (that turn out to still be N-finite random solutions) converging almost surely to a limit in the topology of $L^p(0,T;H^s)$ for any $p>1$ and in the topology of $C(0,T;H^s)$. The work of the third chapter concludes showing that the limit almost surely solves the integral form of the equation of the mixed shell model.
The uniqueness of that "random solution" is a tough topic. The heuristic done for the classic dyadic model for uniqueness doesn't hold in this scenario, since the solution is made from Gaussian values and all positive or negative arguments don't work.

Turbulence tree models

The last part of the thesis is fully dedicated to turbulence tree models. The tree models, first introduced in [21] and then well studied in [9] and others, mimic the chaotic turbulence behaviour where larger eddies tend to split into smaller eddies, with a kinetic energy transfer. So the tree-like structure have eddies as nodes, and a node is child of another node if the corresponding eddy is formed by a split of the corresponding eddy of the father. We denote by J the set of nodes, and if $j \in J$ we call $\mathcal{O}_j$ the set of offspring of j. We made assumption that every eddies has the same biggest eddy as ancestor, so we can classify eddies in "generations" or "levels". Level 0 is made by only the biggest eddy $\emptyset \in J$, level 1 is made by the eddies produced by the one in level 0 and so on. We will denote the generation of an eddy j by $|j|$. The father of an eddy j will be denoted as $\bar{\jmath}$.
To construct the dynamic we associate to each eddy j a non-negative intensity $X_j(t)$. The following one is the classic tree model

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k,$$

and as for dyadic models we have an existence of solutions theorem for finite energy initial condition, since also in this model the energy is formally conservative. The sketch of the proof is the same of the one made for dyadic models, first one introduces a Galerkin approximation of the infinite dimensional tree model where still holds the energy conservation, then we have well posedness for any initial condition of systems of the Galerkin sequence, then one extracts a limit to a sequence of solutions via Ascoli-Arzelà theorem, limit that turns out to solve the equation of the infinite dimensional model in the integral form.

We now want to prove the existence for a more general model where formal energy conservation doesn't hold. The model has a force acting on the first component (or a dissipation, it can be both depending on the values of the parameters)

$$\frac{d}{dt}X_j = \alpha\left(c_j X_{\bar{j}}^2 - X_j \sum_{k\in\mathcal{O}_j} c_k X_k\right) + \beta\left(\tilde{c}_j X_{\bar{j}} X_j - \sum_{k\in\mathcal{O}_j} \tilde{c}_k X_k^2\right)$$

$$+\gamma\left(X_{\bar{j}} \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} \hat{c}_{j,l} X_l - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2}\right),$$

$$\frac{d}{dt}X_\emptyset = f(t, X_\emptyset) - \alpha X_\emptyset \sum_{k\in\mathcal{O}_\emptyset} c_k X_k - \beta \sum_{k\in\mathcal{O}_\emptyset} \tilde{c}_k X_k^2 - \gamma \sum_{k_1\neq k_2, k_i\in\mathcal{O}_\emptyset} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2},$$

$$X(0) = \bar{X},$$

where $f(t,x) \leq c(t) + g(t)|x|$, with $c(t)$ and $g(t)$ positive continuous functions. A standard assumption on the coefficients c_j, $\tilde{c}_j$ and $\hat{c}_{j_1,j_2}$ is that they have an exponential growing depending on their index generation, i.e. $c_j = 2^{\gamma|j|}$, $\tilde{c}_j = 2^{\tilde{\gamma}|j|}$ and $\hat{c}_{j_1,j_2} = 2^{\hat{\gamma}|j_1|}$, with $\gamma, \tilde{\gamma}, \hat{\gamma} > 0$.

The terms are chosen in a way to let each terms to interact to at most other terms with same generation or one generation lower or higher, and it is the most general model if one ask also for the formal conservation of energy outside of the first component. The results we are going to show work for any tree model with dynamic described above and limited number of children for any eddy, i.e. there exists a $M \in \mathbb{N}$ such that $\forall j \in J$ it holds $\sum_{i\in\mathcal{O}_j} 1 \leq M$, hence we add this assumption introducing the system. This system is taken from [10], and consists of a generalization of another system studied in the same work. Adapting the techniques of the work [4] to our model, to prove the existence of solutions for ℓ^2 initial condition we need an energy bound. This is why we ask to the function acting on the first component to be at most linear in the space argument, in this way we get a bound on the norm of the derivative of the energy using Gronwall lemma.
Having an energy bound, we can use the classical method to get the existence for all ℓ^2 initial conditions

- construct a Galerkin approximation,
- use the energy bound to get existence and uniqueness of solutions for each system in the Galerkin approximation,
- extract a limit of a sequence of solutions via Ascoli-Arzelà theorem, using again the energy bound, and show that the limit solves the equation in the integral form.

Last, we work again on random solution. The aim is to choose parameters in our tree model to have the conservation of a Gaussian measure, in this way we would be able to adapt the techniques of the third chapter. From now on we consider $c_j = 2^{|j|}$, $(H^s, \|\cdot\|_{H^s})$ as the Hilbert space of sequences $x \in \mathbb{R}^\infty$ satisfying

$$\|x\|_{H^s} = \sqrt{\sum_n c_n^{2s} x_n^2} < \infty.$$

The tree model then will be

$$\frac{d}{dt}X_j = \alpha \left(c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k \right) - \beta \left(d_{\bar{j}} X_{\bar{j}} X_j - \sum_{k \in \mathcal{O}_j} d_j X_k^2 \right),$$

where $\alpha, \beta, c_i, d_i \in \mathbb{R}$, $\alpha c_j - \beta d_j = 0$ for any $|j| \geq 1$, $c_0 = d_{\bar{0}} = d_0 = 0$. This model is a particular case of the one studied before, so we have existence of solutions for ℓ^2 initial conditions and we can build a Galerkin approximation with existence and uniqueness for any initial condition. Moreover the Gaussian measure

$$\mu_r^Q = \bigotimes_{|j| \leq N} \mathcal{N}(0, r^2)$$

is invariant for the Q-dimensional approximating system of the Galerkin sequence, where $Q = \sum_{|i| \leq N} 1$. Sadly we can not be so general on the coefficients c_j of the tree, we have to ask a geometric dependence from child to father like $c_j = \lambda c_{\bar{j}}$ with $\lambda > 1$, moreover we have to ask a bound on the degree of the tree, i.e. an eddy can have at most M children.
Fixing all this stuff we can use the same heuristic of the third chapter to get the existence of random solutions also for this tree model of turbulence.

Chapter 1

Turbulence and Shell Models

Abstract

We introduce the reader to the link between Navier-Stokes laws and shell models. Then we give a short introduction to the shell models, with related properties and first historical results.

1.1 Navier-Stokes equation

The Navier-Stokes equation, with appropriate initial condition, mimics the dynamic of a fluid. Despite the understanding of NS equation is out of the goal of this work, we give a short introduction to that theory. The Navier-Stokes equation has being studied for decades and it represents one of the most challenging open problems in mathematics.
In the NS model the fluid is thought as a continuum stream identified by a velocity field $v_i(x,t)$, a temperature field $t(x)$, a pressure field $p(x)$ and a density field $\rho(x)$. This means that at each point x_i the fluid is described by pressure, temperature, density and three components of velocity.
The equations that describe the fluid dynamic are derived from mass conservation, momentum conservation, energy conservation and the equation of state. First we may assume that the liquid is incompressible, hence we can get rid of the equation that defines density. Also we may dissociate the temperature from the momentum and continuity equations, so it left the velocity and pressure field to govern the fluid. Hence we may assume that the dynamic is governed by the following equations (NSE)

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i + f_i \tag{1.1}$$

with the continuity equation, that can be view as an overall mass conservation

$$\partial_i v_i = 0. \tag{1.2}$$

The equation (1.1) states that the acceleration of a fluid fragment equals the sum of the forces acting on the fluid fragment per unit mass. The left hand side is composed by the derivative of the velocity and the advection, the right hand side is composed by the pressure gradient force, the viscous friction and a third term that gathers with all other forces.

If those four equation can be solved it would be possible to establish the three components of the fluid velocity v_i and the pressure p, however the solution is still an open problem.
We can do the following transformations

$$x = Lx',\ v = Vv',\ t = \frac{L}{V}t' \tag{1.3}$$

to let the NSE equation to be dimensionless. The outer scale L can be viewed as the length scale of the largest variation in the flow. From L and V we can have the timescale $T = \frac{L}{V}$, which represents the time that takes the fluid to travel distance L with a velocity of V. Since NSE equation derive from the Newton second law it is invariant to the addition of a uniform velocity, this means that only velocity variation really matters. So, if we put transformations (1.3) into equation 1.1 we get NSE in dimensionless form

$$\partial_t v_i + v_j \partial v_i = -\partial_i p + \mathrm{Re}^{-1} \partial_{jj} v_i + f_i, \tag{1.4}$$

where Re is the dimensionless Reynolds number

$$\mathrm{Re} = \frac{VL}{\nu}.$$

In 1.4 viscosity is the order of the inverse of Reynolds number. This means that for small Reynolds number we have an attenuation of the flow by viscosity, for high Reynolds number we get a very chaotic flow.

1.2 Kolmogorov theory and four-fifth law

The first attempt to formalize the theory of turbulence was done by Richardson [28]. Richardson made a description of the flow as a set of larger eddies splitting up into smaller eddies, with a energy cascade going from larger scales to smaller ones, and then disappearing at the viscous scale. The effort made by Richardson inspired Kolmogorov theory.
The theory built by Kolmogorov has the following assumptions:

- The flow is homogeneous, hence it is invariant by translation.
- The flow is isotropic, hence it is rotationally invariant.
- There is a statistical equilibrium, hence the energy released by the force that powers the flow is evened by the energy dissipated by viscosity.

Hence the state of the flow is given by the mean energy dissipation per unit mass ϵ. The velocity difference $\delta v(l)$ at a scale l is given by

$$\delta v(l) = |v(r + l) - v(r)|$$

and it is the velocity related to an eddy of size l. If we consider a smaller eddy, of size l_1 inside the bigger l_2-size eddy we have that the bigger eddy acts on the smaller eddy like the overall flow acts on the bigger eddy. Hence, assuming that the flow is self-similar, we can conjecture that

$$\delta v(l_2) = f(\frac{l_1}{l_2})\delta v(l_1),$$

where f is a universal function. So $\delta v(l)$ is only a function of the scale l and the mean energy dissipation ϵ, hence we put

$$\delta v(l) = f(l, \epsilon),$$

the dimension must be the same in both sides, so f can only depend on the combination of l and ϵ with the same dimension of the other, and it turns out to be

$$\delta v(l) \sim (\epsilon l)^{\frac{1}{3}}.$$

This relation leads us to introduce the structure functions of a velocity field of different orders. A structure function (of order p) $S_p(l)$ is

$$S_p(l) = \langle \delta v(l)^p \rangle,$$

where the brackets represent the statistical average among the scale l. Kolmogorov in [24] focus on the third order structure function and gives directly

$$S_3(l) = \frac{4}{5}\epsilon l. \tag{1.5}$$

This result follows directly from the assumptions, so a turbulence theory that shares the same assumptions must follow the equation (1.5), namely the four-fifth law. The theory of Kolmogorov states also with dimensional counting that

$$S_p(l) \sim l^{\frac{p}{3}},$$

that holds for Gaussian field. More general we can state

$$S_p(l) =\sim l^{\zeta(p)},$$

where we call $\zeta(p)$ the anomalous scaling exponent, and it is related to the intermittency of the flow.

1.3 Fourier transform and energy scaling

We can now move closer to the goal of this work, the turbulence shell models. The first step is to take the Fourier Transform of the velocity field and its inverse

$$\mathcal{F} : \hat{v}_i(y) = \frac{1}{(2\pi)^3} \int e^{-iyx} v_i(x) dx \tag{1.6}$$

$$\mathcal{F}^{-1} : v_i(x) = \int e^{iyx} v_i(y) dy \tag{1.7}$$

and applying them to the equation (1.1) we get

$$\partial_t v_i(y) = -i \int v_j(y - y') y'_j v_i(y') dy' - i y_i p(y) - \nu y_j y_j v_i(y) + f_i(y). \tag{1.8}$$

Now we want to get rid of the pressure from the NSE, so we use the continuity equation, and assuming that $\partial_i f_i = 0$, that means that the force is rotational, and we have only to apply divergence operator to NSE

$$\partial_{ii}p = -\partial v_j \partial_j v_i.$$

Transforming the last equation with (1.6) and using the fact that incompressibility gives $y_j' v_j(y) = 0$ we get

$$-y_j y_j p(y) = -\int (y_i - y_i') v_j(y-y') y_l' v_m(y') dy \delta_{lj} \delta_{mi} - \int (y_j - y_j') v_l(y-y') y_l' v_j(y') dy'$$
$$- \int y_j y_l' v_l(y-y') v_j(y') dy'.$$

Hence we can use the last equation to substitute p in (1.8) to get the spectral NSE

$$\partial_t v_i(y) = -iy_j \int (\delta_{il} - \frac{y_i y_l'}{y_j^2}) v_y(y') v_l(y-y') dy' - \nu y_j^2 v_i(y) + f_i(y). \quad (1.9)$$

Last, we think at the flow as be confined in a box of size L^3 with periodic boundary condition, so in place of a Fourier transform we have a Fourier series and we switch the last equation with

$$\partial_t v_i(n) = -in_j(\frac{2\pi}{L}) \sum_{n'} (\delta_{il} - \frac{n_i n_l'}{n_j^2}) v_y(n') v_l(n-n') - \nu n_j^2 v_i(n) + f_i(n), \quad (1.10)$$

with the wave vectors $y(n) = \frac{2\pi n}{L}$. Hence in place of NSE we get an infinite dimensional system where the nonlinear terms are quadratic in the velocities.

We now use again the Fourier transform to show a relation between the second order structure function and the energy density

$$E = \frac{1}{2} \int v(x)^2 dx = \frac{1}{2} (2\pi)^3 \int_0^\infty v_i(y) \bar{v}_i(y) d_y \quad (1.11)$$
$$= \frac{1}{2} (2\pi)^3 4\pi \int_0^\infty y^2 |v(y)|^2 dy \stackrel{\text{def}}{=} \int E(y) dy,$$

where we have introduce the spectral energy density $E(y)$ as

$$E(y) = (2\pi)^4 y^2 |v(y)|^2.$$

All together with the following

$$S_2(l) = \langle \delta v(l)^2 \rangle = 2 \int [v(x)^2 - v(1+x) v(x)] dx,$$

gives

$$E(y) = \frac{1}{2\pi} y^{-1} \int_0^\infty x \sin x S_2(\frac{x}{y}) dy. \quad (1.12)$$

Last we get from previous dimensional argument that

$$E(y) \sim \epsilon^{\frac{2}{3}} y^{-\frac{5}{3}} \quad (1.13)$$

which is confirmed by various experiments and observation in 3D turbulence.

1.4 From turbulence to shell models

From the Fourier series (1.10) we deduce a transfer of energy from large to small scale. Shell models are a simplified version of the Fourier series of NSE, with the aim of mimic the energy cascade in a infinite dimensional dynamic system where the equations are coupled, this means that the n-th component interacts only with $n-1$-th and $n+1$-th components.
A shell model can be viewed as a division of the space into concentric spheres with expontially growing radius

$$k_n = \lambda^n$$

with $\lambda > 1$. In this environment the n-th shell will be the set of wave numbers contained in the n-th sphere and not in the $n-1$-th one. Formally speaking a shell model is a system like the following

$$\frac{d}{dt}u_n = k_n G_n[u,u] - \nu_n u_n + f_n,$$

where u_n represents the evolution of the velocity over a wavelength of scale k_n, the function G_n can be chosen to preserve energy, helicity or volume in phase space, usually with interaction only to $n-1$th and $n+1$ terms.
Compared to the equation (1.10) we can note that the dynamic of shell models is way simpler. Despite this, shell models are consistent enough with the turbulence theory to significant mimic the energy cascade of NSE.
For a deeper view on the link between turbulence and shell models we refer to [15].

1.4.1 Obukhov-Novikov models

The first shell model introduced as a simplified version of the Navier-Stokes equation was made by Obukhov [27]. Even if it doesn't follows directly from an approximation of NSE it has an energy cascade coherent with the one described by Kolmogorov in his turbulence theory. The model consists in a infinite dimensional dynamic system governed by ordinary differential equation, each equation is non linear and quadratic in the intensities u_n. The intensities can be viewed as the spectral velocity components $v_i(y)$ within a shell of wave numbers $k_{n-1} < |y| < k_n$. The infinite dimensional system is composed by the equation

$$\frac{d}{dt}u_n = k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2 - \nu_n u_n + f\delta_{n,1}. \tag{1.14}$$

The terms in the equation represent respectively the advection, the pressure, the dissipation and a force acting only on the first component. Like in Navier-Stokes equation the advection and the pressure are quadratic in the intensities and the dissipation is linear.

Following the Obukhov first step, Novikov [14] introduced the following model, with a different non-linearity but still a similar cascade behaviour.

$$\frac{d}{dt}u_n = k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n - \nu_n u_n + f\delta_{n,1}. \tag{1.15}$$

The Obukhov and Novikov models suit very well for 3D turbulence, where eddies tend to split in lesser eddies and we have an energy flow from large scale to small scale. For 2D fluids (like the clouds in the sky) we have a different phenomenon, very close eddies tend to merge and form a larger eddy. This causes an inverse energy flow, from small to large length scale.
Hence, to simulate better the picture of a 2D turbulence we should have a different sign in the Obukhov-Novikov non-linearity. This separation works in the spirit of looking at positive initial conditions, where basically uniqueness is possible, as we will see in the next chapter.

1.4.2 The mixed model

The Obukhov and Novikov models have the presence of external forces and viscosity. We can get rid of those and consider a general model that has the following requirements:

1. Non linear quadratic terms.
2. Scale invariance of coefficients.
3. The intensities have direct interaction with only the closest neighbours.
4. Energy conservation.

If we put together requirements (i) and (iii) we have the following equations:

$$\frac{d}{dt}u_n = a_1 u_{n-1}^2 + a_2 u_{n-1}u_n + a_3 u_{n-1}u_{n+1} + a_4 u_n^2 + a_5 u_n u_{n+1} + a_6 u_{n+1}^2.$$

Remark that condition (iv) means that the quantity

$$E(t) = \sum_i^{\infty} u_i^2(t)$$

is constant along the trajectories, i.e. $\frac{d}{dt}E(t) = 0$. So, if we put together condition (ii) and (iv) we find

$$a_1 = \beta_1 k_n,\ a_2 = \beta_2 k_{n-1},\ a_3 = a_4 = 0,\ a_5 = -a_1 k_1,\ a_6 = -a_2 k_1,$$

for some non negative $\beta_1, \beta_2 \geq 0$. Hence we get

$$\frac{d}{dt}u_n = \beta_1(k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n) + \beta_2(k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2).$$

We may note that for $\beta_1 = 0$ we get the non dissipative isolated Obukhov model, for $\beta_2 = 0$ the non dissipative isolated Novikov model.

A reader can wonder why we ask to the model to have a sort of energy conservation. The conservation of the energy leads us to prove directly the existence of solutions of the dynamical system, for initial condition with finite energy. The proof follows using the following scheme:

- Construct an approximation of the system by N-dimensional system in the Galerkin sense.

- Use energy conservation to prove existence and uniqueness for all initial condition of any N-dimensional system.
- Extract a limit of the solution as $N \to \infty$ using Ascoli-Arzelà theorem and again energy conservation that solves the infinite dimensional system.

Hence we start by constructing an approximation of the system that still conserve energy.

$$\begin{cases} \frac{d}{dt}u_n = \beta_1(k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n) + \beta_2(k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2) & \\ u_n(0) = \underline{\mathrm{u}}_n & \text{for } n = 1, \dots, N \\ u_n(t) = 0 & \text{for } n \geq N+1 \end{cases} \tag{1.16}$$

Proposition 1.4.1. *For any $x_0 \in \mathbb{R}^N$ there exists a unique solution of the system (1.17) with initial condition $u_k(0) = x_{0k}$ for $k \leq N+1$.*

Proof. The truncated system satisfies the Cauchy-Lipscithz theorem and then admits a local solution u^N on $[0, \delta]$ for some $\delta > 0$. We can extend the local solution to a global solution using the energy bound:

$$\sum_{n=1}^{N} u_n^2(t) = \sum_{n=1}^{N} u_n^2(0).$$

□

Hence we can go for the proof of existence

Theorem 1.4.2. *The mixed shell model, described by the system*

$$\begin{cases} \frac{d}{dt}u_n = \beta_1(k_n u_{n-1}^2 - k_{n+1}u_{n+1}u_n) + \beta_2(k_{n-1}u_{n-1}u_n - k_n u_{n+1}^2) \\ u_n(0) = \underline{u}_n \end{cases} \tag{1.17}$$

for any initial condition $\underline{\mathrm{u}}$ such that

$$\sum_{k=1}^{\infty} \underline{u}_k^2 < \infty$$

admits a solution $u(t)$ on $[0, T]$.

Proof. We will use the Ascoli-Arzelà theorem. For any $\underline{\mathrm{u}}$, we consider a sequence of solutions of the system (1.17) $u^N(t)$ obtained considering $x_0 = \underline{\mathrm{u}}$.

For every fixed j and $t \in [0, T]$ it holds:

- Uniform boundedness of $\{u_j^N(t)\}_{N \in \mathbb{N}}$ for both N and t:

$$|u_j^N(t)| \leq \|u^N(t)\|_2^2 \leq \|\underline{\mathrm{u}}\|_2^2.$$

- Equi-Lipschitzianity of $\{u_j^N(t)\}_{N\in\mathbb{N}}$ with respect to N:

$$|\frac{d}{dt}u_j^N(t)| \leq |k_j u_{j-1}^2(t)| + |k_{j+1}u_j(t)u_{j+1}(t)| + |k_j u_{j+1}^2| + |k_{j-1}u_{j-1}(t)u_j(t)| \leq$$

$$\leq \|\underline{u}\|_2^2(2k_j + k_{j+1} + k_{j-1}).$$

Ascoli-Arzelà theorem implies for each fixed j the existence of a converging subsequence in $C([0,T])$, i.e. it is possible to find indices $\{N_k^j, k \in \mathbb{N}\}$ such that

$$\sup_{t\in[0,T]} |u_j^{N_k^j}(t) - u_j(t)| \to 0$$

for fixed j as $k \to \infty$.
The sequence $N_\bullet^j$ can be chosen so that $N_\bullet^{j-1}$ is a subsequence of $N_\bullet^j$ itself. By a standard diagonal argument we can extend the convergence to all j. If we consider indices $N_k = N_k^k$, we are extracting a common subsequence such that

$$\sup_{t\in[0,T]} |u_j^{N_k}(t) - u_j(t)| \to 0$$

for all $j \geq 0$, as $k \to \infty$.

Last it is straightforward to check that the limit obtained via Ascoli-Arzelà theorem, $u(t)$, is a solution for system, using the equation in the integral form. □

We will study deeply this model in the third chapter, so we skip for the moment other results about this.

1.4.3 The GOY model

Despite they are out of the aim of this thesis, we mention also the existence of models where doesn't holds the condition on the intensities to interact only with closest neighbours. The model first proposed by Gledzer [19] is the following

$$\frac{d}{dt}u_n = A_n u_{n+1}u_{n+2} + B_n u_{n-1}u_{n+1} + C_n u_{n-2}u_{n-1} - \nu u_n + f_n \qquad (1.18)$$

where $u_{-1} = u_0 = 0$. The coefficients A_n, B_n, C_n can be chosen in way to let

$$E = \sum_n \frac{u_n^2}{2},\ Z = \sum_n \frac{k_n^2 u_n^2}{2}$$

to be invariants, where E and Z correspond to energy and enstrophy. After Gledzer, the model was studied by Okhitani and Yamada, so from this takes the name Gledzer-Okhitani-Yamada model, the GOY model.

Looking at this model for $\nu = f = 0$ we can find what condition we have to put on coefficients A_n, B_n, C_n to let the energy to be invariant and they are

$$A_n = k_n\tilde{a},\ B_n = k_n\tilde{b},\ C_n = k_n\tilde{c},\ k_n\tilde{a} + k_{n+1}\tilde{b} + k_{n+2}\tilde{c} = 0.$$

Considering also $k_n = k_0\lambda^n$ with $\lambda > 1$ we get from the last equation

$$k_n(\tilde{a} + \lambda\tilde{b} + \lambda^2\tilde{c}) = 0,$$

and also putting $a = \tilde{a}$, $b = \lambda\tilde{b}$, $c = \lambda^2\tilde{c}$ we get

$$a + b + c = 0,$$

that represents the first version of the GOY model. Later the it was asked to the GOY model to have complex numbers for intensities, so, if we put * for complex conjugate and b is a free parameter, the final form of GOY is

$$\frac{d}{dt}u_n = i[k_n u_{n+1}u_{n+2} - bk_{n-1}u_{n-1}u_{n+1} + (b-1)k_{n-2}u_{n-2}u_{n-1}]^* - \nu_n u_n + f_n.$$

Later it was noted that for $b < 1$ the model represents more a 3D turbulence, for $b > 1$ a 2D one. For more results on this model we refer to [5] and therein.

1.4.4 The SABRA model

Like the GOY model, the SABRA shell model let the intensities to interact with more than the closest neighbours. Supposing complex intensities, the model is

$$\frac{d}{dt}u_n = i[u_n u^*_{n+1}u_{n+2} - bk_{n-1}u^*_{n+1}u_{n+1} + (1-b)k_{n-2}u_{n-2}u_{n-1}] - \nu_n u_n + f_n. \tag{1.19}$$

The force f_n is acting only to small wave numbers, and we have $u_{-1} = u_0 = 0$. If we consider k_n acting like Fibonacci sequence, i.e.

$$k_n = k_{n-1} + k_{n-2},$$

we have shell spacing of the golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$. Differently, if we consider $k_n = \varphi^n$ we continue having the shell spacing as a free parameter $k_n = \lambda^n$ like in the other models.

Chapter 2

Properties of the classical dyadic model

Abstract

In this chapter we show some results on the inviscid unforced dyadic shell model, with a particular focus on the existence results. The model is very simple compared to the ones studied later, and represents the first building block to the modern theory of shell models. After exposing some properties of a formal solution of the system, we will see that the existence for finite energy initial conditions follows directly by an approximation argument via Ascoli-Arzelà theorem, as we have done for the mixed shell model in the previous chapter. The uniqueness of solution is not trivial, even for finite energy initial condition: we dedicate one section to build more than one solution for the same initial condition. Still we can have a uniqueness result for a certain class of solution, in the last part of the chapter we prove that for positive finite energy initial conditions the related solutions are unique. The results shown in this chapter are taken from the works [6], [7], [8], [11], [12], [13], [18].

2.1 Weak and strong solutions

For dyadic shell model we consider an inviscid and unforced shell model with the Novikov-like type of non-linearity. Consider the following

$$\begin{cases} \frac{d}{dt}X_n = k_{n-1}X_{n-1}^2 - k_nX_nX_{n+1} \\ X_n(0) = x_n \end{cases} \tag{2.1}$$

where $k_n = 2^{\beta n}$ with $\beta > 1$, $X = (X_n)_{n\in\mathbb{N}}$ is a sequence of real functions $X_n = X_n(t)$ with $X_0(t) = 0$ and $x = (x_n)_{n\in\mathbb{N}}$ is the initial condition.

Definition 2.1.1. *A local solution on* $[0,T]$ *is a sequence* $X = (X_n)_{n\in\mathbb{N}}$ *of differentiable functions on* $[0,T]$ *satisfying (2.1).*

Definition 2.1.2. *A weak solution is a sequence* $X = (X_n)_{n\in\mathbb{N}}$ *of differentiable functions con* $[0,\infty)$ *satisfying (2.1).*

Definition 2.1.3. *We define the space H^s as the Hilbert space of sequence $h \in \mathbb{R}^{\mathbb{N}}$ such that the following norm is finite*

$$\|h\|_{H^s} \stackrel{def}{=} (\sum_{n\geq 1}(k_n^s h_n)^2)^{\frac{1}{2}}.$$

With this notation we set $H = H^0 = \ell^2$.

Definition 2.1.4. *A finite energy solution is a weak solution such that $X(t) \in H$ for all $t \geq 0$.*

Definition 2.1.5. *A Leray-Hopf solution is a finite energy solution such that $\|X(t)\|_2$ is a non increasing function of t.*

The following one is a first big result for weak solutions, once a solution become positive it remains positive.

Proposition 2.1.6. *Let X be a weak solution of (2.1), let $n \geq 1$ and $t_0 \geq$, so:*

1. *If $X_n(t_0) > 0$ then $X_n(t) > 0$ for all $t \geq t_0$.*
2. *If $X_n(t_0) \geq 0$ then $X_n(t) \geq 0$ for all $t \geq t_0$.*
3. *If $X_1(t_0) = \ldots = X_n(t_0) = 0$ then $X_1(t) = \ldots = X_n(t)$ for all $t \geq t_0$.*

Proof. By the variation of constant formula, for all $n \geq 1$ and all $0 \leq t_0 \leq t$,

$$X_n(t) = X_n(t_0)e^{-\int_{t_0}^t k_n X_{n+1}(s)ds} + \int_{t_0}^t k_{n-1}X_{n-1}^2(s)e^{-\int_s^t k_n X_{n+1}(\tau)d\tau}ds.$$

So the first two statement are proved since $X_{n-1}^2 \geq 0$. Also if $X_n(t_0) = 0$ and $X_{n-1} = 0$ on $[t_0, t]$ we get $X_n = 0$ on $[t_0, t]$, so the third one follows by induction. $\square$

2.1.1 Properties of weak solutions

Proposition 2.1.7. *(Time Change)*
Let X be a weak solution of (2.1) with initial condition x. Let $a > 0$, let

$$Y(t) = aX(at).$$

Then Y is a weak solution for (2.1) with initial condition ax.

Proof. Computing the derivative of $Y(t)$ we have that if $Y(t)$ is a weak solution it has to solve

$$\frac{d}{dt}Y(t) = aX_n(at) = aX_n(0) + a^2\int_0^t k_{n-1}X_{n-1}^2(as)ds - a^2\int_0^t k_n X_n(as)X_{n+1}(as)ds$$

so we get to verify

$$X_n(at) = X_n(0) + a\int_0^t k_{n-1}X_{n-1}^2(as)ds - a\int_0^t k_n X_n(as)X_{n+1}(as)ds.$$

On the other hand we know it is true

$$X_n(t) = X_n(0) + \int_0^t k_{n-1} X_{n-1}^2(s)ds - \int_0^t k_n X_n(s) X_{n+1}(s) ds$$

and from this we can get the equality above with the transformation $s = au$, $ds = adu$.

$\square$

Proposition 2.1.8. *(Time Inversion)*
Let X be a weak solution of (2.1) with initial condition x. Let $T > 0$, then

$$Y(t) = -X(T - t)$$

is a local solution on $[0, T]$ with initial condition $-X(T)$.

Proof. It follows by the straightforward computation. $\square$

Proposition 2.1.9. *(Forward Shift)*
Let X be a weak solution of (2.1) with initial condition x. Let m a be positive integer and let for all $n \geq 1$

$$Y_n(t) = \begin{cases} X_{n-m}(k_m t) & n > m \\ 0 & n \leq m \end{cases}, \; Z_n(t) = \begin{cases} k_m^{-1} X_{n-m}(t) & n > m \\ 0 & n \leq m \end{cases} \tag{2.2}$$

The $Y = (Y_n)_{n\in\mathbb{N}}$ and $Z = (Z_n)_{n\in\mathbb{N}}$ are weak solutions with shifted and scaled initial conditions

$$y_n = \begin{cases} x_{n-m} & n > m \\ 0 & n \leq m \end{cases}, \; z_n = \begin{cases} k_m^{-1} x_{n-m} & n > m \\ 0 & n \leq m \end{cases}. \tag{2.3}$$

Proof. For Y it is enough to differentiate $X_{n-m}(k_m t)$, remarking that $k_m k_{n-m} = k_n$ and that $Y_m(t) = 0 = X_0(k_m t)$. Then we apply the time change for the result on Z.

$\square$

Proposition 2.1.10. *(Backward Shift)*
Let $m \geq 1$ and let X be a weak solution of (2.1) with initial condition x such that $x_1 = \ldots = x_m = 0$ and $x_{m+1} \neq 0$. Let also

$$Y_n(t) = X_{n+m}(k_m^{-1} t), \; Z_n(t) = k_m X_{n+m}(t). \tag{2.4}$$

Then $Y = (Y_n)_{n\in\mathbb{N}}$ and $Z = (Z_n)_{n\in\mathbb{N}}$ are weak solutions with shifted and scaled initial conditions

$$y_n = x_{n+m}, \; z_n = k_m x_{n+m}.$$

Proof. For $n \geq 2$

$$\begin{aligned} Y_n'(t) = k_m^{-1} X_{n+m}'(k_m^{-1} t) = k_m^{-1} k_{n+m-1} X_{n+m-1}^2(k_m^{-1} t) \\ - k_m^{-1} k_{n+m} X_{n+m}(k_m^{-1} t) X_{n+m+1}(k_m^{-1} t) = \\ = k_{n-1} X_{n+m-1}^2(k_m^{-1} t) - k_n X_{n+m}(k_m^{-1} t) X_{n+m+1}(k_m^{-1} t) = \end{aligned}$$

$$= k_{n-1}Y_{n-1}^2(t) - k_n Y_n(t) Y_{n+1}(t).$$

Hence, by Proposition (2.1.6) we have $X_m(t) = 0$ for all $t \geq 0$, so if we extend the definition of Y_n to $n = 0$ we get

$$Y_0(t) = X_m(k_m^{-1}t) = 0,$$

so the equation above holds also for $n = 1$. □

Proposition 2.1.11. *Let be $m \geq 1$ and let be X a weak solution of the inviscid unforced dyadic shell model with initial condition x such that $x_m \neq 0$ and $x_n = 0$ for all $n < m$. Let be for all $n \geq 1$*

$$Y_n(t) = (-1)^{\delta_{m,n}} X_n(t).$$

Then $Y = (Y_n)_{n \geq 1}$ is a weak solution with initial condition $y_n = (-1)^{\delta_{m,n}} x_n$.

Proof. It follows by straightforward computation. □

2.1.2 Existence of solutions

Definition 2.1.12. *We define the energy of a finite size block the following*

$$E_n(t) = \sum_{i \leq n} X_i^2(t).$$

For the inviscid unforced dyadic model we have the following existence result.

Theorem 2.1.13. *Given $x \in H$ there exists at least one Leray-Hopf solution. Given $x \in \mathbb{R}^{\mathbb{N}}$ with infinetely many non negative components there exists at least one weak solution.*

Proof. From Theorem (1.4.2) we have the existence of a solution X for $x \in H$, we have to prove that that solution is Leray-Hopf, i.e. $\|X(t)\|$ is a non increasing function of t. For all $n \geq 1$, for all k such that $N_k \geq n$ and all $t \geq 0$

$$E_n^{N_k}(t) \leq E_{N_k}^{N_k}(0) \leq \|x\|^2.$$

Hence for $k \to \infty$ we have $E_n(t) \leq \|x\|^2$. Then $\|X(t)\| \leq \|x\|$. Now consider $s \in [0, T]$, let $n \geq 1$ and k such that $N_k \geq n$. If $E_{N_k}(0) \leq E_{N_k}(s)$, then

$$E_n^{N_k}(t) \leq E_{N_k}(0) \leq E_{N_k}(s) \leq \|X(s)\|^2$$

so that if $E_{N_k}(0) \leq E_{N_k}(s)$ for infinetely-many k's, by taking the limit on the subsequence k_m and then in n to get $\|X(t) \leq X(s)\|$.
On the other hand, suppose $E_{N_k}(0) > E_{N_k}(s)$ for $k \geq k_0$. If $E_{N_k}(0) > E_{N_k}(s)$ then the derivative must have been negative for some $t_0 \in [0, s]$. Since

$$E_n' = -2k_n X_n^2 X_{n+1}$$

for Proposition (2.1.6) $X_{N_k+1}(t_0) > 0$ and $X_{N_k+1}(u) > 0$ for all $u \in [s, t]$, in particular $E_{N_k}(t) \leq E_{N_k}(s)$. Since it has to be true for all k we get, taking the limit, $\|X(t)\| \leq \|X(s)\|$.

□

2.2 The lack of uniqueness

Respect to existence, uniqueness is an issue for the inviscid unforced dyadic shell model, as it doesn't holds for all the solution with initial condition in H. From now on we introduce the tools to prove this.

Definition 2.2.1. *A stationary solution is a sequence of real numbers $x = (x_n)_{n\geq 1}$ such that $x \in H$ and, with a given $f \in \mathbb{R}$*

$$\begin{cases} 0 = f - k_1 x_1 x_2 \\ 0 = k_{n-1}x_{n-1}^2 - k_n x_n x_{n+1}, \quad n \geq 2 \end{cases} \tag{2.5}$$

Definition 2.2.2. *A self-similar solution is a finite energy solution X such that there exists a differentiable function φ and a sequence of real numbers $y = (y_n)_{n\geq 1}$ such that*

$$X_n(t) = y_n \varphi(t)$$

for all $n \geq 1$ and $t \geq 0$.

The following one result consists of a big lemma necessary to prove the lack of uniqueness.

Proposition 2.2.3. *There exists a bounded non-decreasing sequence of positive numbers $(r_n)_{n\geq 1}$ such that the following are the only self-similar solutions such that $X_n > 0$ for all $n \geq 1$,*

$$X_n(t) = \frac{k_n^{-\frac{1}{3}} r_n}{t - t_0}$$

where $t_0 < 0$ can be chosen freely. The set of self-similar solution is given by the above ones and their forward shifts and their modifications done with Proposition 2.1.11, i.e.

$$X_n(t) = \begin{cases} 0 & n = 0, 1, \dots, m \\ \pm k_1^{-\frac{1}{3}} r_1 \frac{k_m^{-1}}{t - t_0} & n = m+1 \\ k_{n-m}^{-\frac{1}{3}} r_{n-m} \frac{k_m^{-1}}{t - t_0} & n \geq m+2 \end{cases} \tag{2.6}$$

where $m \geq 1$ and $t_0 < 0$ can be chosen freely.

Proof. With the following substitution $X_n(t) = y_n\varphi(t)$ in the system we get

$$y_n \varphi'(t) = (k_{n-1} y_{n-1}^2 - k_n y_n y_{n+1})\varphi^2(t),$$

with $n \geq 1$ and $t \geq 0$, so we get

$$\varphi'(t) = \lambda \varphi^2(t)$$

and for $n \geq 1$

$$\lambda y_n = k_{n-1} y_{n-1}^2 - k_n y_n y_{n+1}$$

where as usual $y_0 = 0$ and $\lambda \neq 0$ is a free parameter.
The differential equation has the general solution

$$\varphi(t) = \lambda^{-1}(t_0 - t)^{-1}.$$

Since we are looking for a global solution on $[0, \infty)$ we put $t_0 < \infty$ and we choose $\lambda = -1$, hence

$$\varphi(t) = \frac{1}{t - t_0}$$

that is greater than 0 for $t \geq 0$. Rewriting the recurrence

$$-y_n = k_{n-1}y_{n-1}^2 - k_n y_n y_{n+1}$$

for $n \geq 1$. This is very close to the definition of stationary solution, hence we make the following change of variables, setting $r_n = k_n^{\frac{1}{3}} y_n$ we get

$$r_n r_{n+1} = r_{n-1}^2 + s_n r_n \tag{2.7}$$

for $n \geq 1$, where

$$s_n = k_{n-1}^{-\frac{1}{3}} k_n^{-\frac{1}{3}} = 2^{-\frac{2}{3}\beta n + \frac{1}{3}\beta}$$

and $r_0 = 0$.
Thanks to Forward shift, Backward shift and Proposition 2.1.11 we can suppose without loss of generality that $r_1 > 0$, so $r_n > 0$. Hence we can make one more change of variables $z_n = r_n^2$ to get, with $n \geq 1$,

$$\sqrt{z_n}\sqrt{z_{n+1}} = z_{n+1} + s_n\sqrt{z_n}. \tag{2.8}$$

To complete the proof we need lemmas 2.2.4 and 2.2.6 below. □

Lemma 2.2.4. *Let $(s_n)_{n\geq 1}$ be any summable decreasing sequence of positive numbers. Then there exists a bounded non-decreasing sequence of negative numbers $(z_n)_{n\geq 0}$ satisfying 2.8, such that $z_0 = 0$ and $z_1 > 0$.*

Proof. The sketch of the proof is the following, for any $N \geq 1$ we want to construct a solution $z_n^{(N)}$ of 2.8 such that $z_N^{(N)} = z_{N+1}^{(N)}$ to let the convergence be to a non oscillating solution as $N \to \infty$. The construction is possible once $z_N^{(N)} = z_{N+1}^{(N)} = x$ is large enough. So to get also $z_0^{(N)} = 0$ we consider $z_n^{(N)}(x)$ as a function of x, and we find that there is a unique value of x, $a_0^{(N)}$, for which $z_0^{(N)} = 0$ and such that $z_n^{(N)}$ is bounded by $a_0^{(N)}$ for $n = 1, 2, \ldots, N-1$. Hence we find that $a_0^{(N)}$ is uniformly bounded and that the limit for $N \to \infty$ of $z_n^{(N)}(a_0^{(N)})$ is the solution of 2.8.

Step 1: We define for $N \geq 1$ the real numbers

$$0 = a_N < a_{N-1} < \ldots < a_0$$

and functions

$$z_{N-1} \geq z_{N-2} \geq \ldots \geq z_0$$

such that $z_n : [a_{n+1}, \infty) \to \mathbb{R}$ are continuous, equal to 0 in a_{n+1} and a_n, negative on (a_{n+1}, a_n), positive and strictly increasing on (a_n, ∞) with

$\lim_{x\to\infty} z_n(x) = \infty$.
Together with defining $z_N(x) = z_{N+1}(x) = x$ it follows that the functions satisfy 2.8 pointwise on $[a_n, \infty]$ for $n = 1, 2, \ldots, N$.
We prove this inductively starting from the top, so let $n = N, N-1, \ldots, 1$ and

$$z_{n-1}(x) = \sqrt{z_n(x)}(\sqrt{z_{n+1}(x)} - s_n) \ , \ x \geq a_n \tag{2.9}$$

By inductive hypothesis z_n and z_{n+1} are both continuous, non-negative and strictly increasing on $[a_n, \infty)$ (true also for z_N and z_{N+1}). Hence z_{n-1} is well defined and continuous, and $z_{n-1} = x$ if and only if $z_n(x) = 0$ or $z_{n+1}(x) - s_n^2$. Now let

$$a_{n-1} = z_{n+1}^{-1}(s_n^2)$$

which is well defined and larger than a_{n+1}, since z_{n+1} restricted to $[a_{n+1}, \infty)$ is a bijection onto $[0, \infty)$. We can also prove that $a_{n-1} > a_n$. This is straightforward for $n = N$ and follows from inductive hypothesis for $n \leq N-1$ since

$$a_{n-1} = z_{n+1}^{-1}(s_n^2) \geq z_{n+2}^{-1}(s_n)^2 > z_{n+2}^{-1}(s_{n+1}^2) = a_n.$$

Hence $z_{n-1}(x) = 0$ if and only if $x = a_n$ or $x = a_{n-1}$. From (2.9) we have also that z_{n-1} is negative on (a_n, a_{n-1}) and positive on (a_{n-1}, ∞), so z_{n-1} is on the latter interval strictly increasing with infinite limit as $x \to \infty$.
Moreover $z_{n-1} \leq z_n$ is straightforward for $n = N$ and follows from inductive hypothesis for $n \leq N-1$ since

$$z_{n-1}(x) = \sqrt{z_n(x)}(\sqrt{z_{n+1}(x)} - s_n) \leq \sqrt{z_{n+1}(x)}(\sqrt{z_{n+2}(x)} - s_{n+1}) = z_n(x).$$

Step 2: We prove that a_0 is bounded by a quantity not depending on N:
since $z_n \geq z_{n-1}$ by (2.9) we have

$$z_n(x) \geq \sqrt{z_n(x)}(\sqrt{z_{n+1}(x)} - s_n)$$

so, for $n = 1, 2, \ldots, N$

$$\sqrt{z_{n+1}(x)} - \sqrt{z_n(x)} \leq s_n$$

hence

$$\sqrt{z_{N+1}(x)} - \sqrt{z_2(x)} \leq \sum_{i=2}^{N} s_i.$$

So since by $a_{n-1} = z_{n+1}^{-1}(s_n^2)$ we have that the definition of a_0

$$s_1 = \sqrt{z_2(a_0)} \geq \sqrt{a_0} - \sum_{i=2}^{N} s_i.$$

Hence let $S = (\sum_{i=1}^{\infty} s_i)^2$, we ha have proved that $a_0 \leq S$.

Step 3: Now we let N change. We prove that there exists an increasing sequence $(N_k)_{k\geq 1}$ such that taking the limit as $k \to \infty$ in $z_n^{(N_k)}(a_0^{(N_k)})$ we get the sequence required.
For $n = 0$ it is trivial by definition.
For all $N \geq 1$ and all $n \leq N+1$ we have

$$z_n^{(N)}(a_0^{(N)}) \leq a_0^{(N)} \leq S.$$

Hence by compactness, given n and any subsequence N_k, there exists a subsequence N_{k_i} such that $z_n^{(N_{k_i})}(a_0^{(N_{k_i})})$ converges. Hence with a diagonal argument we select a subsequence such that has convergence for all $n \geq 1$. Denoting with $z_n^{(\infty)}$ that limit, by continuity we have that it is not decreasing and it satisfies 2.8, it is bounded and $z_o^{\infty} = 0$ by construction. □

Lemma 2.2.5. *Let x, y be positive numbers such that $\frac{y}{x} \leq R$ for some positive R. Let $\alpha > 1$. Then*

$$\alpha x + y \geq \frac{\alpha + R}{1+R}(x+y).$$

Proof. The function $u \to \frac{\alpha+u}{1+u}$ is monotone decreasing on $u \geq 0$, so

$$\frac{\alpha + \frac{y}{x}}{1+\frac{y}{x}} \leq \frac{\alpha+R}{1+R}.$$

□

Lemma 2.2.6. *Let $(s_n)_{n\geq 1}$ be any summable decreasing sequence of positive numbers. Then there exists a unique sequence of non-negative numbers $z = (z_n)_{n\geq 0}$ satisfying 2.8, such that $z_0 = 0$, $z_1 > 0$ and such that the sequence $y = (y_n)_{n\geq 1}$, defined by*

$$y_n = k_n^{-\frac{1}{3}} \sqrt{z_n}$$

is in H.

Proof. The existence of such a z follows by Lemma 2.2.4, since z bounded implies $y \in H$. For the uniqueness, let z be the sequence of Lemma 2.2.4 and let $\tilde{z}$ be another sequence defined as follows.
We rewrite (2.8) for $n \geq 1$ as

$$\sqrt{z_{n+1}} = \frac{z_{n-1}}{\sqrt{z_n}} + s_n. \tag{2.10}$$

By induction on n, if $z_1 = \tilde{z}_1$ then the sequence would be identical. If $z_1 > \tilde{z}_1$, then $z_n > \tilde{z}_n$ for odd n and $z_n < \tilde{z}_n$ for n even and greater than 2.
So we need to show that the distance between $\tilde{z}_n$ and z_n becomes larger as

$n \to \infty$.
We define $\alpha_n^2 = \frac{\tilde{z}_n}{z_n}$ and we split the proof in three cases, one for each asymptotic behaviour of α_n.

1. Case ($\alpha_n \to 1$): Let $\epsilon_n = |\sqrt{\tilde{z}_n} - \sqrt{z_n}| = |\alpha_n - 1|\sqrt{z_n}$, so that $\epsilon_n \to 0$ since z is bounded. From (2.8) we have

$$\begin{aligned}\tilde{z}_{n-1} = \sqrt{\tilde{z}_n}(\sqrt{z_{n+1}} - s_n) = (\sqrt{z_n} \pm \epsilon_n)(\sqrt{z_{n+1}} \mp \epsilon_{n+1} - s_n) = \\ = z_{n-1} \pm \epsilon_n\sqrt{z_{n+1}} \mp \epsilon_{n+1}\sqrt{z_n} - \epsilon_n\epsilon_{n+1} \mp \epsilon_n s_n.\end{aligned}$$

   Let $\bar{\epsilon}_n = \max\{\epsilon_n, \epsilon_{n+1}\}$, so

$$\epsilon_{n-1}(\sqrt{\tilde{z}_{n-1}} + \sqrt{z_{n-1}}) = |\tilde{z}_{n-1} - z_{n-1}| \leq \bar{\epsilon}_n|\sqrt{z_{n+1}} - \sqrt{z_n}| + \bar{\epsilon}_n^2 + \bar{\epsilon}_n|s_n|.$$

   For any $\delta > 0$, let m be an integer large enough so that for all $n \geq m$ we have $s_n < \delta$, $\bar{\epsilon}_n < \delta$, $|\sqrt{z_{n+1}} - \sqrt{z}_n| < \delta$ and $\sqrt{\tilde{z}_{n-1}} + \sqrt{z_{n-1}} > z_\infty = lim_n z_n$, so

$$\epsilon_{n-1}z_\infty < 3\bar{\epsilon}_n\delta$$

   meaning that for any $n \geq m$

$$\max\{\epsilon_n, \epsilon_{n+1}\} > \frac{z_\infty}{3\delta}\epsilon_{n-1}.$$

   If we take $\delta < \frac{z_\infty}{3}$ by induction we find a sequence $(n_k)_{k\geq 1}$ defined, for $k \geq 1$ by $n_1 = m - 1$ and

$$n_{k+1} = \begin{cases} n_k + 1 & \text{if } \epsilon_{n_k+1} \geq \epsilon_{n_k+2} \\ n_k + 2 & \text{if } \epsilon_{n_k+1} > \epsilon_{n_k+2} \end{cases}$$

   such that ε_{n_k} is unbounded, and that gives a contradiction.

2. Case ($\limsup_n \alpha_n > 1$) : Let $\limsup_n \alpha_n = 1 + 3\epsilon$ and choose δ positive, $\delta \leq \epsilon\frac{z_1}{\sqrt{z_\infty}}$. Remark that $\alpha_n = \sqrt{\frac{\tilde{z}_n}{z_n}}$ is alternatively greater and smaller than 1. Let m be a positive integer such that the parity of m gives $\alpha_m > 1$, $s_n < \delta$ for all $n \geq m$ and $\alpha_{m+2k} > 1 + 2\epsilon$ for all $k \geq 0$. So by 2.10

$$\sqrt{\tilde{z}_{n+1}} = \frac{\tilde{z}_{n-1}}{\sqrt{\tilde{z}_n}} + s_n = \frac{\alpha_{n-1}^2}{\alpha_n}\frac{z_{n-1}}{\sqrt{z_n}} + s_n. \tag{2.11}$$

   Hence we consider this equation for $n = m + 2k + 1$, with $k \geq 0$, so that

$$\frac{\alpha_{n-1}^2}{\alpha_n} = \frac{\alpha_{m+2k}^2}{\alpha_{m+2k+1}} > 1.$$

   Now we note that z_{n-1} is bounded below by z_1 while z_n is bounded above by z_∞ and $s_n \leq \delta$, so

$$\frac{s_n\sqrt{z_n}}{z_{n-1}} \leq \frac{\delta\sqrt{z_\infty}}{z_1} \leq \epsilon.$$

Hence by Lemma 2.2.5

$$\alpha_{n+1}\sqrt{z_{n+1}} = \sqrt{\tilde{z}_{n+1}} \geq \frac{\frac{\alpha_{n-1}^2}{\alpha_n} + \epsilon}{1+\epsilon}\sqrt{z_n+1}$$

we get

$$\alpha_{n-1} \geq \frac{\alpha_{n-1}^2}{1+\epsilon} \text{ or } \frac{\alpha_{n+1}}{1+\epsilon} \geq (\frac{\alpha_{n-1}}{1+\epsilon})^2.$$

Since $\alpha_{n-1} = \alpha_{m+2k} \geq 1+2\epsilon > 1+\epsilon$, inductively we have that α_{m+2k+1} is super-exponential in k and so are the corresponding subsequences of $\tilde{z}$ and $\tilde{y}$, meaning that $\tilde{y}$ cannot be in H, hence it is a contradiction.

3. Case ($\limsup_n \alpha_n = 1$ and $\liminf_n \alpha_n < 1$): We consider again equation 2.11 with n of parity such that $\alpha_{n-1} > 1$ and $\alpha_n < 1$. As before $z_{n-1} \geq z_1$ and $z_n \leq S$. Bounding s_n by the maximum s_1 of the sequence we get

$$\frac{s_n\sqrt{z_n}}{z_{n-1}} \leq \frac{s_1\sqrt{S}}{z_1} = R$$

so applying again Lemma 2.2.5 we get for n odd

$$\alpha_{n+1}\sqrt{z_{n+1}} = \sqrt{\tilde{z}_{n+1}} \geq \frac{\frac{\alpha_{n-1}^2}{\alpha_n} + R}{1+R}\sqrt{z_{n+1}}$$

hence since $\liminf_n \alpha_n < 1$ there exists $\epsilon > 0$ such that

$$\alpha_{n+1} \geq \frac{\frac{\alpha_{n-1}^2}{\alpha_n} + R}{1+R} \geq \frac{\alpha_n^{-1} + R}{1+R} \geq \frac{1+\epsilon+R}{1+R}$$

for all n odd large enough, giving $\limsup_n \alpha_n > 1$ which is a contradiction.

□

Theorem 2.2.7. *For some initial condition in H there exists infinetely many finite energy solutions.*

Proof. By Proposition 2.2.3 there exists a self-similar solution X whose total energy is strictly decreasing. So let $T > 0$, $Y(t) = -X(T-t)$ is a local solution on $[0,T]$ by Time inversion. For any time $s \in [0,T]$, we consider the solution $Y^{(s)}$ obtained ny attaching Y on $[0,s]$ to a Leray-Hopf solution on $[s,\infty)$ given by Theorem 2.1.13 with initial condition $Y(s) = -X(T-s) \in H$. The energy of this solution is strictly increasing on $[0,s]$ and then is non-increasing on $[s,\infty)$, so to different value of s correspond finite energy solutions which are really different, all with the same negative condition $-X(T)$. □

2.3 Uniqueness for positive solutions

Theorem 2.3.1. *Let $x \in H$, with $x_n \geq 0$ for all n. There exists a unique weak solution with initial condition x.*

We spend the all section to introduce the tools necessary to prove this result, and at the end we put all the stuff together for the final proof of the uniqueness for positive finite energy initial conditions.
The first step is to prove uniqueness for the solutions that have a particular property.

Proposition 2.3.2. *Let $x \in \mathbb{R}^{\mathbb{N}}$ with all non-negative components. Suppose that one can prove that for any weak solution X with initial condition x one has*

$$\lim_{n\to\infty} 2^{-n} k_n \int_0^t X_n^3(s)ds = 0,$$

for $t \geq 0$. Then there exists a unique weak solution with initial condition x.

Proof. Let X and Y be two weak solutions with initial condition x. For all $n \geq 1$ we define Z_n and W_n as $Z_n = Y_n - X_n$ and $W_n = Y_n + X_n$. So

$$\begin{aligned} Z_n W_{n+1} + W_n Z_{n+1} &= (Y_n - X_n)(Y_{n+1} + X_{n+1}) + (Y_n + X_n)(Y_{n+1} - X_{n+1}) \\ &= 2Y_n Y_{n+1} - 2X_n X_{n+1}. \end{aligned}$$

So since $Y_n(t) = X_n(t)$ if and only if $Z_n(t) = 0$, it suffices to prove that $Z_n(t) = 0$ for all $t \geq 0$ and $n \geq 1$. Thus

$$Z_n' = Y_n' - X_n' = k_{n-1} Z_{n-1} W_{n-1} - \frac{1}{2} k_n (Z_n W_{n+1} + W_n Z_{n+1}).$$

Hence Z is a weak solution with initial condition 0 for $t \geq 0$ of

$$Z_n' = k_{n-1} Z_{n-1} W_{n-1} - \frac{1}{2} k_n Z_{n+1} W_n - \frac{1}{2} k_n Z_n W_{n+1}.$$

Now, for all $N \geq 1$ let

$$\psi_N(t) = \sum_{n=1}^{N} \frac{Z_n^2}{2^n}.$$

The functions $\psi_N(t)$ are non-negative, non-decreasing in N and such that $\psi_N(0) = 0$. We want to prove that for all $t > 0$, $\lim_{N\to\infty} \psi_N(t) = 0$, to get $\psi_N(t) = 0$ and $Z_n(t) = 0$.

$$\psi_N' = \sum_{n=1}^{N} 2^{-n+1} Z_n Z_n' = -2^{-N} k_N Z_N Z_{N+1} W_N - \sum_{n=1}^{N} 2^{-n} k_n Z_n^2 W_{n+1}.$$

Note that $W_n(t)$ is non-negative for all $n \geq 1$ and all $t \geq 0$, so

$$\begin{aligned} \psi_N' &\leq -2^{-N} k_N Z_N Z_{N+1} W_N = -2^{-N} k_N (Y_N^2 - X_N^2)(Y_{N+1} - X_{N+1}) \\ &\leq 2^{-N} k_N (Y_N^2 X_{N+1} + X_N^2 Y_{N+1}) \leq 2^{-N} k_N (Y_N^3 + X_{N+1}^3 + X_N^3 + Y_{N+1}^3). \end{aligned}$$

So, since $\psi_N(0) = 0$, we have

$$\psi_N(t) \le 2^{-N} k_N \int_0^t [Y_N^3(s) + X_{N+1}^3(s) + X_N^3(s) + Y_{N+1}^3(s)] ds.$$

Applying the hypothesis to both weak solutions X and Y we conclude that $\lim_{N\to\infty} \psi_N(t) = 0$ for all $t > 0$.

□

Proposition 2.3.3. *Let X be any weak solution with initial condition $x \in H$ with all non-negative components. Then there exists a constant c depending only on $\|x\|$ and β such that for any positive non-increasing sequence $(b_n)_{n\ge1}$ the following inequality holds*

$$\mathcal{L}\{t \ge 0 : X_n(t) > b_n \text{ for some } n\} \le c \sum_{n\ge1} \frac{1}{k_n b_n^3}, \tag{2.12}$$

where $\mathcal{L}$ stands for Lebesgue measure. The quantity $c = 34k_1\|x\|^2$ satisfies this theorem.

Proof. We can assume without loss of generality that $\sum_n \frac{1}{k_n b_n^3}$ is finite.
Consider $I = \bigcup_n \{t \in (0, +\infty) | X_n(t) > b_n\}$, this set is open so we can approximate it with finite union of intervals. So the proof is left to verify for all $J = \bigcup_{k=1}^m [u_k, v_k) \subset I$ that

$$\mathcal{L}(J) \le c \sum_{n\ge1} \frac{1}{k_n b_n^3}.$$

Now to any $s \in J$ we can associate a component $n_s \ge 1$ and a time $t_s > s$. We will show that there exists a countable set $S \subset J$ such that $\bigcup_{s\in S}[s; t_s)$ is a covering of J of pairwise disjoint intervals, so that we can estimate the measure of J as

$$\mathcal{L}(J) \le \sum_{s\in S}(t_s - s).$$

So we will find that the measure of each interval is bounded by a quantity depending on n_s. Moreover the cardinality of the set of $s \in S$ such that $n_s = n$ will be bounded with a consideration of the energy flow.

Step 1: The aim of the first step is to define n_s and t_s. For all $s \in J$ let

$$n_s = \min\{n \ge 1 : X_n(s) > b_n, X_n(s) \ge X_{n+2}(s)\}.$$

Note that n_s is well defined since the set on which we ask for the minimum is always non-empty, since $s \in I$ gives that there must be some n for which $X_n(s) > b_n$ and if the second quantity is false for that n it holds for the non-increaseness of b_n that

$X_{n+2}(s) > X_n(s) > b_n \ge b_{n+2}$, so the first inequality is true also for $n+2$. Iterating this argument we get by induction that if the set were empty then $X_{n+2k}(s) \ge b_n$ for all $k \ge 0$ and that would mean $\|X(s)\| = \infty$ that is a contradiction.
To define t_s we study the three different behaviours of the solution around

n_s after time s, either component n_s decrease a lot, or component n_s+2 increases a lot, or X_{n_s} stays high and X_{n_s+2} stays low. So for all $s \in J$ we set

$$t_s = \min\{t > s : X_{n_s}(t) < \frac{1}{2}X_{n_s}(s) \text{ or } X_{n_s+2}(t) > 2X_{n_s}(s) \text{ or } t = s+\frac{1}{k_{n_s+1}b_{n_s}}\}. \tag{2.13}$$

Step 2: The aim of the second step is to prove that there exists a countable set $S \subset J$ such that $\bigcup_{s\in S}[s;t_s)$ is a covering of J of pairwise disjoint intervals. Let $F = \{[s,t_s) : s \in J\}$ and let $\mathcal{F}$ be the family of subsets A of F such that the elements of A are pairwise disjoint intervals and the union of the elements of A is some interval $[\min(J), c)$. The set $\mathcal{F}$ is non-empty since if $s_0 = \min(J)$ and $s_{n+1} = t_{s_n}$ for $n \geq 0$ so $\mathcal{F}$ contains at least $\{[s_0, s_1)\}$, $\{[s_0, s_1), [s_1, s_2)\}$ and so on.
Consider now the partial order $\subseteq$ of $\mathcal{F}$, proving that this is a total order we would get that the union of all the elements of $\mathcal{F}$ is an element F' of $\mathcal{F}$ and so it would be the required covering and we will set $S = \{s : [s,t_s) \in F'\}$. So we have to prove that the order is total, suppose by contradiction that $A, B \in \mathcal{F}$ and neither $A \subseteq B$ or $B \subseteq A$, so that $A \setminus B$ and $B \setminus A$ are both non-empty. So let

$$\alpha = \inf \bigcup_{[s,t)\in A\setminus B} [s,t_s) \text{ and } \beta = \inf \bigcup_{[s,t)\in B\setminus A} [s,t_s),$$

without loss of generality we can suppose $\alpha \leq \beta$.
Since $\alpha \in \bigcup_{[s,t_s)\in A}[s,t_s)$ there exists $[s_A, t_{s_A}) \in A$ such that $\alpha \in [s_A, t_{s_A})$, so since α is the infimum $\alpha = s_A$ and $[s_A, t_{s_A}) \in A \setminus B$. Then since $\alpha \in \bigcup_{[s,t_s)\in B}[s,t_s)$ there exists $[s_B, t_{s_B}) \in B$ such that $\alpha \in [s_B, t_{s_B})$. Hence since $[\alpha, t_\alpha) \in A \setminus B$ it holds $\alpha > s_B$ and so $\beta < \alpha$, which is a contradiction.

Step 3: In this third step we go for energy estimates. We claim the following

$$\begin{cases} E_{n_s}(s) - E_{n_s}(t_s) \geq \frac{3}{4}b_{n_s}^2 & \text{for } X_{n_s}(t_s) = \frac{1}{2}X_{n_s}(s); \\ E_{n_s+1}(s) - E_{n_s+1}(t_s) \geq 3b_{n_s}^2 & \text{for } X_{n_s+2}(t_s) = 2X_{n_s}(s); \\ E_{n_s}(s) - E_{n_s}(t_s) \geq \frac{1}{32k_2}b_{n_s}^2 & \text{for } t_s = s + \frac{1}{k_{n_s+1}b_{n_s}}. \end{cases} \tag{2.14}$$

The first one is true since by monotonicity of E_{n_s-1} we have

$$\begin{aligned} E_{n_s}(s) - E_{n_s}(t_s) &= E_{n_s-1}(s) + X_{n_s}^2(s) - E_{n_s-1}(t_s) - X_{n_s}^2(t_s) \\ &\geq X_{n_s}^2(s) - X_{n_s}^2(t_s) = \frac{3}{4}X_{n_s}^2(s) \geq \frac{3}{4}b_{n_s}^2. \end{aligned}$$

The second one follows by monotonicity of E_{n_s+2}

$$E_{n_s+1}(s) - E_{n_s+1}(t_s) = E_{n_s+2}(s) + X_{n_s+2}^2(s) - E_{n_s+2}(t_s) - X_{n_s+2}^2(t_s)$$

$$\geq X_{n_s+2}^2(s) - X_{n_s+2}^2(t_s) = 3X_{n_s}^2(s) \geq 3b_{n_s}^2.$$

So it lasts to prove the third part of (2.14). By definition of $t_s = s + \frac{1}{k_{n_s+1}b_{n_s}}$ then for all $t \in [s, t_s]$ it holds $X_{n_s}(t) \geq \frac{1}{2}X_{n_s}(s)$ and $X_{n_s+2}(t) \leq X_{n_s+2}(s)$, so

$$\begin{aligned} X'_{n_s+1}(t) &= k_{n_s}X_{n_s}^2(t) - k_{n_s+1}X_{n_s+1}(t)X_{n_s+2}(t) \\ &\geq \frac{1}{4}k_{n_s}X_{n_s}^2(s) - 2k_{n_s+1}X_{n_s}(s)X_{n_s+1}(t). \end{aligned}$$

Hence by $X_{n_s+1} \geq 0$ and Gronwall inequality

$$X_{n_s+1}(t) \geq \frac{1}{4}k_{n_s}X_{n_s}^2(s)\int_s^t e^{-2k_{n_s+1}X_{n_s}(s)(t-s)}ds = \frac{X_{n_s}(s)}{8k_1}(1-e^{-2k_{n_s+1}X_{n_s}(s)(t-s)}),$$

so

$$\begin{aligned} E_{n_s}(s) - E_{n_s}(t_s) &= \int_s^{t_s} 2k_{n_s}X_{n_s}^2(t)X_{n_s+1}(t)dt \\ &\geq \frac{k_{n_s-1}X_{n_s}^2(s)}{16}\int_s^{t_s}(1-e^{-2k_{n_s+1}X_{n_s}(s)(t-s)})dt \\ &\geq \frac{k_{n_s-1}X_{n_s}^2(s)}{16}[(t_s - s) - \int_s^{\infty}(e^{-2k_{n_s+1}X_{n_s}(s)(t-s)})dt] \\ &= \frac{k_{n_s-1}X_{n_s}^2(s)}{16}[\frac{1}{k_{n_s+1}b_{n_s}} - \frac{1}{2k_{n_s+1}X_{n_s}(s)}] \\ &\geq \frac{k_{n_s-1}X_{n_s}^2(s)}{16}\frac{1}{2k_{n_s+1}X_{n_s}(s)} = \frac{X_{n_s}^2(s)}{32k_2} \geq \frac{b_{n_s}^2}{32k_2}. \end{aligned}$$

Step 4: In this final step we estimate $\mathcal{L}(J)$. Combining monotonicity of E_n and first inequality of (2.14) we get

$$\begin{aligned} \|x\|^2 \geq E_n(0) \geq \sum_{s\in S}(E_n(s)-E_n(t_s)) &\geq \sum_{s\in S, n_s=n, X_n(t_s)=\frac{1}{2}X_n(s)}(E_n(s)-E_n(t_s)) \\ &\geq \frac{3}{4}b_n^2\#\{s \in S : n_s = n, X_n(t_s) = \frac{1}{2}X_n(t_s)\}. \end{aligned}$$

For the other two inequality of (2.14) we get

$$\begin{aligned} \|x\|^2 \geq E_{n+1}(0) \geq \sum_{s\in S}(E_{n+1}(s)-E_{n+1}(t_s)) &\geq \sum_{s\in S, n_s=n, X_{n+2}(t_s)=2X_n(s)}(E_{n+1}(s)-E_{n+1}(t_s)) \\ &\geq 3b_n^2\#\{s \in S : n_s = n, X_{n+2}(t_s) = 2X_n(t_s)\}, \end{aligned}$$

and

$$\|x\|^2 \geq E_n(0) \geq \sum_{s\in S}(E_n(s)-E_n(t_s)) \geq \sum_{s\in S, n_s=n, t_s=s+\frac{1}{k_{n+1}b_n}} (E_n(s)-E_n(t_s))$$

$$\geq \frac{1}{32k_2} b_n^2 \#\{s \in S : n_s = n, t_s = s + \frac{1}{k_{n+1}b_n}\}.$$

Thus

$$\#\{s \in S : n_s = n\} \leq \frac{\|x\|^2}{b_n^2}\left(\frac{4}{3} + \frac{1}{3} + 32k_2\right) \leq 34k_2 \frac{\|x\|^2}{b_n^2}.$$

So summing the measure of all intervals $[s, t_s)$ leads

$$\mathcal{L}(J) \leq \mathcal{L}\left(\bigcup_{s\in S}[s,t_s)\right) = \sum_{s\in S}(t_s - s) = \sum_{n\geq 1}\sum_{n_s=n}(t_s - s)$$

$$\leq \sum_{n\geq 1} 34k_2 \frac{\|x\|^2}{b_n^2}\frac{1}{k_{n+1}b_n} \leq 34k_1\|x\|^2 \sum_{n\geq 1}\frac{1}{k_n b_n^3}.$$

Since J is arbitrary, the bound is valid for $\mathcal{L}(I)$.

□

Lemma 2.3.4. *Let X be any weak solution with initial condition $x \in H$ with all non-negative components. Then there exists a constant c such that the following inequality holds for all $n \geq 1$ and $M > 0$*

$$\mathcal{L}(X_n > M) = \mathcal{L}\{t \geq 0 : X_n(t) > M\} \leq \frac{c\|x\|^2}{k_n M^3}.$$

Proof. For fixed $n \geq 1$ let

$$b_i = \begin{cases} L & i < n \\ M & i \geq n. \end{cases}$$

with L integer greater than M. We use Proposition 2.3.3 to get

$$\mathcal{L}(X_n > M) \leq 34k_1\|x\|^2\sum_{i\geq 1}\frac{1}{k_i b_i^3} \leq 34k_1\|x\|^2\left[\sum_{i=1}^{n-1}\frac{1}{k_i L^3} + \sum_{i\geq n}\frac{1}{k_i M^3}\right].$$

If we let $L \to \infty$, we conclude

$$\mathcal{L}(X_n > M) \leq \frac{c(\beta)\|x\|^2}{k_n M^3}.$$

□

Now we have all the tools necessaries to prove the uniqueness for positive solutions.

Proof. Of Theorem 2.3.1
We want to apply Proposition 2.3.2, so let X be a weak solution with initial condition x. Fixing $t \geq 0$ and $n \geq 1$

$$\int_0^t X_n^3(s)ds = \int_0^{\|x\|^3} \mathcal{L}\{s \in [0,t] : X_n^3(s) > y\}dy.$$

For lemma 2.3.4

$$\mathcal{L}\{s \in [0,t] : X_n^3(s) > y\} \leq \min(t, \frac{c\|x\|^2}{k_n y}) = \begin{cases} t & y \leq \frac{c\|x\|^2}{k_n t} \\ \frac{c\|x\|^2}{k_n y} & y > \frac{c\|x\|^2}{k_n t} \end{cases}$$

so

$$\int_0^t X_n^3(s)ds \leq \frac{c\|x\|^2}{k_n} + \frac{c\|x\|^2}{k_n} \int_{\frac{c\|x\|^2}{k_n t}}^{\|x\|^3} \frac{dy}{y}$$

hence

$$2^{-n}k_n \int_0^t X_n^3(s)ds \leq 2^{-n}c\|x\|^2[\log(\|x\|^3) + \log(\frac{k_n t}{c\|x\|^2})]$$
$$= 2^{-n}c\|x\|^2[\log(\|x\|^3) + n\log k_1 \log(\frac{t}{c\|x\|^2})].$$

And letting $n \to \infty$ the proof is complete.

$\square$

Chapter 3

Random solutions for the mixed model

Abstract

The aim of this chapter is to prove an existence result on the mixed shell model extending the classic standard existence results from ℓ^2 initial conditions to μ-almost every initial conditions, where μ is a Gaussian measure on the infinite dimensional space of initial conditions.
The first step consists in the identification of a mixed model that leaves a certain Gaussian measure μ invariant, in spite of the nonlinear character of the equation. This requires a particular choice of the coefficients of the model. Then we find a Galerkin approximation of the infinite dimensional shell model, in a way that every finite dimensional system of the sequence admits a unique solution by Cauchy-Lipschitz theorem and such that every finite N-dimensional system has an invariant measure given by the projection of μ on the first N coordinates.
In the second step we introduce random initial conditions and we obtain fundamental estimates on the norm of random solutions on certain spaces. The invariance of the measure plays an important role in the computation of the norm estimates and our results on these estimates fully depend on it.
In the third step we use a compactness argument to extract a weak limit of the sequence of random solution for the N-dimensional system, based on a combination of Aubin-Lions lemma and Prohorov Theorem.
Last, in the fourth step, we prove that the weak limit obtained from the third step can be extended to an a.s. limit, thanks to Skorokhod representation theorem, that formally solves the integral equation of the infinite dimensional dynamic system.
We want to stress that the techniques shown in this chapter do not fully rely on the type of non-linearity of the model, so also a reader that doesn't work directly on shell models could take advantage of this.
The results of this chapter are taken from [26].

3.1 Suitable coefficients for the model

The mixed shell model that we consider in this chapter is the infinite dimensional dynamic system defined for $t \in [0,T]$ described for each $n \in \mathbb{N} \setminus \{0\}$ by the following equations:

$$\frac{d}{dt}X_n(t) = k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t) - k_n X_{n+1}^2(t) + k_{n-1}X_{n-1}(t)X_n(t),$$

with $k_n = \lambda^n$, $\lambda > 1$ for $n > 1$ and $k_0 = k_1 = 0$.

The mixed shell model is formally conservative, in the sense that an energy quantity $\mathcal{E}(t) = \sum_n X_n^2(t)$ is formally conserved by the equations of the dynamic, as it will be clear later in this section. Thanks to this property we get the existence of solutions for ℓ^2 initial conditions but we are not able at the moment to extend this results for all initial conditions using only the energy conservation. However, the energy conservation is our starting building block to get at least the existence and uniqueness of solutions for the approximating Galerkin sequence of system, since in each finite system we get trivially that every initial condition belongs to the spaces ℓ^2. This is why in this chapter we will look to a (real, not only formal) conservative Galerkin approximation. So from now on the N-dimensional shell model is the N-dimensional dynamic system described by the following equations, for $1 \leq n \leq N$:

$$\frac{d}{dt}X_n(t) = k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t) - k_n X_{n+1}^2(t) + k_{n-1}X_{n-1}(t)X_n(t),$$

with $k_n = \lambda^n$, $\lambda > 1$ for $1 < n < N$ and $k_0 = k_1 = k_N = k_{N+1} = 0$.

The last equations are well defined even if apparently there are terms with no meaning: since the multiplicating coefficient for those terms is 0, actually there is no need to give meaning to non defined terms and the dynamic system has effective dimension of N.

Definition 3.1.1. *With the notation used above, a function $X \in C^1([0,T];\mathbb{R}^N)$ is said to be a solution of the N-dimensional shell model with initial condition $X_0 \in \mathbb{R}^N$ if for each $1 \leq n \leq N$ it satisfies the equation*

$$\frac{d}{dt}X_n(t) = k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t) - k_n X_{n+1}^2(t) + k_{n-1}X_{n-1}(t)X_n(t),$$

and if $X(0) = X_0$.

As promised, we have built the Galerkin approximation to let every system of the approximating sequence to be conservative, and below we left the proof.

Definition 3.1.2. *For any solution of the N- dimensional shell model we define the kinetic energy $\mathcal{E}(t)$ the sum*

$$\mathcal{E}(t) = \sum_{n=1}^{N} X_n^2(t).$$

Proposition 3.1.3. *The kinetic energy $\mathcal{E}(t)$ is invariant for all the solutions of the N-dimensional shell model.*

Proof. We have to show that $\frac{d}{dt}\mathcal{E}(t) = 0$,

$$\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}\sum_{n=1}^{N} X_n^2(t) =$$

$$2\sum_{n=1}^{N}(k_n X_n X_{n-1}^2 - k_{n+1}X_i^2 X_{n+1} - k_n X_n X_{n+1}^2 + k_{n-1}X_{n-1}X_n^2) =$$
$$= 2(k_1 X_0^2 X_1 - k_{N+1}X_N^2 X_{N+1} - k_N X_N X_{N+1}^2 + k_0 X_0 X_1^2) = 0.$$

□

Thanks to the conservation of kinetic energy we can now show the existence (and uniqueness) of solutions of the N-dimensional approximation for any initial condition.

Proposition 3.1.4. *For any $x_0 \in \mathbb{R}^N$ there exists a unique solution of the N-dimensional shell model with initial condition $X(0) = x_0$.*

Proof. The system truncated system satisfies the Cauchy-Lipscithz theorem and then admits a local solution X^N on $[0, \delta]$ for some $\delta > 0$. We can extend the local solution to a global solution using the energy bound:

$$\sum_{n=1}^{N} X_n^2(t) = \sum_{n=1}^{N} X_n^2(0).$$

□

As we mentioned before, the infinite shell model admits a solution for ℓ^2 initial condition.

Theorem 3.1.5. *With the same notation of this chapter, consider the infinite dimensional shell model*

$$\frac{d}{dt}X_n(t) = k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t) - k_n X_{n+1}^2(t) + k_{n-1}X_{n-1}(t)X_n(t),$$

$$X(0) = \bar{X}.$$

So for any initial condition $\bar{X} \in \ell^2$ there exists at least a solution $X(t)$ on $[0, T]$.

Proof. We will use the Ascoli-Arzelà theorem. For any $\bar{X}$, we consider a sequence of solutions of the N-dimensional shell model $\tilde{X}^N(t)$ obtained considering as initial condition $\bar{X}^N$ the first N-entries of $\bar{X}$.
With abuse of notation we consider all the function $X^N(t)$ embedded in the same infinite dimensional space by taking the value 0 on the empty entries. For every fixed j and $t \in [0, T]$ it holds:

- Uniform boundedness of $\{X_j^N(t)\}_{N\in\mathbb{N}}$ for both N and t:

$$|X_j^N(t)| \leq \|X^N(t)\|_2^2 \leq \|\bar{X}\|_2^2.$$

- Equi-Lipschitzianity of $\{X_j^N(t)\}_{N\in\mathbb{N}}$ with respect to N:

$$|\frac{d}{dt}X_j^N(t)| \leq |k_j X_{j-1}^2(t)|+|k_{j+1}X_j(t)X_{j+1}(t)|+|k_j X_{j+1}^2|+|k_{j-1}X_{j-1}(t)X_j(t)| \leq$$

$$\leq \|\bar{X}\|_2^2(2k_j + k_{j+1} + k_{j-1}).$$

Ascoli-Arzelà theorem implies for each fixed j the existence of a converging subsequence in $C([0,T])$, i.e. it is possible to find indices $\{N_k^j, k \in \mathbb{N}\}$ such that

$$\sup_{t\in[0,T]} |X_j^{N_k^j}(t) - X_j(t)| \to 0$$

for fixed j as $k \to \infty$.
The sequence $N_\bullet^j$ can be chosen so that $N_\bullet^{j-1}$ is a subsequence of $N_\bullet^j$ itself. By a standard diagonal argument we can extend the convergence to all j. If we consider indices $N_k = N_k^k$, we are extracting a common subsequence such that

$$\sup_{t\in[0,T]} |X_j^{N_k}(t) - X_j(t)| \to 0$$

for all $j \geq 0$, as $k \to \infty$.

Last it is straightforward to check that the limit obtained via Ascoli-Arzelà theorem, $X(t)$, is a solution for system, using the equation in the integral form. □

Here is where our work takes a deviation from standard existence results on shell models. The only use of energy conservation leads to not more that existence for ℓ^2 initial conditions, so to improve this we introduce an invariant measure.

Definition 3.1.6. *We define for each $r > 0$ the Gaussian measure on $\mathbb{R}^N$*

$$\mu_r^N = \bigotimes_{i=1}^N \mathcal{N}(0, r^2).$$

Before proving that the Gaussian measure μ_r^N is invariant for the trajectories of the N-dimensional shell model we need first a technical lemma. So let $b : \mathbb{R}^N \to \mathbb{R}^N$ be the vector field

$$b_n(x) = k_n x_{n-1}^2 - k_{n+1}x_n x_{n+1} - k_n x_{n+1}^2 + k_{n-1}x_{n-1}x_n,$$

and let $\varphi_t(x)$ be the solution with initial conditions x. Let also P_t and P_t^* be the semigroups:

$$(P_t g)(x) := g(\varphi_t(x))$$

$$\begin{aligned}(P_t^*\mu)(g) &:= \int_{\mathbb{R}^N} g(\varphi_t(x))\,\mu(dx)\\ &= \mu(P_t g)\end{aligned}$$

where g is bounded measurable and μ is a probability measure. We have used the notation $\mu(f) := \int_{\mathbb{R}^N} f(x)\,\mu(dx)$. Notice that $P_t^*\mu$ is a new probability measure.

Lemma 3.1.7. *Let X, Y be two classes of smooth functions $g : \mathbb{R}^N \to \mathbb{R}$ with the following properties:*

$$P_t(X) \subset Y \text{ for every } t \geq 0$$

$$\frac{d}{dt}(P_t^*\mu)(g)\,|_{t=0} = 0 \text{ for all } g \in Y.$$

Then

$$\frac{d}{dt}(P_t^*\mu)(g) = 0 \text{ for all } g \in X \text{ and } t \geq 0.$$

Proof. First notice that holds the following

$$\left(P_{t+\epsilon}^*\mu\right)(g) = (P_\epsilon^*\mu)(P_t g)$$

since we have that

$$(P_\epsilon^*\mu)(P_t g) = \int_{\mathbb{R}^N} (P_t g)(\varphi_\varepsilon(x))\,\mu(dx)$$
$$= \int_{\mathbb{R}^N} g(\varphi_t(\varphi_\varepsilon(x)))\mu(dx) = \int_{\mathbb{R}^N} g(\varphi_{t+\varepsilon}(x))\mu(dx)$$
$$= \left(P_{t+\epsilon}^*\mu\right)(g).$$

Then we get the thesis by the following computation:

$$\begin{aligned}
\frac{d}{dt}(P_t^*\mu)(g) &= \lim_{\epsilon\to 0}\frac{\left(P_{t+\epsilon}^*\mu\right)(g) - (P_t^*\mu)(g)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{(P_\epsilon^*\mu)(P_t g) - \mu(P_t g)}{\epsilon} \\
&= \frac{d}{ds}(P_s^*\mu)(P_t g)\,|_{s=0} = 0.
\end{aligned}$$

□

Hence we can go for the main result of this section:

Proposition 3.1.8. *For all the $r > 0$, the measure μ_r^N is invariant for the N-dimensional shell model.*

Proof. Let $b : \mathbb{R}^N \to \mathbb{R}^N$ be the vector field,

$$b_n(x) = k_n x_{n-1}^2 - k_{n+1}x_n x_{n+1} - k_n x_{n+1}^2 + k_{n-1}x_{n-1}x_n.$$

If $\varphi_t(x)$ is the solution with initial conditions x, it would be sufficient to have for each $g : \mathbb{R}^N \to \mathbb{R}$ regular enough:

$$\frac{d}{dt}\int_{\mathbb{R}^N} g(\varphi_t(x))\mu_r^N(dx) = 0,$$

to prove that μ_r^N in an invariant measure for the system. Note that for lemma 3.1.7 it is sufficient to look the equality for $t = 0$. We have

$$\frac{d}{dt}\int_{\mathbb{R}^N} g(\varphi_t(x))\mu_r^N(dx) =$$

$$\int_{\mathbb{R}^N} \bigtriangledown g(\varphi_t(x)) \frac{d}{dt}\varphi_t(x)\mu_r^N(dx) = 0,$$

looking for $t = 0$

$$\int_{\mathbb{R}^N} \bigtriangledown g(x)b(x)\mu_r^N(dx) = 0,$$

let $f(x)$ be such that $\mu_r^N(dx) = f(x)dx$, so

$$\int_{\mathbb{R}^N} \bigtriangledown g(x)b(x)f(x)dx = 0,$$

and applying Gauss-Green formula we have

$$\int_{\mathbb{R}^N} g(x)\operatorname{div}(b(x)f(x))dx = 0,$$

that gives

$$\operatorname{div}(b(x)f(x)) = 0.$$

For Gaussian measures μ_r^N we have $f(x) = ce^{\frac{-\|x\|^2}{r^2}}$ for some constant $c > 0$, so we have to show that

$$\operatorname{div}(b(x)f(x)) = \sum_{n=1}^{N} \frac{\partial}{\partial x_n} b_n(x)f(x) =$$

$$ce^{\frac{-\|x\|^2}{r^2}} \sum_{n=1}^{N} [(-2x_n)(k_n x_{n-1}^2 - k_{n+1}x_n x_{n+1} - k_n x_{n+1}^2 + k_{n-1}x_{n-1}x_n) -$$

$$-k_{n+1}x_{n+1} + k_{n-1}x_n] = 0.$$

This gives us two different conditions,

1. $\sum_{n=1}^{N} k_n x_n x_{n-1}^2 - k_{n+1}x_i^2 x_{n+1} - k_n x_n x_{n+1}^2 + k_{n-1}x_{n-1}x_n^2 = 0,$
2. $\sum_{n=1}^{N} -k_{n+1}x_{n+1} + k_{n-1}x_{n-1} = 0.$

The first condition follows directly from $\frac{d}{dt}\mathcal{E}(t) = 0$.

Computing the second condition we have:

$$\sum_{n=1}^{N} -k_{n+1}x_{n+1} + k_{n-1}x_{n-1} =$$

$$k_0x_0 + k_1x_1 - k_Nx_N - k_{N+1}x_{N+1} + \sum_{n=2}^{N-1}(-k_n + k_n)x_n = 0.$$

$\square$

The reader can note that the coefficients of the mixed shell model we took cannot be more general to let to a certain Gaussian measure to be invariant. We mean that if one considers this dynamic system

$$\frac{d}{dt}X_n(t) = (k_n X_{n-1}^2(t) - k_{n+1}X_n(t)X_{n+1}(t)) - (h_n X_{n+1}^2(t) + h_{n-1}X_{n-1}(t)X_n(t)),$$

and does the computation to have a Gaussian invariant measure for the Galerkin approximating sequence as done above, necessarily gets $\alpha = \beta$ and $k_n = h_n$:

$$\operatorname{div}(b(x)f(x)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)f(x)) =$$

$$ce^{\frac{-\|x\|^2}{r^2}} \sum_{i=1}^{n} [(-2x_i)(k_i x_{i-1}^2 - k_{i+1}x_i x_{i+1} - h_i x_{i+1}^2 + h_{i-1}x_{i-1}x_i) - k_{i+1}x_{i+1} + h_{i-1}x_i] = 0.$$

Since we want the energy conservation along trajectories, we may put

$$h_0 = k_1 = h_N = k_{N+1} = 0,$$

so it holds

$$\sum_{i=1}^{N} (k_i x_i x_{i-1}^2 - k_{i+1}x_i^2 x_{i+1} - h_i x_i x_{i+1}^2 + h_{i-1}x_{i-1}x_i^2) = 0,$$

and we have only to check the following condition

$$\sum_{i=1}^{N} (-k_{i+1}x_{i+1} + h_{i-1}x_{i-1}) = 0,$$

hence:

$$\sum_{i=1}^{N} (-k_{i+1}x_{i+1} + h_{i-1}x_{i-1}) = h_0 x_0 + h_1 x_1 - k_N x_N - k_{N+1}x_{N+1} + \sum_{i=2}^{N-1} (-k_i + h_i)x_i.$$

This leads to $k_i = h_i$ for every i and $k_0 = k_1 = k_N = k_{N+1} = 0$. Note that in this way the terms x_0 and x_{N+1} disappear from the equations.

3.2 Random initial conditions

The first aim of this section is to put a random environment where the definition of random solution of a deterministic equation takes sense. Then, once we have built the environment we need, we get fundamental estimates for the next section on the norm of the random solution.

Definition 3.2.1. *Let $(\Omega, \mathcal{F}, P)$ be an abstract probability space, for every N and for $r > 0$ let Y_r^N be a random variable*

$$Y_r^N : (\Omega, \mathcal{F}, P) \to (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^{\mathbb{N}})),$$

with law μ_r^N.

Definition 3.2.2. *A set $(\Omega, \mathcal{F}, P, U_r^N)$ is said to be a N-finite random solution if U_r^N is defined on the abstract probability space $(\Omega, \mathcal{F}, P) \times [0,T]$ to $\mathbb{R}^\infty$, all k-coordinates of U_r^N are almost surely for each time $t \in [0,T]$ equal to 0 if $k > N$ and for $k \le N$ almost surely*

$$U_r^N{}_{(k)}(\omega, t) = F_r^N{}_{(k)}(\omega, t),$$

where, for $\omega \in \Omega$, the function

$$F_r^N(\omega) : [0,T] \to \mathbb{R}^N$$

is the unique solution of the N-dimensional shell model with initial conditions

$$X(0) = Y_r^N(\omega).$$

Remark that F_r^N is still a random variable from the abstract space $(\Omega, \mathcal{F}, P)$.

Proposition 3.2.3. *Let $(\Omega, \mathcal{F}, P, U_r^N)$ be a N-finite random solution. The law of $U_r^N(t)$ is, for any $t \in [0,T]$,*

$$\tilde{\mu}_r^N = \mu_r^N \otimes \bigotimes_{N+1}^{\infty} \delta_0.$$

Proof. It follows directly from the definition of $U_r^N(t)$ and the invariance of μ_r^N along the trajectories of the N-dimensional shell model.

□

At this point it should be clear to the reader that the choice of the abstract space $(\Omega, \mathcal{F}, P)$ is totally arbitrary, it matters only the law of the random variables defined on said space. This opens the door to a future use of the Skorokhod representation theorem to get an almost surely limit in place of a weak limit. Our next issue to solve is to bring to the space of random solutions a suitable norm, in order to be able to speak about convergence in that topology. A natural norm to work with could be the H^s norm defined below, since, roughly speaking, we expect any limit of the finite random solution sequence at time 0 to almost surely have said norm finite, as we will prove in the next section.

Definition 3.2.4. *We define $(H^s, \|\cdot\|_{H^s})$ as the Hilbert space of sequences $x \in \mathbb{R}^\infty$ satisfying*

$$\|x\|_{H^s} = \sqrt{\sum_n k_n^{2s} x_n^2} < \infty.$$

The following proposition on the H^s norms will be significant in the next section to pass from a compactness result in $L^p(0,T; H^s)$ for any $p > 1$ to a compactness result in $C(0,T; H^s)$.

Proposition 3.2.5. *Let $s_1 < s < s_0 < 0$. Then there exists $\theta \in (0,1)$ such that*

$$\|x\|_{H^s} \le \|x\|_{H^{s_0}}^{\theta} \|x\|_{H^{s_1}}^{1-\theta},$$

for every $x \in H^{s_0}$.

Proof. For all $a \in (2s, 0)$, $b \in (0, 2)$, it holds

$$\|x\|_{H^s}^2 = \sum_n k_n^{2s} x_n^2 = \sum_n k_n^a |x_n|^b k_n^{2s-a} |x_n|^{2-b} \leq$$

we use Hölder inequality for generic $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

$$\leq (\sum_n (k_n^a |x_n|^b)^p)^{\frac{1}{p}} (\sum_n (k_n^{2s-a} |x_n|^{2-b})^q)^{\frac{1}{q}}.$$

We can now let $b = \frac{2}{p}$ in order to have $bp = 2$ and $(2-b)q = 2$, that gives $p = \frac{2}{b}$ and $q = \frac{2}{2-b}$, since $\frac{2}{p} \in (0, 2)$ and since it holds

$$(\frac{2}{b})^{-1} + (\frac{2}{2-b})^{-1} = \frac{b}{2} + \frac{2-b}{2} = 1.$$

Hence we get

$$\|x\|_{H^s}^2 \leq (\sum_n k_n^{ap} x_n^2)^{\frac{1}{p}} (\sum_n k_n^{(2s-a)q} x_n^2)^{\frac{1}{q}}.$$

Looking at the statement we want to prove, it lasts to show that we can choose $a \in (2s, 0)$, $p > 1$, $q > 1$ such that $ap = 2s_0$, $(2s-a)q = 2s_1$ and $\frac{1}{p} + \frac{1}{q} = 1$. So solving

$$\begin{cases} ap = 2s_0 \\ (2s-a)q = 2s_1 \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

we get

$$p = \frac{s_0 - s_1}{s - s_1},$$

hence $p > 1$, hence $q > 1$. The last check we have to do is for a and we get $a = \frac{2s_0}{p} \in (2s, 0)$ since $s_0 < s$ and $p > 1$. So we have proved the statement for

$$\theta = \frac{1}{p} = \frac{s - s_1}{s_0 - s_1}.$$

□

Here, with the specific non-linearity of the system, is where the choice of the measure plays a central role. The following estimates holds for the Gaussian measure μ_r^N and a quadratic non-linearity, so a reader that is not strictly focused on shell models can skip this part. Outside of the following two estimates the work is pretty independent of the type of non-linearity of the system, so to generalize the method to something else one musts first check the existence of estimates similar to the following ones.
The following estimates guarantee the existence of a family of compact set in the topology of $L^p(0, T; H^s)$ suitable to apply Prohorov theorem, as we will see in the next section.

Lemma 3.2.6. *Let k_n be the coefficients of the shell model, $l \in \mathbb{N} = \{0, 1, \dots\}$, $q \in \mathbb{R}$ such that $q + l < 0$. Let φ be a $\mathcal{C}^\infty$ function such that $\varphi(0) \neq 0$. Then*

$$\sum_n \log \varphi(t k_n^{2q+2l})$$

is h-times differentiable in $t = 0$ for any $h \in \mathbb{N}$ and the derivative operation commutes with the sum, we mean that for any h $\frac{d^h}{dt^h} \sum_n \log \varphi(t k_n^{2q+2l}) = \sum_n \frac{d^h}{dt^h} \log \varphi(t k_n^{2q+2l})$.

Proof. Step1 Let $\zeta(t) = \varphi(t)^{h_0} \varphi'(t)^{h_1} \dots \varphi^{(k)}(t)^{h_k}$ be a monomial in the variables $\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t)$ of degree $z = \sum_{i=0}^k h_i$. Then his derivative $\frac{d}{dt}\zeta(t)$ is an homogeneous polynomial of degree z in the variables $\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t), \varphi^{(k+1)}(t)$.
This follows by a straightforward computation:

$$\frac{d}{dt}(\varphi(t)^{h_0} \varphi'(t)^{h_1} \dots \varphi^{(k)}(t)^{h_k}) =$$
$$= \sum_i \sum_{|h_i|} h_i \varphi^{(i+1)}(t) \varphi^{(i)}(t)^{h_i - 1} \prod_{j \neq i} \varphi^{(j)}(t)^{h_j}.$$

Step2 We want to prove by induction the following:

$$\frac{d^k}{dt^k} \log \varphi(t) = \frac{P_k(\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t))}{\varphi(t)^{2^k}}$$

where P_k is an homogeneous polynomial of degree 2^k .
For $k = 1$ we have $\frac{d}{dt} \log \varphi(t) = \frac{\varphi'(t)}{\varphi(t)}$. Assuming the thesis true for k we make the computation

$$\frac{d^{k+1}}{dt^{k+1}} \log \varphi(t) = \frac{d}{dt}\left(\frac{P_k(\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t))}{\varphi(t)^{2^k}}\right) =$$
$$= \frac{(\frac{d}{dt} P_k(\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t)))\varphi(t)^{2^k} - P_k(\varphi(t), \varphi'(t), \dots, \varphi^{(k)}(t))(2^k - 1)\varphi(t)^{2^k - 1}\varphi'(t)}{\varphi(t)^{2^{k+1}}}$$

and using the Step1 we have the thesis.

Step3 For a generic function ξ it holds

$$\frac{d^k}{dt^k} \xi(\lambda t) = \lambda^k \xi^{(k)}(\lambda t).$$

Hence, looking for $t = 0$, it holds

$$\frac{d^k}{dt^k} \xi(\lambda t)_{|t=0} = \lambda^k \frac{d^k}{ds^k} \xi(s)_{|s=0}.$$

Step4 To conclude the lemma we have to show that for every $h \in \mathbb{N}$,

$$\sum_n \left(\frac{d^h}{dt^h} \log \varphi(t k_n^{2q+2l})\right)_{|t=0} < \infty.$$

Indeed

$$\sum_n \left(\frac{d^h}{dt^h}\log\varphi(tk_n^{2q+2l})\right)_{|t=0} = \sum_n (k_n^{2q+2l})^h \frac{P_h(\varphi(s),\dots,\varphi^{(h)}(s))}{\varphi(s)^{2^h}}_{|s=0}$$

and since we have a bound on $\frac{P_h(\varphi(s),\dots,\varphi^{(h)}(s))}{\varphi(s)^{2^h}}_{|s=0}$ not depending on n we have the proof.

∎

Proposition 3.2.7. *Let $(\Omega, \mathcal{F}, P, U_r^N)$ be a N-finite random solution. For every $s<0$, $r>0$, $p>1$, $\epsilon>0$ there exists a constant $C_\epsilon>0$, not depending on N, such that*

$$P(\|U_r^N\|_{L^p(0,T;H^s)} \leq C_\epsilon) > 1-\epsilon,$$

for each $N \in \mathbb{N}$.

Proof. We want to prove that for any $p>1$ and for any $\varepsilon>0$ exists $R \in \mathbb{R}^+$ such that

$$P(\|U_r^N\|_{L^p(0,T;H^s)} > R) < \varepsilon.$$

Hence

$$P(\|U_r^N\|_{L^p(0,T;H^s)} > R) = P(\|U_r^N\|^p_{L^p(0,T;H^s)} > R^p) \leq$$

now we apply Markov inequality

$$\leq \frac{1}{R^p} E[\|U_r^N\|^p_{L^p(0,T;H^s)}] =$$

$$= \frac{1}{R^p}\|U_r^N\|^p_{L^p(\Omega\times[0,T],H^s)} = \frac{1}{R^p}\int_0^T E[\|U_r^N(t,\omega)\|^p_{H^s}]ds =$$

here, thanks to proposition 3.2.3, we use the time invariance for the law of U_r^N

$$= \frac{T}{R^p} E[\|U_r^N(0,\omega)\|^p_{H^s}].$$

Hence it is sufficient to show that for any $p>1$ exists $C \in \mathbb{R}^+$ such that

$$E[\|U_r^N(0,\omega)\|^p_{H^s}] < C,$$

for any N, since the proof will follow letting $R \to \infty$. Note that for each N holds

$$E[\|U_r^N(0,\omega)\|^p_{H^s}] \leq E[|(\sum_{n=1}^\infty k_n^{2s} r^2 W_n(\omega))^{\frac{p}{2}}|]$$

where $W_i \sim \chi^2(1)$, with $\{W_i\}_i$ iid.
So it is sufficient to prove that the random variable

$$Z = \sum_{n\geq 1} k_n^{2s} r^2 W_n,$$

has a moment generating function derivable infinite times in 0, this would imply that it has finite p-moment for any $p \geq 1$ and this is would give the uniform bound in N for $L^p(0, T; H^s)$ norm we need.

The moment generating function of Z, $\psi(t)$ is

$$\psi(t) = E[e^{t \sum k_n^{2s} r^2 W_n(\omega)}].$$

Note that

$$\log E[e^{t \sum_{n=1}^m k_n^{2s} r^2 W_n(\omega)}] = \sum_{n=1}^{m} \log \varphi(t k_n^{2s} r^2)$$

for every $m \in \mathbb{N}$, where $\varphi(t k_n^{2s} r^2)$ is the moment generating function of $k_n^{2s} r^2 W_n$. If we define the random variables $Z_m = e^{t \sum_{n=1}^m k_n^{2s} r^2 W_n(\omega)}$ we have that for $t \geq 0$ Z_m is an increasing sequence of random variable, and for $t < 0$ it is dominated by 1. So for all t we can have $E[\lim_{m\to\infty} Z_m] = \lim_{m\to\infty} E[Z_m]$, hence

$$\log \psi(t) = \sum_n \log \varphi(t k_n^{2s} r^2).$$

It lasts to show that $\sum_n \log \varphi(t k_n^{2s} r^2)$ is differentiable infinite times in $t = 0$, and this is true for lemma 3.2.6.

□

Proposition 3.2.8. *Let $(\Omega, \mathcal{F}, P, U_r^N)$ be a N-finite random solution. For every $s < -1$, $r > 0$, $p > 1$ $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$P(\|U_r^N\|_{W^{1,p}(0,T;H^s)} \leq C_\epsilon) > 1 - \epsilon,$$

for each $N \in \mathbb{N}$.

Proof. Again we have that

$$P(\|U_r^N\|^p_{W^{1,p}(0,T;H^s)} > R^p) \leq \frac{1}{R^p} E[\|U_r^N\|^p_{W^{1,p}(0,T;H^s)}] =$$

$$= \frac{1}{R^p} E[\|\frac{d}{dt} U_r^N\|^p_{L^p(0,T;H^s)}].$$

Let $\{W_{1,k}, W_{2,k}, W_{3,k}, W_{4,k}, W_{5,k}, W_{6,k}\}_{k\in\mathbb{N}}$ be a set of iid Gaussian random variables, with $W_{i,j} \sim \mathcal{N}(0, r^2)$. For each N holds

$$E[\|\frac{d}{dt} U_r^N(0,\omega)\|^p_{H^s}] \leq$$

$$\leq E[(\sum_{n\geq 1} (k_n {W_{1,n}}^2 - k_{n+1} W_{2,n} W_{3,n} - k_n {W_{4,n}}^2 + k_{n-1} W_{5,n} W_{6,n})^2 k_n^{2s})^{\frac{p}{2}}].$$

Moreover, exists a constant $D > 0$ such that

$$E[(\sum_{n\geq 1} (k_n {W_{1,n}}^2 - k_{n+1} W_{2,n} W_{3,n} - k_n {W_{4,n}}^2 + k_{n-1} W_{5,n} W_{6,n})^2 k_n^{2s})^{\frac{p}{2}}] \leq$$

$$\leq E[(D \sum_{n\geq 1} k_n^{2+2s} W_{1,n}^4)^{\frac{p}{2}}].$$

So it is sufficient to prove that the random variable

$$Z = \sum_{n\geq 1} k_n^{2+2s} W_{1,n}^4,$$

has a moment generating function differentiable infinite times in 0, this would imply that it has finite p-moment for any $p \geq 1$.
Using the same argument of Proposition 3.2.7 we have that, if $\psi(t)$ is the moment generating function of Z,

$$\log \psi(t) = \sum_n \log \varphi(t k_n^{2+2s} r),$$

where $\varphi(t k_n^{2+2s} r)$ is the moment generating function of $k_n^{2+2s} {W_{1,n}}^4$. Moreover, for Lemma 3.2.6, $\sum_n \log \varphi(t k_n^{2+2s} r)$ is derivable infinite times in $t = 0$.

□

3.3 A compactness result

Considering a N-finite random solution $(\Omega, \mathcal{F}, P, U_r^N)$, we need a compactness criterion for the family of laws $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ to extract a converging subsequence $\lim_{K\to\infty} U_r^{N_k} = U_r^\infty$ in law, and then without loss of generality we would have a limit almost surely $\lim_{K\to\infty} U_r^{N_k} = U_r^\infty$ up to changing the abstract space $(\Omega, \mathcal{F}, P)$ via Skorokhod Theorem.

As anticipated in the previous section, it is natural to extract the limit in the topology of $L^P(0,T;H^s)$ for $s < 0$, since if $s < 0$ we have $U_r^\infty(0) \in H^s$ P - almost surely, since for Markov inequality and monotone convergence

$$P(\|U_r^\infty(0)\|_{H^s} < C) \leq \frac{1}{C} E[\|U_r^\infty(0)\|_{H^s}] = \frac{1}{C} \sum r^2 \lambda^{(n-1)2s}$$

and the series on the right converges since it is geometric with $\lambda > 1$ and $s < 0$.

To prove the convergence of a subsequence we want to use the Prohorov compactness theorem. Thanks to the estimates done on the previous section, we satisfy the condition of the classical Aubin-Lions theorem, that guarantees the existence of proper compact sets, necessaries to apply Prohorov theorem. The following is the Aubin-Lions theorem in the form we will use.

Theorem 3.3.1. *Aubin-Lions*
Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B and a continuous embedding of B into B_1. Let $p, r \in (1, \infty)$. Let X be the space

$$X = L^p(0,T;B_0) \cap W^{1,r}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

More words can be spent about this theorem, the form of the Aubin-Lions theorem we use is the classical one, but it would work also for some weakened hypothesis: one can be more sharp and ask to the space to have a $W^{\alpha,p}$ regularity, with $\alpha \in (0,1)$ instead of $\alpha = 1$. More precisely, we mention this adaptation of Flandoli-Gatarek [17]

Theorem 3.3.2. *Let $B_0 \subset B \subset B_1$ be Banach spaces, with B_0, B_1 reflexive, a compact embedding of B_0 into B and a continuous embedding of B into B_1. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

For a proof of this we refer to Theorem 2.1 of [17].

In the previous section we spent a lot of effort to prove the estimates for all $p \in (1,\infty)$. This has done for a future use of a combination of Proposition 3.2.5 together with the following result from Simon, in order to extend later a relative compactness in $L^p(0,T;H^s)$ for all $p \in (1,\infty)$ to a relative compactness in $C(0,T;H^s)$.

Proposition 3.3.3. *Suppose we have $X \subset B \subset Y$ Banach spaces, with a compact embedding $X \to Y$. Suppose also there exists a $\theta \in (0,1)$ and a M such that*

$$\|v\|_B \leq M \|v\|_X^{1-\theta} \|v\|_Y^{\theta},$$

for any $v \in X \cap Y$. Let F be bounded in $L^{p_0}(0,T;X)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^{r_1}(0,T;Y)$, with $1 \leq p_0 \leq \infty$, $1 \leq r_1 \leq \infty$.
If $\theta(1-\frac{1}{r_1}) > \frac{1-\theta}{p_0}$ then F is relatively compact in $C(0,T;B)$.

For a proof of this result we refer to Corollary 8, section 10 of [31].

We get finally all the tools and estimates to conclude the section with our main compactness result about the sequence of laws of the random solutions.

Theorem 3.3.4. *For fixed $r > 0$, let $(\Omega, \mathcal{F}, P, U_r^N)$ be a N-finite random solution. The family of law $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ is tight in $L^P(0,T,H^s)$ for every $p > 1$, $s < 0$. Moreover, $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ is tight in $C(0,T;H^s)$.*

Proof. We use Aubin-Lions theorem with $B_0 = H^s$, $s < 0$, $B_1 = H^{s_1}$, $s_1 < -1$, $B = H^{s^*}$, $s_1 < s^* < s$ and $p = r$.

So the set

$$K_{R_1,R_2} = \{X | \|X\|_{L^p(0,T;H^s)} \leq R_1, \|X\|_{W^{1,p}(0,T;H^{s_1})} \leq R_2\}$$

is relatively compact in $L^p(0,T;H^{s^*})$.

From Proposition 3.2.7 and 3.2.8 we have that for every $\varepsilon > 0$ there exists a constant C_ε such that :

- $P(\|U_r^N\|_{L^p(0,T;H^s)} \leq C_\varepsilon) \geq 1 - \varepsilon$ for all $N \in \mathbb{N}$,

- $P(\|U_r^N\|_{W^{1,p}(0,T;H^{s_1})} \leq C_\varepsilon) \geq 1-\varepsilon$ for all $N \in \mathbb{N}$.

So, given $\varepsilon > 0$ there exist $R_1(\varepsilon)$, $R_2(\varepsilon)$ such that the family of laws satisfies

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(L^p(0,T;H^s)) | \mu(K^c_{R_1,R_2}) \leq \varepsilon\},$$

hence, since $\bar{K}_{R_1,R_2} \supseteq K_{R_1,R_2}$ we have $\bar{K}^c_{R_1,R_2} \subseteq K^c_{R_1,R_2}$ and then

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(L^p(0,T;H^s)) | \mu(\bar{K}^c_{R_1,R_2}) \leq \varepsilon\},$$

so the family of laws $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ is tight in the topology of $L^p(0,T;H^s)$ for any $p \geq 1$.

For the tightness in $C(0,T;H^s)$ we can use Proposition 3.3.3 with $X = H^{s_0}$, $B = H^s$, $Y = H^{s_1}$ and θ from Proposition 3.2.5. If we let $p_0, r_1 \to \infty$ the condition of Proposition 3.3.3

$$\theta(1 - \frac{1}{r_1}) > \frac{1-\theta}{p_0}$$

is trivial, so we have that there exist p_0 and r_1 such that the set

$$K_{R_1,R_2} = \{X | \|X\|_{L^{p_0}(0,T;H^s)} \leq R_1, \|X\|_{W^{1,r_1}(0,T;H^{s_1})} \leq R_2\}$$

is relative compact in $C(0,T;H^s)$. Hence again, using Proposition 3.2.7 and 3.2.8 we have that for every $\varepsilon > 0$ there exists a constant C_ε such that :

- $P(\|U_r^N\|_{L^{p_0}(0,T;H^s)} \leq c_\varepsilon) \geq 1-\varepsilon$ for all $N \in \mathbb{N}$,
- $P(\|U_r^N\|_{W^{1,r_1}(0,T;H^{s_1})} \leq c_\varepsilon) \geq 1-\varepsilon$ for all $N \in \mathbb{N}$.

So, given $\varepsilon > 0$ there exist $R_1(\varepsilon)$, $R_2(\varepsilon)$ such that the family of laws satisfies

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(C(0,T;H^s)) | \mu(K^c_{R_1,R_2}) \leq \varepsilon\},$$

hence, since $\bar{K}_{R_1,R_2} \supseteq K_{R_1,R_2}$ we have $\bar{K}^c_{R_1,R_2} \subseteq K^c_{R_1,R_2}$.

$$\{\mathcal{L}(U_r^N)\} \subset \{\mu \in Pr(C(0,T;H^s)) | \mu(\bar{K}^c_{R_1,R_2}) \leq \varepsilon\},$$

so the family of laws $\{\mathcal{L}(U_r^N)\}_{N\in\mathbb{N}}$ is tight in the topology of $C(0,T;H^s)$.

□

Corollary 3.3.4.1. *For $s < 0$, there exists a subsequence $n_k \subset \mathbb{N}$ such that the laws of $\{U_r^{n_k}\}$ converge with the topology of $L^p(0,T;H^s)$ for every $p \geq 1$ and with the topology of $C(0,T;H^s)$.*

Proof. We have only to apply Prohorov Theorem to the results of Theorem 3.3.4.

□

3.4 Existence of random solution

In this section we glue all the stuff together to have the existence of solutions for almost every initial condition, where almost every is respect to the probability measure μ_r on the infinite dimensional space of initial conditions. So first we have to establish what is for us a "random solution" for the infinite dimensional model, and then prove that the limit extracted in the previous section, after having changed it from a limit in law to a limit almost surely thanks to Skorokhod theorem, effectively fits the definition of random solution.

Definition 3.4.1. *A set $(\Omega, \mathcal{F}, P, X)$ is said to be a random solution for the infinite shell model if $(\Omega, \mathcal{F}, P)$ is an abstract probability space, $X : (\Omega, \mathcal{F}, P) \times [0, T] \to \mathbb{R}^\infty$ and for almost every $\omega \in \Omega$, $X(\omega, t)$ satisfies for every $i \in \mathbb{N}$ and $t \in [0, T]$*

$$X_i(\omega, t) = X_i(\omega, 0) + \int_0^t k_i X_{i-1}^2(\omega, s) ds - \int_0^t k_{i+1} X_i(\omega, s) X_{i+1}(\omega, s) ds$$
$$- \int_0^t k_i X_{i+1}^2(\omega, s) ds + \int_0^t k_{i-1} X_{i-1}(\omega, s) X_i(\omega, s) ds.$$

The following theorem represents the goal of the work of the chapter.

Theorem 3.4.2. *For fixed $r > 0$, consider a sequence $\{(\Omega, \mathcal{F}, P, U_r^N)\}_{N \geq 1}$ of N-finite random solutions, up to replace the abstract space $(\Omega, \mathcal{F}, P)$ with another probability space $(\Omega', \mathcal{F}', P')$, there exists a subsequence $n_k \in \mathbb{N}$ such that P'-almost surely $U_r^{n_k}$ converges to a function U_r^∞ in $L^p(0, T; H^s)$ for every $p > 1$ and in $C(0, T; H^s)$, for every $s < 0$. Moreover, $(\Omega', \mathcal{F}', P', U_r^\infty)$ is a random solution for the infinite shell model.*

Proof. From corollary 3.3.4.1 we have the existence of a subsequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that the sequence of laws of the random variables $U_r^{n_k}$ converges in the topology of $C(0, T; H^s)$ and $L^p(0, T; H^s)$ for every $p \geq 1$.
Since we have the convergence for any $s < 0$, we can consider the space

$$H^{0-} = \bigcap_{s<0} H^s,$$

endowed with the metric generated by the distance

$$d(x, \tilde{x}) = \sum_{n=1}^{\infty} 2^{-n} (\|x - \tilde{x}\|_{H^{-\frac{1}{n}}} \wedge 1).$$

Note that with this metric we have that $x_n \to^d x \Leftrightarrow x_n \to^{H^s} x$ for any $s < 0$.
We can assume, using Skorokhod representation Theorem, that almost surely $\tilde{U}_r^{n_k}$ converges to U_r^∞, up to replace the abstract space $(\Omega, \mathcal{F}, P)$ where $\tilde{U}_r^N$ are defined with another abstract probability space $(\Omega', \mathcal{F}', P')$, in the topology of $C(0, T; H^{0-})$ and $L^p(0, T; H^{0-})$ for every $p \geq 1$.
The new sequence of random variables $\tilde{U}_r^{n_k}$ has the same law of $U_r^{n_k}$, this means that for every φ measurable function it holds $E[\varphi(\tilde{U}_r^{n_k})] = E[\varphi(U_r^{n_k})]$.
Hence, considering the operator

$$F_i(x(\omega, t)) = x_i(\omega, t) - x_i(\omega, 0) - \int_0^t k_i x_{i-1}^2(\omega, s) ds + \int_0^t k_{i+1} x_i(\omega, s) x_{i+1}(\omega, s) ds$$

$$+\int_0^t k_i x_{i+1}^2(\omega,s)ds - \int_0^t k_{i-1}x_{i-1}(\omega,s)x_i(\omega,s)ds,$$

we have that $E[|F_i(\tilde{U}_r^{n_k})|] = E[|F_i(U_r^{n_k})|]$, hence $\tilde{U}_r^{n_k}$ is still almost surely a solution of the finite dimensional shell model. So for now on we consider without loss of generality $U_r^n = \tilde{U}_r^n$.

Now let $\{Y^N\}$ and Y^∞ such that for each $\varepsilon > 0$ there exists a N_0 such that for every $N > N_0$ we have

$$\sup_{[0,T]} \sum_{n\geq 1} k_n^{2s}(Y_n^N - Y_n^\infty)^2 < \varepsilon,$$

this implies that

$$\sup_{[0,T]} |Y_n^N - Y_n^\infty| < (\frac{\varepsilon}{k_n^{2s}})^{\frac{1}{2}} \doteq \varepsilon_n'.$$

Hence we have:

1.
$$\int_0^t |Y_n^N(s)^2 - Y_n^\infty(s)^2|ds \leq \int_0^t 2|Y_n^N(s)||Y_n^N(s) - Y_n^\infty(s)| + |Y_n^N(s) - Y_n^\infty(s)|^2 ds$$
$$\leq \int_0^t 2\sup_{s\in[0,T]}|Y_n^N(s)| \sup_{s\in[0,T]}|Y_n^N(s) - Y_n^\infty(s)| + \sup_{s\in[0,T]}|Y_n^N(s) - Y_n^\infty(s)|^2 ds$$
$$\leq t(2\sup_{s\in[0,T]}|Y_n^N(s)|\varepsilon_n' + {\varepsilon_n'}^2).$$

2.
$$\int_0^t |Y_n^N(s)Y_{n-1}^N(s) - Y_n^\infty(s)Y_{n-1}^\infty(s)|ds \leq$$
$$\int_0^t |Y_n^N(s) - Y_n^\infty(s)||Y_{n-1}^N(s) - Y_{n-1}^\infty(s)| + |Y_{n-1}^N(s) - Y_{n-1}^\infty(s)||Y_n^N(s)| +$$
$$|Y_n^N(s) - Y_n^\infty(s)||Y_{n-1}^N(s)|ds \leq$$
$$\int_0^t \sup_{s\in[0,T]}|Y_n^N(s) - Y_n^\infty(s)| \sup_{s\in[0,T]}|Y_{n-1}^N(s) - Y_{n-1}^\infty(s)| +$$
$$\sup_{s\in[0,T]}|Y_{n-1}^N(s) - Y_{n-1}^\infty(s)| \sup_{s\in[0,T]}|Y_n^N(s)| +$$
$$\sup_{s\in[0,T]}|Y_n^N(s) - Y_n^\infty(s)| \sup_{s\in[0,T]}|Y_{n-1}^N(s)|ds \leq$$
$$t(\varepsilon_n'\varepsilon_{n-1}' + \sup_{s\in[0,T]}|Y_n^N(s)|\varepsilon_{n-1}' + \sup_{s\in[0,T]}|Y_{n-1}^N(s)|\varepsilon_n').$$

Since almost surely $\sup_{s\in[0,T]} U_n^N(\omega,s) < \infty$ for every n, N, we have that almost surely

$$\int_0^t k_i({U_{i-1}^{n_k}}^2(\omega,s) - {U_{i-1}^\infty}^2(\omega,s))ds - \int_0^t k_{i+1}(U_i^{n_k}(\omega,s)U_{i+1}^{n_k}(\omega,s) - U_i^\infty(\omega,s)U_{i+1}^\infty(\omega,s))ds$$
$$-\int_0^t k_i({U_{i+1}^{n_k}}^2(\omega,s) - {U_{i+1}^\infty}^2(\omega,s))ds + \int_0^t k_{i-1}(U_{i-1}^{n_k}(\omega,s)U_i^{n_k}(\omega,s) - U_{i-1}^\infty(\omega,s)U_i^\infty(\omega,s))ds$$

goes to 0 as $k \to \infty$, so U_r^∞ is a random solution for the infinite shell model.

□

Chapter 4

Existence results on the tree models

Abstract

In this chapter we work on tree models. First we briefly introduce turbulence tree models, remarking the existence of solutions for ℓ^2 initial conditions in the first Katz-Pavlovic model. Then we work on a forced tree model, way more general than the Katz-Pavlovic one, where the force acts only on the first component. We take the coefficients of the model as much general as possible, and from this we build an existence result for solutions of the system with ℓ^2 initial conditions. Differently from other shell models, this forced ones is not conservative, in the sense that the quantity $\sum X_i^2(t)$ is not constant along the trajectories. So to achieve our goal we will need a bound on the behaviour of the ℓ^2 norm of the solution, hence the result will follow from a standard Galerkin approximation argument via Ascoli-Arzelà theorem. Last we consider particular mixed cascade tree model. As done in the last chapter, the coefficients are chosen in a way to give to the system the existence of a class of invariant Gaussian measures. Then, we apply all techniques used in the last chapter to get an existence result that improves the ℓ^2 existence result that we have done for the more general model. The results of this chapter are taken from [4], [10] and [26].

4.1 Introduction to the tree models

The turbulence dyadic shell model comes from a more general model that has a tree structure. This model consists of an infinite system of nonlinear differential equations and was introduced to mimic 3D Euler and Navier-Stokes equations in wavelet decomposition.
One can think of a fluid composed by various eddies, where dynamic causes eddies to split into smaller ones with a kinetic energy transfer. So the tree-like structure have eddies as nodes, and a node is child of another node if the corresponding eddy is formed by a split of the corresponding eddy of the father. We denote by J the set of nodes, and if $j \in J$ we call $\mathcal{O}_j$ the set of offspring of j. We made assumption that every eddies has the same biggest eddy as ancestor, so we can classify eddies in "generations" or "levels". Level 0 is made by only

the biggest eddy $\emptyset \in J$, level 1 is made by the eddies produced by the one in level 0 and so on. We will denote the generation of an eddy j by $|j|$. The father of an eddy j will be denoted as $\bar{\jmath}$.
To construct the dynamic we associate to each eddy j a non-negative intensity $X_j(t)$. The first tree model studied in [9], adapted from the model that formally is not yet a tree that is introduced by Katz and Pavlovic in [21] is the following

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k,$$

with positive coefficients c_j. This model mimics the energy cascade from an eddy to his offspring, with speed of the flow c_j from an eddy to his children, and c_j is such that there exists a $\lambda > 0$ such that $c_j = 2^{\lambda|j|}$.
The Katz-Pavlovic model is formally conservative, as will be more clear later. Using the standard techniques we can have an existence result starting from a Galerkin approximation of the infinite dimensional system. We consider then for any $Q = \sum_{|k|\leq N} 1$ the following Q-dimensional approximation

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k,$$

with $c_k = 0$ for $k = 0$ and $|k| \geq N$.

Proposition 4.1.1. *The quantity*

$$\sum_j X_j(t)^2$$

is invariant along the trajectories of the Q-dimensional approximation.

Proof. By a straightforward computation we get

$$\frac{d}{dt}\sum_i X_j^2(t) = 2\sum_j X_j(t)\left(c_i X_{\bar{\jmath}}^2(t) - \sum_{k\in\mathcal{O}_j} c_k X_j(t) X_k(t)\right) =$$

$$= 2\sum_i \left(c_j X_{\bar{\jmath}}^2(t) X_j(t) - \sum_{k\in\mathcal{O}_j} c_k X_j^2(t) X_k(t)\right),$$

and since each non zero term appears in the sum once as father and once as son, it is all equal to 0.

□

Thanks to the conservative property we can have the existence and uniqueness for the Q-dimensional approximation.

Proposition 4.1.2. *Let c_j be the coefficients of the Q-dimensional approximation. Then following system*

$$\begin{cases} \frac{d}{dt}X_j(t) = c_j X_{\bar{\jmath}}^2(t) - \sum_{k\in\mathcal{O}_j} c_k X_j(t) X_k(t) & \\ X_j(0) = \underline{u}_j & \text{for } j = 1,\dots,Q \\ X_j(t) = 0 & \text{for } j \geq Q+1 \end{cases}$$

with $t \in [0,T]$ admits a unique global solution.

Proof. The system satisfies Cauchy-Lipschitz theorem and admits a local solution $(X_j(t)^{(Q)})$ on $[0,\delta]$ for a certain $\delta > 0$. We can extend the local solution to a global solution thanks to the estimates of the previous proposition, since for every $t \in [0,T]$

$$\|X^{(Q)}(t)\|^2 = \|X^{(Q)}(0)\|^2.$$

□

Then we can have the existence of solution in the Katz-Pavlovic model for ℓ^2 initial conditions.

Theorem 4.1.3. *With the same coefficients of the Katz-Pavlovic model, consider the following*

$$\frac{d}{dt}X_j(t) = c_j X_j^2(t) - \sum_{k\in\mathcal{O}_j} c_k X_j(t)X_k(t),$$

$$X(0) = \bar{X}.$$

So for any initial condition $\bar{X} \in \ell^2$ there exists at least a solution $X(t)$ on $[0,T]$.

Proof. We will use the Ascoli-Arzelà theorem. For any $\bar{X}$, we consider a sequence of solutions of the Q-dimensional approximation $\tilde{X}^Q(t)$ obtained considering as initial condition $\bar{X}^Q$ the first Q-entries of $\bar{X}$.
With abuse of notation we consider all the function $X^Q(t)$ embedded in the same infinite dimensional space by taking the value 0 on the empty entries. For every fixed j and $t \in [0,T]$ it holds:

- Uniform boundedness of $\{X_j^Q(t)\}$ for both Q and t:

$$|X_j^Q(t)| \leq \|X^Q(t)\|_2^2 \leq \|\bar{X}\|_2^2.$$

- Equi-Lipschitzianity of $\{X_j^Q(t)\}$ with respect to Q:

$$|\frac{d}{dt}X_j^Q(t)| \leq c_j |X_j^Q(t)^2| + \sum_{k\in\mathcal{O}_j} c_k |X_j^Q(t)X_k^Q(t)| \leq$$

$$\leq \|\bar{X}\|_2^2 (c_j + \sum_{k\in j} c_k).$$

Ascoli-Arzelà theorem implies for each fixed j the existence of a converging subsequence in $C([0,T])$, i.e. it is possible to find indices $\{N_k^j, k \in \mathbb{N}\}$ such that

$$\sup_{t\in[0,T]} |X_j^{N_k^j}(t) - X_j(t)| \to 0$$

for fixed j as $k \to \infty$.
The sequence $N_\bullet^j$ can be chosen so that $N_\bullet^{j-1}$ is a subsequence of $N_\bullet^j$ itself. By a standard diagonal argument we can extend the convergence to all j. If we consider indices $N_k = N_k^k$, we are extracting a common subsequence such that

$$\sup_{t\in[0,T]} |X_j^{N_k}(t) - X_j(t)| \to 0$$

for all $j \geq 0$, as $k \to \infty$.

Last it is straightforward to check that the limit obtained via Ascoli-Arzelà theorem, $X(t)$, is a solution for system, using the equation in the integral form. $\square$

To have a more complete introduction to tree model one can see the work [9]. Our model will be way more general from the Katz-Pavlovic one, which represents a particular case of our system.

We consider the following dynamic system on the tree, where $\emptyset$ is the root.

$$\text{Infinite System} \tag{4.1}$$

$$\frac{d}{dt}X_j = \alpha\left(c_j X_{\bar{j}}^2 - X_j \sum_{k\in\mathcal{O}_j} c_k X_k\right) + \beta\left(\tilde{c}_j X_{\bar{j}} X_j - \sum_{k\in\mathcal{O}_j} \tilde{c}_k X_k^2\right) + $$
$$+\gamma\left(X_{\bar{j}} \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} \hat{c}_{j,l} X_l - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2}\right),$$

$$\frac{d}{dt}X_\emptyset = f(t, X_\emptyset) - \alpha X_\emptyset \sum_{k\in\mathcal{O}_\emptyset} c_k X_k - \beta \sum_{k\in\mathcal{O}_\emptyset} \tilde{c}_k X_k^2 - \gamma \sum_{k_1\neq k_2, k_i\in\mathcal{O}_\emptyset} \hat{c}_{k_1,k_2} X_{k_1} X_{k_2},$$

$$X(0) = \bar{X},$$

where $f(t,x) \leq c(t) + g(t)|x|$, with $c(t)$ and $g(t)$ positive continuous functions. A standard assumption on the coefficients c_j, $\tilde{c}_j$ and $\hat{c}_{j_1,j_2}$ is that they have an exponential growing depending on their index generation, i.e. $c_j = 2^{\eta|j|}$, $\tilde{c}_j = 2^{\tilde{\eta}|j|}$ and $\hat{c}_{j_1,j_2} = 2^{\hat{\eta}|j_1|}$, with $\eta, \tilde{\eta}, \hat{\eta} > 0$.

The aim of this work is to prove the existence of solutions of the dynamic system with $\bar{X} \in \ell^2$. To do this the first step is to introduce a truncated version of the system (1), for any $N \in \mathbb{N}$ we consider the following finite-dimensional dynamic system:

$$\text{Truncated System} \tag{4.2}$$

$$\frac{d}{dt}X_j = \alpha\left(b_j X_{\bar{j}}^2 - X_j \sum_{k\in\mathcal{O}_j} b_k X_k\right) + \beta\left(\tilde{b}_j X_{\bar{j}} X_j - \sum_{k\in\mathcal{O}_j} \tilde{b}_k X_k^2\right) + $$
$$+\gamma\left(X_{\bar{j}} \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} \hat{b}_{j,l} X_l - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} \hat{b}_{k_1,k_2} X_{k_1} X_{k_2}\right),$$

$$\frac{d}{dt}X_\emptyset = f(t, X_\emptyset) - \alpha X_\emptyset \sum_{k\in\mathcal{O}_\emptyset} b_k X_k - \beta \sum_{k\in\mathcal{O}_\emptyset} \tilde{b}_k X_k^2 - \gamma \sum_{k_1\neq k_2, k_i\in\mathcal{O}_\emptyset} \hat{b}_{k_1,k_2} X_{k_1} X_{k_2},$$

$$X(0) = \bar{X}^N,$$

where $b_k, \tilde{b}_k, \hat{b}_{k,k_1} = 0$ for any $|k| > N$, $b_k, \tilde{b}_k, \hat{b}_{k,k_1} = c_k, \tilde{c}_k, \hat{c}_{k,k_1}$ for any $|k| \leq N$.

The truncated system represents an approximation, in the Galerkin sense, of the infinite system. Moreover it's easier to study the growth of the kinetic energy in the truncated system.

4.2 Existence in forced dyadic system

To get the existence result on the tree we look at the techniques shown in the work [4]. In that work a dyadic system with a noise on the first component is considered

$$\begin{cases} du_0 = -u_0 u_1 dt + \sigma dW(t) \\ du_j = (-2^{cj} u_j u_{j+1} + 2^{c(j-1)} u_{j-1}^2) dt \\ u(0) = \underline{\mathrm{u}}, \end{cases}$$

with $t \in [0, T]$, $\sigma \in \mathbb{R}^+$, $c \in [1, 3]$, $\underline{\mathrm{u}} \in \ell^2$, $\underline{\mathrm{u}}_j \geq 0$ for every $j \geq 1$ and where $\{W(t)\}_{t \geq 0}$ is a one dimensional Brownian motion.
At first look this system has many differences from the one we are studying, first of all it is a dyadic model instead a tree one, the second major difference is the noise instead of a deterministic forced first component. Despite that, the existence relies only on an energy bound and we can use the same techniques in our model.

The second equation is formally conservative, in the sense that

$$\sum_{j=0}^{\infty} u_j (2^{c(j-1)} u_{j-1}^2 - 2^{cj} u_j u_{j+1}) = 0,$$

so the obstacle to conclude with an energy argument is given by the perturbation on the first component.

The system can be studied as a deterministic system considering for fixed $\omega \in \Omega$

$$\begin{cases} u_0(t) = u_0(0) - \int_0^t u_0(s) u_1(s) ds + \sigma dw(t) \\ du_j = (-2^{cj} u_j u_{j+1} + 2^{c(j-1)} u_{j-1}^2) dt \\ u(0) = \underline{\mathrm{u}}, \end{cases}$$

where $t \in [0, T]$, $\sigma \in \mathbb{R}^+$, $c \in [1, 3]$, $\underline{\mathrm{u}} \in H^+$ and $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$. Here H^+ is the set of sequences which are positive away from the component $j = 0$ that is

$$H^+ = \{u \in \ell^2 : u_j \geq 0, j \geq 1\}.$$

This is set to let $u(t) \in H^+$ for all $t \in [0, T]$. Proving the existence for fixed element $\omega \in \Omega$ is equivalent to prove pathwise existence for the stochastic system, hence we give the definition of solution for the deterministic version.

Definition 4.2.1. *We say that u is a solution for the deterministic system on $[0,T]$ with initial condition $\underline{u} \in H^+$ and noise $w \in C([0,T],\mathbb{R})$ with $w(0)=0$, if u satisfies the system, $u_j \in C([0,T],\mathbb{R})$ for all j and $u(t) \in H^+$ for all $t \in [0,T]$.*

As usual all starts with the truncated version of the infinite dimensional system, so for any $N \in \mathbb{N}$ we define the following system

$$\begin{cases} u_0^{(N)}(t) = u_0^{(N)}(0) - \int_0^t u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma dw(t) & \\ du_j^{(N)}(t) = (-2^{cj}u_j^{(N)}(t)u_{j+1}^{(N)}(t) + 2^{c(j-1)}(u_{j-1}^{(N)}(t))^2)dt & \text{for } j=1,\dots,N \\ u_j^{(N)}(0) = \underline{u}_j & \text{for } j=1,\dots,N \\ u_j^{(N)}(t) = 0 & \text{for } j \geq N+1 \end{cases}$$

with $t \in [0,T]$ and $\underline{u} \in H^-$. The following proposition plays the central role in the existence result of this section

Proposition 4.2.2. *Let $\|\cdot\|$ denote the ℓ^2 norm, $a = \|\underline{u}\| + 2\sigma\|w\|_\infty$. Then*

$$\|u^{(N)}(t)\|^2 \leq (a^2T + 2a)^2.$$

Proof. First we show that the component $u_0^{(N)}(t)$ is uniformly bounded in N, so we consider the following equation

$$u_0^{(N)}(t) = u_0^{(N)}(0) - \int_0^t u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma w(t).$$

So we can assume

$$\max_{t\in[0,\delta]} |u_0^{(N)}(t)| = u_0^{(N)}(t^*) > 0,$$

for some $t^* \in [0,\delta]$. Then let $\bar{t} = \sup\{t \leq t^* : u_0^{(N)}(t) = 0\}$; if the set $\{t \leq t^* : u_0^{(N)}(t) = 0\}$ is empty $\bar{t} = 0$. Hence

$$\begin{aligned} u_0^{(N)}(t^*) &\leq |u_0^{(N)}(0)| - \int_{\bar{t}}^{t^*} u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma[w(t^*) - w(\bar{t})] \\ &\leq |u_0^{(N)}(0)| + 2\sigma \sup_{t\in[0,T]} |w(t)|. \end{aligned}$$

Hence for every t it holds

$$|u_0^{(N)}(t)| \leq a.$$

Looking at all the components we get

$$\|u^{(N)}(t)\|^2 = \sum_{j=0}^N (u_j^{(N)}(t))^2 \leq a^2 + \sum_{j=1}^N (u_j^{(N)}(t))^2.$$

Now it we take the derivative of the second sum we have

$$\frac{d}{dt}[\sum_{j=1}^N (u_j^{(N)}(t))^2] = 2(u_0^{(N)}(t))^2 u_1^{(N)}(t) \leq 2a^2 u_1^{(N)}(t) \leq 2a^2 \sqrt{\sum_{j=1}^N (u_j^{(N)}(t))^2}.$$

Hence it follows

$$\sum_{j=1}^{N}(u_j^{(N)}(t))^2 \le (a^2 t + \sqrt{\sum_{j=1}^{N}(u_j^{(N)}(0))^2})^2 \le (a^2 t + a)^2.$$

So finally it holds

$$\|u^{(N)}(t)\|^2 \le a^2 + (a^2 t + a)^2 \le (a^2 T + 2a)^2.$$

□

With the energy bound we can first have the existence and uniqueness of solutions for the truncated system.

Proposition 4.2.3. *The N-dimensional system*

$$\begin{cases} u_0^{(N)}(t) = u_0^{(N)}(0) - \int_0^t u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma dw(t) & \\ du_j^{(N)}(t) = (-2^{cj}u_j^{(N)}(t)u_{j+1}^{(N)}(t) + 2^{c(j-1)}(u_{j-1}^{(N)}(t))^2)dt & \text{for } j = 1, \dots, N \\ u_j^{(N)}(0) = \underline{u}_j & \text{for } j = 1, \dots, N \\ u_j^{(N)}(t) = 0 & \text{for } j \ge N+1 \end{cases}$$

with $t \in [0,T]$ admits a unique global solution.

Proof. The system satisfies Cauchy-Lipschitz theorem and admits a local solution $(u_n^{(N)})_{n\in\mathbb{N}}$ on $[0,\delta]$ for a certain $\delta > 0$. We can extend the local solution to a global solution thanks to the estimates of the previous proposition, since for every $t \in [0,T]$

$$\|u^{(N)}(t)\|^2 \le (a^2 T + 2a)^2.$$

□

Having the existence for the truncated system, we can go for the main result of this section, the existence for the infinite dimensional system.

Theorem 4.2.4. *For every $T \in [0,+\infty)$, $\sigma \in \mathbb{R}^+$, $c \in [1,3]$, $\underline{u} \in H^+$ and $w \in C([0,T],\mathbb{R})$ with $w(0) = 0$, the system*

$$\begin{cases} u_0(t) = u_0(0) - \int_0^t u_0(s)u_1(s)ds + \sigma dw(t) \\ du_j = (-2^{cj}u_j u_{j+1} + 2^{c(j-1)}u_{j-1}^2)dt \\ u(0) = \underline{u}, \end{cases}$$

admits at least a solution.

Proof. As usual we want to apply Ascoli-Arzelà theorem.
For every fixed j and $t \in [0,T]$ it holds:

- Uniform boundedness of $(u_j^{(N)}(t))_{N\in\mathbb{N}}$ in both N and t:

$$|u_j^{(N)}(t)| \le \|u^{(N)}(t)\| \le a^2 T + 2a;$$

- Equi-Lipschitzianity of $(u_j^{(N)}(t))_{N\in\mathbb{N}}$ with respect to N:

$$|\frac{d}{dt}u_j^{(N)}(t)| \leq 2^{cj+1}\|u^{(N)}(t)\|^2 \leq 2^{cj+1}(a^2T+2a)^2.$$

Ascoli-Arzelà theorem implies for each fixed j the existence of a converging subsequence in $C([0,T])$, i.e. it is possible to find indices $\{N_k^j, k\in\mathbb{N}\}$ such that

$$\sup_{t\in[0,T]} |u_j^{(N_k^j)-u_j(t)}| \to 0$$

for fixed j as $k\to\infty$.
The sequence $N_\bullet^j$ can be chosen so that $N_\bullet^{j-1}$ is a subsequence of $N_\bullet^j$ itself. By a standard diagonal argument we can extend the convergence to all j. If we consider indices $N_k = N_k^k$, we are extracting a common subsequence such that

$$\sup_{t\in[0,T]} |u_j^{(N_k)} - u_j(t)| \to 0$$

for all $j\geq 0$, as $k\to\infty$.
Also it is straightforward to check that the limit is a solution for the integral representation of the equation.

□

4.3 Energy Bound

We come back to the forced tree model, as shown in the last section we need an energy bound to get the overall existence result. As usually we defined the energy in the classic way.

Definition 4.3.1. *Kinetic Energy*
For system 4.1 we define Kinetic Energy the sum

$$\mathcal{E}_c(t) = \sum_{|j|=0}^{\infty} X_j^2(t) = \|X(t)\|_2^2.$$

For system 4.2 we define Kinetic Energy the sum

$$\mathcal{E}_c^N(t) = \sum_{|j|=0}^{N} X_j^2(t).$$

By this definition the energy is formally telescoping, except for the first term:

Proposition 4.3.2. *It holds*

$$\frac{d}{dt}\mathcal{E}_c^N(t) = 2X_\emptyset(t)f(t,X_\emptyset(t)).$$

Proof. By a straightforward computation we get

$$\frac{d}{dt}\mathcal{E}_c^N(t) = 2\sum_{|j|=1}^{N}\left(X_j(t)\alpha(b_jX_{\bar{j}}^2(t) - X_j(t)\sum_{k\in\mathcal{O}_j} b_kX_k(t))\right) +$$

$$+2\sum_{|j|=1}^{N}\left(X_j(t)\beta(\tilde b_j X_{\bar j}(t)X_j(t)-\sum_{k\in\mathcal{O}_j}\tilde b_k X_k^2(t))\right)+$$

$$+2\sum_{|j|=1}^{N}\left(X_j(t)\gamma\left(X_{\bar j}(t)\sum_{l\neq j,l\in\mathcal{O}_{\bar j}}\left(\hat b_{j,l}X_l(t))-\sum_{k_1\neq k_2,k_i\in\mathcal{O}_j}(\hat b_{k_1,k_2}X_{k_1}(t)X_{k_2}(t)\right)\right)\right)+$$

$$+2X_\emptyset(t)(f(t,X_\emptyset(t))-\alpha X_\emptyset\sum_{k\in\mathcal{O}_\emptyset}(b_kX_k(t))-\beta\sum_{k\in\mathcal{O}_\emptyset}(\tilde b_k X_k^2(t))-$$

$$\gamma\sum_{k_1\neq k_2,k_i\in\mathcal{O}_\emptyset}(\hat b_{k_1,k_2}X_{k_1}(t)X_{k_2}(t))),$$

and each term outside of $2X_\emptyset(t)f(t,X_\emptyset(t))$ appears once as father and once as son with inverted sign, so they all elide and we get the thesis.

□

We can finally get the energy bound we need using Gronwall lemma.

Proposition 4.3.3. *Let* $\|\bar X^N\|_2\le a_N$, $\sup_{[0,T]}c(t)=d$, $\sup_{[0,T]}g(t)=f$. *So*

$$|\mathcal{E}_c^N(t)|\le(a_N+Td)e^{T(d+2f)}.$$

Proof. From Proposition 4.3.2 we have

$$\frac{d}{dt}\mathcal{E}_c^N(t)\le 2X_\emptyset(t)f(t,X_\emptyset(t))\le 2\sqrt{\mathcal{E}_c^N}(c(t)+g(t)\sqrt{\mathcal{E}_c^N})\le 2c(t)\sqrt{\mathcal{E}_c^N}+2g(t)\mathcal{E}_c^N,$$

so, since $\sqrt{\mathcal{E}_c^N}\le\frac{1+\mathcal{E}_c^N}{2}$ we get

$$\frac{d}{dt}\mathcal{E}_c^N(t)\le c(t)+(c(t)+2g(t))\mathcal{E}_c^N,$$

now we define

$$\tilde{\mathcal{E}}_c^N(t)=\mathcal{E}_c^N(t)-\int_0^t e^{\int_s^t c(u)+2g(u)du}c(s)ds,$$

so since

$$\frac{d}{dt}\int_0^t e^{\int_s^t c(u)+2g(u)du}c(s)ds=c(t)+(c(t)+2g(t))\int_0^t e^{\int_s^t c(u)+2g(u)du}c(s)ds,$$

it holds

$$\frac{d}{dt}\tilde{\mathcal{E}}_c^N(t)\le(c(t)+2g(t))\tilde{\mathcal{E}}_c^N(t),$$

then by Gronwall Lemma

$$\tilde{\mathcal{E}}_c^N(t)\le\tilde{\mathcal{E}}_c^N(0)e^{\int_0^t c(s)+2g(s)ds},$$

thus

$$|\mathcal{E}_c^N(t)|\le|\mathcal{E}_c^N(0)|e^{\int_0^t c(s)+2g(s)ds}+\int_0^t e^{\int_s^t c(u)+2g(u)du}c(s)ds.$$

□

4.4 Existence of solutions

We are left to prove the existence of solutions on the forced tree model for ℓ^2 initial conditions. As usual we first prove the existence in the truncated N-dimensional model.

Proposition 4.4.1. *For any N and any initial condition $\bar{X}^N$ the truncated system 4.2 admits a solution $X^N(t)$ on $[0,T]$.*

Proof. The system 4.2 satisfies the Cauchy-Lipschitz theorem and then admits a local solution X^N on $[0,\delta]$ for some $\delta > 0$. We can extend the local solution to a global solution using the energy bound of Proposition 4.3.3:

$$\|X^N(t)\|_2^2 \leq (a_N + Td)e^{T(d+2f)},$$

so from this bound on $\|X^N(t)\|_2$ we get the global existence of solutions. $\square$

At this point it lasts only to glue all the result together and use Ascoli-Arzelà theorem to get the existence for the infinite dimensional forced tree model.

Theorem 4.4.2. *In system 4.1 for any initial condition $\bar{X} \in \ell^2$ there exists at least a solution $X(t)$ on $[0,T]$.*

Proof. We will use the Ascoli-Arzelà theorem and the results of Proposition 4.3.3. For any $\bar{X}$, we consider a sequence of solutions of system 4.2 $\tilde{X}^N(t)$ obtained considering as initial condition $\bar{X}^N$ the first k_N-entries of $\bar{X}$, where k_N is the dimension of system 4.2 for fixed N. For every fixed j and $t \in [0,T]$ it holds:

- Uniform boundedness of $\{X_j^N(t)\}_{N\in\mathbb{N}}$ for both N and t:

$$|X_j^N(t)| \leq \|X^N(t)\|_2^2 \leq (a_N + Td)e^{T(d+2f)} \leq (\|\bar{X}\|_2^2 + Td)e^{T(d+2f)}.$$

- Equi-Lipschitzianity of $\{X_j^N(t)\}_{N\in\mathbb{N}}$ with respect to N:

$$|\frac{d}{dt}X_j^N(t)| \leq |\mathcal{E}_c^N(t)||(\alpha(b_j + \sum_{k\in\mathcal{O}_j} b_k) + \beta(\tilde{b}_j + \sum_{k\in\mathcal{O}_j} \tilde{b}_k) +$$

$$\gamma(\sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} \hat{b}_{j,l} + \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} \hat{b}_{k_1,k_2})| + |c(t) + g(t)\sqrt{\mathcal{E}_c^N(t)}|,$$

  and since $|\mathcal{E}_c^N(t)| \leq (\|\bar{X}\|_2^2 + Td)e^{T(d+2f)}$ we have for a constant $M_j = (\alpha(b_j+\sum_{k\in\mathcal{O}_j} b_k)+\beta(\tilde{b}_j+\sum_{k\in\mathcal{O}_j}\tilde{b}_k)+\gamma(\sum_{l\neq j,l\in\mathcal{O}_{\bar{j}}}\hat{b}_{j,l}+\sum_{k_1\neq k_2,k_i\in\mathcal{O}_j}\hat{b}_{k_1,k_2})$ not depending on N:

$$|\frac{d}{dt}X_j^N(t)(\|\bar{X}\|_2^2 + Td)e^{T(d+2f)}M_j + d + f\sqrt{(\|\bar{X}\|_2^2 + Td)e^{T(d+2f)}}.$$

Ascoli-Arzelà theorem implies for each fixed j the existence of a converging subsequence in $C([0,T])$, i.e. it is possible to find indices $\{N_k^j, k \in \mathbb{N}\}$ such that

$$\sup_{t\in[0,T]} |X_j^{N_k^j}(t) - X_j(t)| \to 0$$

for fixed j as $k \to \infty$.
The sequence $N_\bullet^j$ can be chosen so that $N_\bullet^{j-1}$ is a subsequence of $N_\bullet^j$ itself. By a standard diagonal argument we can extend the convergence to all j. If we consider indices $N_k = N_k^k$, we are extracting a common subsequence such that

$$\sup_{t\in[0,T]} |X_j^{N_k}(t) - X_j(t)| \to 0$$

for all $j \geq 0$, as $k \to \infty$.

We last have to show that the limit obtained via Ascoli-Arzelà theorem, $X(t)$, is a solution for system 4.1. This follow by a straightforward computation, for fixed component $j \in \mathbb{N}$ let N_k be big enough such that

$$\sup_{[0,T]} |X_j^{N_k}(t) - X_j(t)| < \varepsilon,$$

so

$$|X_j(t) - X_j(0) - \int_0^t [\alpha(c_j X_{\bar{j}}^2(s) - X_j(s) \sum_{k\in\mathcal{O}_j} c_k X_k(s)) + \beta(\tilde{c}_j X_{\bar{j}}(s) X_j(s) - \sum_{k\in\mathcal{O}_j} \tilde{c}_k X_k^2(s)) +$$

$$+\gamma(X_{\bar{j}}(s) \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} (\hat{c}_{j,l} X_l(s)) - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} (\hat{c}_{k_1,k_2} X_{k_1}(s) X_{k_2}(s)))ds]| =$$

$$= |X_j(t) - X_j(0) - \int_0^t [\alpha(c_j X_{\bar{j}}^2(s) - X_j(s) \sum_{k\in\mathcal{O}_j} c_k X_k(s)) + \beta(\tilde{c}_j X_{\bar{j}}(s) X_j(s) - \sum_{k\in\mathcal{O}_j} \tilde{c}_k X_k^2(s)) +$$

$$+\gamma(X_{\bar{j}}(s) \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} (\hat{c}_{j,l} X_l(s)) - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} (\hat{c}_{k_1,k_2} X_{k_1}(s) X_{k_2}(s)))ds] -$$

$$-X_j^{N_k}(t) + X_j^{N_k}(0) + \int_0^t [\alpha(c_j (X_{\bar{j}}^{N_k})^2(s) - X_j^{N_k}(s) \sum_{k\in\mathcal{O}_j} c_k X_k^{N_k}(s)) +$$

$$+\beta(\tilde{c}_j X_{\bar{j}}^{N_k}(s) X_j^{N_k}(s) - \sum_{k\in\mathcal{O}_j} \tilde{c}_k (X_k^{N_k})^2(s)) +$$

$$+\gamma(X_{\bar{j}}^{N_k}(s) \sum_{l\neq j, l\in\mathcal{O}_{\bar{j}}} (\hat{c}_{j,l} X_l^{N_k}(s)) - \sum_{k_1\neq k_2, k_i\in\mathcal{O}_j} (\hat{c}_{k_1,k_2} X_{k_1}^{N_k}(s) X_{k_2}^{N_k}(s)))ds]| \leq$$

$$|X_j(t) - X_j^{N_k}(t)| + \alpha c_j \int_0^t |X_{\bar{j}}^2(s) - {X_{\bar{j}}^{N_k}}^2(s)| ds +$$

$$+\alpha \int_0^t \sum_{k\in\mathcal{O}_j} c_k |X_j(s) X_k(s) - X_j^{N_k}(s) X_k^{N_k}(s)| ds +$$

$$+\beta\tilde{c}_j \int_0^t |X_{\bar{j}}(s) X_j(s) - X_{\bar{j}}^{N_k}(s) X_j^{N_k}(s)| ds + \beta \sum_{k\in\mathcal{O}_j} \tilde{c}_k \int_0^t |X_k^2(s) - {X_k^{N_k}}^2(s)| ds +$$

$$+\gamma\sum_{l\neq j,l\in\mathcal{O}_{\bar{j}}}\hat{c}_{j,l}\int_0^t|X_{\bar{j}}(s)X_l(s)-X_{\bar{j}}^{N_k}(s)X_l^{N_k}(s)|ds+$$

$$\gamma\sum_{k_1\neq k_2,k_i\in\mathcal{O}_j}\hat{c}_{k_1,k_2}\int_0^t|X_{k_1}(s)X_{k_2}(s)-X_{k_1}^{N_k}(s)X_{k_2}^{N_k}(s)|ds\leq$$

$$\varepsilon+\alpha c_j t(\varepsilon^2+2\varepsilon\sup_{s\in[0,t]}|X_j(s)|)+\alpha t\sum_{k\in\mathcal{O}_j}c_k(\varepsilon^2+\varepsilon\sup_{s\in[0,t]}|X_j(s)|+\varepsilon\sup_{s\in[0,t]}|X_k(s)|)+$$

$$+\beta\tilde{c}_j t(\varepsilon^2+\varepsilon\sup_{s\in[0,t]}|X_j(s)|+\varepsilon\sup_{s\in[0,t]}|X_{\bar{j}}(s)|)+\beta\sum_{k\in\mathcal{O}_j}\tilde{c}_k t(\varepsilon^2+2\varepsilon\sup_{s\in[0,t]}|X_k(s)|)+$$

$$+\gamma t\sum_{l\neq j,l\in\mathcal{O}_{\bar{j}}}(\hat{c}_{j,l}(\varepsilon^2+\varepsilon\sup_{s\in[0,t]}|X_{\bar{j}}(s)|+\varepsilon\sup_{s\in[0,t]}|X_l(s)|)+$$

$$+\gamma\sum_{k_1\neq k_2,k_i\in\mathcal{O}_j}\hat{c}_{k_1,k_2}t(\varepsilon^2+\varepsilon\sup_{s\in[0,t]}|X_{k_1}(s)|+\varepsilon\sup_{s\in[0,t]}|X_{k_2}(s)|),$$

and it all goes to 0 as $\varepsilon\to 0$. A similar computation works for $X_\emptyset(t)$. □

We want to stress to the reader that this result relies almost only on the energy bound, as long as one can prove that the limit obtained can solve the integral equation.

4.5 Invariant measure method on the tree

In this section we show how one can replicate the work of the previous chapter on a tree shell model.

We will slightly modify the model of this chapter to let to a certain class of Gaussian measures to be invariant for the system. The following system is the one we want to explore in this section:

$$\frac{d}{dt}X_j=\alpha(c_jX_{\bar{j}}^2-\sum_{k\in\mathcal{O}_j}c_kX_jX_k)-\beta(d_{\bar{j}}X_{\bar{j}}X_j-\sum_{k\in\mathcal{O}_j}d_jX_k^2),$$

where $\alpha,\beta,c_i,d_i\in\mathbb{R}$, $\alpha c_j-\beta d_j=0$ for any $|j|\geq 1$, $c_0=d_{\bar{0}}=d_0=0$.
As usual we consider the truncated version of the infinite dimensional system. So, for $N\in\mathbb{N}$ we define:

$$\frac{d}{dt}X_j=\alpha(c_jX_{\bar{j}}^2-\sum_{k\in\mathcal{O}_j}c_kX_jX_k)-\beta(d_{\bar{j}}X_{\bar{j}}X_j-\sum_{k\in\mathcal{O}_j}d_jX_k^2),$$

where $\alpha,\beta,c_i,d_i\in\mathbb{R}$, $\alpha c_j-\beta d_j=0$ for any $1\leq|j|\leq N-1$, $c_0=d_{\bar{0}}=d_0=0$ and if $|j|\geq N$, $c_j=d_j=0$.

For both system (infinite and truncated ones) we ask that there exists an $M\in\mathbb{N}$ such that $\sum_{k\in\mathcal{O}_j}1\leq M$ for every $j\in\mathbb{N}$ and there exists a $\lambda>1$ such that

$$c_j=\lambda c_{\bar{j}}$$

for every $j \geq 2$ for the infinite system, for every $2 \leq j \leq Q$ for the truncated one and $c_j = \lambda$ for $|j| = 1$.

For now on we put $Q = \sum_{|k|\leq N} 1$.

Theorem 4.5.1. *The Q-dimensional dynamic system defined above is conservative, in the sense that the kinetic energy $\sum_{|k|\leq N} X_k^2$ is preserved.*

Proof.

$$\frac{d}{dt}(\sum_{|j|\leq N} X_j^2) = 2\sum_j \alpha(c_j X_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j^2 X_k) - \beta(d_{\bar{j}} X_{\bar{j}} X_j^2 - \sum_{k\in\mathcal{O}_j} d_j X_j X_k^2) =$$

the two sums are telescoping since every term compares once as father and once as son, so

$$= 2(\alpha c_0 X_0 X_{\bar{0}}^2 - \sum_{|j|=N, k\in\mathcal{O}_j} \alpha c_k X_j^2 X_k - \beta d_0 X_0^2 X_{\bar{0}} + \sum_{|j|=N, k\in\mathcal{O}_j} \beta d_j X_j X_k^2) = 0.$$

□

Proposition 4.5.2. *For $N \in \mathbb{N}$ the system:*

$$\frac{d}{dt} X_j = \alpha(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k) - \beta(d_{\bar{j}} X_{\bar{j}} X_j - \sum_{k\in\mathcal{O}_j} d_j X_k^2),$$
$$X(0) = X_0$$

where $\alpha, \beta, c_i, d_i \in \mathbb{R}$, $\alpha c_j - \beta d_j = 0$ for any $1 \leq |j| \leq N-1$, $c_0 = d_{\bar{0}} = d_0 = 0$ and if $|j| \geq N$, $c_j = d_j = 0$ admits a solution for any initial condition X_0.

Proof. The proof follow directly by proposition 4.4.1.

□

As proved in theorem 4.4.2 the infinite dimensional tree model defined in this section admits a solution for any ℓ^2 initial condition. On the tree model considered in this section we want to improve the existence result from an ℓ^2 initial condition result to an almost every initial condition, with respect to a Gaussian measure to infinite dimensional space of initial conditions.

Theorem 4.5.3. *The product Gaussian measure*

$$\mu_r^Q = \bigotimes_{|j|\leq N} \mathcal{N}(0, r^2)$$

is invariant for the system.

Proof. Consider the dynamic system in the following form:

$$\dot{X} = b(x).$$

As proved for the dyadic model it suffices to have

$$\operatorname{div}(b(x)f(x)) = 0.$$

For Gaussian measures μ_r^Q we have $f(x) = ce^{\frac{-\|x\|^2}{r^2}}$ for some constant $c > 0$, so

$$\operatorname{div}(b(x)f(x)) = \sum_{i=1}^{Q} \frac{\partial}{\partial X_i} b(x)f(x)_i =$$

$$ce^{-\frac{\|X\|^2}{2}}[\sum_j \alpha(c_j X_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j^2 X_k) - \beta(d_{\bar{j}} X_{\bar{j}} X_j^2 - \sum_{k\in\mathcal{O}_j} d_j X_j X_k^2)] +$$

$$+ce^{-\frac{\|X\|^2}{2}}[\sum_{|j|\leq N} -\alpha(\sum_{k\in\mathcal{O}_j} c_k X_k) - \beta d_{\bar{j}} X_{\bar{j}}],$$

where the first term of the sum is equal to 0 as proved in the previous theorem, the second term is equal to

$$ce^{-\frac{\|X\|^2}{2}}(-\beta d_{\bar{0}} X_{\bar{0}} - \beta d_0 X_0 - \alpha \sum_{|j|=N, c\in\mathcal{O}_j} c_k X_k - \sum_{1\leq|j|\leq N-1} \alpha c_j + \beta d_j) = 0.$$

□

Definition 4.5.4. *Let $(\Omega, \mathcal{F}, P)$ be an abstract probability space, for every N and for $r > 0$ let Y_r^Q be a random variable*

$$Y_r^Q : (\Omega, \mathcal{F}, P) \to (\mathbb{R}^Q, \mathcal{B}(\mathbb{R}^Q)),$$

with law μ_r^Q.

Definition 4.5.5. *A set $(\Omega, \mathcal{F}, P, U_r^Q)$ is said to be a Q-finite random solution if U_r^Q is defined on the abstract probability space $(\Omega, \mathcal{F}, P) \times [0, T]$ to $\mathbb{R}^\infty$, all k-coordinates of U_r^Q are almost surely for each time $t \in [0, T]$ equal to 0 if $k > Q$ and for $k \leq Q$ almost surely*

$$U_r^Q{}_{(k)}(\omega, t) = F_r^Q{}_{(k)}(\omega, t),$$

where, for $\omega \in \Omega$, the function

$$F_r^Q(\omega) : [0, T] \to \mathbb{R}^Q$$

is the unique solution of the Q-dimensional truncated tree model with initial conditions

$$X(0) = Y_r^Q(\omega).$$

Remark that F_r^Q is still a random variable from the abstract space $(\Omega, \mathcal{F}, P)$.

Proposition 4.5.6. *Let $(\Omega, \mathcal{F}, P, U_r^Q)$ be a Q-finite random solution. The law of $U_r^Q(t)$ is, for any $t \in [0, T]$,*

$$\tilde{\mu}_r^Q = \mu_r^Q \otimes \bigotimes_{Q+1}^{\infty} \delta_0.$$

Proof. It follows directly from the definition of $U_r^Q(t)$ and the invariance of μ_r^Q along the trajectories of the Q-dimensional truncated tree model.

□

We give to the space a similar norm to the one of the previous chapter.

Definition 4.5.7. *We define $(H^s, \|\cdot\|_{H^s})$ as the Hilbert space of sequences $x \in \mathbb{R}^\infty$ satisfying*

$$\|x\|_{H^s} = \sqrt{\sum_n c_n^{2s} x_n^2} < \infty.$$

Lemma 4.5.8. *Let c_n be the coefficients of the tree model, $k \in \mathbb{N} = \{0, 1, \dots\}$, $q \in \mathbb{R}$ such that $q + k < 0$. Let φ be a $\mathcal{C}^\infty$ function. Then*

$$\sum_n \log \varphi(t c_n^{2q+2k})$$

is h-times differentiable in $t = 0$ for any $h \in \mathbb{N}$ and the derivative operation commutes with the sum, we mean that for any h $\frac{d^h}{dt^h} \sum_n \log \varphi(t c_n^{2q+2k}) = \sum_n \frac{d^h}{dt^h} \log \varphi(t c_n^{2q+2k})$.

Proof. The proof is the same done for the dyadic case..

□

The following two proposition are the ones causing the coefficients c_k to be so restrictive. Until now one can have put $\sum_{i \in \mathcal{O}_j} c_i = \lambda c_j$, having a result on a more general tree. Sadly the two following estimates don't work with a non-increasing c_k coefficients and we need all c_k increase by a geometric factor from a generation to another.

Proposition 4.5.9. *Let $(\Omega, \mathcal{F}, P, U_r^N)$ be a Q-finite random solution. For every $s < 0$, $r > 0$, $p > 1$, $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, not depending on N, such that*

$$P(\|U_r^Q\|_{L^p(0,T;H^s)} \leq C_\epsilon) > 1 - \epsilon,$$

for each $Q = \sum_{|i| \leq N} 1$ with $N \in \mathbb{N} \setminus \{0\}$.

Proof. We want to prove that for any $p > 1$ and for any $\varepsilon > 0$ exists $R \in \mathbb{R}^+$ such that

$$P(\|U_r^Q\|_{L^p(0,T;H^s)} > R) < \varepsilon.$$

Hence

$$P(\|U_r^Q\|_{L^p(0,T;H^s)} > R) = P(\|U_r^Q\|^p_{L^p(0,T;H^s)} > R^p) \leq$$

now we apply Markov inequality

$$\leq \frac{1}{R^p} E[\|U_r^Q\|^p_{L^p(0,T;H^s)}] =$$

$$= \frac{1}{R^p} \|U_r^Q\|^p_{L^p(\Omega \times [0,T], H^s)} = \frac{1}{R^p} \int_0^T E[\|U_r^Q(t,\omega)\|^p_{H^s}] ds =$$

here, thanks to proposition 4.5.6, we use the time invariance for the law of U_r^Q

$$= \frac{T}{R^p} E[\|U_r^Q(0,\omega)\|^p_{H^s}].$$

Hence it is sufficient to show that for any $p > 1$ exists $C \in \mathbb{R}^+$ such that

$$E[\|U_r^Q(0,\omega)\|_{H^s}^p] < C,$$

for any Q, since the proof will follow letting $R \to \infty$. Note that for each Q holds

$$E[\|U_r^Q(0,\omega)\|_{H^s}^p] \le E[|(\sum_{n=1}^{\infty} c_n^{2s} r^2 W_n(\omega))^{\frac{p}{2}}|]$$

where $W_i \sim \chi^2(1)$, with $\{W_i\}_i$ iid.
So it is sufficient to prove that the random variable

$$Z = \sum_{n \ge 1} c_n^{2s} r^2 W_n,$$

has a moment generating function derivable infinite times in 0, this would imply that it has finite p-moment for any $p \ge 1$ and this is would give the uniform bound in Q for $L^p(0,T;H^s)$ norm we need.

The moment generating function of Z, $\psi(t)$ is

$$\psi(t) = E[e^{t \sum c_n^{2s} r^2 W_n(\omega)}].$$

Note that

$$\log E[e^{t \sum_{n=1}^{m} c_n^{2s} r^2 W_n(\omega)}] = \sum_{n=1}^{m} \log \varphi(tc_n^{2s} r^2)$$

for every $m \in \mathbb{N}$, where $\varphi(tc_n^{2s} r^2)$ is the moment generating function of $c_n^{2s} r^2 W_n$. If we define the random variables $Z_m = e^{t \sum_{n=1}^{m} c_n^{2s} r^2 W_n(\omega)}$ we have that for $t \ge 0$ Z_m is an increasing sequence of random variable, and for $t < 0$ it is dominated by 1. So for all t we can have $E[\lim_{m\to\infty} Z_m] = \lim_{m\to\infty} E[Z_m]$, hence

$$\log \psi(t) = \sum_n \log \varphi(tc_n^{2s} r^2).$$

It lasts to show that $\sum_n \log \varphi(tc_n^{2s} r^2)$ is differentiable infinite times in $t = 0$, and this is true for lemma 4.5.8.

$\square$

Proposition 4.5.10. *Let $(\Omega, \mathcal{F}, P, U_r^Q)$ be a Q-finite random solution. For every $s < -1$, $r > 0$, $p > 1$ $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$P(\|U_r^Q\|_{W^{1,p}(0,T;H^s)} \le C_\epsilon) > 1 - \epsilon,$$

for each $Q = \sum_{|i| \le N} 1$ with $N \in \mathbb{N} \setminus \{0\}$.

Proof. Again we have that

$$P(\|U_r^Q\|_{W^{1,p}(0,T;H^s)}^p > R^p) \le \frac{1}{R^p} E[\|U_r^Q\|_{W^{1,p}(0,T;H^s)}^p] =$$

$$= \frac{1}{R^p} E[\|\frac{d}{dt} U_r^Q\|_{L^p(0,T;H^s)}^p].$$

Let $\{W_{1,k}, \dots, W_{2M+2,k}\}_{k\in\mathbb{N}}$ be a set of iid Gaussian random variables, with $W_{i,j} \sim \mathcal{N}(0, r^2)$. For each N holds

$$E[\|\frac{d}{dt}U_r^Q(0,\omega)\|_{H^s}^p] \leq$$
$$\leq E[(\sum_{n\geq 1}(c_n({W_{1,n}}^2 + \dots + {W_{M+2,n}}^2))^2 c_n^{2s})^{\frac{p}{2}}].$$

Moreover, exists a constant $D > 0$ such that

$$E[(\sum_{n\geq 1}(c_n({W_{1,n}}^2 + \dots + W_{M+2,n}))^2 c_n^{2s})^{\frac{p}{2}}] \leq$$
$$\leq E[(D\sum_{n\geq 1} c_n^{2+2s} W_{1,n}^4)^{\frac{p}{2}}].$$

So it is sufficient to prove that the random variable

$$Z = \sum_{n\geq 1} c_n^{2+2s} W_{1,n}^4,$$

has a moment generating function differentiable infinite times in 0, this would imply that it has finite p-moment for any $p \geq 1$.
Using the same argument of Proposition 4.5.9 we have that, if $\psi(t)$ is the moment generating function of Z,

$$\log\psi(t) = \sum_n \log\varphi(tc_n^{2+2s}r),$$

where $\varphi(tc_n^{2+2s}r)$ is the moment generating function of $c_n^{2+2s}{W_{1,n}}^4$. Moreover, for Lemma 4.5.8, $\sum_n \log\varphi(tc_n^{2+2s}r)$ is derivable infinite times in $t = 0$.

□

From now on the guideline is the same of the last two sections of the previous chapter, having estimates from proposition 4.5.9 and 4.5.10 we can extract tightness for the laws of the Q-finite random solution and then with Prohorov theorem we can have a converging subsequence of said laws, both in $L^p(0,T;H^s)$ and $C(0,T;H^s)$ topology.

Theorem 4.5.11. *For fixed $r > 0$, let $(\Omega, \mathcal{F}, P, U_r^Q)$ be a Q-finite random solution. The family of law $\{\mathcal{L}(U_r^Q)\}$ is tight in $L^P(0,T,H^s)$ for every $p > 1$, $s < 0$. Moreover, $\{\mathcal{L}(U_r^Q)\}$ is tight in $C(0,T;H^s)$.*

Proof. We use Aubin-Lions theorem with $B_0 = H^s$, $s < 0$, $B_1 = H^{s_1}$, $s_1 < -1$, $B = H^{s^*}$, $s_1 < s^* < s$ and $p = r$.

So the set

$$K_{R_1,R_2} = \{X | \|X\|_{L^p(0,T;H^s)} \leq R_1, \|X\|_{W^{1,p}(0,T;H^{s_1})} \leq R_2\}$$

is relatively compact in $L^p(0,T;H^{s^*})$.

From Proposition 4.5.9 and 4.5.10 we have that for every $\varepsilon > 0$ there exists a constant C_ε such that :

- $P(\|U_r^Q\|_{L^p(0,T;H^s)} \leq C_\varepsilon) \geq 1-\varepsilon$ for all $Q = \sum_{|i|\leq N} 1$ with $N \in \mathbb{N} \setminus \{0\}$,
- $P(\|U_r^Q\|_{W^{1,p}(0,T;H^{s_1})} \leq C_\varepsilon) \geq 1-\varepsilon$ for all $Q = \sum_{|i|\leq N} 1$ with $N \in \mathbb{N}\setminus\{0\}$.

So, given $\varepsilon > 0$ there exist $R_1(\varepsilon)$, $R_2(\varepsilon)$ such that the family of laws satisfies

$$\{\mathcal{L}(U_r^Q)\} \subset \{\mu \in Pr(L^p(0,T;H^s))|\mu(K^c_{R_1,R_2}) \leq \varepsilon\},$$

hence, since $\bar{K}_{R_1,R_2} \supseteq K_{R_1,R_2}$ we have $\bar{K}^c_{R_1,R_2} \subseteq K^c_{R_1,R_2}$ and then

$$\{\mathcal{L}(U_r^Q)\} \subset \{\mu \in Pr(L^p(0,T;H^s))|\mu(\bar{K}^c_{R_1,R_2}) \leq \varepsilon\},$$

so the family of laws $\{\mathcal{L}(U_r^Q)\}$ is tight in the topology of $L^p(0,T;H^s)$ for any $p \geq 1$.

For the tightness in $C(0,T;H^s)$ we can use Proposition 3.3.3 with $X = H^{s_0}$, $B = H^s$, $Y = H^{s_1}$ and θ from Proposition 3.2.5. If we let $p_0, r_1 \to \infty$ the condition of Proposition 3.3.3

$$\theta(1 - \frac{1}{r_1}) > \frac{1-\theta}{p_0}$$

is trivial, so we have that there exist p_0 and r_1 such that the set

$$K_{R_1,R_2} = \{X | \|X\|_{L^{p_0}(0,T;H^s)} \leq R_1, \|X\|_{W^{1,r_1}(0,T;H^{s_1})} \leq R_2\}$$

is relative compact in $C(0,T;H^s)$. Hence again, using Proposition 4.5.9 and 4.5.10 we have that for every $\varepsilon > 0$ there exists a constant C_ε such that :

- $P(\|U_r^Q\|_{L^{p_0}(0,T;H^s)} \leq c_\varepsilon) \geq 1-\varepsilon$ for all $Q = \sum_{|i|\leq N} 1$ with $N \in \mathbb{N} \setminus \{0\}$,
- $P(\|U_r^Q\|_{W^{1,r_1}(0,T;H^{s_1})} \leq c_\varepsilon) \geq 1-\varepsilon$ for all $Q = \sum_{|i|\leq N} 1$ with $N \in \mathbb{N}\setminus\{0\}$.

So, given $\varepsilon > 0$ there exist $R_1(\varepsilon)$, $R_2(\varepsilon)$ such that the family of laws satisfies

$$\{\mathcal{L}(U_r^Q)\} \subset \{\mu \in Pr(C(0,T;H^s))|\mu(K^c_{R_1,R_2}) \leq \varepsilon\},$$

hence, since $\bar{K}_{R_1,R_2} \supseteq K_{R_1,R_2}$ we have $\bar{K}^c_{R_1,R_2} \subseteq K^c_{R_1,R_2}$ and then

$$\{\mathcal{L}(U_r^Q)\} \subset \{\mu \in Pr(C(0,T;H^s))|\mu(\bar{K}^c_{R_1,R_2}) \leq \varepsilon\},$$

so the family of laws $\{\mathcal{L}(U_r^Q)\}$ is tight in the topology of $C(0,T;H^s)$.

□

Corollary 4.5.11.1. *For $s < 0$, there exists a subsequence $n_k \subset \mathbb{N}$ such that the laws of $\{U_r^{n_k}\}$ converge with the topology of $L^p(0,T;H^s)$ for every $p \geq 1$ and with the topology of $C(0,T;H^s)$.*

Proof. We have only to apply Prohorov Theorem to the results of Theorem 4.5.11.

□

Definition 4.5.12. *A set $(\Omega, \mathcal{F}, P, X)$ is said to be a random solution for the tree model if $(\Omega, \mathcal{F}, P)$ is an abstract probability space, $X : (\Omega, \mathcal{F}, P) \times [0, T] \to \mathbb{R}^\infty$ and for almost every $\omega \in \Omega$, $X(\omega, t)$ satisfies for every $i \in \mathbb{N}$ and $t \in [0, T]$*

$$X_j(\omega, t) = X_j(\omega, 0) + \alpha \int_0^t (c_j X_{\bar{j}}^2(\omega, s) - \sum_{k \in \mathcal{O}_j} c_k X_j(\omega, s) X_k(\omega, s)) ds -$$

$$\beta \int_0^t (d_{\bar{j}} X_{\bar{j}}(\omega, s) X_j(\omega, s) - \sum_{k \in \mathcal{O}_j} d_j X_k^2(\omega, s)) ds.$$

Theorem 4.5.13. *For fixed $r > 0$, consider a sequence $\{(\Omega, \mathcal{F}, P, U_r^Q)\}$ of Q-finite random solutions, up to replace the abstract space $(\Omega, \mathcal{F}, P)$ with another probability space $(\Omega', \mathcal{F}', P')$, there exists a subsequence $n_k \in \mathbb{N}$ such that P'-almost surely U_r^{nk} converges to a function U_r^∞ in $L^p(0, T; H^s)$ for every $p > 1$ and in $C(0, T; H^s)$, for every $s < 0$. Moreover, $(\Omega', \mathcal{F}', P', U_r^\infty)$ is a random solution for the tree model.*

Proof. From corollary 4.5.11.1 we have the existence of a subsequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that the sequence of laws of the random variables $U_r^{n_k}$ converges in the topology of $C(0, T; H^s)$ and $L^p(0, T; H^s)$ for every $p \geq 1$.
Since we have the convergence for any $s < 0$, we can consider the space

$$H^{0-} = \bigcap_{s<0} H^s,$$

endowed with the metric generated by the distance

$$d(x, \tilde{x}) = \sum_{n=1}^{\infty} 2^{-n} (\|x - \tilde{x}\|_{H^{-\frac{1}{n}}} \wedge 1).$$

Note that with this metric we have that $x_n \to^d x \Leftrightarrow x_n \to^{H^s} x$ for any $s < 0$.
We can assume, using Skorokhod representation Theorem, that almost surely $\tilde{U}_r^{n_k}$ converges to U_r^∞, up to replace the abstract space $(\Omega, \mathcal{F}, P)$ where $\tilde{U}_r^N$ are defined with another abstract probability space $(\Omega', \mathcal{F}', P')$, in the topology of $C(0, T; H^{0-})$ and $L^p(0, T; H^{0-})$ for every $p \geq 1$.
The new sequence of random variables $\tilde{U}_r^{n_k}$ has the same law of $U_r^{n_k}$, this means that for every φ measurable function it holds $E[\varphi(\tilde{U}_r^{n_k})] = E[\varphi(U_r^{n_k})]$.
Hence, considering the operator

$$F_i(x(\omega, t)) = x_i(\omega, t) - x_i(\omega, 0) - \alpha \int_0^t (c_j x_{\bar{j}}^2(\omega, s) - \sum_{k \in \mathcal{O}_j} c_k x_j(\omega, s) x_k(\omega, s)) ds +$$

$$+\beta \int_0^t (d_{\bar{j}} x_{\bar{j}}(\omega, s) x_j(\omega, s) - \sum_{k \in \mathcal{O}_j} d_j x_k^2(\omega, s)) ds$$

we have that $E[|F_i(\tilde{U}_r^{n_k})|] = E[|F_i(U_r^{n_k})|]$, hence $\tilde{U}_r^{n_k}$ is still almost surely a solution of the truncated tree model. So for now on we consider without loss of generality $U_r^n = \tilde{U}_r^n$.

It is then straightforward to check that the limit solves the equation in the integral form, as done for theorem 4.4.2, so U_r^∞ is a random solution for the tree model.

$\square$

Sintesi in italiano

L'obiettivo principale della tesi è studiare l'esistenza di soluzioni di sistemi dinamici derivanti da Shell Model di turbolenza, ossia Shell Model che mimino la cascata energetica della dinamica di un fluido. Tali modelli, benché costituiscano una versione più semplificata di fluidodinamica rispetto alle equazioni di Navier-Stokes, presentano una dinamica sufficientemente ricca da permettere parallelismi con lo studio delle equazioni Navier-Stokes vere e proprie, di base molto più inaccessibili.
I primi due capitoli della tesi sono dedicati a ripercorrere la derivazione degli Shell Model dalla decomposizione in serie di Fourier dell'equazioni di Navier-Stokes e a mostrare lo stato dell'arte dei risultati di esistenza e unicità di soluzioni per quello che riguarda il modello diadico classico, che rappresenta il mattoncino base nello studio degli Shell Model. Il percorso porterà a mostrare come nel modello base vi sia esistenza per soluzioni con condizione iniziale a energia finita, con dimostrazione di unicità per soluzioni con condizioni iniziali che hanno tutte le componenti positive, e dimostrazione di esistenza di più soluzioni per la stessa condizione iniziale nel caso di infinite componenti negative.
Nel terzo capitolo viene ripercorso il lavoro di un articolo in preparazione dell'autore su un modello diadico misto. L'obiettivo conseguito è andare oltre i risultati di esistenza per condizioni iniziali a energia finita e ottenere l'esistenza di una soluzione (componente per componente) per quasi ogni condizione iniziale, dove per quasi ogni è inteso rispetto a una misura Gaussiana nello spazio infinito dimensionale delle condizioni iniziali. Tale soluzione è ottenuta come limite, per cui dopo aver approssimato il sistema dinamico con sistemi finito dimensionali, attraverso un argomento di compattezza si estrapola un limite in legge delle variabili aleatorie che data una condizioni iniziale casuale restituiscono la soluzione rispetto alla condizione iniziale (variabili indicizzate sui sistemi approssimanti e quindi limite ottenuto col crescere della dimensione). La legge di tali variabili aleatorie è invariante per il sistema dinamico e questo sarà fondamentale per ottenere le stime che garantiscono la compattezza. Utilizzando il teorema di rappresentazione di Skorokhod è quindi possibile passare da un limite in legge a uno quasi certo e ottenere il risultato cercato. Per quel che riguarda invece l'unicità, la presenza di soluzioni con componenti Gaussiane centrate non rende applicabile il metodo usato per il modello diadico classico, e fa inoltre pensare a una maggiore analogia con i casi in cui non c'è unicità di soluzioni.
Infine nell'ultimo capitolo vengono approfonditi i modelli ad albero, ovvero i modelli indicizzati su un albero anziché sui numeri naturali. Questi modelli rappresentano in modo più appropriato la dinamica di divisione dei vortici in vortici più piccoli con rilascio di energia e conseguente cascata energetica. Inizialmente viene dimostrato un risultato di esistenza di soluzioni per condizioni

iniziali a energia finita su un modello generale di albero con termini interagenti con al più quelli di una generazione precedente o successiva, dopodiché passando a un modello più specifico con coefficienti tali da permettere l'invarianza a una certa classe di misure Gaussiane, viene riprodotto lo stesso schema dimostrativo del capitolo precedente per ottenere l'esistenza di soluzioni per quasi ogni condizione iniziale, rispetto alla fissata misura Gaussiana.

English Summary

The focus of the thesis is to study well-posedness, with respect to generic Gaussian distributed initial data, in turbulence shell models. In the of state-of-the-art results we have existence of solution for any finite energy initial conditions. Here we show the generic existence of solutions with respect to initial data distributed as Gaussian invariant measures, in "mixed" dyadic and tree-like shell models, extending the classical deterministic results. The existence is given thanks to compactness argument and techniques similar to the ones used by Albeverio and Cruzeiro for Euler equation (and more recently with a different approach by F. Flandoli), adapted to our model. Uniqueness is not provided, and the natural oscillating behaviour of the solutions obtained may suggests that it doesn't hold at all.
Chapter 1 and 2 are fully dedicated to introduce shell models and give the state-of-the-art about existence in the dyadic case, Chapter 3 presents the novel result on the mixed dyadic model and Chapter 4 the one on the mixed tree model.

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