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Geometric prediction: a model to analyze a cognitive process in geometrical problem-solving

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Abstract

The purpose of the research is to study cognitive aspects of how geometric predictions are produced during problem-solving activities in Euclidean geometry. The process of geometric prediction is seen as a specific visuo-spatial process involved in geometrical reasoning. Indeed, when solvers engage in solving a geometrical problem, they can imagine the consequences of transformations of the figure; such transformations can be more or less coherent with the theoretical constraints given by the problem, and the products of such transformations can hinder or promote the problem-solving process.

Previous research has stressed the dual nature of geometrical objects, intertwining a conceptual component and a figural component. Interpreting geometrical reasoning in terms of a dialectic between these two aspects (Fischbein, 1993), this study aims at gaining insight into the cognitive process of *geometric prediction*, a process through which a figure is manipulated, and its change is imagined, while certain properties are maintained invariant.

This process is described through a model of prediction-generation elaborated cyclically by observing, analyzing through a *microgenetic approach*, and re-analyzing solvers' resolution of *prediction open problems* in a paper-and-pencil environment and in a Dynamic Geometry Environment (DGE).

The prediction open problems designed were proposed during *task-based interviews* to participants selected on a voluntary basis. Participants were a total of 37 Italian high school students and undergraduate, graduate and PhD students in mathematics. Data are composed of video and audio recordings, transcriptions, solvers' drawings.

The final version of the model provides a description of the prediction processes accomplished by a solver who engages in the resolution of *prediction open problems* proposed in this study; it provides a lens through which solvers' productions can be analyzed and it provides insight into prediction processes. In particular, it sheds light onto the key role played by theoretical elements that are introduced by the solvers during the resolution process and the key role played by the solver's theoretical control.

The study has implications for the design of activities, especially at the high school level, with the educational objective of fostering students' geometrical reasoning and in particular their theoretical control over the geometrical figures.

Breve descrizione

La ricerca mira a studiare gli aspetti cognitivi coinvolti nella produzione di previsioni geometriche durante la risoluzione di problemi nell'ambito della Geometria Euclidea.

Si può considerare il processo di previsione geometrica come uno specifico processo visuo-spaziale coinvolto nel pensiero geometrico. Infatti, durante il processo di risoluzione di un problema geometrico, un solutore può immaginare diverse trasformazioni della figura e i loro effetti; tali trasformazioni possono essere più o meno coerenti con i vincoli teorici dati dal problema. Inoltre, i prodotti di tali trasformazioni possono inibire o supportare il processo risolutivo.

Ricerche precedenti hanno evidenziato e posto l'attenzione sulla natura degli oggetti geometrici, considerando sia la componente concettuale che la componente figurale. Interpretando il pensiero geometrico in termini di dialettica tra questi due aspetti (Fischbein, 1993), lo studio mira a comprendere il processo di *previsione geometrica*, inteso come un processo attraverso il quale una figura viene manipolata, i suoi cambiamenti immaginati, mentre alcune proprietà vengono mantenute invarianti.

Il processo di previsione geometrica verrà descritto attraverso un modello di generazione di previsioni elaborato ciclicamente: osservando e analizzando a più riprese, secondo un *approccio microgenetico*, il comportamento di diversi solutori durante la risoluzione di *problemi aperti di previsione* proposti sia in ambiente carta e penna che in un Ambiente di Geometria Dinamica (AGD).

I problemi aperti di previsione progettati per lo studio sono stati proposti durante interviste *task-based* a solutori coinvolti su base volontaria. Hanno preso parte allo studio un totale di 37 solutori italiani tra studenti di scuola secondaria di secondo grado, studenti di laurea magistrale e di dottorato in Matematica. I dati constano di registrazioni video e audio, trascrizioni delle interviste, disegni dei solutori.

La versione finale del modello descrive i processi di previsione di un solutore coinvolto nella risoluzione dei *problemi aperti di previsione* proposti nello studio. Inoltre, il modello fornisce una lente teorica utile per analizzare le produzioni dei solutori e comprendere più profondamente i processi di previsione. In particolare, il modello chiarisce il ruolo cruciale sia degli elementi teorici introdotti dal solutore durante il processo risolutivo, sia del controllo teorico che i solutori esercitano.

Lo studio ha implicazioni didattiche utili in particolar modo per la scuola secondaria di secondo grado, per la progettazione di attività volte a promuovere il pensiero geometrico degli studenti e il loro controllo teorico sulle figure geometriche.

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1. Introduction

This study focuses on cognitive aspects involved in the generation of predictions within the specific domain of geometrical reasoning, when the reasoning is carried out within the context of Euclidean Geometry on the plane.

When a solver engages in a geometrical problem on a geometrical figure, she can imagine the consequences of transformations of the figure before they were physically accomplished; such transformations can be more or less coherent with the theoretical constraints depending on the expertise of the solver. We seek to gain insight into this process.

Producing and transforming images seems to be a common behavior among mathematicians. Indeed, Polya himself described how an image can have a heuristic role when it is sketched out in a drawing (1957):

> Figures are not only the object of geometric problems but also an important help for all sorts of problems in which there is nothing geometric at the outset. Thus, we have two good reasons to consider the role of figures in solving problems. (ibid., p. 103)

Other mathematicians like Hadamard (1945) and Poincaré (1952) describe how imagining images seems to be a typical activity of many mathematicians. This is widely documented, for example, by Hadamard (1945) who describes a personal anecdotical example of how he uses mental pictures, even in mathematical contexts other than geometry. He considers the following task: "we have to prove that there is a prime greater than 11" (ibid., p. 76). He describes the steps of the proof and the corresponding images that he produced along the way (Figure 1).

STEPS IN THE PROOF	MY MENTAL PICTURES
I consider all primes from 2 to 11, say 2, 3, 5, 7, 11.	I see a confused mass.
I form their product $2 \times 3 \times 5 \times 7 \times 11 = N$	N being a rather large number, I imagine a point rather remote from the confused mass.
I increase that product by 1, say N plus 1.	I see a second point a little beyond the first.
That number, if not a prime, must admit of a prime divisor, which is the required number.	I see a place somewhere between the confused mass and the first point.

Figure 1 Steps of the proof and Hadamard's images used for proving that there is a prime greater than 11 (Hadamard, 1945, pp. 76-77)

Although there exist individual differences, Hadamard (1945) highlights that mathematicians use images and that these images very often are of a geometric nature; when immersed in thought, mathematicians often avoid using words or symbols (algebraic or others), as they prefer instead to focus on images. Moreover, Einstein wrote to Hadamard:

> Words and language, written or oral, seem not to play any role in my thinking. The psychological constructs which are the elements of thought are certain signs or pictures, more or less clear, which can be reproduced and combined at liberty. (ibid., p. 82).

More recently, in interviews with mathematicians, Sfard (1994, 2008) found that they all relied heavily on imagery.

> As an aside, let me remark that one of my studies produced much evidence in support of Jacques Hadamard's claim that the majority of mathematicians use visual imagery even in the most advanced and abstract of discourses. These pictures are sometimes actually drawn and sometimes just imagined. (Sfard, 2008, p. 150)

According to Wheatley (1997), images also play a central role in students' mathematical reasoning:

> Students who used images in their reasoning were more successful in solving nonroutine mathematics problems than those who approached the task procedurally […] The difference between good and poor problem solvers is often the extent to which they use imagery (ibid., p. 281-295).

The competence of treating images in a way that supports problem-solving in Geometry has recently been referred to as having a *mathematical eye* by Mariotti and Baccaglini-Frank (2016). The skills which support the mathematical eye include the ability to predict, i.e. the identification of particular properties or configurations of a new figure, arising from a manipulation process.

Other researchers talk about *anticipation*. Borrowing and adapting Boero's definition given in a different mathematical domain but very suitable here as well, we can say that *anticipating* means imagining the consequences of certain choices operated on mathematical objects (Boero, 2001). Moreover, imagining the consequence of a set of transformations of a figure has a fundamental heuristic role within the geometrical problem-solving.

> In order to direct the transformation in an efficient way, the subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. (ibid., p. 12)

Manipulation of images is a topic that has been addressed also by the research on *visuo-spatial abilities* carried out by cognitive psychologists (see, for example, Cornoldi & Vecchi, 2004). In a previous study (Miragliotta & Baccaglini-Frank, 2017; Miragliotta, Baccaglini-Frank & Tomasi, 2017), we tried to analyze students' geometrical thinking starting from theoretical constructs which belong to the research field of Cognitive Psychology. However, this approach revealed to be unsuitable to understand students' predictions during the geometrical problem solving. We believe that the main reason of such unsuitability resides in the key role played by theoretical elements during the transformation of a geometrical figure. However, the reference to a mathematical theory (there Euclidean Geometry) is not acknowledged within the psychological research on visuo-spatial abilities. In line with our findings, the strong involvement of the reference to a precise theoretical domain (described below as "rigorous analytical thought processes") in prediction processes is also highlighted by Presmeg (1986):

> Especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful. (ibid., p. 45)

So, besides the transformations that a solver can imagine on the figural or spatial domain, we need to consider the theoretical constraints given by the mathematical theory of reference. Indeed, processes of *anticipation* have a particular nature, that is different from both deduction and induction. These processes rely on *transformational reasoning* (Simon, 1996).

> Transformational reasoning involves envisioning the transformation of a mathematical situation and the results of that transformation. (ibid., p. 207)

These are the kinds of processes that we want to study.

More specifically, restricting our focus on the Theory of Euclidean Geometry on the plane, our aim is to gain insight into the processes of prediction within the domain of the geometrical reasoning. Starting from explaining both the potential and the limits of a pure psychological perspective, applying the categories of visuo-spatial abilities, in the description of the geometric prediction process, the study aims at achieving an integration between well-established results and categories developed within the research field of neurosciences and a long standing tradition of research in mathematics education concerning *geometrical reasoning* and *visualization*. Our goal is to:

- describe a general process of prediction within the domain of geometrical reasoning;

- propose a model of *geometric prediction* which highlights the cognitive aspects involved in the process;
- develop analytical tools that provide access to the figural and the theoretical components of the solvers' products of prediction.

In Chapter 2 we situate the study within the literature on *visualization* and *spatial reasoning* in Geometry; we present an overview of the main definitions provided by researchers that address these two topics; we introduce the notion of *imagery* and describe how it was approached by researchers in Mathematics Education. Moreover, we present the semiotic approach provided by Duval. Finally, we touch on Fischbein's *Theory of Figural Concepts* and on the possible role of a *Dynamic Geometry Environment* (DGE) in studying prediction processes; these last two aspects will be described in greater depth in the next chapter.

The theoretical framework of the study is presented in Chapter 3. Initially we introduce the theoretical framework of *visuo-spatial abilities* from a cognitive point of view (Cornoldi & Vecchi, 2004), that was developed by cognitive psychologists for explaining the interaction between solvers and images of different kinds, including those that address geometric figures. Then, we focus on the theoretical constructs that belong to the field of research in Mathematics Education, explaining how the *Theory of Figural Concepts* (Fischbein, 1993) well describes the specific nature of geometrical objects. Since this perspective constitutes our main interpretative lens, we describe in greater depth the theoretical constructs that we borrow from it and adapt their definitions to this study. In this chapter we also provide a first definition of *geometric prediction* (*GP*) and describe how this theoretical construct could interact with *anticipatory intuitions*. Moreover, starting from the notion of *open problem* (Arsac et al., 1998; Silver, 1995), we present a kind of geometric problems that are suitable for stimulating productive thinking: *prediction open problems*. Finally, we focus on the role of the exploration of a geometrical task within a *Dynamic Geometry Environment* (DGE) regarded as a source of additional windows onto the processes of prediction and onto its products.

In Chapter 4, we summarize our working hypotheses and theoretical assumptions. Moreover, in light of the theoretical framework, we list our three research questions.

Chapter 5 is devoted to the description of our methodological choices. We introduce the methodological tools of *clinical interviews* (Ginsburg, 1981) and *task-based interviews* (Goldin, 2000), explaining the rationale for our choices; we describe the methodological approach for collecting and analyzing data, introducing in particular the *microgenetic methods*. Moreover, we illustrate the experimental design of our study and explain how our data were collected; we also provide an a priori analysis of the *prediction open problems* proposed during the interviews. Finally, we introduce the tools that we developed for the analyses, according to a microgenetic approach, and describe in greater depth how they will be used according to our research aims.

In the four subsequent chapters, we present our findings emerging from data analyses. In particular, in Chapter 6 we describe the observable characteristics of processes and products of GP. In Chapter 7 we focus on the findings emerging from the "funnel" tool of analysis. More specifically, we use the funnels to highlight the role that the theoretical elements recalled or introduced by the solvers play within the processes of GP. Chapter 8 is devoted to second level of findings, focusing on how several GP processes, or their products, can interact within the resolution of the given tasks. Moreover, we describe the general and local obstacles that could inhibit the solvers in accomplishing prediction processes or that lead them to coherent products. In Chapter 9 we analyze what happens when a solver who has undertaken GP processes in a paper-and-pencil environment moves to the DGE *GeoGebra*.

In the concluding chapter, we summarize our results and answer the research questions, providing a final version of the model of *geometric prediction processes*. This model describes the prediction processes accomplished by a solver who engages in the resolution of *prediction open problems* proposed in this study. Moreover, it provides a lens through which we can analyze solvers' productions and gain further insight into the prediction processes. More specifically, it sheds light onto the role of theoretical elements introduced by the solvers. After answering the research questions, highlighting the theoretical contributions that this study offers, we contextualize our findings within the existing literature, and finally we describe possible educational implications and directions for further research.

2. Literature review

In this chapter we contextualize our study within the literature, describing how it is situated within the wide domain of research on *visualization* and *spatial reasoning* in Geometry.

Because of the huge number of research studies on these two topics, we discuss only the most significant ones with respect to our aims, presenting an overview of the main definitions provided that address visualization and spatial reasoning.

Because of the connection between visualization and mental imagery that has been established in several studies, we introduce the notion of imagery and describe how it has been approached by in Mathematics Education.

The topic of visualization has addressed from different points of view. Since it was widely used within the research on visualization, we present the semiotic approach provided by Duval. We also introduce Fischbein's theory of figural concepts and the role of the Dynamic Geometry Environment. These last two aspects will be described in greater depth in the next chapter.

2.1 Visualization and spatial reasoning

Despite the long hegemony of Psychology (e.g. Kosslyn, 1980; McGee, 1979; Lohman, 1979; Paivio, 1971; Piaget & Inhelder, 1971; Pylyshyn, 1973; Richardson, 1977; Shepard & Metzler, 1971; Thomas, 1989; Thurstone, 1938), today *visualization* and *spatial reasoning* have become two important topics within research in Mathematics Education. According to Presmeg (2006), the seminal work of Alan Bishop (1980) provided a foundation for research on visualization in Mathematics Education.

Recently interest on this topic has grown, mainly for two reasons: on the one hand, it has been recognized as an important part in studies and careers which involved Sciences, Technology, Engineering, and Mathematics (STEM) (Wai, Lubinski, & Benbow, 2009; Newcombe & Shipley, 2015); on the other, there has been increasing evidence that spatial reasoning is learnable, it is related skills that are malleable at any age, and that are strongly correlated to achievement not only in STEM but in all subject areas (Newcombe, 2010). Furthermore, researchers believe that skills related to spatial reasoning could be useful also for everyday life (Mulligan, 2015).

Despite a general scientific interest on these topics, the terms *visualization* and *spatial reasoning* have been used in various ways within the literature. We clarify the definitions that have been advanced that are the most significant with respect to this study.

Below are definitions of *visualization* given in the field of research in Mathematics Education.

According to Gutiérrez (1996) *visualization in mathematics* is "the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical"(ibid., p. 9), and it is used to solve a problem or prove properties. It is composed of four elements: mental images, external representations, processes of visualization, and abilities of visualization. A *mental image* is

> a mental representation of a mathematical concept or property containing information based upon pictorial, graphical or diagrammatic elements. "Visualisation", or visual thinking, is a kind of reasoning based on the use of mental images". (Gutierrez, 1996, p. 6)

Mental images are considered as the basic elements in visualization:

> A "process" of visualization is *a mental or physical action where mental images are involved.* (ibid., p. 10, italics in the original)

The involvement of mental images is also stressed by Clement (2014):

> visualization is something which someone does in one's mind—it is a personal process that assumes that the person involved is developing or using a mental image. (ibid., p. 181)

According to Presmeg (2006),

> […] visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (ibid, p. 206)

We can notice that, in this perspective, visualization is considered explicitly linked to mathematical activity and could involve mental images.

Zimmermann and Cunningham (1991) use the term *visualization* for referring to a process of both producing and using

> representations of mathematical concepts, principles or problems, whether hand drawn or computer generated. (ibid., p. 3)

According to Hershkowitz et al. (1989), it is considered an

> ability to represent, transform, generalize, communicate, document, and reflect on visual information (ibid., p. 75).

Arcavi (2003) blends and paraphrases the definitions given by other authors (Zimmermann and Cunningham, 1991; Hershkowitz et al., 1989). He provides a wide definition which considers visualization both as an ability and as a process:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (Arcavi, 2003, p. 217)

Visualization is also considered a *product* of the thought activity on images (even mental) for several purposes, including their reproduction and on developing new understandings.

Duval (1998) highlights the connection between visualization and mathematical activities, according to a semiotic perspective. He includes visualization among cognitive processes involved in Geometry. Visualization is based on the production of a *semiotic representation* and it makes visible all that is not accessible to vision (Duval, 1999). He claims:

> Visualization is the recognition, more or less spontaneous and quick, of what is mathematically relevant in any visual representation given or produced (Duval, 2014, p. 160)

Based on actions and cognitive operations that can be performed with or within a visual representation, he distinguishes two kinds of visualization: *visualization based on perception* and *visualization based on the construction of configurations* using tools.

Concerning the several definitions of *spatial reasoning,* we find a similar heterogeneity. According to researchers of the group called *Spatial Reasoning Study Group*, *spatial reasoning* includes visualization and is

> the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects (Bruce et al., 2017, p. 146)

Dindyal (2015) speaks of *spatial visualization* which includes

> building and manipulating mental representations of two-and three-dimensional objects and perceiving an object from different perspectives (ibid., p. 521)

Kinach (2012) adds on the definition also the use of representations:

> Spatial thinking takes a variety of forms, including building and manipulating two-and-three-dimensional objects; perceiving an object from different perspectives; and using diagrams, drawings, graphs, models, and other concrete means to explore,

> investigate, and understand abstract concepts such as algebraic formulas or models of the physical world. (ibid., p. 535)

In general, researchers have used several terms for referring to theoretical constructs similar to visualization and spatial reasoning; moreover, they used several definitions. Nevertheless, almost invariably, the definitions:

- are related to the construct of *imagery*
- have in common the activity of imagining objects and interacting with them through mental transformations (for example, rotation, stretch, reflection).

In this study, we will refer to *visualization* as defined by Presmeg (2006) and *spatial reasoning* as defined by (Bruce et al., 2017).

As stressed by Gal and Linchevski (2010), the resolution of geometrical tasks involves, explicitly or implicitly, a "seeing" process which includes several successive phases, beginning with perception and ending with "higher cognitive processes". *Mental imagery* seems to play a role in this process. According to Finke (1989):

> "Mental imagery" is defined as the mental invention or recreation of an experience that in at least some respects resembles the experience of actually perceiving an object or an event, either in conjunction with, or in the absence of, direct sensory stimulation. (ibid., p. 2)

Following this perspective, we can consider *mental images* as inner figural representations that possess similar features of inner representations generated by the cognitive process of perception (Mariotti, 2015).

2.2 Bishop's IFI and VP

Processes of manipulation of visual images have been widely explored. Bishop (1983) separates the knowledge required to represent the problem from the abilities required to solve it in its context. With this aim, he proposes two very different kinds of abilities in what he calls the *spatial-mathematical interface*: the ability for *interpreting figural information* (IFI) and the ability for *visual processing* (VP).

He describes these abilities as follows:

> IFI involves knowledge of the visual conventions and spatial "vocabulary" used in geometric work, graphs, charts, and diagrams of all types. Mathematics abounds with such forms and IFI includes the "reading" and interpreting of these. […] VP, on the other hand, involves the ideas of visualisation, the translation of abstract relationships and non-figural data into visual terms, the manipulation and

> extrapolation of visual imagery, and the transformation of one visual image into another. (ibid., p. 177)

So, *IFI* refers to the interpretations of images involved in a geometrical activity. It relies on the interpretation of content and context and is connected to the particular form of the stimulus; whereas *VP* refers to the creation and manipulation of visual images, it is "an ability of process", and is a more dynamic ability than *IFI* (Presmeg, 2008, p. 85). Furthermore, Bishop (1983) stresses that *VP* is the most difficult ability to test, and the most important to emphasize in teaching mathematics, because it is an ability that is difficult to develop. Indeed, according to Bishop (1983) and Krutetskii (1976), *VP* is teachable although individual preferences remain.

IFI seems related to the ability to see and interpret, whereas *VP* seems to address visualization, defined as "the ability to interpret, transform, generalize, communicate, document, and reflect on visual information" (Gal & Linchevski, 2010).

2.3 Imagery strategies

A large contribution to the research on visualization is provided by Presmeg's works. Presmeg's research was influenced by other works (Suwarsono, 1982; Krutetskii, 1976) about learners' preferences for using visualization in a problem-solving situation. Individuals differ in their preferred mode of reasoning (Krutetskii, 1976; Premeg, 1986): some preferring logical reasoning and others spatial reasoning. In particular, in pupils at school levels who are especially gifted in mathematics, Krutetskii (1976) identifies three *mathematical casts of mind*: a tendency to interpret the world mathematically. Here is the list of *mathematical casts of mind*, as reported by Presmeg (1997, p. 307).

> *Analytic*: very strong verbal-logical component predominating over weak visual-pictorial one; spatial concepts weak; they operate easily with abstract schemes; they have no need for visual supports for visualizing objects or patterns in problem-solving.
>
> *Geometric*: very strong visual-pictorial component, predominating over an above average verbal-logical component; spatial concepts very good; they use visual supports in problem solving; they feel a need to interpret visually an expression of an abstract mathematical relationship.

Harmonic: relative equilibrium of strong verbal-logical and visual-pictorial components; spatial concepts well developed. This type seems to have two subtypes:

Subtype A (*abstract harmonic*): inclination for mental operations without the use of visual-pictorial; they can use visual supports in problem solving, but prefers not to.

Subtype B (*pictorial harmonic*): inclination for mental operations with the use of visual-pictorial schemes; they can use visual supports in problem solving and prefers to do so.

Even if this classification is based on studies with mathematically talented individuals, Presmeg's research (1986, 1997) confirms that it would extend to all ability levels.

She was very interested in characterizing students' preferred mode in solving given tasks and how the teacher facilitates this or otherwise. In particular, Presmeg (1994) examined the mathematical use of *imagery*. The definition is the following:

A visual image is a mental construct depicting visual or spatial information. (Presmeg, 1997, p. 303)

At the end of her research, she classifies teachers and students into two categories: *visualizers* and *non-visualizers*.

Visualisers are individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods. Nonvisualisers are individuals who prefer not to use visual methods when attempting such problems. (Presmeg, 1986, p. 42)

Visual imagery used by *visualizers* were classified (ibid., p. 43) as follows:

- Concrete, pictorial imagery (pictures-in-the-mind)
- Pattern imagery
- Memory images of formulae
- Kinesthetic imagery
- Dynamic imagery

In particular, the solvers who made use of *dynamic imagery* manifest a mental manipulation of the figure:

Paul explained that after seeing that the rectangle would be two, he "slid" the parallelogram up into the rectangle, using a moving image. (ibid., p. 44)

Among the interviews, Presmeg (1997) did not find a large number of instances of dynamic imagery. Nevertheless, she stresses:

> [...] these dynamic forms were especially powerful in facilitating the mathematical problem solving of the students who used them (ibid., p. 305).

However, the imagery alone is not sufficient in problem solving:

> […] imagery was marked by an interplay between concrete perceptual visualization, on the one hand, and a relentless drive toward abstract, aesthetic principles of symmetry or invariance on the other. […] If imagery is to be useful in problem solving, it needs to be controllable. (Presmeg, 1997, p. 301-306)

The classification of imagery provided by Presmeg (1986) was recalled and adapted by Owens (1999) in her studies on children's visualization in the specific domain of Geometry. In particular, she finds several instances of *dynamic imagery* in young children's actions and words.

> For example, one child imagined a square becoming a rectangle and then becoming thinner as he developed his rectangle concept. Another dynamically changed a trapezium into a parallelogram (ibid., p. 221)

Moreover, she identifies five groupings of strategies. These include:

- *Perceptual strategies*, used by solvers who attend to spatial features, relying on what they can see or do;
- *Pictorial imagery strategies*, which involves pictorial imagery, developing mental images associated with concepts;
- *Pattern and dynamic imagery strategies* used by solvers who make use of "pattern and movement in their mental imagery and developing conceptual relationship". (ibid., p. 224)

In particular, this last strategy is explicitly connected with predictions. Indeed, one of the descriptions is the following:

> predicts changes by mentally modifying shapes and their attributes using motion or pattern analysis (ibid., p. 225)

All these contributions stress the existence of a dynamic dimension of the visual imagery and that it can be particularly helpful in problem-solving.

2.4 Cognitive apprehensions

A wide contribution in the field research on visualization was given by Duval, who followed a different approach. Duval (1999) considers *vision* as a psychological construct connected to visual perception. As perception, it involves two cognitive functions: epistemological function and synoptic function. The first one consists in giving “direct access” to any physical object. According to Duval, “nothing is more convincing than what is seen” (ibid., p. 10). The second one concerns the simultaneous and immediate apprehension of several objects. In this perspective, according to its first function, vision is different from representation.

Paraphrasing Peirce’s definition of *sign* or *representamen* (Peirce, CP, 2.228), Duval (1999) defines *representation* as “something which stands instead of something else” (ibid., p. 10). Because of the second function, vision is opposite to discourse and deduction “which requires a sequence of focusing acts on a string of statements” (ibid., p. 10), neither immediate nor simultaneous.

Furthermore, visual perception provides a direct access to objects. For this reason, it is different from visualization. Indeed, *visualization* is based on the production of *semiotic representations* (Duval, 1999), which show relations and organization of relations between *representational units*. *Visualization* is one of the three strictly connected cognitive processes involved in doing geometry: *visualization*, *construction*, *reasoning* (Duval, 1998). Visualization is the intrinsically semiotic one and it allows to grasp *at once* a complete apprehension of any organization of relations. Elsewhere, Duval (2014) stresses:

> Visualization is the recognition, more or less spontaneous and quick, of what is mathematically relevant in any visual representation given or produced. […] Visualization, like understanding is always a jump that corresponds to a new awareness in which everything is completely reorganized in an obvious way (ibid., p. 160).

Duval (1999) considers visualization as an *intentional* process which “bring[s] into play a semiotic system”, different from *automatic* processes (ibid., p. 6).

In order to explain interactions with a geometrical representation, Duval (1995) talks about *cognitive apprehensions*, rather than abilities or skills, as other authors do. His key idea is that there exist several ways of looking at a drawing or at a visual stimulus. For functioning as a geometrical figure, a drawing must evoke one or more cognitive apprehension. Indeed, a geometric figure “always associates both discursive and visual representations” (Duval, 2006). Duval (1994, 1995) distinguishes four cognitive apprehensions whose fusion is involved in the use of geometrical figures. He characterizes each one separately but stresses that the solution of a geometrical problem frequently requires their interaction.

Perceptual apprehension is the most immediate one (Duval, 1994); it allows us to identify and immediately recognize and *"at first glance"* a form or an object, on a plane or in depth. It happens by means of cognitive processes performed automatically and therefore unconsciously (ibid., p. 124). Indeed, what we perceive depends on *figural organization laws* and *pictorial cues*. The first one refers to the Gestalt principles of perceptual organization. Citing Coren et al. (1979), Duval stresses what he means by *cue*:

> […] a *cue* is a signal that prompts an action from an actor *automatically* […] a *clues* suggests *conscious* consideration that leads to the detection of the correct response. (Duval, 1995, p. 145, emphasis in original)

Sequential apprehension is involved when we construct a figure or describe its geometrical construction. In this case, the organization of *elementary figural units* depends on technical constraints of the tools used to construct the figure and on the mathematical properties of the figure. Cues and perceptual laws are not involved in this process.

Furthermore, Duval highlights that a drawing without a denomination or explicit hypothesis is an ambiguous representation. Indeed, without discursive elements, different perceivers who see the same drawing could observe different properties. Perceptual apprehension is not enough to recognize mathematical properties:

> some must first [be] given through speech (denomination and hypothesis) and others can be derived from the given properties (Duval, 1995, p. 146).

This is the domain of *discursive apprehension*. Indeed, what the figure shows could be different from what it represents. Perceptual apprehension allows us to perceive a figure without conscious analysis, by recognizing shapes and objects; it consists of the speech acts (denomination, definition, primitive commands in a software) which determine what the perceived figure represents.

The fourth one is *operative apprehension*; it has a heuristic function, which means it allows to gain an insight into a solution of a geometrical problem looking at a figure. Through operative apprehension we can transform (mentally or physically) the given figure in different ways:

- we can divide up the whole figure into parts and combine them in another figure (*mereologic way*);
- we can make the figure larger or narrower or slanted (*optic way*);
- we can change its position or its orientation (*place way*).

These modifications could be accomplished using different operations, respectively: reconfiguration; size variation and plane variation; variation of orientation, like rotation or translation.

Moreover, Duval (1995) talks about the "operative apprehension of a given figure" (ibid., p. 147). He seems to argue that a figure can or cannot offer heuristic help to solve a geometric problem, because of its particular representation or because of the particular task design. Consequently, possible operations could be more or less visible and natural for a solver.

Duval (1994) explains that operative apprehension is different from discursive apprehension, because the latter is subordinate to the use of definitions and theorems and the former concerns operations which could be accomplished freely, without theoretical or technical constraints. For this reason, operative apprehension is also different from sequential apprehension, because it is free from constraints related to any construction tools.

The operations used to transform a figure could be accomplished only in the register of the figure. Thus mathematical knowledge is not involved in operative apprehension, even if some transformations could be congruent with mathematical ones. For this reason, operative apprehension is close to perceptual apprehension. They share the same figural organization laws, but at a different level: in the second case it is an automatic and immediate process, in the first one it is a conscious process and could require a long time.

2.5 Fischbein's perspective

According to Fischbein *geometrical objects* are completely described and controlled by an axiomatic system of definition and theorems, but at the same time they maintain certain figural aspects of images. We will not devote a long discussion on the Fischbein's perspective on visualization and geometrical reasoning because the *Theory of Figural Concepts* (Fischbein, 1993) will be widely described in the next chapter.

In this chapter we only want to present what is relevant for our review of Fischbein's main paper on the *figural concepts* in order to draw from his claims how he conceives the following notions: images, images in geometry, geometrical figure and figural concepts (Table 1).

<table>
<tr><th>Claims within the paper “The theory of figural concepts” (Fischbein, 1993)</th><th>pp.</th><th>Our review</th></tr>
<tr><td>Concepts and mental images are usually distinguished in current psychological theories</td><td>139</td><td rowspan="7">An image:
- is a mental entity;
- is different from concepts;
- is a sensorial representation of an object;
- could be turned and moved;
- is a picture in the head;
- interacts with concepts;
- is based on the perceptive-sensorial experience;
- could be perceptive or mental.</td></tr>
<tr><td>In all the actual cognitive theories, concepts and images are considered two basically distinct categories of mental entities. […] The image of a metallic object is the sensorial representation of the respective object (including color, magnitude, etc.).</td><td>139</td></tr>
<tr><td>In contrast [respect to concept], an image (we refer here to mental images) is a sensorial representation of an object or phenomenon.</td><td>139</td></tr>
<tr><td>Concepts do not turn, do not move, and images, as such, do not possess the perfection, the generalization, the abstractness, the purity which are supposed when performing the calculations.</td><td>141</td></tr>
<tr><td>We are so used to distinguishing between images, as "pictures in the head", and concepts, i.e., general, non-sensorial ideas, that it is very difficult to accept a construct which would have, simultaneously, conceptual and imaginative spatial qualities.</td><td>143</td></tr>
<tr><td>[…] images and concepts interact intimately.</td><td>144</td></tr>
<tr><td>There is extensive experimental evidence concerning the reciprocal role played by images and concepts in learning and solving activities […] But in this interplay, images and concepts are considered distinct categories of mental entities.</td><td>144</td></tr>
</table>

It [a real cube] is a **sensorial image** like so many images which come into mind as an effect of our practical experience: the house in which I live, the room in which I use to work, representations of relatives, friends, students, etc. Beyond that image there is **another image not sensorially perceived but *thought***, the genuine object of our geometrical reasoning. This is the image to which we refer when performing a mathematical operation.	143	In geometry, an *image*: - is a spatial representation under the control of a definition; - may be exhaustively controlled by concepts; - could be mentally manipulated; - could be thought, not only perceived by senses, and this is specific of mathematics.
As a matter of fact, the triangle to which we refer and its elements cannot be considered either pure concepts or mere **common images**.	140	
[…] mathematical-logical operations manipulate only a purified version of the image, the **spatial-figural content** of the image.	148	
It [the meaning of circle] is an image entirely controlled by a definition. Without this **type of spatial images**, geometry would not exist as a branch of mathematics.	148	
[…] the considered figure is, *from the beginning,* not an **ordinary image** but an already logically controlled structure.	143	
[…] a **geometrical figure** is *not* a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the **mental representation of space property**.	141	
(a) a **geometrical figure** is *a mental* image, the properties of which are completely controlled by a definition;	148	

<table>
<tr><td>[…] (c) the mental image of a geometrical figure is, usually, the representation of the materialized model of it.</td><td></td><td rowspan="6">Geometrical figure:
- is a mental representation of space properties;
- is a mental image, the properties of which are completely controlled by a definition;
- could be mentally manipulated;
The mental image of a geometrical figure is the representation of the materialized model of it.</td></tr>
<tr><td>One has, then, to consider three categories of mental entities when referring to geometrical figures: the definition, the image (based on the perceptive-sensorial experience, like the image of a drawing) and the figural concept.</td><td>148</td></tr>
<tr><td>[…] the scientist manipulates the images according to the respective concepts. The difference between empirical sciences and geometry, in this respect, is that in geometry the images may be exhaustively controlled by concepts while in empirical sciences they are not.</td><td>148</td></tr>
<tr><td>To manipulate an image, a spatial representation under the strict but also intrinsic control of a definition would not be possible if only two independent processing codes would exist. When solving a geometrical problem we manipulate geometrical figures as if they were homogeneous mental entities, not combinations of two categories of heterogeneous mental constructs.</td><td>153</td></tr>
<tr><td>[…] the student has to learn to mentally manipulate geometrical objects by resorting simultaneously to operations with figures and to logical conditions and operations. Such a type of activities, already referred to in the present paper, consists of (a) asking the students to draw the image obtained by unfolding a geometrical body (actually perceived or mentally represented); …</td><td>158</td></tr>
<tr><td>Though the circle is an image, a spatial representation, its existence and its properties</td><td>156</td></tr>
</table>

are entirely imposed by an abstract, formal definition.		
We deal here with figural concepts because every part of the **image** (angles, sides, points, the circle, the arc) are simultaneously images and concepts, the images being **controlled by the respective definitions**. But, in the dynamics of the reasoning process, the image by itself seems to be unable to answer the question.	157	
The term "figural concept", introduced by us, is intended to emphasize the fact that we deal with a particular type of mental entities which are not reducible, neither to **usual images** - perceptive or entencephalic - nor to genuine concepts.	160	A figural concept: - is an image intrinsically controlled by concepts; - could be manipulated during the process of mathematical invention.
But, usually *in the process of mathematical invention* we try, we experiment, we resort to analogies and inductive processes by **manipulating** not **crude images** or pure, formal axiomatic constraints, but ***figural concepts***, *images intrinsically controlled by concepts.*	160	

Table 1 A review of the paper *"The theory of figural concepts"* (Fischbein, 1993). The first column contains the main claims about images, geometrical figures, and figural concepts; the corresponding page is shown in the second column; the last column contains the critical features of each theoretical construct that can be inferred from the paper.

In particular, Table 1 shows that Fischbein talks about the "mental manipulation" of several objects: images, geometrical figures, and figural concepts.

Moreover, such manipulation can be considered as an anticipatory device during geometrical problem-solving (Fischbein, 1993, p. 159). Nevertheless, he does not clarify how a solver can accomplish manipulations during the resolution of a geometrical task.

2.6 The Dynamic Geometry Environment

The role of the modern digital technologies as powerful tools in supporting the teaching and learning of Geometry is widely recognized (Battista, 2008; Bruce et al, 2011; Clements & Sarama, 2011; Highfield & Mulligan, 2007; Sinclair, de Freitas & Ferrara, 2013; Sinclair & Moss, 2012). In particular, research stresses the importance of the visual and kinetic interaction that a student can accomplish within a Dynamic Geometry Environment. Indeed, the use of a DGE is particularly relevant because of the specificity of the interaction between the user and the microworld.

> The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner's input, and the feedback of the environment is provided in the proper register allowing its reading as a mathematical phenomenon. (Balacheff & Kaput, 1996, p. 470).

Moreover, Noss and Hoyles (1996) conceive the interaction between a solver and the microworld as a window on processes of meaning-making. They stress that the computer can be a channel through which communication can happen and a window through which this can be seen. In the next chapter we will illustrate how a DGE can be conceived as a microworld and how the solver’s interaction with a DGE is integrated into our study.

Within the wide pool of research on the role of DGE, we want to focus on two findings:

- the transformational-saliency hypothesis advanced by Battista (2007)
- the *maintaining dragging modality as a psychological tool* (Mariotti & Baccaglini-Frank, 2011; Baccaglini-Frank & Antonini, 2016)

The dynamism of the figure is actually recognized as one of the main features of the DGE. In particular, *dragging* is the function that allows direct manipulation of the figure on the screen (Laborde & Strässer, 1990), inducing transformations that can be perceived as a movement of the figure. In this context, geometrical properties are interpreted as *invariants* (Laborde, 2005) and exploring a dynamic figure can become a search for such invariants (Laborde et al., 2006; Hölzl, 1996; Arzarello et al., 2002; Healy & Hoyles, 2001; Baccaglini-Frank et al., 2009).

An interesting point of view on the effectiveness of dragging within the exploration of a geometrical figure is provided by Battista (2007). He starts from two assumptions. First, “the relationships are established with unconscious visual transformations” (Battista, 2007, p. 860).

> For example in a parallelogram, seeing the relationship that opposite sides are parallel might be based on mentally (but unconsciously) translating one side onto the opposite side. Similarly, seeing that opposite angles in a parallelogram are congruent may be based on an unconscious 180° rotation. (ibid., p. 860)

The second assumption, which he called the *transformational-saliency hypothesis* is closely connected to dragging. This hypothesis essentially states that people notice invariance. For Battista (2008), it is not just that one *might see* invariance in dragging but that one cannot help but notice it.

He thus conjectured that investigating shapes through the transformations of a dynamic figure in a DGE "make the essence of the properties more psychologically salient to students than simple comparing examples of shapes as in traditional instruction" (Battista, 2008, p. 152).

Dragging thus changes the way shapes are perceived, moving from a static visual apprehension to that of a temporal attention on what remains invariant.

From an educational point of view:

> Because of the nature of this kind of digital technology, which enables continuous transformation through dragging in which only non-critical attributes of a shape can change, but critical ones are preserved, an a priori analysis of DGE affordances suggests that they could both (1) help learners see and make a large example space of geometric shapes such as long, skinny triangles, and (2) help learners appreciate aspects of the inclusive relations in the sense that it is possible to transform a constructed parallelogram, for example, into a rectangle. (Sinclair & Bruce, 2015, p. 325)

The analyses on the use of dragging from a cognitive point of view is documented by several research studies (Arzarello et al., 2002; Olivero, 2002), which focused on the ways in which dragging may affect students' reasoning process and led to a first classification of *dragging modalities* that students might use in solving open problems.

A new insight into the first classification was provided by Baccaglini-Frank (2010) who, focusing on dragging for conjecture-generation, added a new category: *maintaining dragging*, i.e. dragging a base point so that the dynamic figure maintains a certain property.

More specifically,

> we consider maintaining dragging (MD) the mode in which a base point is dragged, not necessarily along a pre-conceived path, with the specific intention of the user to maintain a particular property. (Baccaglini-Frank and Mariotti, 2010, p. 230)

This dragging modality is used when a solver, who has recognized a configuration of the dynamic figure as interesting, attempts to induce a particular recognized property to become an *invariant under dragging*.

Indeed, as stressed by Leung et al. (2013), “a figure can be seen as a set of affordances that the dragger perceives” (ibid., p. 441) through interaction with the figure, which allows her to discover invariants. In this perspective, the notion of affordance is “the possibility of actions we have not actually undertaken.” (Neisser, 1989 p. 12)

Moreover, researchers (Mariotti & Baccaglini-Frank, 2011; Baccaglini-Frank & Antonini, 2016) advance the hypothesis that the scheme of *maintaining dragging* can be internalized, leading to a *phycological tool* (in the perspective of Vygotsky, 1978), freed from the physical support of the DGE.

Once it becomes a psychological tool, *internalized maintain dragging* seems to support the process of discovery of geometrical properties (Mariotti & Baccaglini-Frank, 2011; Baccaglini-Frank & Antonini, 2016).

3. Theoretical framework

In this chapter we will present the fundamental theoretical constructs that other researchers in Mathematics Education have developed and that will be used in this study as theoretical lenses to analyze solvers' thinking during the resolution of geometrical tasks. Moreover, we build on a subset of these constructs to develop tools that are more suitable for this study.

First, we describe a theoretical framework used within research in Cognitive Psychology for explaining the interaction between solvers and images of different kinds, including those that address geometric figures (Section 3.1). We highlight the shortcomings that we found in the effort of applying this cognitive theoretical framework to the domain of geometrical reasoning.

From Section 3.2 on, we focus on the theoretical constructs that belong to the field of research in Mathematics Education. In Section 3.2, we explain how the *Theory of Figural Concepts* well describes the specific nature of the geometrical objects. Since this perspective constitutes the main interpretative lens of this study, we describe in greater depth the theoretical constructs that we will use in this study.

Section 3.2.1 contains the definition of the *figural concept*, followed by the introduction of the notion of *geometric figure* in Section 3.2.2.

In Section 3.2.3 we distinguish between the figural component and the conceptual component of a figural concept and interpret geometrical reasoning in terms of a dialectic between these two aspects. In Section 3.2.4 we explain the role that *conceptual control* and the *prototype effect* can have within the resolution of a geometrical task. Finally, we provide some definitions of theoretical constructs that belong to the *Theory of Figural Concept* in a formulation that is more suitable for this study (Section 3.2.5).

A preliminary definition of *geometric prediction* is proposed in Section 3.3. In particular, we refer to geometric prediction as mental processes that interacts with the following constructs: the theoretical elements and the figural elements of a geometric configuration, the solver’s theoretical control, the figural transformations of the figural components. All the fundamental notions and terminology are explained.

Having provided an initial formulation of *geometric prediction*, we touch on another related theoretical construct: *intuition*. In Section 3.4, we introduce and describe features of *intuitive knowledge*. In particular, we highlight the connections with processes of prediction, focusing mostly on *anticipatory intuition*.

In Section 3.5, we describe *open problems* as a kind of geometric problems that are suitable for stimulating productive thinking and we introduce *prediction open problems* as useful tools for eliciting prediction processes. Moreover, we highlight the theoretical perspective following which we will analyze solvers' productions, in particular focusing on the role of gestures as a window onto the process.

In Section 3.6, we focus on the exploration of a geometrical task within a *Dynamic Geometry Environment* (DGE) regarded as a source of additional windows onto the processes of prediction and onto its products. More specifically, we advance hypotheses on the possible role of *surprise* in processes of prediction.

3.1 Visuo-spatial abilities

From the perspective of Cognitive Psychology, generating and processing *mental images* take place within a complex process of acquisition and use of abilities, including those denoted *visuo-spatial abilities*.

An analysis of the literature on this topic reveals that a shared definition of these abilities does not exist yet. Nevertheless, a list of visuo-spatial abilities appears in (Cornoldi & Vecchi, 2004, p. 16):

- *visual organization*, the ability to organize incomplete, not perfectly visible or fragmented patterns;
- *planned visual scanning*, the ability to scan a visual configuration rapidly and efficiently to reach a particular goal;
- s*patial orientation*, the ability to perceive and recall a particular spatial orientation or be able to orient oneself generally in space;
- v*isual reconstructive ability*, the ability to reconstruct a pattern (by drawing or using elements provided) on the basis of a given model;
- *imagery generation ability*, the ability to generate vivid visuo-spatial mental images quickly;
- *imagery manipulation ability*, the ability to manipulate a visuospatial mental image in order to transform or evaluate it;
- *spatial sequential short-term memory*, the ability to remember a sequence of different locations;

- *visuo-spatial simultaneous short-term memory*, the ability to remember different locations presented simultaneously;
- *visual memory*, the ability to remember visual information;
- *long-term spatial memory*, the ability to maintain spatial information over long periods of time.

As we can guess, these abilities are also related to geometrical activities, and, indeed, clinicians evaluate students' performances in visuo-spatial tasks to assess their possible difficulties in learning school geometry.

A previous study (Miragliotta & Baccaglini-Frank, 2017) focused on these cognitive abilities: we tried to give an operational definition of some visuo-spatial abilities in the specific domain of geometrical reasoning.

Our findings revealed that two of the visuo-spatial abilities were particularly relevant in geometrical reasoning; these were described as: *imagery generation* and *imagery manipulation*.

However, providing operational definitions of the abilities within the domain of geometry and using them to analyze student's reasoning reveled to be quite difficult. In particular, we reached the conclusion that visuo-spatial abilities alone as listed above are not enough to explain the complex processes involved in geometrical problem solving. In particular, a process that seems to be often carried out by the solvers is to imagine the consequence of their (mental) manipulation on a figure consistently with given theoretical constraints. The reference to a mathematical theory (there Euclidean Geometry), to which the constraints belong, is neglected within the psychological research on visuo-spatial abilities.

However, a frequent process for mathematicians involves imagining consequences of transformations on a geometrical object which are consistent with a set of theoretical constraints (given or induced by a step-by-step construction).

Moreover, we observed some instances of this kind of process in verbal and gesture productions of students involved in solving a geometrical task. Such a process can be carried out through the use of certain abilities listed above, but there is more to it. Let us consider an explanatory example to further argue this point.

During a previous study (Miragliotta, Baccaglini-Frank & Tomasi, 2017), the students were shown a figure that was constructed ahead of time in a DGE (Figure 2). The construction in Figure 2 below is that of a *robust* parallelogram accomplished starting from three points on the screen (E, F, G), connected by segments. Vertex H is constructed as the intersection point between two lines: the first is parallel to EF (through G); the second is parallel to FG (through E).

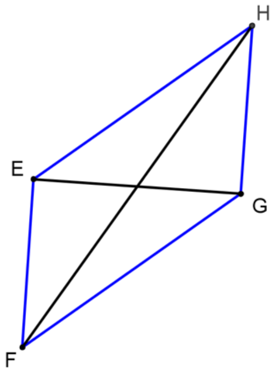


Figure 2 An instance of the dynamic figure of a robust parallelogram used within a previous study (Miragliotta, Baccaglini-Frank & Tomasi, 2017)

The students had previously discovered the properties of several quadrilaterals, including the parallelogram. The solvers were asked whether the given quadrilateral could become a square by moving one of its sides. When the interviewer asked the question and while the solvers were talking, the figure was static on the screen. Here is an interesting excerpt.

Interviewer: Do you think the parallelogram could become a square by moving just side EF?

Elena: I think so, because I said [before] that **the angles can vary**.

Domenico: I say no. Because I don't see how the square could come out by moving EF. I mean, if I move it [he refers to EF] **I don't see the square being born**.

We tried to analyze this excerpt using some of visuo-spatial abilities listed before.

Domenico appears to only be using the *imagery generation ability.* The term "*to be born*" (the Italian "*nascere*" in the original) suggests just something that is generated. It seems that such "generation" happens only in his mind because the student speaks about the transformation while the figure is static on the screen. Moreover, the linguistic expression used suggests that he is observing (in his mind) a generative process, and therefore that he is using his *imagery manipulation ability*. Nevertheless, the student's discourse reveals not only a transformation of the figure as it appears, but also a transformation that is strictly connected with the theoretical constraints. It seems that he (mentally) transforms the figure in order to observe whether he can give it properties which he considers necessary for it to become a square (eventually according to what he considers a square). At the end of or during the manipulation, he seems to know whether the figure has the properties of a square. This point is hard to be explained if we can use only the visuo-spatial abilities proposed in psychology.

In solving the task, Elena also appears to be using theoretical elements ("*the angles can vary*"), which also guide her answer. Elena's answer cannot be explained at all if we are to use only the visuo-spatial abilities in psychology, because of her theoretical considerations.

Other examples like these suggested that visuo-spatial abilities are not sufficient for explaining what happens in these kinds of situations, mainly because we need to be able to see how theoretical elements come into play.

The main shortcomings in the use of visuo-spatial abilities listed above for interpreting students' behaviors in geometrical problem solving are the following:

- a lot of interpretation is frequently involved in establishing which abilities are used during each process analyzed;
- none of them explicitly deal with a geometrical theory of reference (e.g., Euclidean Geometry) when the reasoning is carried out in this context.

What could explain the solvers' behaviors during the resolution of a geomatical tasks such as the one shown above is a model that considers elements that belong both to the solvers' visual experience and to their *theoretical awareness* of the geomatical objects. Indeed, when we are using representations of geometrical objects a conceptual component is strongly involved. This point will be widely explained in the next section.

3.2 The Theory of Figural Concepts

Among the theoretical perspectives which address the topic of spatial reasoning within the domain of geometry, we have chosen the *Theory of Figural Concepts* (Fischbein, 1993). It carefully considers and well describes the multifaced nature of the geometrical objects and stresses how the different components of it are strongly intertwined. It seems to be the most suitable theoretical perspective for investigating how different elements of thinking and perception intervene during a prediction process within the domain of geometry.

As other mathematical domains, geometry is a logical system made up of definitions and theorems, and whose objects are ideal. However, geometry maintains a strong connection with material objects (solids or drawings), at least at an initial stage of learning. Geometry as a mathematical theory is a cultural artifact that, at the beginning, relies on a more natural conceptualization of space. At a more advanced stage, *geometry as a logical structure* acquires a broader sense, without the necessity of "a real environment" as a basis (Hershkowitz et al., 1989), but its objects can continue sharing attributes of both conceptions of geometry.

3.2.1 The figural concept

This point is a key issue within the *Theory of Figural Concepts*, which considers geometrical objects as having a dual nature. *Geometrical objects* are completely described and controlled by an axiomatic system of definition and theorems, but at the same time they maintain certain figural aspects of images. This is a specific characteristic of geometry. Let us describe in more detail the crucial elements of this theory.

Starting from the main findings of research in Cognitive Psychology (see, for example, Kosslyn, 1996), Fischbein makes a distinction between images and concepts.

> What then characterizes a concept is the fact that it expresses *an idea*, a general, ideal representation of a class of objects, based on their common features. In contrast, an image (we refer here to mental images) is a *sensorial* representation of an object or phenomenon. (Fischbein, 1993, p. 139, italics in the original)

In contrast to an *image* that is used as a synonym of "picture in the head", what Fischbein calls *concept* is ideal, abstract, perfect and universal (idib., p. 141). Although he recognizes the interaction between images and concepts in problem-solving activities, he considers the distinct category of *mental entities*. The resolution of geometrical tasks is considered to be a very special case, where the distinction between images and concepts is not so clear cut. The main assumption of Fischbein's theory is reported below.

> What we assume is that, in the special case of geometrical reasoning, one has to do with a third type of mental objects which simultaneously possess both conceptual and figural properties. (Fischbein, p. 144)

This third multifaced object, different from pure concepts and pure images, is called the *figural concept*. It realizes the fusion between the conceptual and the figural component of a geometrical object.

For the sake of clarity, let us consider the following task (Fischbein, 1993, p. 139):

> Consider an isosceles triangle ABC, with AB = AC (Figure 2). We want to prove that $\angle B = \angle C$.

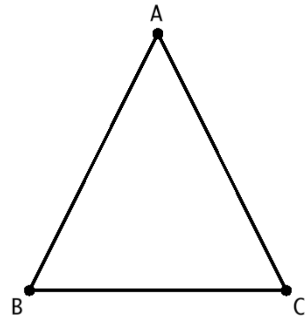


Figure 3 The drawing of an isosceles triangle

Although for the sake of accuracy here we propose a revised version of a possible resolution of the task, we maintain the author's same approach. We can consider an additional line through *A* and perpendicular to *BC*. We name the intersection point *H* and consider the triangles *ACH* and *ABH*. A solver can imagine overlapping *ACH* on *ABH* with a rotation around the line within the space; the rotation is similar to what can be obtained through paper folding.

> AC will coincide perfectly with AB on the left side. […] As a consequence, the angle ∠B and ∠C must be equal. (ibid., pp. 139-140)

Although the resolution proposed by Fischbein is not a rigorous geometrical proof, it can be considered a possible first idea of the solution that has to be followed by an analytical approach. A possible rigorous proof can make use of the triangles criteria of congruence.

The example was shown in order to stress that the objects to which we refer during the resolution process are points, sides, angles and the operations with them. They have a conceptual nature and an ideal existence. Nevertheless, they share a figural nature, without which we cannot conceive operations like detaching, reversing or superposing. So, the triangle and its elements cannot be considered either pure concepts or mere images, but they share a two-fold nature: conceptual and figural.

Manipulability is a fundamental characteristic of Fischbein's *figural concepts*:

> But, usually *in the process of mathematical invention* we try, we experiment, we resort to analogies and inductive processes by manipulating not crude images or pure, formal axiomatic constraints, but *figural concepts, images intrinsically controlled by concepts.* (Fischbein, 1993, p. 160, italics in the original)

Figural concepts "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities – like ideality, abstractness, generality, perfection" (ibid., p. 143). From the developmental point of view, initially the visual aspect is dominant, and gradually the role of formal constraints becomes more important, until the construction of the figural concept is reached (Mariotti, 2005). Now, it is clear that the figural concepts are not natural or innate, but their construction is a learning achievement that has to be carefully supported by adequate geometrical learning experiences (Fischbein & Mariotti, 1997).

Although the analysis of the developmental aspects of the figural concept is not the focus of our study, this is an important point because it highlights the malleability of the construct that can change during the time and, consequently, can be supported by an effective teaching and learning activity.

3.2.2 Figural concepts and geometrical figures

The issue of giving a working definition of *geometrical figure* is addressed twice in (Fischbein, 1993). A first attempt is reported below.

> A geometrical figure may, then, be described as having *intrinsically* conceptual properties. Nevertheless, a geometrical figure is *not* a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property. (ibid., p. 141, italics in the original).

A complete characterization of the *geometrical figure* is provided by the following list.

> (a) a geometrical figure is a *mental* image, the properties of which are completely controlled by a definition;
>
> (b) a drawing is not the geometrical figure itself, but a graphical or a concrete, material embodiment of it;
>
> (c) the mental image of a geometrical figure is, usually, the representation of the materialized model of it.
>
> The geometrical figure itself is only the corresponding i*dea* that is the abstract, idealized, purified figural entity, strictly determined by its definition.
>
> (ibid., p. 149, italics in the original)

The difference between *geometrical figure* and *figural concept* is not so clearly stressed. Nevertheless, the last sentence allows us to infer a small difference. It seems that the geometrical figure is strictly connected with the definition of a particular figure that is shared by the community of mathematicians; acquiring a corresponding figural concept that mirrors the properties of the geometrical figure becomes a goal of the teaching and learning process.

Indeed, the figural concept is individually constructed during the learning experiences; it evolves simultaneously, and it does not necessarily coincide with the corresponding geometrical figure, at least at an initial stage.

This interpretation is supported by another definition of geometrical figure: elsewhere it is considered as a triadic structure composed by "the definition, the image (based on the perceptive-sensorial experience, like the image of a drawing) and the figural concept" (ibid., p. 148).

So, it seems that the geometrical figure and the figural concept interact, but they are different. The important point for this study is that figural concepts and geometric figures have a conceptual and a figural nature.

3.2.3 Conceptual components and figural components

A geometrical figure and a figural concept are made up of two fundamental components: the figural component and the conceptual component.

The *conceptual components* refer to the theoretical status of a figural concept: the definition, the properties, the theorems. They may be affected by logical fallacies (Fischbein, 1993, p. 145). In the present study, the conceptual components are part of the *Theory of Euclidean Geometry* (TEG).

The *figural components* refer to the spatial properties of a figural concept. They may be influenced by the Gestalt theory of perception (ibidem).

In principles, the two aspects are strongly intertwined and blended. When the fusion is complete, we talk about a *harmony* between the two components. However, the fusion between the conceptual and the figural components is not always complete, and the two components can be in contrast. A conflict between the conceptual and the figural component or an erroneous interpretation of one of them may cause a break of the harmony between these two aspects.

This is an important point for interpreting solvers' mistakes or incoherent interpretations of a geometrical task. Indeed, the correctness and the effectiveness of reasoning reveal the harmony between the two components; instead, the mistakes may be interpreted as a break of the harmony (Mariotti, 1995).

Operationally, the *geometrical reasoning* can be interpreted in terms of a dialectic between these two aspects (Mariotti, 1995).

Let us consider the following example proposed by Fischbein (1993).

In a circle with its center at O we draw two perpendicular diameters AB and CD. We chose arbitrary a point M and we draw the perpendiculars MN and MP on the two diameters. What is the length of PN?

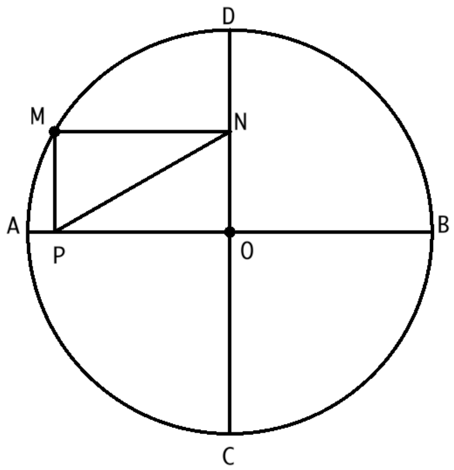


Figure 4 A possible drawing sketching the problem of the circle

At a first glance, the problem seems quite complex and difficult to be solved. Indeed, the length of the segments *MP* and *MN* changes and functionally depends on the particular position of *M*. However, a solver can focus on *MPON* and observe that it is a rectangle; the segment *MO* is a diagonal of that rectangle, as is *MP*. Consequently, *PN* has the same length of *MO* which is the radius of the circle. So, the length of *PN* is the same of the radius of the circle; if we know the length of the radius, we can easily find an answer to the question. The fundamental point is stressed by Fischbein:

> The equality of the diagonals is not questioned, the equality of the radiuses is not questioned. These relationships do not depend on the drawing itself. They are imposed by definitions and theorems. (ibid., p. 142)

The key point is that the solver does not find a solution by considering separately the figural aspects and the theoretical constraints, but by accomplishing a process in which "a distilled figure is considered, revealing logical relationship" (ibid., p. 142). In this case, the fusion between figure and concept is complete.

Nevertheless, not all solvers have harmonically fused the two components and one of the two can dominate the solution process.

3.2.4 Conceptual control

As highlighted, the rules which control and influence the figural components and the conceptual components have a different nature. Mariotti (1992) explains this difference talking about two different systems of control:

> The figural control system suggests transforming the drawing, moving (translating, rotating, reflecting, …) the pieces, changing their places […] But, only the conceptual control system can affirm the possibility and the correctness of this procedure. Thus only a dialectic interaction between these two systems can make it possible to reach the solution through this way. (ibid., p. 15)

So, during the resolution of a geometric task, the harmony between the conceptual and the figural components of a figural concept is reflected in an effective dialogue between the figural control and the conceptual control.

The visuo-spatial abilities defined in the domain of cognitive psychology (e.g. Cornoldi & Vecchi, 2004) can play a role within the first system. However, during the resolution of a geometric task, the figural transformations are also controlled by the conceptual control system. This is a huge difference between the transformation that we can imagine in a general spatial domain and in the figural domain of Euclidean Geometry.

The researchers (e. g., Fischbein, 1993; Mariotti, 1995; Mariotti & Fischbein, 1997; Mariotti & Baccaglini-Frank, 2018) stressed the fundamental role of the latter system of control, in particular when a solver needs to rearrange the figural components of a given drawing coherently with respect to a reference mathematical theory. More specifically:

> It is under the conceptual control that the solver may imagine certain properties as logically dependent upon others. […] Furthermore, in the paper-and-pencil environment, no element of the figure is privileged with respect to others, and reasoning on a specific unique drawing that represents a class of figures requires a high harmonization between the figural component and the conceptual component. (Baccaglini-Frank, 2010, p. 28)

So, conceptual control system becomes fundamental during all the phases of the resolution of a geometric task.

However, very often the figural components may escape the conceptual control and they can dominate the interpretation of a drawing of a geometrical figure. This could lead to an interpretation that is consistent within the figural control system, but completely incoherent with the conceptual constraints within the reference mathematical theory. Indeed:

> The figural component tends to liberate itself from the formal control and to behave autonomously in conformity with Gestalt pattern. (Fischbein, 1993, p. 154)

This is recognized as one of the main obstacles in geometrical reasoning.

Nevertheless, there is another phenomenon connected with the figural components of the figural concepts that could affect an effective resolution of a geometrical task: the *prototype effect* (Mariotti, 1993, 1995) or *prototype phenomenon* (Hershkowitz, 1989).

There exist some figural components connected to a specific figural concept that are very common among students. For example, usually an isosceles triangle is sketched as "standing on its base" and a right triangle is shown with the legs in the vertical-horizontal position on a sheet of paper (ibid., p. 68); the height of a triangle is only the perpendicular segment that is "inside" the triangle. The features of a prototype depend on the geometrical experience of the students.

The *prototype effect* arises when a stereotyped image of a figural concept is assimilated or becomes the concept itself. This influences the resolution of a geometrical task. Indeed, Mariotti (1995) stresses that not only there are figural components of a geometric figure that are "more popular" or common than others among students, but their influence during the resolution process can be so strong that it overcomes any conceptual control. So, for analyzing student reasoning during the resolution of a geometrical task, the existence of standard or stereotyped figural arrangement of a geometrical figure and the consequent prototype effect have been considered.

3.2.5 The *Theory of Figural Concept* within the present study

Summarizing, Fischbein's *Theory of Figural Concept* gives us an adequate theoretical lens for analyzing the components of the geometrical reasoning that can interact with the prediction processes, which will be described in detail in the next section.

It provides an operational definition of *geometrical reasoning* in terms of harmony between conceptual and figural components and of interaction between two different systems of control.

According to the aim of this study, the operational distinction between figural components and conceptual components of a figural concept is useful in order to unveil the role that they play during a process of prediction.

However, we prefer to use a "softer" definition of the two components. Indeed, possibly the figural concept that the solvers are referring may not be so transparent, while they are involved in the resolution of a geometric task. This is particularly evident when the solvers perform a step-by-step construction which produces a not necessarily known or defined geometrical figure.

Let us consider the following situation (Figure 5):

On the plane there is a point A and a line r; another point B is constructed as the symmetric point of A with respect to the line r.

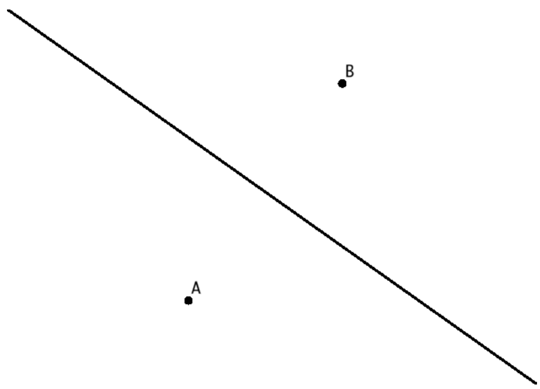


Figure 5 A possible drawing sketching out the given problem

In this case, there is not a proper geometrical figure to which we can refer; it is more likely an arrangement of geometrical objects (like points, segments, lines) that are connected by some given theoretical constraints. In such an arrangement the solvers may or may not elaborate a particular figural concept, but we cannot be sure. In any case, it is a geometrical object itself, so it has the same twofold nature of figural concepts. According to the Fischbein's distinction, we can identify:

- *figural elements* of a geometrical configuration;
- *theoretical elements* of a geometrical configuration.

These are definitions that could refer both to a geometrical figure and to a simpler geometrical configuration. We make use of the term *theoretical* in order to stress the connection of these elements with the properties and the constraints given within a particular mathematical reference theory. In this study, the reference theory is always the *Theory of Euclidean Geometry* (*TEG*) conceived as an axiomatic system of definition and theorems. So, "theoretical" stands for "given by or deduced from the *TEG*".

For referring to the control that the solvers can use on a geometrical arrangement or figure in the specific domain of the *TEG*, we will use the expression *theoretical control* intending

> [the act of] mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (Mariotti & Baccaglini-Frank, 2018, p. 156).

We will say that the solvers show *good theoretical control over a figure* if, during the resolution of a geometrical task, they are able to consider all the theoretical constraints of the figure at any time. Otherwise, we will identify a *lack of theoretical control over the figure*. We can observe these two constructs only by looking at the solvers' productions and explanations.

The key idea of this theoretical lens and one of our assumptions is that the interaction between solvers and geometrical objects cannot be reduced to a figural transformation of a figure sketched in a drawing.

3.3 A preliminary definition of GP as a process

The main goal of this study is to describe, from a cognitive point of view, a process of generation of prediction during the resolution of particular geometrical tasks within the context of Euclidean Geometry. The elaboration of a preliminary definition of the process and the subsequent observation of instances of predictions, if possible, seemed to be the best way of finding an answer to accomplish this. In this section, we introduce a first definition of the process we intend to focus on, and we present our working hypothesis and theoretical assumptions.

Let us consider a problem presented in (Fischbein, 1999, p. 51) that addresses a well-known result of the *TEG* (i.e. the *Varignon's Theorem on quadrilaterals*). A possible drawing is sketched below (Figure 6).

Consider the quadrilateral ABCD and the midpoints of its sides PQRS.

Prove that the quadrilateral PQRS is a parallelogram.

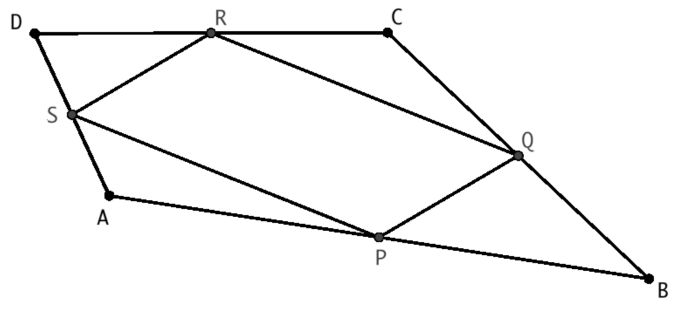


Figure 6 A possible drawing of the two quadrilaterals presented in the task

The resolution of a task like this requires the solvers to recall their figural concepts of quadrilateral, parallelogram, midpoints. To reach a first idea of the solution they need to dominate the conceptual components of all these figural concepts, but also their figural counterparts. The author makes a comment on the resolution as follows:

> To solve such a problem, the student has to learn to manipulate the figural concepts freely, imaginatively, and constructively – but the student must do so under the strict control of formal constraints. The symbiosis between rigor and constructive liberty is specifically mathematical. (ibid., p. 51)

The author stresses the important role that the handling of figural concepts plays in the resolution process. In this formulation of the task, the goal is explicitly given: the solver has only to find why the quadrilateral *PQRS* is always a parallelogram. Nevertheless, in order to be aware of the truth of the conclusion, the solver can operate figural changes of the configuration, strictly guided by the conceptual control. The transformation of a geometrical figure that a solver can perform is considered a specific feature of the resolution of a geometrical task. This point is also stressed by other authors (e. g. Mariotti, 1992; Presmeg, 1997).

Let us consider a different formulation of the problem, as follows:

> *Consider the quadrilateral ABCD and the midpoints of its sides PQRS.*
>
> *What can you say about the quadrilateral PQRS?*

We have simply changed the question: in this formulation there is the only information that *PQRS* is a quadrilateral. With a small change, we have obtained a completely different task. The property of *PQRS* of being a parallelogram is itself a finding to be reached by the solver. To do so the solver has to evaluate several dispositions of the quadrilateral *ABCD*, even when considering the most general quadrilateral. The solver can accomplish the process of evaluation discretely, which means considering several instances (possibly different cases) of the same geometrical situation; otherwise, the solver can continuously figurally transform the figure, considering the several cases as instances of the same dynamic object. The latter case is very close to the transformation of a dynamic figure under dragging in a Dynamic Geometry Environment (Baccaglini-Frank & Antonini, 2016).

The solution of the problem evidently involves figural transformations of the figure that the solver is able to perform under the control of the conceptual component (i.e. theoretical control). Indeed, the solvers can imagine or perform figural changes of the quadrilateral *ABCD,* and therefore they have to properly consider the effect of this changing on the quadrilateral *PQRS*. In this way, the conceptual components that define the figural concept of "parallelogram" arise as invariant properties which can lead the solver to recognize *PQRS* as a parallelogram[1].

We notice that this process is complex but very close to the discovery experiences of mathematicians. Clearly for a solver who knows and recalls *Varignon's Theorem on quadrilaterals* the task is quite simple, and the problem can resort to the problem of proving that *PQRS* is a parallelogram.

We consider the figural changings of a geometrical figure that the solver can consider and accomplish under the theoretical control as a *prediction* on the possible behavior of the figural concept. The corresponding drawing represents a possible outcome of the prediction, used by the solvers for communicating their findings.

The term *prediction* was chosen in order to intend the feeling or belief that something will go in a certain way. We started from the definition of "*to forecast*" as contained in the *Oxford Dictionary*:

> to say what you think will happen in the future based on information that you have now.

Since this verb is generally used in association with the weather forecast, we preferred to use one of its synonyms: to *predict* or to *make a prediction*.

Fischbein (1993) also addresses the issue of making predictions in solving geometrical tasks, referring to *prediction* as an *ability* to be improved in order to make the students more capable of handling figural concepts in geometrical reasoning (ibid., p. 159). At the moment we prefer to talk about prediction as a *process* that could be supported by abilities or approaches that belong specifically to the figural control system and/or to the conceptual control system.

A first attempt to define the construct was made in a previous study (Miragliotta & Baccaglini-Frank, 2017), where a *geometric prediction* was defined as

> the identification of certain properties or configurations of a new figure, arising from a process of manipulation.

---

[1] It is known that another way to solve the problem is to use the well-established heuristic of adding auxiliary lines (the diagonals of the quadrilateral ABCD); then the *Midpoint Theorem* can be used for several subfigures of the given one. However, this is a useful approach when the solver already has a first idea of the solution and she wants to prove it. Instead, we are more interested in analyzing how such a first idea could or could not arise and how the theoretical elements, that the solver recalls, could influence this process.

With the expression *manipulation of a geometrical figure* we intend the figural transformation of its figural components. Possible transformations are translating, rotating, reflecting, lengthening, shortening. The action of manipulation can be only imagined or accomplished on a physical support (for example a drawing on a sheet of paper or a dynamic figure in a DGE). This is consistent with the constructs of *dynamic imagery* (Presmeg, 1986) and *visual processing* (Bishop, 1983). In the following, the term *manipulation* will be used according to this interpretation.

The manipulations can be coherent or incoherent with respect to the TEG. The solvers may or may not be aware of such coherence and be able to perform coherent manipulations, according to their *theoretical control*.

From the start, one of our hypotheses was that the transformation of the figural components of a figural concept is strictly connected with the generation of a prediction. Indeed, while the transformation is accomplished, the solvers seem to generate figural expectations on the final figure and they constitute a product of the prediction process.

Because of the strict connection we hypothesized between manipulation and prediction, we use the notion of *invariant of a figure* from the literature on research in DGEs. The terms "invariants", "geometrical invariants", "invariant properties" of a figure have been used to refer to certain properties that are maintained when some transformations on the figure are performed (e.g. Yerushalmy et al., 1993; Goldenberg et al., 1998; Hadas et al., 2000). Indeed:

> A geometric property is an invariant satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions. (Laborde, 2005, p. 22).

Our assumption is that, also outside the DGE, the solvers can perform manipulations on a drawing or imagine manipulations that allow them to maintain and perceive some geometrical properties as invariants. Preliminary observations and the pilot study seemed to confirm our hypothesis.

Referring to Varignon's problem described above, while a solver is manipulating the figure or after a manipulation, the parallelism of the opposite sides can be recognized as an invariant property of the figure.

Now we can provide the first formulation of *geometric prediction* (GP):

> [*Geometric prediction* is] a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant. (Mariotti & Baccaglini-Frank, 2018, p.157).

The outcome of such a process will be called in the following a *product of GP*. We expected that it is a geometrical object and therefore has a twofold nature. Products of GP may or may not be coherent with theoretical constraints, as well. So the construct of GP interacts with the two components of a geometrical figure and with the theoretical control.

Operationally, we identify instances of GP when we can recognize in or infer from the solvers' productions:

- the existence of figural expectations over the figure;
- a set of theoretical constraints that the solvers are maintaining over the figure.

Since the term "*prediction*", by definition, refers both to a statement that expresses a forecast and the act of making such a statement, in the following we make a distinction.

- We use the expression "*geometric prediction*" or briefly "*GP*", when we refer to the process of predicting something.
- We use the expression "*product of geometric prediction*" or "*product of GP*", when the focus is on the outcome of the process.

Because of the explorative nature of our study, in the following, we prefer to talk about *GP processes* because a priori we do not know whether there can be different processes of GP with certain commonalities (this is what we expect) or a single common process.

Let us consider another example.

Read and perform the following step-by-step construction:

- *segment AB;*
- *a point C on the plane;*
- *the triangle ABC.*

If C has to stay on a line r parallel to AB, what can you say about the configuration?

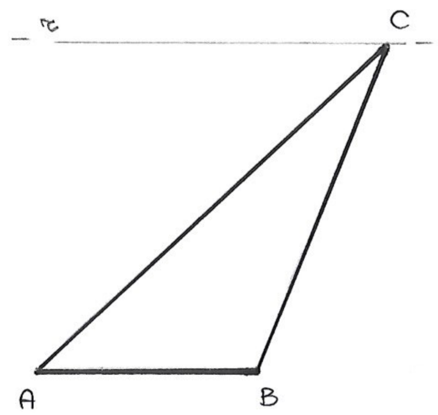


Figure 7 The picture of a possible drawing obtained following the given step-by-step construction

A possible sequence of coherent answers is reported below[2].

Solver: I have a triangle, its base [AB] and a point C.

Solver: Ok, if C has to stay on a parallel line, I can imagine moving C back and forth on this line [*r*]. In this way, I will obtain several triangles.

Solver: Also right triangles and an isosceles triangle.

Solver: And…I can consider a hypothetical height which is moved according to the point [C].

Solver: Through all of these and other triangles…

Solver: Ok, in this way, it could be always the same. I mean, the height has always the same length.

Looking at the solver's utterances and drawings (Figure 8) we infer that:

- several instances of the given geometrical configuration (a triangle and a line) are considered;
- the instances of the triangle are obtained considering the figure as a dynamic object upon which the solver can perform several figural transformations (i.e. manipulations);
- during such manipulations, the height of the triangle and its length, which are spontaneously introduced by the solver, arise as invariant properties;

[2] Although the discourse is artificially constructed, it is constructed by trying to make explicit the resolution process of the researchers' who work on this study. Moreover, we have borrowed some expressions (for example "hypothetical") that the students usually use during the resolution of tasks like the reported one.

- the resolution process shows the figural transformation, eventually imagined and then sketched out in a drawing, that the solver theoretically controls strongly.

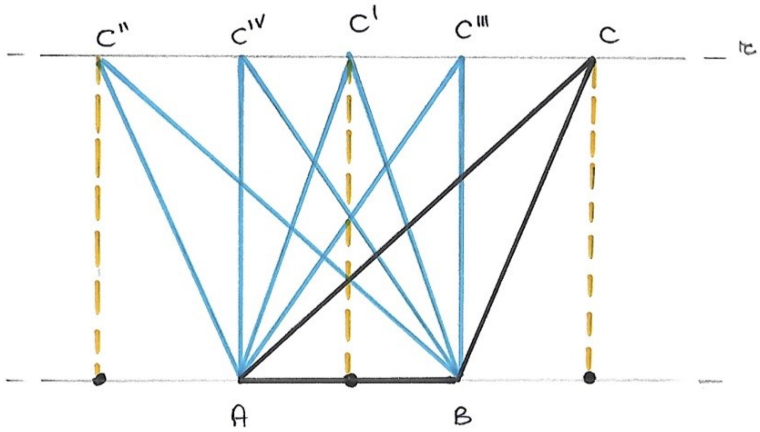


Figure 8 The picture of the drawings that the solver performs during the resolution of the given task

According to our interpretation, in this excerpt we identify instances of several prediction processes; they lead the solver to communicate some products of GP, among which the one that implies advanced theoretical control is the invariance of the height.

Since the definition of GP refers to a mental process, it is clear that we want to study a process that is not step-by-step observable by the researcher. Our assumption is that the features of the process could be inferred and then described looking at the products of a solver who is placed in a situation that elicits the GP processes. In our case, we choose to observe the solvers while they are solving a geometric task. This point will be presented in Section 3.5.1 and described in greater depth in Chapter 5.

In the next section, we will introduce another theoretical construct that could play an important role during the prediction processes.

3.4 The theoretical construct of *intuition*

A construct that seems very close to GP is *intuition*. In most of his work on this topic, Fischbein (1987) highlights that "intuition" is a controversial term that is used with several meanings in the literature. So, he stresses that he will use intuition as a synonymous of *intuitive knowledge*. In this perspective, intuition is a kind of cognition, characterized by self-evidence and immediacy, different from perception. It implies an extrapolation beyond the directly accessible information. Indeed:

> […] intuitions refer to self-evident statements which exceed the observable facts. (ibid., p. 14)

It is a form of *immediate knowledge*, i.e. "a form of cognition which seems to present itself to a person as being self-evident" (ibid., p. 6). Moreover, it is not a pure theory:

> […] it is a theory expressed in a particular representation using a model: a paradigm, an analogy, a diagram, a behavioral construct etc. (ibid., p. 50)

In sum, intuitive knowledge is

> […] a kind of knowledge which is not based on sufficient empirical evidence or on rigorous logical arguments and, despite all this, one tends to accept it as certain and evident. (ibid., p. 26)

The characteristics of intuition are listed below.

- *Self-evidence*: the feeling that some statements or a relationship "are true by themselves without the need for any justification".
- *Intrinsic certainty:* the intuitive facts are accepted as certain. It implies that robust intuitions (correct or not) "tend to survive even when contradicted by systematic formal instruction."
- *Perseverance:* intuition could be so robust that erroneous intuitions could coexist with correct interpretations.
- *Coerciveness:* an intuition imposes itself as an absolute and unique interpretation.
- *Theory status:* an intuition is not a skill or a particular perception, it is a theory because it expresses an invariant property guessed through a certain experience.
- *Extrapolativeness:* intuition exceeds the given facts.
- *Globality:* intuition is a global and synthetic view of a situation, opposed to analytical thinking. Intuition could be more or less structured and then more or less stable.
- *Implicitness*: intuition could be the surface of tacit and subjacent processes.

(ibid., pp. 43-56)

Intuition is also related to the construct of *overconfidence*. It is a selection activity aimed to preserve "those data which seem to support a certain conception and at the same time, to ignore those contradicting it." (ibid., p. 33). This implies another feature of intuition: its resistance to change and its reluctance to admit alternatives.

Fischbein describes intuition as a natural attitude of human beings that is used in order to avoid uncertainty. Moreover, he stresses that intuitions can be a potential source of errors. This suggests that humans are induced to make mistakes because of their intuition. Fischbein clarifies this point, highlighting that *expert* solvers are able to follow a *control stage* once the entire sequence of thoughts has been accomplished.

> They know *in principle* that they may be wrong but they go on reasoning as if they were convinced that they are correct at every step. (ibid., p. 37, italics in the original)

Instead, young students are not equipped for this kind of intellectual duplicity. This clarifies that, according to Fischbein, intuitions are not innate, they are

> […] learned cognitive capacities in the sense that they are always the product of an ample and lasting practice in some field of activity. (ibid., p. 69)

From a developmental point of view, our basic intuitions will never disappear, but they become less influential thanks to the individual *conceptual control* (ibid., p. 172; ibid., p. 174).

Our interest in intuition lies in the resonance that this theoretical construct shows with the construct of geometric prediction.

- They share the reference to objects that, at a certain moment, are not actually present and to relationships that are not ever empirically evident.
- The products of intuition and GP go beyond the directly accessible information and contain theoretical elements.
- Both are connected with the activity of recognizing a property as an invariant.
- They interact with the construct of theoretical control.

Following our research aims, another interesting element is the role of intuition within the problem-solving process. Indeed, intuition can have an anticipatory role. Considering the relationship between intuitions and solutions, intuitions may be grouped into *affirmatory, conjectural, anticipatory* and *conclusive* intuitions.

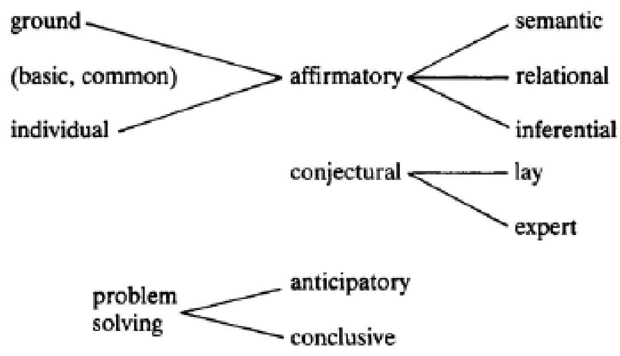


Figure 9 Original picture from (Fischbein, 1987, p. 64) that describes the classification of intuitions based on roles

Affirmatory intuitions are representations or interpretations of facts accepted as certain, self-evident, and self-consistent.

They are classified into:

- *semantic* (referring to the meaning of concepts);
- *relational* (expressed in an apparently self-evident statement, referring to the meaning of a relationship or a statement);
- *inferential* or *logical* (with an inductive or deductive structure).

Generalization is a type of *inferential affirmatory intuition*. Making use of this kind of intuition, one affirms or claims something.

Conjectural intuitions are assumptions about future events or the course of a certain phenomenon.

> Such a conjecture is an intuition only if it is associated with a feeling of confidence (ibid., p. 60)

They could be lay or expert conjectural intuitions.

Anticipatory and conclusive intuition are grouped as problem-solving intuitions.

> *During* the solving endeavor itself, they may appear as, subjectively, as moments of illumination, as certain, evident, definitive, globally grasped truths. These are anticipatory intuitions. (ibid., p. 62, italic in the original)

Anticipatory intuition is different from affirmatory. Due to the latter, we accept as evident a certain notion or a certain statement; instead, anticipatory intuition:

> […] appears as a discovery, as a solution to a problem and the (apparently) sudden result of a previous solving endeavor. (ibid., p. 61)

What distinguishes conjectural and anticipatory intuition is that

> […] anticipatory intuitions represent a phase in the process of solving a problem (necessarily followed by an analytical endeavor), while conjectural intuitions are, more or less, ad hoc evaluations and predictions generally not included in a systematic solving activity. (ibid., p. 61)

Anticipatory intuition and conjectural intuition are classified separately in order to stress that the first one belongs explicitly to a problem-solving activity. Indeed, it is a global view of a solution, which precedes the analytical solution. Nevertheless, there is not a clear-cut distinction:

> In fact, we have to consider a continuum from affirmatory to anticipatory intuitions passing through conjectural ones. (ibid., p. 61)

Moreover, *anticipatory intuitions* seem to be inspired, directed, stimulated or blocked by existing *affirmatory intuitions*, more related to semantic processes.

Conclusive intuitions sum in a global and structured vision the global solution to a problem previously reasoned upon. The global view could be expressed in verbal terms, in an image, in a gesture or in a combination of these.

The table summarizes some examples of intuitions reported in (Fischbein, 1987).

Affirmatory intuition	Sentence like: - *Two points determine a straight line* - *The whole is bigger than each of its parts*
Affirmatory semantic intuition	Concepts like "point" or "straight line" have a non-intuitive axiomatic meaning, but several intuitive meanings.
Affirmatory relational intuition	Sentence like: - *Through a point outside a line one may draw one and only one parallel to that line* - *The whole is bigger than each of its parts* - *A heavier object falls faster than a lighter one*
Affirmatory inferential intuition	From A=B and B=C one deduces *directly* as a self-evident conclusion that A=C. In a syllogism, the conclusion is determined by the premises. But the validity of the syllogism as a method of deducing a truth from previously accepted premises cannot be proved.
Conjectural intuition	Lay: *"I will invest my money in that business, I am sure that it will be successful."* Experts are able to make decisions in their domain on the ground of an apparently minimal amount of information

Table 2 Some categories of intuitions and examples, as presented in Fischbein (1987)

We do not find explicit examples of anticipatory and conclusive intuitions, probably because they are more likely to be possible phases or moments of a resolution process. Indeed, Fischbein (1999) explains that when the solvers accroach a problem, they can invest much effort in trying various strategies. These strategies can be ineffective but, at a certain point, something happens.

> *Suddenly*, he has the feeling that he has found the solution. He does not possess, yet, all the elements of the solution, that is, the formal, analytical, deductively justified steps of the solution. What he has in mind, during the first moment, *is a global idea, a global representation of the main direction leading to the solution*. This is also an intuition, an anticipatory intuition, called, sometimes, the 'illumination' moment. What characterizes such an intuition is, first of all, the fact that it represents *a moment in a solving endeavor*. Secondly, such an intuition is associated with a feeling of deep conviction, a feeling of certitude, *before* the entire chain of the formal – analytical basis of the solution has been established by the solver. For a mathematician, the solving process is not concluded before he is able to invoke explicitly *all* the arguments supporting the initially guessed solution. (ibid., p. 34, italic in the original)

So, an anticipatory intuition:

- arises suddenly during the resolution process, usually but not ever after the solver has investigated the problem for a while;
- reveals a global view of the solution;
- are perceived by the solver as certain, even if a detailed justification or proof is yet to be found;
- precedes a more analytical phase.

We are not interested in all these groups of intuitions, but only in those that are directly involved in problem-solving where GP processes are supposed to occur. In particular, *anticipatory intuition* seems to be strongly connected with the process of GP. Indeed, anticipatory intuitions, as GP processes, can intervene during the solver's investigation and can suddenly give a new insight into the solution.

Furthermore, Fischbein (1987) recognizes that intuition could produce some *intuitive representations* which could be mathematically correct or not. Indeed, the role of intuitive knowledge

> […] is to offer behaviorally meaningful representations, internally structured, of intrinsic credibility, *even if these qualities do not, in fact, exist in the given situation*. It is highly possible that the process of rendering intuitive will produce a distorted

representation of the original reality and the predictions made could be totally or partially wrong. (ibid., p. 12, italic in the original)

In this quotation we find a connection with the products of GP.

3.5 Looking for windows onto the GP processes

Consistently with our objectives and starting from the assumption that the features of the processes of GP could be inferred looking at the solvers' productions, we have chosen the particular situations that can elicit the processes. In this section, we describe the kind of geometrical tasks that will be used in the study and present the theoretical perspective that has allowed us to construct our tools of analysis. How we use such tools will be described in greater depth in Section 5.4.

Within the field of research in Mathematics Education, the analysis of discourse is well documented (see for example, Pimm, 1987; Sfard, 2008) and analysis of drawings also has a long tradition (see, for example, Polya, 1988; Mesquita, 1998; Duval, 2000, 2006). Since our intention is not to conduct an exhaustive analysis of solvers' discourse, drawings and gestures exclusively but instead use them as channels to get access to solvers' thinking during the resolution of prediction open problems, in the following sections we present only a few essential theoretical elements that allow us to make use of these as windows onto GP.

3.5.1 Open problems in Geometry

The examples of geometric problems reported in the previous sections suggest that the particular formulation of the task can elicit, more easily than others, the communication of products of GP. For this reason, we carefully look at the literature in order to find the best kind and formulation of problems that can help us trigger processes of GP.

We are not interested in observing if the solvers reach *the solution* of a geometric problem or if they succeed in proving a given statement. Instead, we are interested in observing:

- the processes that lead the solvers to *their solution* of the task;
- if and when the solvers are able to communicate *a solution*;
- which is the role of the prediction process.

Following these premises, the most suitable problem in the literature is the *open problem*. Indeed, open problems within the geometry domain are suitable to stimulate productive thinking (Mogetta et al., 1992) and useful for analyzing how the students produce a result.

The expression *open problem* (Arsac et al., 1998; Silver, 1995) refers to a task stated in a form such that the solvers have not specific instructions to be followed: they are left free to explore the problem and draw their own conclusions. The question does not suggest, reveal or anticipate the solution or a possible answer.

In particular in Geometry, the general structure of an open problem is characterized as follows:

> The statement is short, and does not suggest any particular solution methods or the solution itself. It usually consists of a simple description of a configuration and a generic request of a statement about relationships between elements of the configuration or properties of the configuration.
>
> The questions are expressed in the form "which configuration does…assume when…?" "which relationship can you find between…?" "What kind of figure can…be transformed into?". These requests are different from traditional closed expressions such as "prove that…", which present students with an already established result. (Mogetta et al., 1999, pp. 91-92)

It is not a problem reduced to a mere implementation of an already known procedure or routine. The solvers have to carefully choose a solution path and they could have a real discovery experience. Moreover, the result may not be unique.

Often the resolution process could lead the solver to the formulation of a conditional statement after a (physical or mental) exploration of the situation (Mariotti et al., 1997). Otherwise, the production of conjectures could be an explicit request (Boero et al., 1996, 2007; Olivero, 2000; Arzarello et al., 2002). In these cases, literature talks about *conjecturing open problems*, to make a distinction between them and other types of open problems.

When a conjecturing open problem is proposed to the solvers, they communicate one or more sentences with a premise, a conclusion and a relationship between them. The sentence may express some conditionality and, if it is the case, the conditional statement constitutes the formulation of a conjecture. The production of a conjecture may lead to the production of a theorem, conceived as the system of statement, proof and mathematical theory (Mariotti et al., 1997).

Concerning open problems, our assumptions is that:

- the prediction process could, but not necessarily, lead the solvers to the communication of a conditional statement;

- the prediction processes can intervene at an initial stage of the resolution process;
- a careful formulation of the task can expand this initial stage, allowing us to observe what happens.

Consistently with our aims and considering these assumptions, we designed another kind of open problems: *prediction open problems*. These are a particular kind of open problem in which the solver is asked to describe possible alternative arrangements of a geometric configuration (imagined, given by a drawing and/or by a step-by-step construction) maintaining given properties. Predictions could be asked for explicitly or not.

We added the adjective "prediction" because, among the set of open problems, we want to make a distinction with respect to other kinds of open problems.

A first example of *prediction open problem* is provided by the task reported in Section 3.2 (see the second formulation of the Varignon's problem); another is reported in Section 3.3. Other typical questions are the following:

- *What can you say about…(the point, the segment, the configuration)?*
- *Make a prediction: what can you say about the obtained configuration?*
- *Make a prediction: which …(quadrilaterals)… could it become?*

A complete list of the questions proposed in this kind of problem is reported in Chapter 5.

During the resolution of a geometrical task and for communicating their reasoning the solvers are allowed to speak, make drawings and gestures. We have considered all of these forms of communication as windows onto the solvers' solution trajectories and prediction processes.

3.5.2 Gestures

During the last two decades, there has been increasing interest in analyzing gestures and their role in shaping and communicating thinking. Indeed, according to McNeill (1992), gestures also play a role in human thought:

> gestures do not just reflect thought but have an impact on thought. Gestures, together with language, help constitute thought. (ibid., p. 245)

This topic is addressed by the works of McNeill (1992; 2005) and Kendon (1988; 2004), where the authors analyze the close relationship between thoughts, gestures, and speech. Based on these studies also researchers in Mathematics

Education (for example, Arzarello et al., 2009; Arzarello et al., 2015; Chen & Herbst, 2013) decided to investigate relationships between gestures and mathematical thinking, pointing out how gestures can be analyzed in order to gain insight into cognitive processes. Indeed:

> By virtue of idiosyncrasy, co-expressive, speech-synchronized gestures open a "window" onto thinking that is otherwise curtained. (McNeill & Duncan., 2000, p. 143)

In this perspective, gesture and speech are considered jointly as an enhanced window onto the cognitive process. Goldin-Meadow (2003) also highlights a strong connection between speech and gestures. Empirical evidence shows that speech and gesture are *co-expressive* (McNeill & Duncan, 2000, p. 142-143): they express the same idea but do not necessarily highlight the same aspects of it.

> Each modality, because of its unique semiotic properties, can go beyond the meaning possibilities of the other, and this is the foundation of our use of gesture as an enhanced window into mental processes. (ibid., pp. 143-144)

So, looking jointly at both we can infer the features of an idea better than if we only observe one of them.

Then, in order to gain insight into the cognitive process of GP without neglecting any important information, the analysis of gestures becomes a fundamental step. Moreover, since gesture and speech are tightly connected, it is important to study the gestures coupled with the speech performed by the solver.

3.5.3 Kinds of gestures considered in the present study

Among the several definitions of gestures, in this study use the following: *gestures* are "idiosyncratic spontaneous movement[s] of the hands and arms accompanying speech" (McNeill, 1992, p. 37). However, we are not interested in *all* the gestures produced by the solvers. For example, practical gestures like taking a pen from the desk or playing with a necklace are not relevant for our purposes. We are interested in those gestures "being done for the purposes of expression" (Kendon, 2004, p. 15).

In order to clarify which kinds of gestures we will consider, we refer to the *Kendon's Continuum*:

Gesticulation –> Language-like Gestures –> Pantomimes –> Emblems –> Sign Language

Looking at the drawing from left to right, the continuum shows how the presence of speech declines and idiosyncratic gestures are replaced by conventional signs. In this study we will consider:

- *gesticulations*, gestures that occur with speech;
- *language-like gestures*, gestures that complete a verbal utterance;
- *pantomimes*, gestures that do not accompany speech.

We are particularly interested in the gestures produced by solvers during the resolution of geometrical tasks. So, we focus especially on gesture that address a geometrical semantic content.

In the following sections, we will use the term *gesture* to talk about: spontaneous movements of the hands and arms that accompanying speech and are used in order to communicate geometrical meanings.

Moreover, depending on the semantic content of speech, there are several dimensions of gestures (McNeill, 1992, pp. 38-39). The most relevant for our study are the following:

- *Iconic*: are gestures that embodied picturable aspects of semantic content which refers to a concrete object, event or action.
- *Metaphoric*: are quite similar to iconic gestures, but the semantic content refers to an abstract object, event or action.
- *Deictic*: are used to indicate a concrete or abstract object. The most common are the pointing gestures.

These categories are not clear-cut. Although the corresponding utterance can clarify the most prominent dimension, a gesture can simultaneously have more than one dimension.

Since we move within a mathematical field, the identification of metaphoric gestures could be difficult. Although the geometrical objects share some aspects of image and could have a physical embodiment, they are defined within an axiomatic system of definition and theorems. So, their nature is not concrete: strictly speaking, it is not possible for a gesture to refer to a mathematical object, so an iconic gesture does not exist. For this reason, Edwards (2009) distinguishes between *iconic-physical* and *iconic-symbolic* gestures.

> Rather than referring to a concrete object in and of itself, the [iconic-symbolic] gesture refers to a symbolic, written inscription, which in turn represents a specific mathematical entity or procedure. (ibid., p. 138)

The *iconic-physical* gestures refer to a concrete object, so its use is replied by gestures. Moreover, we take Krause's perspective (2016) and consider *metaphoric* those gestures that represent a mathematical idea but are not iconic, neither physical nor symbolic.

We also make use of a phenomenon and of a theoretical construct defined within the literature on gestures: mismatch and catchment.

A speech-gesture *mismatch* is identified when the gesture conveys a meaning that is different from that expressed in speech (Goldin-Meadow, 2003, pp. 25-29). Research reveals that in this case the students rely on gestures rather than on speech (McNeill, 2005, p. 137; Krause, 2016).

When a feature of the gesture recurs in at least two gestures, not necessarily performed closely spaced, McNeill (2005) talks of a *catchment*. It is defined as "thematic discourse unit realized in an observable thread of recurring gestural imagery" (McNeill, 2005, p. 18). It shows a cohesion within the discourse and reveals a recurrence in solver's thinking.

> By discovering the catchments created by a given speaker, we can see what this speaker is combining into larger discourse units – what meanings are being regarded as similar or related and grouped together, and what meanings are being put into different catchments or are being isolated, and thus are seen by the speaker as having distinct or less related meanings. (McNeill et al., 2001, p. 10)

Summarizing, according to the aim of shedding light onto the GP processes and starting from the hypothesis that elements which belong to the TEG play a role within the processes, we will construct tools for analyses that allow us to identify and isolate the main interacting elements: the theoretical elements, the figural elements and the solvers' theoretical control.

The tasks will be designed in order to elicit the GP processes and to allow us to observe the role of these components within the GP processes. The tools for analyses will be conceived in order to highlight the products of GP, the theoretical and the figural elements, the theoretical control, and their interactions. To this aim, they must be constructed to easily conduct both a diachronic and synchronic analysis of all the solvers' productions. The task design and the operative tools for analyses will be presented and discussed in Chapter 5.

3.6 The role of the DGEs within the present study

The crucial role that the interaction between the students and a dynamic figure in a DGE plays in developing and supporting mathematical thinking is widely recognized (see, for example, Laborde & Strässer, 1990; Hadas et al., 2000; Mariotti, 2005; Baccaglini-Frank, 2010; Leung et al., 2013). Moreover, studies have well documented the use of DGE for the exploration of open problems within the domain of the TEG (see, for example, Goldenberg et al., 1998; Laborde, 2000; Olivero, 2000; Arzarello, 2002; Mariotti, 2005; Baccaglini-Frank, 2019).

We are not interested in the processes of prediction that a solver can undertake within a DGE, but in a particular interaction. We focus on what happens when a solver who has undertaken GP processes outside the DGE (for example but not only, in a paper-and-pencil environment) moves to the resolution of the same task in a DGE.

A Dynamic Geometry Environment is a particular kind of *microworld*.

> A microworld consists of the following interrelated essential features: a set of primitive objects, and rules expressing the ways the operations can be performed and associated, which is the usual structure in the *formal system* in the mathematical sense; a *domain of phenomenology* that relates objects and actions on the underlying objects to phenomena at the 'surface of the screen'. This domain of phenomenology determines the type of feedback the microworld produces as a consequence of user actions and decisions. (Balacheff & Kaput, 1996, p. 471)

In the case of a DGE like *Cabri Géomètre* or *GeoGebra* the "objects" are the geometrical objects (like points, lines, circles, and so on) that can be constructed and transformed according to the "rules" of the Theory of Euclidean Geometry[3]. A geometrical figure within these microworlds is called *dynamic figure*.

3.6.1 Dynamic figures

A *dynamic figure* has the following feature:

- it satisfies the set of theoretical conditions used for constructing the figure within the DGE;
- it satisfies and maintains all the theoretical constraints that logically derive from the given ones.

[3] For the sake of completeness, we stress that a DGE like *GeoGebra* allows the user to explore also other mathematical domains, but in this study our focus is restricted on the TEG.

Moreover, all those properties remain invariant under *dragging*. In other words, the theoretical constraints are maintained invariant also when the solver figurally changes the dynamic figure using the drag modality.

The dynamism of the figure is actually recognized as one of the main features of the DGE. Speaking of a specific DGE (*Cabri Géomètre*), but potentially referring to other DGEs with the same rational, Laborde (1993) states:

> The nature of the graphical experiment is entirely new because it entails movement. The movement produced by the drag mode is the way of externalising the set of relations defining a figure. The novelty here is that the variability inherent in a figure is expressed in graphical means of representation and not only in language. A further dimension is added to the graphical space as a medium of geometry: the movement. (p. 56).

In this context, geometrical properties are interpreted as *invariants* (Laborde, 2005). Dynamism is not only a "nice feature" of the DGE, but it can influence the students' interaction and construction of figural concepts. Indeed, dynamic figures are considered as scaffolding the drawing-figure gap:

> in the sense that it remains a material object (albeit virtual on the screen), but the invariance it carries in dragging can represent the basic properties. (Sinclair & Robutti, 2013, p. 574)

Moreover, the possibility of acting on a dynamic figure shifted the idea of a geometrical figure "from something that is, to something that can become"(Sinclair & Moss, 2012, p. 43).

> In this sense, the tool allows quick and precise changes, changes in the nature of objects the students are talking about, promoting each such object from the status of a specific concrete thing (e.g. a drawing) to that of a class of discursive objects. (ibid., p. 43)

Since dragging changes the figural components of the dynamic figure but not the conceptual components, it can foster students' access to the world of the theory of reference, in our case the Theory of Euclidean Geometry (Mariotti, 2006), and it can help the solver maintain the set of given theoretical constraints.

According to this perspective, the interaction between the solvers and a dynamic figure can be viewed as an additional window onto the GP processes because:

- the manipulations imagined by the solvers can be actualized and seen on the screen also for the researcher;

- the dynamic figure on the screen has the same theoretical constraints for the solver and for the researcher even if these may be interpreted differently;
- if the solvers have accomplished a GP process before moving the figure, they can have figural expectations. The way in which the solvers describe the possible behavior of the dynamic figure can make the product of GP more understandable for the researchers. This allows them to confirm, reject or refine the inferences on the solvers' products of GP.

3.6.2 DGE and paper-and-pencil environment

When the solvers explore a geometrical problem within a DGE, the context is different from the paper-and-pencil environment.

As stressed in the previous sections, in a paper-and-pencil environment when the solvers explore a geometrical situation sketched out in a drawing, they can manipulate the figure, consider a particular figural change and perform another drawing of this situation. During this process, for maintaining the theoretical elements of the geometrical configuration as invariants, the solvers must keep track of them and make sure that these are all present in the new drawing (Baccaglini-Frank, 2010). In other words, they have to exercise theoretical control over the figure. Eventually, the solvers can manipulate the figure continuously, as if they were using *maintaining dragging* in a DGE (Baccaglini-Frank & Antonini, 2016; Baccaglini-Frank & Sinclair, 2017). Nevertheless, whether the solvers conceive the figure dynamically or discretely, the interaction with the drawing follows the path described above.

On the other hand, in a DGE, the transformation can be easily performed continuously and

> each new figure will automatically exhibit all the properties according to which the original figure was constructed. In this manner the solver does not have to keep track of all the conceptual components and reconstruct the figure after each move. Instead s/he can observe change and invariance through small perturbations of the figure, that is, dragging a base point "only a little" to explore the figure. (Baccaglini-Frank, 2010, p. 30)

Furthermore, as stated by Laborde,

> the computer not only enlarges the scope of both possible experimentation and visualization but modifies the nature of the feedback. The feedback is visual on the surface, but it is controlled by the theory underlying the environment (Laborde, 2002).

The different nature of the interaction with a static drawing and a DGE is clear. The coherence of the figural transformations to the TEG is a solvers' responsibility to check; the transformations of a dynamic figure are always coherent with the given theoretical constraints and the theoretical control can be transferred onto the software.

3.6.3 The possible role of *surprise*

What activates human sense-making is the *disturbance* that can be experienced as *surprise* (Mason, 2004). More specifically:

> A person is surprised when something occurs unexpectedly, when it is in contradiction to expectations. […] In the study of mathematics we are dealing with intellectual surprise: that is the discovery that of some unforeseen truth. (Moshovits-Hadar, 1988, p. 34)

Surprise can also play a role in doing mathematics, to the extent that researchers (Moshovits-Hadar, 1988; Nunokawa, 2001; Mason, 2004) suggest the teachers to provoke surprise during the mathematical activity in order to support students' learning. Indeed, intellectual surprise usually gives us "a drive to find some more" (Moshovits-Hadar, 1988, p. 35).

Generally speaking, surprise occurs when facts do not fit with their expectations (Nunokawa, 2001). More specifically, there are several recognized sources of surprise. Among others there are three that seem strictly connected with the interaction with a DGE and with predictions: "A common property in a random collection of objects", "Unexpected existence, and non-existence of the expected", "Refutation of a conjecture obtained inductively" (Moshovits-Hadar, 1988, p. 35).

Since a source of surprise is the contrast between the expected and the unexpected, a priori we glimpse a connection between surprise and solver figural expectations, and therefore predictions. Since the interaction with a dynamic figure can reveal the alignment or the misalignment between a product of GP and the DGE feedbacks, DGE seems to be a powerful tool for inducing surprise.

We hypothesize that having an idea of the possible figural behavior of the configuration under dragging (i.e. having accomplished GP processes), that are supported by figural expectations, may induce the solver to be surprised if such expectations are not aligned with the DGE's feedbacks.

The surprise can induce the necessity to discover why or to explore again the situation, activating a new resolution process. Otherwise, if the figural expectations are aligned with the DGE feedbacks, the solver can refine and explore the situation further.

In both cases, the solvers have the opportunity to demand part of the theoretical control over the figure to the DGE logical system. In this way, they can discover invariants that they have not grasped before.

Since it is not the DGE on its own that necessarily elicits surprise but a careful design of the geometrical problems, we pay attention to this aspect during the task-design (see Chapter 5). In particular, tasks that concern impossible constructions and locus constructions seem to be particularly fruitful (Baccaglini-Frank & Sinclair, 2017) and can be formulated as open problems.

Summarizing, the solvers' interaction with a dynamic figure within the analyses of prediction processes seems to be useful both for solvers and for researchers. From the solvers' point of view, the interaction with a dynamic figure:

- is useful for refuting or confirming a product of GP;
- can support further undertaking of processes of GP;
- allow them to carry out the process transferring onto the software part of the theoretical control.

From the researchers' point of view, the solvers' interaction with and description of a dynamic figure:

- reveals the features of the products of GP previously communicated and inferred by the researcher;
- reveals a possible misalignment between the products of GP and the actual behavior of the figure.

4. Research questions

In this chapter we will summarize the theoretical assumptions derived from the theoretical framework and our working hypothesis. Moreover, we list the research questions of this study.

4.1 Theoretical assumptions, working hypotheses, and research questions

Summarizing, in the present study we take the theoretical assumptions listed below.

- The geometrical objects have a dual nature that is both conceptual and figural.
- The *Theory of Figural Concepts* gives us an adequate theoretical lens and operational tools for analyzing students' thinking in geometrical problem solving. In particular, the operational tools that belong to this theory can allow us to explain the processes of prediction that a solver can accomplish during the resolution of a geometrical task.
- A *figural concept* is not a stable construct, but it has a developmental nature and a personal status.
- The expression *geometrical figure* is used with the meaning it has within the *Theory of Figural concept*;
- In this perspective, *geometrical reasoning* can be interpreted in terms of a dialectic between conceptual components and figural components.
- A drawing is not a geometrical figure and the interaction between solvers and geometrical objects cannot be reduced to the manipulation of the figure as it is sketched out in a drawing.

Operationally, we start from the working hypothesis listed below.

- During the resolution of a geometrical task and according to their *theoretical control*, the solvers can interact with figural concepts accomplishing figural transformations. We are interested in learning more about these sorts of processes.

- In some cases, such transformations may have a dynamic nature that is quite close to the continuous transformations of a dynamic figure within a DGE.
- The transformations of the figural components of a geometrical figure are strictly connected with the generation of a prediction; indeed, while the transformation is accomplished, the solvers can generate figural expectations on the final figure that constitute a product of the prediction process.
- The features of the process could be inferred and then described looking at the products of a solver who is placed in a situation that elicits the GP processes.
- *Prediction open problems* elicit processes of GP.
- The theoretical construct of *intuition,* and in particular of *anticipatory intuition,* can interact with the processes of GP during the resolution of the given prediction open problems.
- The elaboration of a preliminary definition of the GP processes and the subsequent observation of instances of predictions, if possible, seemed to be the best way of gaining insight into the GP processes.
- The GP process can be observed looking at the solvers' productions like utterances, gestures, and drawings.
- The dual nature of the geometrical objects plays a role within the prediction process.
- When the solvers who have undertaken GP processes outside the DGE move to the resolution of the same task in a DGE, the presence or the absence of solvers' surprise during the exploration can be an additional way for shedding light into the features of the products of GP.

4.2 Research questions

The theoretical framework presented in the previous chapter allows us to formulate the research questions of this study, that are reported below.

When a solver engages in the resolution of *prediction open problems* proposed in this study, she seems to go through certain processes that lead to the production of figural expectations.

1. How can these processes be modeled?
2. What insight into students' actual processes of GP can be gained when our model is used for analyzing solvers' figural expectations?
3. In particular, what are the roles of the theoretical elements, of the figural elements, and of theoretical control?

Moreover, based on our model, we will revise the definition of *geometric prediction*.

5. Methodology

In this chapter we will describe our methodological choices for the study. In particular, in Section 5.1 we will briefly introduce the methodological tools of *clinical interviews* and *task-based interviews*, explaining the rationale for our choice. In this section we will also describe the methodological approach used for our data analyses, discussing how we used a *microgenetic method*. In this section we will also illustrate the research design.

In Section 5.2, we will explain how our data were collected, describing in detail how we made use of the chosen methodological tools. In this section, we describe how we carried out the semi-structured task-based interviews in the pilot study and in the final study.

In Section 5.3, we provide an a priori analysis of the four tasks proposed during the interviews.

Finally, in Section 5.4 we describe the data collected and how they were analyzed, focusing on the outcomes of the different ways in which they were analyzed.

5.1 The qualitative approach

Our study aims at investigating and describing particular cognitive processes that lead to the production of figural expectations. The explorative nature of our aim and the research questions led us to use a qualitative approach. In particular, we needed to be able to observe solvers during an activity that could elicit GP processes. Moreover, when the observation did not give sufficient insight, we needed to also be able to interact with the solver. Such an interaction could depend on the solver's answers and therefore it must be modeled depending on the particular situation. This motivated our choice of using *clinical interviews* (Ginsburg, 1981).

Since literature has stressed how *open problems* within the domain of geometry are suitable to stimulate productive thinking, we have chosen the *open problem* (Arsac et al., 1998; Silver, 1995) as a kind of task that can elicit processes of prediction and we designed a particular kind of open problem: the *prediction open problem*. The prominent role of this task leads us to use a particular kind of clinical interview: the *task-based interview* (Goldin, 2000).

Finally, since we aim at investigating cognitive processes, we needed to be able to observe each process as it occurs and to analyze it in fine grain. This motivated our choice of using *microgenetic methods* (Schoenfeld, Smith & Arcavi, 1993; Chinn & Sherin, 2014; Lewis, 2017) for the analysis of our data.

5.1.1 Clinical interviews

The *clinical interview* is a research methodology that was originally developed by Piaget (1929) as an instrument for psychological research; it is "a flexible method of questioning intended to explore the richness of children's thought, to capture its fundamental activities, and to establish the child's cognitive competence" (Ginsburg, 1981, p. 4). The clinical interview was used in order to gain a deeper understanding of children's mathematical thinking (Goldin, 2000). It is aimed at describing a local process, that is what might be going on in the child's mind *at the specific time of the interview*.

A clinical interview can be described as "a one-to-one encounter between an interviewer, who has a particular research agenda, and a subject" (diSessa, 2007, p. 525). It is designed to allow the interviewee to expose her "natural" ways of thinking about the situation at hand (Ginsburg, 1981). The topic of a clinical interview could be of various kinds, indeed:

> The interviewer proposes usually problematic situations or issues to think about and the interviewee is encouraged to engage these as best he/she can. The focal issue may be a problem to solve, something to explain, or merely something to think about. An interviewer may encourage the subject to talk aloud while thinking and to use whatever materials may be at hand to explore the issue or explain his/her thinking. (diSessa, 2007, p. 525)

The role of the interviewer is that of an "*active observer*": since she does not have a direct access to what happens in the subject's mind, she can observe the subject's outcomes and try to make inferences about the subject's thinking; she may ask for clarification, elaboration, and confirmation in order to make accurate inferences. It is important to stress that, during a clinical interview, the interviewer may attempt to perform minimal interventions, trying to avoid affecting the subjects' answers. Moreover, the interviewer may intervene in a flexible way, adapting the inquiry to the interviewee's answers (Ginsburg, 1981; diSessa, 2007).

For these reasons, the interviewer needs to prepare in advance a set of questions and possible prompts. Indeed, if a researcher intends to observe as much as possible and infer what she cannot directly observe about the interviewee's cognitive processes, she needs to interact with the interviewee. This is in line with one of the underlying assumptions concerning clinical interviews: human knowledge and activity patterns are *generative* (diSessa, 2007).

> Generativity emphasizes that knowledge is only useful if it is adaptive. People learn much of the time, and a significant part of the knowledge that they have will be directed toward generating new knowledge and new ways of behaving. Generativity may show in short-term adaptation to a particular problem or even to a particular prompt from the interviewer, and it will most certainly show in longer-term adaptive patterns of development. (ibid., p. 530)

The clinical interview is developed to investigate and describe precisely the thought processes by drawing a "clear description of mind" (Ginsburg, 1981, p. 6), so it is appropriate for discovering and analyzing particular cognitive processes (Clement, 2000). Our aim is to construct a cognitive model describing processes that might go on in the mind of a solver who is engaging in a particular kind of open problems. Therefore we chose a particular kind of clinical interview as the main methodological tool for our study: the *task-based interview* (Goldin, 2000).

5.1.2 Task-based interviews

The *task-based interviews* are a kind of clinical interviews and they have been used in qualitative research in mathematics education to observe, interpret and, in general, gain knowledge about students' mathematical behaviors. Since task-based interviews can serve as research instruments for making systematic observations in the psychology of doing mathematics and solving mathematical problems, they provide a suitable context for observing GP processes.

Indeed, as stressed by Goldin (2000):

> […] task-based interviews make it possible to focus research attention more directly on the subjects' processes of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce. Thus, there is the possibility of delving into a variety of important topics more deeply than is possible by other experimental means. (ibid., 520)

An underlying assumption of the use of the task-based interview is the following:

> [we] cannot observe subjects' thinking, reasoning, cognitive processes, internal representations, meanings, knowledge structures, schemata, affective or emotional states, and the like. […] Through research one can hope (at best) to make inferences about them, using what can be observed to infer what cannot. (ibid., p. 527)

According to Goldin (2000), there are three main elements that characterize a *task-based interview*: a solver, a researcher, and a task environment.

> The task-based interviews for the study of mathematical behavior involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way. The latter component justifies the term task-based, so that the subjects' interactions are not merely with the interviewers, but with the task environments. (ibid., p. 519)

In our case, the task environment is composed of a set of given *prediction open problems* to be solved making use of a sheet of paper and a pen or a DGE or just imaging the solution. The details of the tasks will be presented and discussed in Section 5.3.

Although group interviews with two or more solvers can also be conducted, we preferred to conduct one-to-one interviews because this seemed to be more suitable for the purpose of gaining insight into processes that might present individual differences (Shoenfeld, 1985).

Generally, the task-based interviews are video recorded, and the solvers' productions are collected in order to later analyze what takes place during the interview.

> By analyzing verbal and nonverbal behavior or interactions, the researcher hopes to make inferences about the mathematical thinking, learning, and/or problem solving of the subjects. From these inferences, we hope to deepen our understanding of various aspects of mathematics education. We may aim to test one or more explicit hypotheses, using qualitative analyses of the data; we may seek merely to obtain descriptive reports about the subjects' learning and/or problem solving; or we may hope to achieve an intermediate goal, such as refining or elaborating a conjecture. (Goldin, 2000, p. 519)

Moreover, the design of the task-based interviews is strictly intertwined with the research purposes: in our case, it is describing and gaining deeper insight into a cognitive process.

A variety of techniques are used within a task-based interview. Among the others there are thinking aloud and open-ended prompting (Clement, 2000). Moreover, the interview can be *structured*, with a detailed protocol determining in advance the interviewer's interaction and questions; other interviews are *semi-structured*, with a protocol that can be modified depending on the researcher's judgment. Since we want to observe GP processes both spontaneous and prompted, we opt for the latter structure for having the opportunity to provide previously planned prompts and interventions depending on the solvers' approach to the task. We will describe questions and prompts used during the interviews in Section 5.2. The specificity of the task-based interview is that hints, prompts, or new questions should be provided only after ample opportunity has been given for free problem solving (Goldin, 2000).

Data collected using a *clinical semi-structured task-based interview* will be analyzed in fine grain following to a *microgenetic approach* (Schoenfeld, Smith & Arcavi, 1993; Chinn & Sherin, 2014; Lewis, 2017) that will be presented in the next section.

5.1.3 Microgenetic methods

As reported by Chinn and Sherin (2014), the microgenetic methods were originally developed by Gestalt psychologists; then were used for the first time for investigating the learning process by Soviet psychologists including Vygotskij and Luria, as well as by Piaget and scholars influenced by Piaget. Theorists such as Werner (1948, 1957) and Vygotskij (1962, 1978) viewed short-term change as a miniature version of long-term change, generated by similar underlying processes. Recently an approach to microgenetic work has been formulated by developmental psychologists like Robert Siegler (2006). Even if microgenetic methods originally come from disciplines different from mathematics education, it has been adapted to the learning sciences by Schoenfeld et al. (1993) and diSessa (2014).

The main underlying assumptions of microgenetic methods (Chinn & Sherin, 2014, p. 171) are that:

- learning "occurs continuously, and in small steps, with every moment of thought";
- "learning does not occur in a straight line, from lesser to greater understanding; it occurs parallel on multiple fronts";
- "there are multiple kinds of learning, each requires its own study";
- finally, "we learn from our environment, which includes, most critically, the cultural tools other individuals provide to us".

The main goal of microgenetic methods is to gain insight into the processes as they occur and to do so in such a way as to permit strong inferences about the process. This method provides detailed information about an individual over a period of transition.

According to Siegel (2006), the microgenetic method has three essential features:

- Observations span the period of rapidly changing competence.
- Within this period, the density of observations is high, relative to the rate of change.
- Observations are analysed intensively, with the goal of inferring the representations and processes that gave rise to them.

(ibid., p. 469)

The last feature implies that the researcher should try to make inferences about the solvers' cognitive processes involved, going beyond the superficial behaviors. This supports our choice of using this method for the present study.

Moreover, Parnafes and diSessa (2013) highlighted several characteristics of the microgenetic analysis:

- *Theory-focused*: the main aim is to generate and improve theories concerning learning.
- *Fine-grained*: the standard is having a moment-by-moment explanatory account of learning in a particular context.
- *Open consideration of relevant aspect of data*: any feature of the process might be potentially relevant.

These characteristics imply that verbal language is important as well as gestures. For this reason, video recording is a minimal requirement. Because of the density of the observation and the complexity of the analysis, a study that follows the microgenetic method is not long-lasting and it cannot include too many participants. Indeed, most microgenetic studies have examined individual learning, typically in interview settings. Although it is also possible to examine collaborative groups, the larger the group, the more likely it is that observations will miss critical events.

Our methodological choices have their roots in the microgenetic method, which is:

1) <u>theory-focused.</u> The analyses use the existing *Theory of Figural Concepts* (Fischbein, 1993) and improve it by applying it to the generation of geometric predictions.
2) <u>fine-grained</u>. The data consist of 37 sessions of about 1 hour each of which a small number of hours were analyzed in detail. However, the close examination of short sequences (2 or 3 minutes) contains huge amounts of relevant detail to be accounted for and provided opportunities for unveiling the prediction processes.

3) <u>involving open considerations of data</u>. In observing solvers while they were solving a prediction open problem, solvers' verbal expressions were central, but also their gestures and interaction with drawings.

Moreover, following the Vygotskian perspective, in the present study we consider speech, drawings and gestures as *signs* that supported the human thinking and communication.

> The following can serve as examples of psychological tools and their complex systems: language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs; etc. (Vygotskij 1981, p. 137)

Indeed,

> The invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose, and so on) is analogous to the invention and use of tools in one psychological respect. The sign acts as an instrument of psychological activity in a manner analogous to the role of a tool in labour. (Vygotskij, 1978, p. 52)

Our tools for analyses will be described in greater depth in Section 5.4.

5.1.4 The research design

We first conceived a preliminary definition and model of GP processes to test and refine during a pilot study, using semi-structured task-based interviews (Ginsburg, 1981; Goldin, 2000; diSessa, 2007) which proposed several formulations of prediction open problems.

The pilot study was conducted interviewing 18 Italian high school students (ages 14-18), undergraduates and graduate students majoring in mathematics (ages 19-33), during the months of November and December 2017. We set out to investigate the effectiveness in terms of eliciting GP processes and providing windows onto the processes of 15 geometrical open problems. Each problem was proposed and tested in three or four formulations, varying the dominance of the text or of the pictures and the exploration environment (paper-and-pencil or DGE).

We used every interview to test and refine our definitions and our model of GP processes as well as prompts and formulations of the tasks. This cyclic and continuous process of refinement has been successfully used by other research that adopted a qualitative approach and involved the construction and refinement of a theoretical framework or of a model (Hadas et al., 2000; Steffe & Thompson, 2000).

The interviews were video recorded and subsequently analyzed through the lens of our theoretical framework under refinement.

After the pilot study, we chose and reformulated 6 tasks for the final data collection. We chose to use the DGE *GeoGebra*, developed by Markus Hohenwarter (2001-2002), because it is quite common to find it in Italian middle and high schools.

5.2 How data were collected

As described in Section 5.1.2, the problems proposed were open-ended tasks (we will discuss our specific open problem in Section 5.3). This form of problem was designed to give the solver the opportunity to display her "natural inclination" (Piaget, 1929) and it seems to provide windows onto the solver's thinking by maximizing the opportunity for observing, reflecting upon, discussing, and testing alternative interpretations of a student's response (Hunting, 1997). In the next sections, we will describe the researcher's preparation for the interviews and how they were conducted.

5.2.1 Semi-structured task-based interview

At the very beginning of each session with a solver, the interviewer provided some introductory indications listed below.

- The interviewer (that is the researcher) said that she would give the solver some tasks; she explained that the solver could solve each task as she believed appropriate. Moreover, the interviewer explicitly stated that there are no "right or wrong" answers but only "the solver's own" answers.
- The interviewer specified that she could repeat the question as much as the solver liked or needed and that, if necessary, the solver could make use of sheets of paper and colored pens. Moreover, she stated that the solver was allowed to ask for explanations about unknown words or forgotten meanings.
- The interviewer explained that any time she would ask "*why?*" it did not mean that the solver was wrong (Hunting, 1997; diSessa, 2007), but that she was seeking for an explanation.
- The solver is only asked to explain out aloud as much as possible what she is thinking for solving the task (Schoenfeld, 1985a).

This introductory phase was refined after the pilot study for putting the interviewee at more ease (Ginsburg, 1981; diSessa, 2007).

Each interview followed a common structure. After the interviewer introduced the session, she proposed several prediction open problems to the solver. Each task had to be accomplished first in a paper-and-pencil environment and then within the *GeoGebra* environment. So, the material supports given to the solver were:

- sheets of paper and colored pens, during the first part of the interview;
- the DGE *GeoGebra*, during the second part of the interview.

Typical requests to a solver were to explain an action, to describe what she was looking at or trying to accomplish, to provide clarification or elaboration of a statement she made (diSessa, 2007). However subsequent prompts and requests would be formulated using the solvers' language, in an attempt to confirm an interpretation or test an alternative one (Ginsburg, 1981).

Moreover, we elaborated some questions and prompts that we would use when a solver seemed to "get stuck". We were aware of the fact that certain prompts might change the solver's thought processes and actions (diSessa, 2007), however we wanted to be able to observe certain types of processes even if they did not occur spontaneously.

Taking into account the possible difficulties that the solvers had encountered during the pilot study we prepared general questions and prompts for the interviews. We also developed a series of prompts that could be used interchangeably when the solvers seemed to have trouble in making a prediction.

The questions and prompts are the following (Table 3).

Question	Function
"What are you thinking about?"	used for inducing the solvers to make explicit their thinking
"Make a prediction: do you think that [a certain point] can occupy other positions?"	used for triggering GP processes, if they were not spontaneously undertaken
"Imagine moving…[the point]… and make a prediction: …" and then the	used for triggering GP processes

interview repeats the question of the task.	
The first and main question of the task.	the question can be repeated in the same or another formulation, after a long exploration of the problem that does not conduct to the communication of a product of GP
"Show me how" or *"Why?"*	used for inducing the solvers to make explicit their predictions
"Make a drawing of what you have imagined"	used for making explicit the solver's figural expectations when the GP processes are carried out only imagining the outcomes
"Would you like to change your previous answers?"	used for observing if the solvers change their mind at a certain point (e.g., after they have performed a drawing or dragged a point of a dynamic figure in the DGE)
"Does the configuration behave as you expected?"	used in the DGE for testing the coherence between the behavior of the dynamic figure and the solver's expectation
"Before moving [a point of the dynamic figure], what do you expect to see on the screen?"	used in the DGE for making explicit the predictions on a dynamic figure

Table 3 List of questions that the interviewer may use during the interviews and their function

Other questions related to the specificity of the tasks will be presented in Section 5.3. Moreover, the interviewer can rephrase some questions using the solver's words as she has made explicit in previous answers (Ginsburg, 1981; Hunting, 1997).

In preparing for the interviews, we took into consideration the issue of the length of each interview (Hunting, 1997; diSessa, 2007). After the pilot study, we decided that the ideal time in order to optimize the collection of significant data with the participants of this study and the type of activities used was one hour per solver.

5.2.2 Data collection

The final data collection had involved a convenience sample of 37 Italian solvers who took part in the study as volunteers. Among them there are:

- high school students (ages 14-18);
- undergraduate, graduate and PhD students in mathematics (ages 23-35).

For design, we decided to interview solvers with a mathematical background: all the solvers of our sample are supposed to have been exposed – even if in different levels of depth because of the different ages – to the geometrical knowledge that is important to manage for solving the given tasks. These inferences were drawn looking at the list of achievements that should be developed by students attending Italian schools as reported in the government's document *Indicazioni Nazionali* (MIUR, 2010). Only for the high school students, we interviewed their mathematics teachers about the mathematical knowledge that her students of a certain grade were supposed to know. This way we could check that all the underlying *mathematics facts* (Schoenfeld, 1985b) of the proposed prediction open problems had been introduced to the solvers.

Data were collected during the months of February and May 2018. The problems were designed to elicit processes of GP and they were used within *task-based interviews* (Goldin, 2000). So, the first question was always the same; then there was a sequence of questions defined a priori and a set of stimuli in order to stimulate solvers' comments.

The tasks of the final set have a common structure, composed of two parts as described below.

- The interviewer explained the geometrical situation, showing a paper with the text of the task. The task had to be performed imagining or drawing the geometrical configuration in a paper-and-pencil environment. In the following, we will refer to this part as "the first part of the interview".
- Once the solver had proposed a solution or stated that she was not able to find one, the interviewer opened a DGE file and asked the solver to move some points of the dynamic figure, consistently with her prediction, and to

explore it in order to reach another or more complete solution. In the following, we will refer to this part as "the second part of the interview".

All interviews were carried out in a quiet room and each solver spends 60 minutes with the interviewer and works through as many tasks as she can. Interviews are video recorded using a camera next to the solver and a screen capturing software when the computer is used. Data are composed of video recordings, audio recordings, transcriptions, solvers' drawings.

Before the interviews, the interviewees (or their parents or guardians in the case of minors) signed a consent form concerning the use of data for scientific purposes (see Appendix A).

For the screen capturing we used an open-source software called *Open Broadcaster Software (OBS)*; for the transcriptions, we used the digital media transcription software *InqScribe*.

The order of the tasks to be proposed was decided during the interview depending on the time constraints and the solver's attitude. In retrospect, we decided to conduct a detailed analysis only of four tasks, which are the ones that were proposed the most during the interviews: Task 2, Task 4, Task 5, and Task 6. In the following we will maintain these labels.

5.3 Prediction open problems for the interviews and a priori analyses

As described in Chapter 3, the expression *open problem* (Arsac et al., 1998; Silver, 1995) refers to a task stated in a form such that the solvers are not given specific instructions to follow: they are left free to explore the problem and draw their own conclusions. The question does not suggest, reveal or anticipate the solution or a possible answer. We designed a particular kind of open problems: *prediction open problems*. These are a particular kind of open problem in which the solver is asked to describe possible alternative arrangements of a geometric configuration (imagined, given by a drawing and/or by a step-by-step construction) maintaining given properties. We added the adjective "prediction" because, among the set of open problems, we wanted to make a distinction with respect to other kinds of open problems.

More specifically, we decided to construct *prediction open problems* that provide a step-by-step construction to be imagined or accomplished on a sheet of paper. This design choice was made because we wanted to make explicit to the solvers the theoretical elements that characterized the geometrical configuration to be reasoned upon. As mentioned in Chapter 3, tasks that concern impossible constructions and locus constructions seem to be particularly fruitful for observing prediction processes and possible instances of surprise. Moreover, the use of locus problems as resources for observing and fostering the geometrical reasoning is suggested also by other authors, like Fischbein (1993) and de Finetti (1967). Moreover, we designed the step-by-step constructions so that the dynamic figure, that the solvers would use during the exploration of the problem within the DGE, corresponded to the given or sketched out figure, and therefore it was constructed following the same step-by-step contrition. For this reason the step-by-step constructions were designed in order to *robustly* (Healy, 2000) maintain the given constraints when accomplished in a DGE, and *softly* the constraints that the solver must recognize and maintain for coherently solving the problem.

The list of instructions was made available throughout the interview so that the solver did not have to memorize or remember it, but rather so that she could make use of it whenever necessary.

Since we set out to focus on prediction processes, we did not push the solvers to produce conditional statements or proofs. Nevertheless, if the conjecture-generation or the proof processes were undertaken spontaneously, we left the solvers free to explore further in these directions.

In general, we adopted a non-interventionist position: when the solvers communicated contradictory answers or products of GP that were incoherent with respect to the formal Theory of Euclidean Geometry (TEG), we did not point out to the solvers the contradictions of their answers.

In the next sections we describe the text of the four tasks that were analyzed (each is in a box), followed by the specific preplanned script that the interviewer could use during the interview. Moreover, we a priori analyzed each task, highlighting the possible trajectory that the solver might follow in order to reach a solution that is coherent with respect to the reference mathematical theory (in our case the Theory of Euclidean Geometry). We developed the problems for the study so that the generation of predictions would potentially lead to coherent solutions in the most direct and simple way.

5.3.1 Task 2

First part of the interview (paper and pencil)

> **Task 2**
>
> Read and perform the following step-by-step construction:
>
> - fix two points *A* and *B*;
> - connect them with a segment *AB*;
> - choose a point *P* on the plane;
> - connect *A* and *P* with a segment *AP*;
> - construct *M* as the midpoint of *AP*;
> - construct the segment *MB* and name its length *d*.
>
> *A* and *B* are fixed, and the length of *MB* must always be *d*.

First question: *What can you say about the point P?*

Possible further questions:

a) *Make a prediction: do you think that P can occupy other positions?*

b) *Make a prediction: do you think that P can occupy other positions so that MB remains of length d?*

c) *Imagine moving P and make a prediction: do you think that P can occupy other positions so that MB remains of length d?*[4]

For each of these questions:

- if the solver's answer is "Yes", the interviewer asks "*Which?*" or "*How?*"
- if the solver's answer is "No", the interviewer asks "*Why?*"

Finally, if the solver has trouble because of the label "d", the interviewer can ask:

Imagine you have measured the length of MB and it is 3 cm. Make a prediction: do you think that P can occupy other positions so that MB remains of length 3 cm?

---

[4] *Question (b)* and *Question (c)* are examples of the prompts that the interviewer might use when a solver seemed to "get stuck" or to have trouble in making a prediction. This is the reason why *Question (c)* is so focused on a particular investigation: what happens after a possible movement of P. In this way, the interviewer tries to prompt a prediction and suggests an exploration that the solver might not have imagined or taken into account. This is a useful strategy in order to observe whether the suggestion influences the prediction generation process.

5.3.2 Task 2: a priori analysis

In order to provide an answer to the task, the following theoretical elements are important to be noticed:

- *MB* must always have the same length, so the locus of *M* is a circle centered at *B* and with radius *BM;*
- *M* is the midpoint of *AP*, so the length of *AM* is always equal to 1/2 of the length of *AP* or the length of *AP* is equal to *2*AM*.

The theoretical element "*fixed length*", which is expressed in the statement "*the length of the segment MB must always be d*", is crucial information in order to reach a solution. The solvers could use this information almost in two ways.

The solvers could use the fixed length of MB for drawing several positions of M: these are positions for which the length of MB remains constant. In particular, the solvers could use a new theoretical element in order to find other positions of M: line symmetry of the point M with respect to AB. The solvers might start looking at the position of M only after having traced some positions of P which maintain MB of the same length. Then, looking at the positions of M, they could recognize that M traces out a circle (see Figure 10).

Otherwise, the theoretical element "*fixed length*" may foster recollection of the definition of circle, leading to immediate recognition of the locus of M as a circle.

In both cases, we expected that solvers produce a first product of GP, like this: "*the locus of M is a circle centered at B and with radius d*".

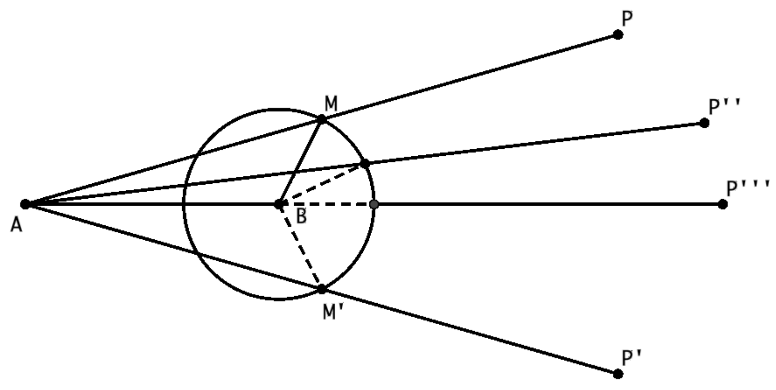


Figure 10 An instance of a possible drawing produced by the solvers during the resolution of Task 2

Using the theoretical element "*M is the midpoint*", the solvers can imagine moving M along the circle, or draw it, and observe different positions of P, discovering that also P lies on a circle. Here the solvers may produce another product of GP, like this: "*the locus of P is a circle*".

Using the relationship *AP* = *2*AM*, they can view the locus of P as the circle corresponding to the locus of M through a homogenous dilation of factor 2. Furthermore, this theoretical element may also help the solvers find the center and the radius of the locus of P: the center is a point O on the line through A and B, satisfying the relationship *AO=2*AB*; the radius has length 2*MB (see Figure 11). Otherwise, the solvers can trace the center O only intersecting a line through P parallel to MB and a line through AB.

Depending on the kind of theoretical elements that the solvers recalled, they may communicate more or less mathematical detailed products of GP.

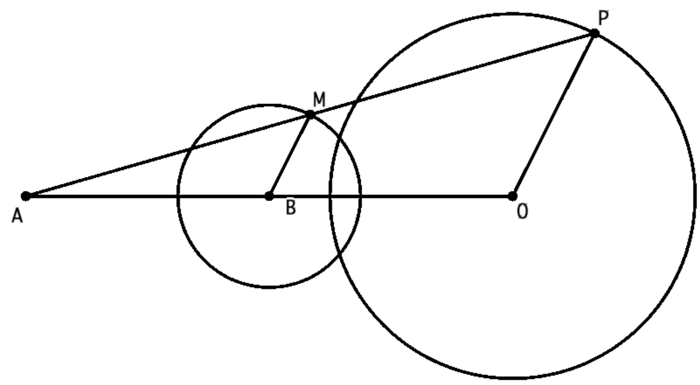


Figure 11 An instance of a possible drawing that shows the solution of Task 2

5.3.3 Task 4

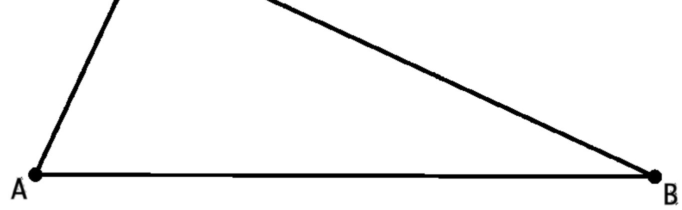
First part of the interview (paper and pencil)

Task 4

Consider the right triangle in the figure, with the hypotenuse of fixed length.

A and B are fixed.

The length of AB has to always be the same.



First question: *What can you say about the vertex with the right angle?*

Possible further questions:

a) *Make a prediction: do you think that it (or point C or the vertex with the right angle) can occupy other positions?*

b) *Make a prediction: do you think that it (or point C or the vertex with the right angle) can occupy other positions so that the angle stays right?*

c) *Imagine moving the vertex C and make a prediction: do you think that it can occupy other positions so that the angle stays right?*

For each of these questions:

- if the solver's answer is "Yes", the interviewer asks "*Which?*" or "*How?*"
- if the solver's answer is "No", the interviewer asks "*Why not?*"

The interviewer can refer to the vertex with the right angle as "it" or "C" or "the vertex with the right angle" depending on the solver's answers.

5.3.4 Task 4: a priori analyses

In order to provide an answer to the task, the following theoretical elements are important to be considered:

- the hypotenuse of the triangle is AB;
- the given theoretical constraints are the length of the hypotenuse and the right angle at C;
- there are no constraints on the lengths of the legs.

So, first of all, the solvers need to correctly recognize the hypotenuse of the triangle in the provided non-stereotyped drawing of the right triangle ABC.

Coherently considering all the theoretical constraints and applying several line symmetries, the solvers can generate several predictions about the possible positions of C and consequently they can consider other triangles that are right at C. For example, the solvers can imagine a line symmetry with respect to AB (see Figure 12a) and a line symmetry with respect to the axis of AB (see Figure 12b).

These positions address two different products of GP: "*C at a symmetric position with respect to AB*" and "*C at a symmetric position with respect to the axis of AB*". By combining the two symmetries, the solvers may consider four possible positions of C that maintain all the given constraints (see Figure 12c). Placing C at these positions, the solvers maintain not only the given constraints but also the length of the two legs.

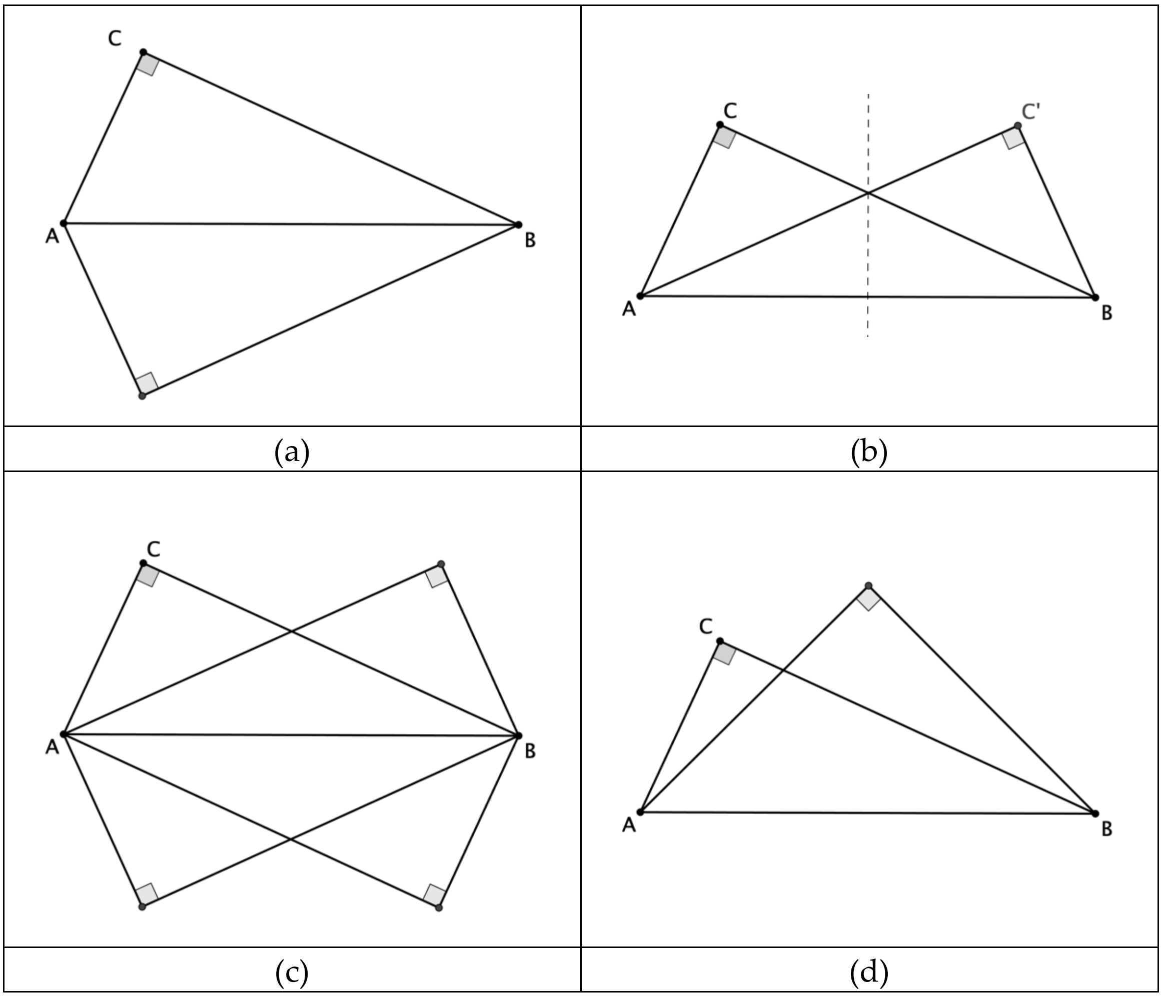


Figure 12 Several instances of possible drawings that report the solvers' products of GP concerning Task 4

Recalling another new theoretical element (the isosceles triangle) the solvers may consider another position for C. Indeed, a triangle can be right and isosceles at the same time and the solvers can consider the position of C that realizes these two conditions (see Figure 12d). In this way, the solvers can generate another product of GP. Combining the new position with the theoretical element "*line symmetry with respect to AB*", the solver can obtain another position for C.

Approaching the task in this way, the solvers will obtain six possible positions for C (Figure 13).

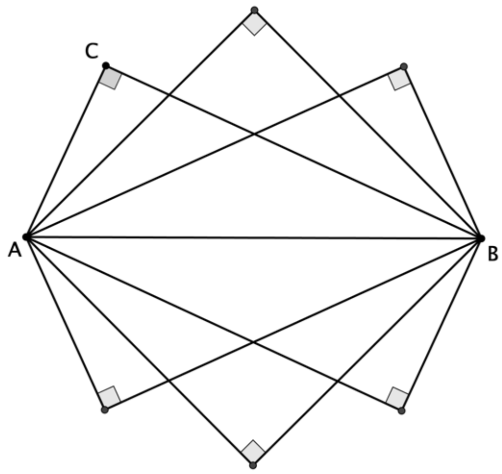


Figure 13 An instance of a possible drawing that reports the investigated positions of C

Now, the solvers can investigate what happens when they imagine moving C between these six positions and reach the locus of C as a circle.

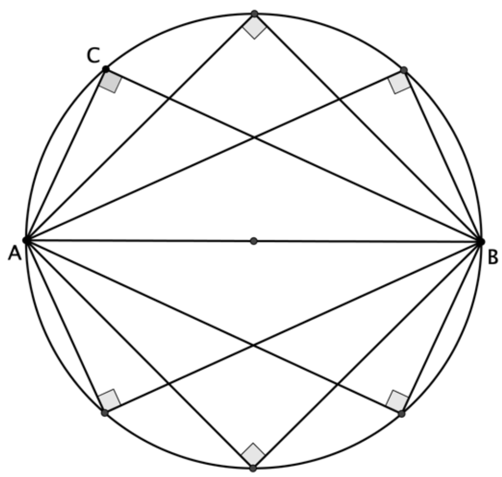


Figure 14 An instance of a possible drawing that sketches a complete solution of Task 4

One of the possible ways to identify the circle looking at the six positions is recognizing the invariance of the distance between C and the midpoint of AB. This could be another product of GP.

Otherwise, right after the first question the solvers can recall an already known Theorem: a right triangle is inscribable in a circle with its hypotenuse as its diameter. Recalling this Theorem allows the solvers to immediately recognize a circle as the locus of C. This is a realistic possibility in the interviews of the students from high school up: during the pilot study indeed it happened that 9th-grade students recalled such a mathematical result, even though not always in a formal way.

5.3.5 Task 5

First part of the interview (paper and pencil)

Imagine a triangle ABC.
Consider the midpoint of the side AB and name it M.
Imagine tracing the segment CM.
Imagine that A and B are fixed.

First question: *Make a prediction: is it possible that CM is congruent to CB?*

- if the solver's answer is "Yes", the interviewer asks "*How?*"
- if the solver's answer is "No", the interviewer asks "*Why not?*"

If the solvers are not able to answer the first question, the interviewer can suggest that they draw only the segment AB and its midpoint. Then the first question is repeated.

Further questions:

1) *Make a drawing of what you imagined.*
2) *Would you like to change your previous answer?*
3) If the answer to the first question was "*Yes*" without any other details, the interviewer asks: *Show me in the drawing how CM could be congruent to CB.*

   If the answer to the first question was "*No*", the interviewer asks: *why not?*
4) *Are there other ways in which CM can be congruent to CB?*
5) *Imagine moving point C. Do you think that there are other positions for point C so that CM is congruent to CB?*

5.3.6 Task 5: a priori analyses

The first part of this task is composed of two subparts. In the beginning the situation has to be only imagined; then the solver is offered the opportunity to make a drawing. Depending on the solvers' *theoretical control*, they may solve the task with or without making use of drawings.

In both cases, the main theoretical elements to be identified and recalled are the following:

- ABC is a general triangle, there are no particular theoretical constraints on the triangle;
- since M is the midpoint of AB, CM is a median of the triangle ABC.

First of all it is important to consider ABC as the most general kind of triangle, in order to avoid adding new theoretical constraints that are not logical consequences of the given theoretical elements. Then the solvers may focus on the sub-configuration composed of the segments CM, CB, and MB; they may recognize a triangle in this sub-configuration; they may consider the theoretical element "*CM is congruent to CB*" and translate it into the constraint "*CMB is an isosceles triangle*".

If they have considered ABC as the most general triangle, it is quite easy to realize that, starting from the given constraints, CMB can always be an isosceles triangle. In this way, they can communicate a first product of GP like "*C at the vertex of an isosceles triangle*" and show the corresponding position of C (see Figure 15).

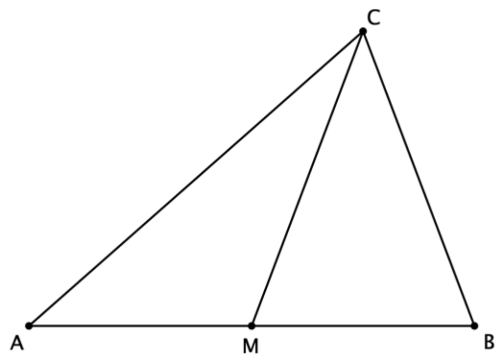


Figure 15 An instance of a possible drawing that shows a product of GP on C concerning Task 5

Otherwise, the solvers can first recognize an interesting position for C so that CM is equal to CB and only later recognize that the triangle CMB is isosceles.

Starting from this first position of C, the solvers may explore further configurations. For example, they can introduce the theoretical element "*line symmetry*" and find another position for C (see Figure 17) that is the symmetric position of C with respect to AB. Moreover, they may restrict the exploration on one of the half-planes marked by the segment AB and investigate other positions of C that maintain the given constraints (see Figure 16). These are additional products of GP.

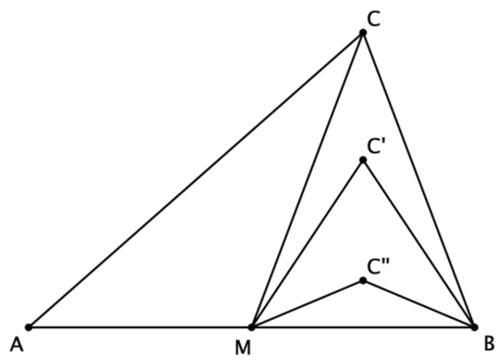


Figure 16 Two instances of possible drawings that show the solvers' products of GP concerning Task 5

Imagining moving C between these positions, the solvers can recognize an entire locus for C (see Figure 17) and communicate a corresponding product of GP, like this: "*C on a line perpendicular to AB and passing through the midpoint of MB*". Depending on the solver's *theoretical control*, the midpoint can be recognized using both a top-down or a bottom-up process.

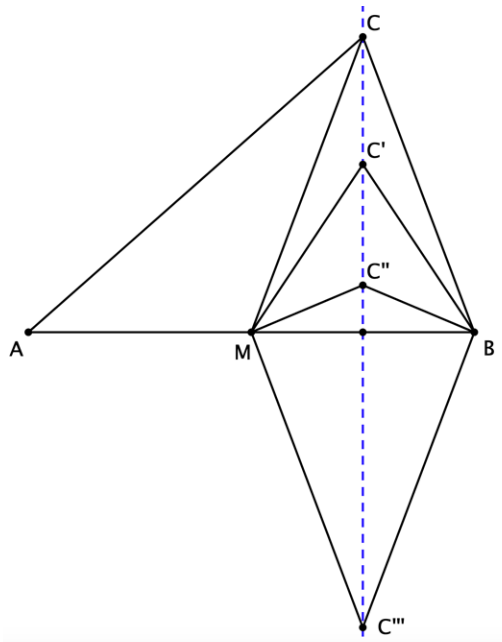


Figure 17 A possible drawing which shows the locus of C performed during the resolution of Task 5

Otherwise, the solvers may directly recall a more mathematical advanced theoretical element, that is "*the axis of a segment*", and its definition as the locus of points equidistant from two given points (i.e. M and B).

5.3.7 Task 6

First part of the interview (paper and pencil)

Read and perform the following step-by-step construction: - two points A and P; - construct the symmetric point of P with respect to A and name it Q.

1) *Imagine moving the point P and make a prediction: what happens to the configuration?*
2) *Imagine moving the point P along a line and make a prediction: what happens to the configuration?*
3) *Imagine moving the point A along a line and make a prediction: what happens to the configuration?*
4) *Imagine moving the point P along a circle and make a prediction: what happens to the configuration?*

If the solver does not draw anything more than A and P, the interviewer can suggest *"Make a drawing of…."* and then asks *"Would you like to change your previous answers?"*

5.3.8 Task 6: a priori analyses

Task 6 can be considered as composed of four sub-tasks that the solver may perceive as connected or disconnected problems.

The underlying theoretical element is always the same: the point symmetry centered at A. Each question adds an additional theoretical constraint to the configuration: P on a line, P on a circle, A on a line. Since A and P are points on the plane and Q is constructed as the symmetric point of P with respect to A, the point symmetry affects only the behavior of Q. After the short step-by-step construction, the solvers have drawn on the sheet of paper three points (Figure 18).

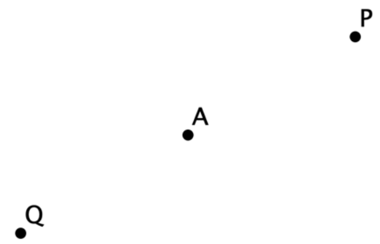


Figure 18 An instance of the configuration obtained following the given step-by-step construction (Task 6)

Let us consider separately the questions about P and about A. In all cases, the solvers may imagine continuous movements of a certain point and consequently they can construct continuously the loci of the other points; otherwise, they may imagine several discrete positions of the point on a particular path and construct discretely the symmetric points. The loci that the solvers obtain are products of GP, as well as the different positions of the points.

<u>Questions 1 – 2 – 4</u>

Answering the first question, the solvers may spontaneously consider several trajectories for P. If it is not the case, Question 2 and Question 3 should help the solvers to focus on particular loci.

For example, the solvers may imagine moving P on lines that are oriented differently on the plane. In this case, point A is fixed, because it is the center of the point symmetry; since Q depends on P through a point symmetry, Q will describe the same trajectory of P but following an opposite direction of the movement (Figure 19a). A coherent product of GP with respect to the given constraints could be "*Q on a line parallel to the line of P*".

Moreover, the solvers may imagine moving P on several circles that leave A outside the path. In this case, point A is fixed, and Q will describe the same trajectory of P in the same direction (Figure 19b). The solvers may consider moving P on circles centered at A and with radius AP. In this case, while P describes a semicircle, Q will describe a symmetric semicircle (Figure 19c). Coherent products of GP could be "*Q on a circle congruent to the circle of P*" or "P and *Q on a circle centered at A*".

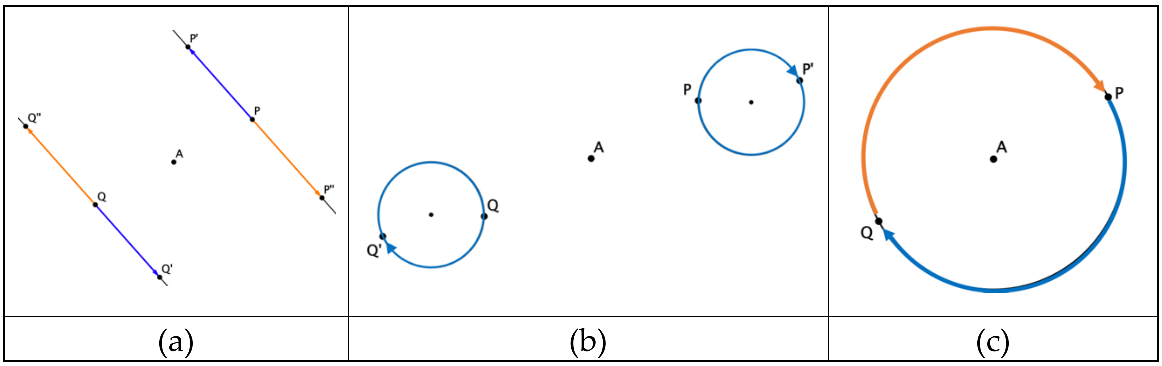


Figure 19 Three instances of the possible loci described by P and Q during the resolution of Task 6

The solvers may also consider curvilinear paths different from circles.

Question 3

This question shifts the focus onto A. Taking into account how the configuration composed of the three points was obtained, when the solvers imagine moving A they can realize that the point P is fixed on the plane and Q follows the movement of A according to a point symmetry. If A is moving on a line, Q traces a line in the same direction (Figure 20).

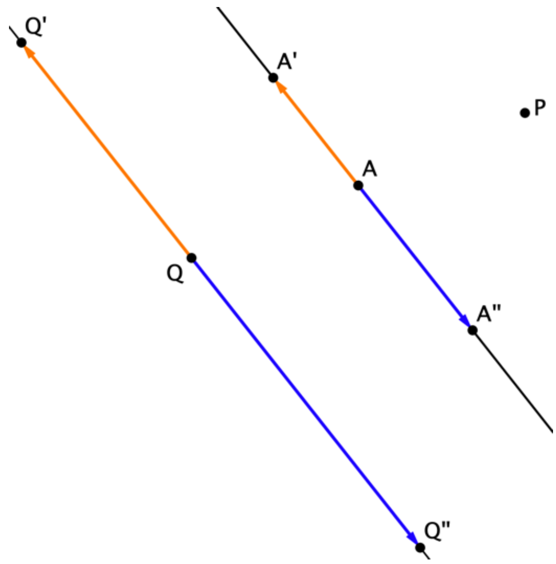


Figure 20 An instance of a possible configuration obtained moving A on a line (Task 6)

A coherent product of GP could be "*Q on a line parallel to the line of A*".

5.3.9 Second part of the interview (*GeoGebra* file)

The second part of the interview follows a common structure. The interviewer invites the solvers to open a previously prepared *GeoGebra* file and explains that they will find a configuration that was constructed in advance following the same provided step-by-step construction.

Then the interviewer asks which points the solver thinks that can be moved. The answer to this question could be a window onto the solvers' theoretical control over the dynamic figure.

The interviewer invites the solver to move the points of the dynamic figure, according to her predictions. Eventually unexpected feedback of the DGE should induce solver's surprise.

After the solver has tried to move some points, the interviewer asks if the dynamic figure behaves as she expected. The answers to this question provide additional windows onto the solver's products of GP because they can better reveal what she was expecting.

Then the interviewer asks if the solver wants to change the previously stated answers or to say something more about the configuration.

If the solver was not able to provide an answer during the first part of the interview, the dynamic exploration can trigger a new resolution process and eventually new processes of GP. Also an unexpected behavior of the dynamic figure under dragging could induce the solver to change her mind and undertake a new resolution process. In these cases, the interviewer can propose again the very first question of the task.

We expected that, during the exploration of the dynamic figure, the solvers would make use of different *dragging modalities* (Arzarello et al., 2002; Olivero, 2002; Baccaglini-Frank & Mariotti, 2010) and in particular *maintaining dragging* (Baccaglini-Frank, 2010) in order to maintain certain predicted properties.

As highlighted in Chapter 3, dragging changes the figural components of the dynamic figure but not the conceptual components. So, it can help the solver maintain the set of given theoretical constraints. Moreover, since the transformations of a dynamic figure under dragging are always coherent with the given theoretical constraints, the solver's theoretical control can be transferred onto the software. For these reasons, during the exploration of the dynamic figure the solver can reach new solutions to the problem, refine or reject their products of GP previously communicated.

5.4 Data analyses

In this section, we will explain how data were analyzed. More specifically, we describe the tools for analysis that we have developed and provided the "rules" for the transcription coding.

According with our research aims, in order to gain insight and describe the GP process, we set out to analyze solvers' interviews at several levels of depth, as described in the next sections.

5.4.1 Level 0

The first step was to transcribe the interview from the videos, using the digital media transcription software *InqScribe*. In particular, it allowed to easily annotate timecodes and speakers' information.

The second step was to organize the transcription in a table structured as follows:

Time	*Who*	*What is said*	*What is done*	*Comment*

Table 4 Structure of the table used for transcription

In the first column we show the exact instant (from the beginning of the video it is part of) when an utterance was produced. The time format is [minutes : seconds . tenths].

The second column shows who is speaking: the interviewer (i.e. *Int*) or the solver (i.e. *Stud*).

In the third column we list the solver and interviewer's spoken words as they were produced. Sometimes solvers do not repeat or explicitly explain the subject or the object of the sentence and the researcher needs to infer these elements watching at the video. In order to make a distinction between spoken words and inferred ones, we used square brackets for the latter.

The interviews were conducted in Italian and then translated into English to be reported in the present study.

We start transcribing the spoken words produced by a speaker in a box, and we switch to a new box in the following cases:

- if the speaker changes;
- if the speaker is the same, but there is a long pause;
- if the speaker does not pronounce words (empty box), but meanwhile performs a gesture or a drawing or simply she is looking at a particular object.

We use the punctuation using the criteria listed above:

- we use ellipses (i.e. "…") in the same box between two words when there is a short pause into the same sentence;
- we use a full stop when the pause is longer and divides two periods;
- we distinguish two sentences depending on the meaning and on the tone of the voice;
- we use ellipses at the end of a sentence if it seems that the speaker would continue to talk, but did not complete the sentence;
- we use a comma to show a different tone of the voice in the same sentence;

- we use ellipses at the beginning of the sentence, if the utterance is connected with the previous one of the same speaker. We infer this connection based on the tone of the voice.

The fourth column contains a description of the solver's behavior and productions that are different from the discursive ones: gestures, drawings, long silences and so on. Moreover, these boxes contain snapshots of the solver's gestures and drawings. Sometimes, in order to give the idea of movement, red arrows or trajectories are added on snapshots.

The last column is devoted to our comments. At this level of analysis, it remains empty.

5.4.2 Level 1

After the transcription phase, we start the first round of analysis, aimed at identifying instances of GP in the solvers' productions. To this end, we created a list of labels, useful for recognizing instances of GP processes in the solvers' discourse, gestures, and drawings. The list of labels is reported below.

- **Product of GP**

  In the paper and pencil environment, we recognize the product of a prediction process:

  - if the solver refers to elements (a geometrical object or part of it) which are not present in the drawing;
  - if the solver describes the behavior of elements which are not present in the drawing;
  - if the solver refers to a new arrangement of the configuration without drawing anything.

  In the DGE, we recognize the product of a prediction process:

  - if the solver refers to elements (a geometrical object or part of it) which are not present on the screen at that moment;
  - if the solver refers to a new arrangement of the configuration without dragging anything.

  We give a progressive number to the products of GP and we assign a number from 0 to 2 which refers to the degree of interpretation that the researcher has to use in order to describe the product of GP in detail. 0-effort is assigned if the product is well described by the solver and clear to the researcher. 2-effort if the product is ambiguous or fuzzy[5] and if it is not described clearly, in which case the researcher identifies the product of the GP only through considerable interpretation. A new product of GP appears in the following format: "*GP_Number_(0-1-2)*".

For example, let us compare two sentences like these:

1. *But another circle will be created with point P, because, that's it, it is not fixed, so...*
2. *That is, therefore I could have this circle – now I don't do it for not making a mess, but – a circle centered at P with this, at C with this radius AB over 2 multiplied by d.*

The first one is an example of a 1-effort product of GP, because the locus of P is named (a circle) but it is not described in detail. The second is an example of 0-effort product of GP, because the solver gives the essential details in order to describe a circle: the center and the radius.

Furthermore, we marked in round brackets whether the solver communicates a new product of GP in a gestural and/or discursive way. The order of the adjectives replaces the chronological order. For example, if we find first a discursive instance of GP and then a gestural one, we write in brackets: "*discursive – gestural*".

- **Window discourse**

  This is a statement or the use of certain words to communicate a product of GP. We do not use this as a label in the last column, but only mark it with bold type in the words in the third column.

  Using the previous example, we mark words in this way:

  1. *But* ***another circle will be created with point P****, because, that's it, it is not fixed, so...*

[5] The "fuzziness" is a useful category in order to see at a glance (a number) the amount of details that an observer can grasp looking at the solver's productions; in this perspective, a *fuzzy product of GP* is opposite to a *detailed product of GP*. Since this is an interpretative category, a product of GP could be fuzzy *for the observer* and not necessarily for the solver; in this case, it means that the solver did not make explicit all the details that are useful to communicate her prediction. The more a product of GP is detailed, the more the observer is able to encode the product of GP without too much interpretation.

2. *That is, therefore I could have this circle – now I don't do it for not making a mess, but –* ***a circle centered*** *at P with this,* ***at C with this radius AB over 2 multiplied by d****.*

- **Window gesture**

This is a gesture that communicates a product of GP. Gestures may or may not be coupled with discursive elements.

For example, the following are window gestures related to the process of prediction of a particular circle:

Figure 21 Instances of gestures which communicate a product of GP

The classification of gestures used in this study was provided in Section 3.5.3.

- **Theoretical element**

Elements that belong to the formal Theory of Euclidean Geometry. They include: all the properties that a solver gives to the configuration or that she gives to part of it; theorems and mathematical results.

- **Figural element**

Elements that belong to the figural domain in a specific moment (seen on the drawings or on the screen), related to the configuration in front of the solver.

- **Anticipatory Intuition**

According to Fischbein's definition (1987), we label *anticipatory intuition* moments in which a solver produces a sentence or a gesture suddenly without an explicit link to the previous process of solution and which led to new insight into the problem. At such a moment, we infer that an intuition has occurred, and we can observe an evidence of it.

- **Instance of surprise**

Gestural or discursive expressions which reveal a surprise.

For the sake of completeness, it is important to clarify that in this study the term "window" is used metaphorically, borrowing this use from Noss and Hoyles (1996). The term was chosen to highlight a twofold issue: since the study is focused on observing a personal cognitive process, the researchers can indirectly access this process only *looking through* particular *windows* (for example, gestures or statements that communicate a product of GP); consequently, the open-ended tasks are designed to *open windows onto* the processes of GP.

We need to make a concluding remark on the transcription analyses. Since the communicated products of GP can be more or less convincing for the solvers, they could make use of linguistic expressions that communicate "plausibility rather than certainty" (Rowland, 1995, p. 332). Literature refers to these expressions as *hedges*,

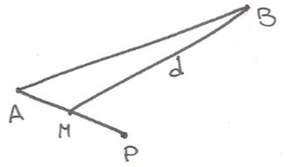
> examples of which include *about, around, maybe, think, normally, suppose, (not) sure, (not) exactly*. (ibid., p. 333)

Our aim is not to conduct a wide analysis of the possible hedges in solvers' utterances according to the taxonomy of hedges; we are only interested in making use of this theoretical construct as a marker of fuzziness into the solvers' discursive productions.

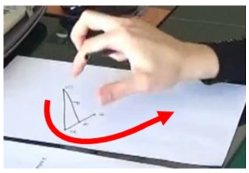
5.4.3 Labels: an example

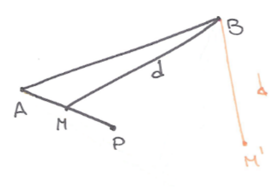
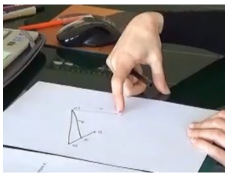
As an example of how we use the labels listed before, we report some excerpts of Tiziana's interview (Task_2).

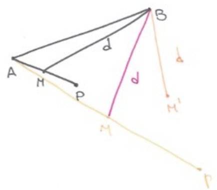
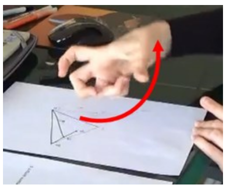
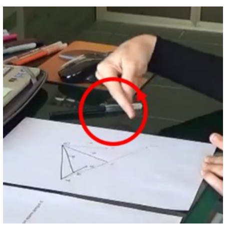
Excerpt: *Tiziana_MD_T2_P1_(01:17-05:15)*

Time	*Who*	*What is said*	*What is done*	*Comment*
01:17.10	Int	What can you say about P?		
01:23.05	Stud	So, in the meantime the first thing that I thought is that I decided to draw it externally with respect to the **segment AB**, but actually I could have also chosen it inside segment AB.	 *Drawing 1a*	
01:35.21	Stud	Because it says "Choose a point **P on the plane**" and…so,		She recalls a theoretical element

		let's say, it's up to me.		
01:44.08	Stud	Ok.		
01:47.13	Stud	So…can I…	She moves her hand to ask if she can continue talking	
01:48.17	Int	Yes yes, tell me all that you are thinking about regarding P.		
01:53.05	Stud	So based on where I will choose P, the position of M changes and **consequently** also the distance of M from B.		Theoretical elements It seems that M depends on P as a functional relation: M(P)
02:01.09	Int	Ok. Make a prediction: do you think that P can occupy other positions so that MB remains with length d?		
02:14.05	Stud	Yes.		
02:15.05	Int	Which?		
02:16.11	Stud	For example if I take **the symmetric point with respect to AB, I expect** to have the same distance between... uhm...	She moves the pen quickly on the drawing to indicate an axial symmetry with respect to AB, moving P along a segment perpendicular to AB 	**GP_1_(0) (discursive – gestural): the symmetric point of P with respect to AB** She uses the pen as an extension of her hand. She dynamically constructs the new point.
02:28.17	Stud	So now I was thinking something, but I do not actually know if it is the correct one, I mean I was thinking of	She is pointing with the pen in the place where she expected to see the symmetric point of P and then	It seems that she recalls the step-by-step construction, but starting from another position of the point P (the

		constructing the **symmetric part** on the other side, but I don't...uhm...	the symmetric point of M.	symmetrical one with respect to AB): she points to the symmetric point and then to the corresponding midpoint.
02:41.06	Stud	Yes, let's say another point M', would that work?		**Window gesture**
02:44.04	Int	Yes, yes.		
02:45.17	Stud	Ok. Yes, for example if **I take the symmetric point P with respect to AB on this other side**…I will have again…**I will find a midpoint M' that will be symmetric to M with respect to AB**, so **I expect** that then the distance from B to this point M' will coincide with d.	She is pointing the pen where she wants to put the symmetric point.	**GP_2_(0) (discursive – gestural): the symmetric point of M with respect to AB** **Window gesture**
03:05.12	Int	Do you think that there are other positions for point P, so that MB remains of length d?		
03:13.07	Stud		She is looking at the Drawing 1a	
03:22.14	Stud	So…uhm…ok…	She is pointing at B	
03:29.21	Stud	What am I thinking? The problem could be this: starting from…from this MB and creating others within the plane, I mean…can I? I will use a different color.	She is pointing at MB and movers two fingers as a compass: 	**GP_3_(1) (gestural): M on an arc of a circle** **Window gesture** The gesture suggests that the point M is on an arc of a circle.

03:43.04	Int	Yeah, you can do whatever you want. Also how many you want.		
03:47.16	Stud	So what I was thinking is more **intuitive**, but if I for example measure it.	She adds graphic elements on her drawing using orange.	GP3 is a product of GP: she seems to perceive her answer more as an intuition than as an analytical solution.
03:48.29	Int	Yes.		
03:55.00	Stud	This is always d. Ok.	 *Drawing 1b*	
03:58.21	Stud	This could be another point M', then whenever I draw AM' I can **attach here** the other seg...**the other half M'P**. At this point, P can also be in another position, **any other position in the plane that verifies this condition**.	She is pointing at the endpoints of AM', without tracing the segment. Then she mimics with a gesture the segment M'P: 	**Window gesture** She constructs the point P starting from the position of the point M'. **GP_4_(2) (gestural – discursive):** **P on the plane constructed using AM=MP**
04:24.06	Stud	Now here I chose it too…too far to be able to create it again, but maybe if I do something like this.	She draws a new segment MB with another orientation	She predicts that the drawing will not fit on the sheet of paper. This suggests that the GP4 is actually a product of GP and it is not only a vague answer recalling the theoretical condition, in which figural elements are unclear.
04:32.23		There are other sheets of paper as		

		you want. Take as many as you want.		
04:34.17	Stud	Alright.		
04:37.24	Stud	They are always, I don’t know…let’s say this is **always d**, a little bit closer.		
04:42.22	Stud	Then in the moment at which...	She traces the segment AM	
04:47.22	Stud	...this turns into the new M, it is enough to take P at this position and...	She traces the segment MP  *Drawing 1c*	
04:56.14	Stud	...I will mark P again, I will mark M and I could have the same distance d.		
05:00.28	Stud	This is when, say, I don’t have a criterion to define it in any case, in the sense that I can’t say ahead of time where to put P, but I can start from B and...		
05:15.05	Stud	...create the **circle**, so, basically: I use **d as a sort of radius** and all the points that are on the **circle with center B and radius d** are points M that can be...that can verify this property.	She moves her finger as a compass:  She moves a finger in a circle: 	She conjectures a circle as the locus of M, only having drawn two instances of the possible positions of the point M. So, the circle seems to appear as an **Anticipatory Intuition** **GP_5_(0) (discursive – gestural): M on a circle C(B, d)**

Each drawing is labeled using a number and a letter. The number is an ordinal number which shows the chronological order in which the drawings are produced. The letter marks the various subsequent snapshots of the same drawing. In this example, when the solver adds a segment BM′ on the *Drawing 1a*, we label the new picture *Drawing 1b*, because it is the same drawing (number 1) with a new graphical element.

Solvers can interact with a drawing making use of two different processes that the literature (Anderson, 1995; Gal & Linchevski, 2010) distinguishes into: *top-down* and *bottom-up processes*.

> Bottom-up processing uses information from the sensory physical stimulus for pattern recognition […]. Top-down processing occurs when context or general world knowledge guides perception. (ibid., p. 170)

Analyzing the transcriptions, we can also observe and mark instances of these two approaches to the drawings.

5.4.4 Level 2

In order to analyze in greater depth the prediction processes one by one during the same interview, we created another tool of analysis.

During the interview, the solvers usually elaborate several configurations of the geometrical object in focus. Generally, the first configuration is the given one or the constructed one following a geometrical step-by-step construction. The subsequent configurations are produced by the solver using a drawing and/or a gesture.

We defined *configuration* an instance of a geometrical figure (according to Fischbein's definition) expressed by a drawing and/or a gesture.

A *new configuration* may be:

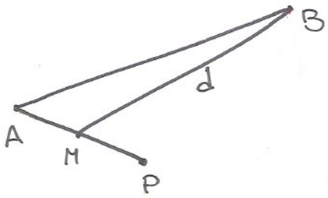
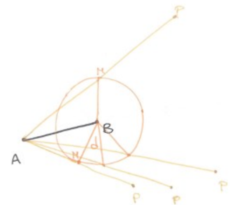
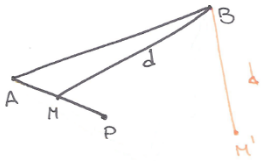
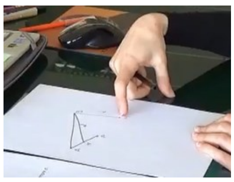
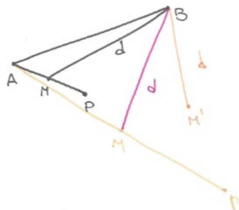
a) a new drawing;

b) new details on the same drawing;

c) a gesture related to a drawing which communicates a new geometrical object or a new relationship between the objects seen on the drawing;

d) a combination of a drawing and a gesture.

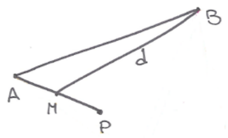
In the following table, we show some examples of possible new configurations, starting from the same first configuration.

<table>
<tr><td rowspan="4">Example of a starting configuration:

</td><td>Example of a new configuration (Type a):
</td></tr>
<tr><td>Example of a new configuration (Type b):

</td></tr>
<tr><td>Example of a new configuration (Type c):
</td></tr>
<tr><td>Example of a new configuration (Type d):
 and 
</td></tr>
</table>

After the transcription phase and the labelling phase, we read through each transcription again in order to mark the configurations that the solver has elaborated. Then we divide the transcription into segments. A segment starts from one configuration and ends with the subsequent configuration. We take into account only the segments in which one or more products of GP are communicated by the solver. Practically, we look at the transcription and see whether between two subsequent configurations there are any “products of GP” (these were labelled before). Then, we analyze each segment with another tool, trying to identify elements that belong to the students’ conceptual component and to their figural component of the geometrical objects in focus. We collect these elements in a sort of "funnel"[6] which shows the previously recognized products of GP in gestural and/or discursive form.

The general diagram of the tool is the following.

Configuration 1 (*Drawing 1a*):



	Theoretical elements	1	Figural elements
(01:23)			[P]
			AB
	Point on the plane		P
			P
			M
	Distance		MB
	Line symmetry		P and AB
	Constant/fixed distance		
	GP_1_0 (discursive – gestural): the symmetric point of P with respect to AB		

Configuration 2 (Gesture):

Figure 22 An example of solvers' funnel

We start with the initial configuration: in this case (*Configuration 1* in Figure 22) it is the first drawing produced by the solver following a step-by-step construction. At the end we list the new configuration produced by the solver: in this case (*Configuration 2* in Figure 22) it is a gesture that communicates a new position of the point P.

In the middle there is the actual funnel, composed of 3 columns. In the first and in the third there are, respectively, *theoretical elements* and *figural elements*, that the solver referred to in a gestural or discursive way, or that were inferred by us (in square brackets in this case). As stressed in Chapter 3, theoretical elements and

[6] The term "funnel" is metaphorically used. It was chosen to highlight that several elements flow into a GP process; these elements belong to different categories (figural and theoretical), but they interact dynamically, like two fluids in a funnel; such an interaction produces a new object that is more condensed than the sum of the original elements, but it maintains some traces of both of them.

figural elements are two constructs defined according to Fischbein's (1993) distinction between conceptual and figural components of geometrical objects. We identify these elements, using the description given above in the list of labels. In the central column, we write the funnel number and we use two different colors to highlight which element of the two columns is expressed by the solver at a specific moment: time proceeds from top to bottom; the vertical order of boxes follows the chronological sequence in which solvers made elements explicit. Furthermore, we added an "X" when the element was mathematically incorrect or incoherent with respect to the given geometrical construction. At the end of the funnel, there is the product of the GP. Each funnel represents the sequence of the observable steps that produce a new product of GP.

The funnel in Figure 22 is constructed using a part of the excerpt *Tiziana_MD_T2_P1_(01:17-05:15)* shown in the previous section. In particular, it is constructed looking at what happens from time 01:23 to time 02:16 of the interview. As an example, here we show the construction of the funnel in Figure 22. We selected the solver's utterance and gestures; then we marked figural elements (blue) and theoretical elements (green), as follows.

01:23.05	Stud	So, meanwhile, the first thing I thought was that I decided to draw it outside the segment AB, but actually, I could have chosen it even within the segment AB.
01:35.21	Stud	Because it says "Choose a point *P* on the plane" and…so, let's say that it is up to my discretion.
[…]		
01:53.05	Stud	So, based on where I choose P it changes the position of M and consequently also varies the distance of M from B.
[…]		
02:16.11	Stud	For example, if I take the symmetric point of P with respect to AB, I expect to get the same distance between ... um ...

We also look at the gestures, in order to make our inferences as close as possible to the solver's intention.

In this excerpt, we recognize some figural elements (blue), like the segment AB, the point P, the segment MB, the point M. The first figural element (point P) is inferred, because the solver uses "it" in the utterance, but looking at the transcription we can see that the subject (the point P) was already expressed in the question by the interviewer. We use green to highlight the identifiable theoretical elements, like point on the plane, distance, line symmetry. We recognize a product of a GP in the last statement of the student: "*I take the symmetric point of P with* *respect to AB".* The gesture confirms that the product of GP is another point P′ constructed as the symmetric point of P with respect to the line through AB.

Furthermore, in the first and in the third column of the funnel, we use colors (green and blue) in order to highlight new theoretical elements or new figural elements. A *new theoretical element* is a theoretical element that is not explicitly reported in the given step-by-step construction and therefore it is inferred, deduced or generally introduced by the solver. For example, the line symmetry transforming the point P is a new element which allows the solver to reach another position for the point P. A *new figural element* is a figural element introduced by the solver which is not included in a certain initial configuration of a funnel.

Summarizing, we have four categories of theoretical elements, expressed by two dichotomies. A theoretical element could be:

- mathematically *coherent* or *incoherent* with respect to the given geometrical construction, within the formal Theory of Euclidean Geometry (TEG);
- *new* or *already-known*.

Here are some examples of hypothetical utterances that contain theoretical elements of each category. The utterances are referred to Task 2.

Already-know	New
P is a point on the plane M is the midpoint of AP	I take P symmetric with respect to AB AP is equal to twice AM

The utterances in the first column contain theoretical elements reported in the given step-by-step construction, so we can say that the solver should know these properties. They do not add more information about the geometrical situation. In the second column the opposite happens: in the first utterance, the new point is constructed as the symmetric point of P, because the solver decides to do so. She introduces for the first time the line symmetry into the solution process. So, we define as "new" the theoretical element "line symmetry" referred to the point P and the segment AB. Also the second utterance contains the new theoretical elements "double length". This is new because the solver had to deduce this relation from the given property "M is the midpoint of AP".

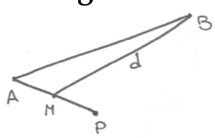
Coherent	Incoherent
I take P symmetric with respect to AB M is on a circle centered at B	I take P symmetric with respect to A MP is a radius and M a center

The utterances in the first column contain theoretical elements like: the line symmetry, the circle, the center of a circle. They are referred to the suitable figural elements and are coherent with the geometrical situation within the TEG. Indeed, they respect the given constraints: AB is fixed, the length of MB is invariant. Instead, the first utterance in the second column contains an incoherent theoretical element: the point symmetry of P. Indeed, if P is placed at a symmetric position with respect to A, it does not maintain the given constraints. We can say the same for the second utterance: if MP is the radius of a circle centered at M, MB does not maintain the same length.

Thus, the theoretical elements are not coherent or incoherent in an absolute way, but they are coherent or not with respect to (1) the invariance of the given constraints, (2) the figural elements they are referred to, (3) the compliance with laws of the Theory of Euclidean Geometry.

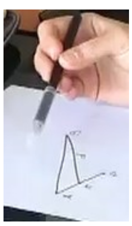
Sometimes the product of a GP seems to be related to a previous one. For example, in the excerpt we are looking at, after the GP_1 the solver undertakes another GP, strictly connected with the theoretical elements identified before. Indeed, she starts talking about another position for the point M, only using the line symmetry and without the interviewer's prompts. We highlight this connection using a blue arrow between the two funnels, as follows.

Configuration 1 (*Drawing 1a*):



	Theoretical elements	**1**	**Figural elements**
(01:23)			[P]
			AB
	Point on the plane		P
			P
			M
	Distance		MB
	Line symmetry		P and AB
	Constant/fixed distance		
	GP_1_0 (discursive – gestural): the symmetric point of P with respect to AB		

Configuration 2 (Gesture):



Theoretical elements		2	Figural elements
(02:28)	[Line] symmetry		
	Symmetric point		M’
	Symmetric point		P’
	Line symmetry		AB
	Line symmetry		M’ and AB
	Distance		BM’=d
	GP_2_0 (discursive – gestural): the symmetric point of M with respect to AB		

Configuration 3 (Gesture):

Figure 23 An example of two connected funnels

Finally, we summarize the funnels’ connections between one another in a diagram, as follows.

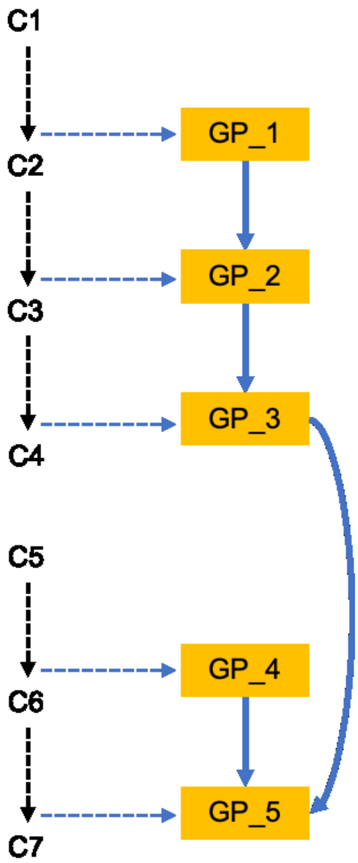


Figure 24 An example of the diagram which summarizes the connections between funnels

We label "C_n" the configurations produced by the solver. The label "GP_N" identifies the products of GP communicated by the solvers when they pass form the configuration C_n to the configuration C_{n+1}. The vertical blue arrows show the funnels' connection.

We recognize a connection between funnels in the cases described below.

a) If a product of GP adds some details to a previously communicated product.

   For example, the following products of GP belong to two connected funnels:

   GP_1_(0) (discursive – gestural): *the locus of C is a semicircle*

   GP_2_(0) (discursive – gestural): *the locus of C is a semicircle centered at the midpoint of AB*

   Using the GP_1 the solver communicates that the point C is on a semicircle. Then, in the GP_2 she describes in detail the features of the locus, making explicit the center of the circle.

b) If a product of GP or some part of it is recalled during a new GP process, in a gestural or discursive way.

   For example, Tiziana's *Funnel 3* and *Funnel 5* are connected also because she repeats the same gesture which is referred to the arc of a circle traced by the point M (see Figure 25).

Figure 25 Two instances of the same gesture, respectively at time 03:36 and 05:15 of the excerpt *Tiziana_T2_P1_(01:17-05:15)*

c) If the solver makes use of theoretical elements already introduced in a previous GP.

   In the previous excerpt, Tiziana uses for the first time the line symmetry for obtaining a new position of the point P (see *Funnel 1*). Then she recalls the theoretical element "line symmetry" in order to predict a new position for the point M. So, *Funnel 1* and *Funnel 2* are connected.

d) If we find linguistic expressions that reveal that the solver was referring to a product of GP or theoretical/figural elements previously communicated.

   Examples of such a kind of linking words are: "*like before*", "*once again*", "*as I said before*".

Sometimes the linking words are only a single expression. Looking at Tiziana's excerpt and in particular at the utterances at time 05:00 and 05:15, we can see that the conjunction "but" underlies the connection between *Funnel 4* and *Funnel 5*.

Finally, we clarify how we labelled the excerpts that we will report in the following. The format is the following:

Name of the solver_Grade_Number of the task_Part of the interview_(from time:till time)

So, an excerpt from the first part of the interview of a 9th-grade solver named Lucy while she is solving Task 4 looking at what happen from time 00:01 till time 02:00, will be label like "*Lucy_G9_T4_P1_(00:01 – 02:00)*".

In the excerpt reported in Section 5.4.3, the solver is named Tiziana and, when the interview was video recorded, she has already reached a master's degree in mathematics. So, we labelled the excerpt from her interview in this way: *Tiziana_MD_T2_P1_(01:17-05:15)*.

In general, we used the following labelling:

- *Name of the solver_GNumber*: when the interview was video recorded, the solver was a "*Number*"th-grade student;
- *Name of the solver_MS*: when the interview was video recorded, the solver has already reached a bachelor's degree (the Italian "*Laurea Triennale*") and was attending the postgraduate classes in Mathematics (the Italian "*Laurea Magistrale*");
- *Name of the solver_MD*: when the interview was video recorded, the solver has already reached a master's degree in mathematics (the Italian "*Laurea Magistrale*");
- *Name of the solver_PhD*: when the interview was video recorded, the solver was a PhD student in Mathematics.

5.5 Concluding remarks

From the analyses of solver's productions we observed emerging characteristics of the process and the products of GP; then we reached a characterization of the features that was later refined through further rounds of analysis of the excerpts.

Overall the next chapters allow us to answer the research questions, providing both the elements needed to reach a model of the GP processes and the description of the complex interaction between these elements.

In particular, Chapter 6 contains the description of the characteristics of the products of GP, it explains how motion, immediacy and intuitions could be integrated into the GP processes. Chapter 7 focuses on the role of theoretical elements recalled by the solver and how they are strongly involved in the processes of GP. Chapter 8 describes some additional characteristics of the processes of GP that only the second level of analyses revealed. Chapter 9 reports on the preliminary findings concerning a line of research that was not widely explored: what happens when a solver who has undertaken GP processes in a paper-and-pencil environment moves to a Dynamic Geometry Environment (DGE); nevertheless, it was included in this study because these preliminary findings provide additional windows onto the process of GP and integrate the observer's inferences about the products of GP.

Each chapter contains some "telling" example from data analyses. Among the analyzed excerpts, we chose the excerpts that, in a quite short time-span, show a certain feature or a subset of certain features of the GP process in the most meaningful way in terms of number and clarity of the windows (gesture and/or discourse) onto the processes.

6. Characteristics of GP

In this chapter, we will describe several features of processes and products of GP, emerging from data analysis.

In particular, in Section 6.1 and Section 6.2 we describe the characteristics of the products of GP. In Section 6.1, we explain the emerging dichotomy between *coherent* and *incoherent* products of GP. The criterion of coherence is determined according to the compliance with the reference mathematical theory: the Theory of Euclidean Geometry (*TEG*) on the plane, in our case. Moreover, we provide examples of the process that led to *incoherent* products of GP and we explain why they are incoherent. In Section 6.2, we describe the difference between *detailed* and *fuzzy* products of GP and explain how this dichotomy is reflected in a different interplay between solvers and drawings during the GP processes.

In Section 6.3, we focused on how motion could be integrated into the GP processes. Indeed, the solvers could explore alternative arrangements of the configuration making use of motion or not. Depending on how motion is integrated into the process of GP, a process of GP may or may not have a dynamic dimension. We explain how we can observe dynamism within the solvers' exploration and provide several examples from data of processes of GP with and without an observable dynamic dimension.

In Section 6.4, we analyze the kinds of gestures that the solvers perform during GP processes. Moreover, we describe the role of gestures within the prediction processes and what we can infer about solvers' predictions looking at gestures.

Section 6.5 address the topic of immediacy. It is the quality of a process of GP that is undertaken without a strong intervention of the interviewer. We consider immediacy as an indication of the naturalness of the GP processes. Data reveals a process of GP can be carried out in an immediate way, but not all the processes of GP are immediate; immediacy seems to be a feature of the GP processes that are undertaken by expert solvers.

Finally, in Section 6.6 we report about the possible intervention within the resolution process of *anticipatory intuitions* that can interact with the processes of GP. Moreover, we highlight the commonalities and differences between the theoretical construct of intuition and geometric prediction.

6.1 Coherent and incoherent products of GP

As highlighted in the Chapter 5, there are not correct or wrong answers to the given tasks. Solvers can describe whatever they know, imagine, infer or deduce about the given geometrical configuration. However, each step-by-step construction lists several geometrical constraints that characterize the problem. So the solvers' answer could be more or less coherent to these constraints. Consequently, during the solution process, when the solver communicates a product of GP, it could be coherent or incoherent with respect to the geometrical figures that are actually possible within the TEG given the specific set of constraints.

The criterion of coherence is determined according to the compliance with the reference Theory: the Theory of Euclidean Geometry (TEG) on the plane, in our case.

In order to be *coherent* a product of GP must respect:

- the given properties and theoretical constraints;
- the theorems of the Theory of Euclidean Geometry.

Otherwise, if the given constraints are not preserved within the reference Theory, the product is *incoherent*.

In our sample, according to how the solver makes use of the given constraints, we find several cases of incoherent products of GP:

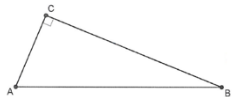
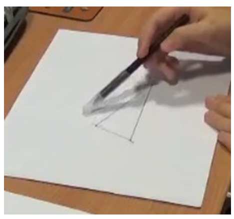
a) the given constraints are modified, and the solver obtains one or more new constraints;
b) the given constraints are maintained, but the solver adds one or more constraints;
c) one of the given constraints is completely removed or neglected.

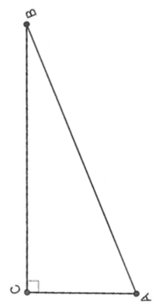
In all cases, the incoherence is evident to the researcher only: the solvers do not seem to be aware of such an inconsistency, or at least in their productions we do not find instances that suggest such an awareness; sometimes, they simply show some uncertainties.

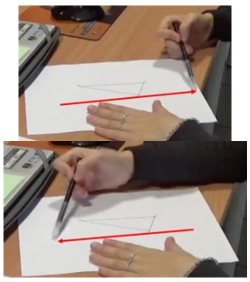
In the subsequent sections, we provide examples of the process that led to incoherent products of GP and we explain why they are incoherent. For the sake of clarity, we show excerpts which belong to the same open problem: Task 4.

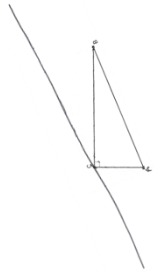
6.1.1 Case (a): the given constraints are modified

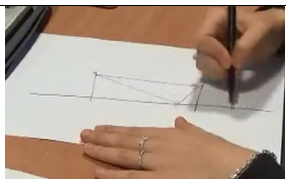
The following excerpt is the whole first part of Ilaria's interview: *Ilaria_G9_T4_P1*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:01.02	Int	Consider the right triangle in the figure. With hypotenuse...		
00:07.11	Stud	Of a given length.	She is reading the step-by-step construction.	
00:08.19	Int	Mm.		
00:09.17	Stud	A and B are fixed, the length of AB has to always be the same.	 *Given drawing*	
00:17.25	Stud	Hypotenu...well, then...		
00:23.20	Stud	So...given...so **no** point of this triangle **can be moved**.		
00:29.06	Stud	Because...if A and B are fixed...if **A B are fixed** and then **C and B are fixed,** too.	She is pointing to the points she mentioned	She seems to see the configuration as a right triangle with hypotenuse CB.
00:41.25	Stud	Eh, no the length, the **hypotenuse CB is fixed**, also the length of AB is fixed and this is a **right triangle**, also moving C **so that…that…CB has the same length, it will no longer remain a right triangle.**	"it is a right triangle": she is pointing at the angle C. Starting from C, she is moving the pen on a small curve trajectory: 	She confirms that our inference about the hypotenuse is right. **GP_1_(1) (discursive): if C is moving, the triangle is not right** GP_1 is spontaneously communicated: the interviewer did not ask the question. **Window gesture** The gesture seems quite fuzzy. It is not so

				clear where she intends to move point C. It seems that she only tries to move C in a position such that BC is equal in length.
01:02.28	Stud	And it is as if **everything** were **fixed**, I mean, I practically cannot do anything.		
01:08.01	Int	Ok. Which do you think is the hypotenuse of the triangle?		
01:11.21	Stud	Hypotenuse...ah, this one!	She is pointing at AB	
01:14.09	Int	Ok.		
01:15.04	Stud		She laughs.	
01:15.23	Int	So consider the hypotenuse with a fixed length. A and B fixed.		
01:21.28	Stud	Ok.	She rotates the sheet of paper: 	It seems that she places the triangle in an orientation that is more suitable for her.
01:22.04	Int	The question I ask you is: what can you say about the vertex with the right angle?		
01:26.20	Stud	The vertex with the right angle.		
01:29.09	Stud	If this...this is fi...is the fixed one…	She is pointing at AB and looking at the step-by-step construction	
01:33.01	Int	Yes...		

01:35.19	Stud	So, I **can’t move it anyway**, so...so nothing changes.		GP_1 (discursive)
01:42.08	Stud	Because if...	She is pointing to the segment AB	
01:47.19	Stud	Because to obtain the hypotenuse usually you use a theorem.		She recalls a theoretical element
01:52.12	Stud	In which the length C...C and B and A and B is calculated, is used. So if this moves, **if the length of C and A or C and B is changed, also the length of the hypotenuse changes according to this Theorem.**	She is pointing to AC and CB	She recalls the Pythagorean Theorem in order to justify GP_1.
02:10.28	Int	Ok. Make a prediction.		
02:15.13	Stud	Mm, let’s see.		
02:17.08	Int	Think...mmm...imagine to move this vertex C.		
02:21.29	Stud	C, ok.		
02:22.24	Int	Do you think that it can occupy other positions so that the right angle remains?		
02:28.11	Stud	Yes.		**Anticipatory Intuition** (C was a fixed point till now)
02:29.08	Int	Which?		
02:30.14	Stud	Because if... there is... mmm... a **perpendicular line** to AB, a line.	She is moving the pen describing a trajectory parallel to the segment AB and passing through the point C: 	**GP_2_(0) (gestural): C on a straight line parallel to the segment AB** Perpendicularity and parallelism are confused. **Window gesture** The gesture comes before the locus description.

02:39.02	Stud	Ehm, **I can move C along all the points of this line without it stopping from being…right**. If I am not mistaken.		GP2 (discursive)
02:52.08	Int	Ok, show me which.		
02:56.07	Stud	Can I?		
02:56.23	Int	Certainly.		
02:59.25	Stud		She draws a line that is parallel to AB and passes through C:  *Drawing 1a*	
03:04.01	Stud	Yes, more or less straight…		
03:05.25	Int	Mm mm.		
03:06.22	Stud	I can move C here, **it should** come out always right.	She is pointing to a position on the line:  Using the pen (without drawing anything) she traces the connection segments between the new point C and A and between the new points C and B.	GP2 (discursive – gestural)
03:12.19	Stud	Except when I move it too far away.	She is pointing to a position on the line: 	She refines the GP_2 and restricts the movements of C within a segment on the line.
03:16.01	Stud	Like this length and this length I can move it	She deaws two segments:	She refines the GP_2

		practically where I want.		
03:24.20	Stud	It should be **perpendicular** but these are...are details that I am not good at drawing.	She points to the line	She seems to be inverting perpendicularity and parallelism.
03:31.19	Stud	Moving it along this r, on this line r.		
03:35.12	Int	Mm mm.		
03:37.08	Stud	r...ehm...C...I can move on **any point of this line r**...between the **projections** of...of B and of A on this line ...and it should remain...C...and the angle C always right.	 *Drawing 1b*	**GP_3_(0) (discursive): C on a line parallel to AB and between two projections of A and B on this line**

As usual, the interviewer starts presenting the task. Immediately Ilaria continues to read the step-by-step construction by herself. At time 00:23 she starts talking about the possible new arrangement of the configuration without an interviewer's hint. It seems that she is wondering about the possible motions of the points A, B, and C. Even if the interviewer did not intervene, we cannot be sure that the implicit question about the motion of the points arises spontaneously for the solver. Ilaria approaches Task 4 after Task 2, so she was already exposed to questions about possible new positions for the given points. Anyway, we can say that the first impression about the configuration is that the whole triangle is fixed: she concludes that it is not possible to move its vertexes. We will see how this first idea will affect and inform Ilaria's predictions.

She explains her claim. The subsequent utterance (at 00:29) starts with "because" and it clarifies that there is a misunderstanding about the properties of the triangle. While she is pointing to the given drawing, Ilaria states that C and B are fixed as well as A and B. This reveals that she is seeing CB as the hypotenuse. Our inference is confirmed by the utterance at time 00:41, where she explicitly talks about CB as the hypotenuse. Here we find the first product of GP, which is communicated in a discursive way:

GP_1: *if C is moving, the triangle is not right.*

The window gesture associated with the GP_1 is quite fuzzy: it is not clear how she intends to move C in order to verify if the triangle could remain right or not.

GP_1 is a first simple example of incoherent product of GP. It is incoherent because: point C is not fixed; there exists more than one position for C in order for the hypotenuse to be fixed and the triangle to be right. Probably an imprecise interpretation of the constraints causes this incoherent product of GP. We can explain the solver's mistaken interpretation recalling one of the typical drawings of a right triangle in many Italian textbooks (Figure 26) that may have become a prototype for Ilaria.

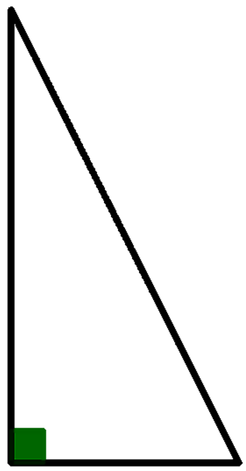


Figure 26 An instance of a prototypical drawing of a right triangle

Often in Italian mathematical textbooks, a right triangle is represented as follows: one of the legs is drawn as a horizontal segment; the other leg is placed perpendicularly to the first one as a vertical segment; generally, the hypotenuse is a segment that is neither horizontal nor vertical. In particular, this last feature of a prototypical right triangle could explain why the solver considers the segment CB as a hypotenuse: CB is placed where she is expected to find the hypotenuse.

The interviewer guesses the mistaken interpretation, so at time 01:08 she asks the solver to explicitly identify the hypotenuse. Driven by the question, Ilaria correctly points to the segment AB and laughs. The laughing could reveal an emergent awareness of the mistake. When the interviewer invites the solver to consider the hypotenuse fixed (time 01:15), as reported in the step-by-step construction, she turns the sheet of paper around. Probably in this way, she can see the drawing in a more suitable orientation and recognize the hypotenuse. This gesture confirms our hypothesis on the influence of a prototypical drawing on her process of GP.

Finally, at time 01:22 the interviewer asks the first question: "*what can you say about the vertex with the right angle?*". From this moment a new GP is undertaken, and it leads to another incoherent product of GP.

At time 01:35, Ilaria repeats that the whole triangle is fixed. Later, at time 01:47 and 01:52 she explains why she is so convinced that it is so. She starts her utterance with "because" and recalls a theoretical element: the *Pythagorean Theorem*. She does not explicitly recall the name of the Theorem but looking at her utterances we can infer that she is using this mathematical result. Paraphrasing what she claims at time 01:52, we can say that: it is impossible to move C and consequently to change the positions of AC and BC, because the movement will change the length of AC and BC; according to the Pythagorean Theorem, if the length of the squared sides changes, also the length of the hypotenuse changes; so the constraint "hypotenuse is fixed" is no longer maintained. It seems that, recalling the Theorem, she provides only a theoretical answer which is not connected with a particular new configuration. For this reason, we did not label the utterance as a product of GP. Furthermore, she does not produce gestures or new drawings.

Then, following the list of questions for the interview, the interviewer asks explicitly to imagine point C moving. It seems that this question triggers an *anticipatory intuition* about the solution: so far C was considered as a fixed point; after the question, Ilaria finds an entire locus for C; it happens suddenly without mentioning any other figural or theoretical elements. At time 02:30 and 02:39 she communicates an incoherent product of GP:

GP_2: *C on a straight line parallel to the segment AB.*

She performs a gesture and then she claims that, if the point C is moving on a line, the triangle is still right. We can easily observe that there is lack of clarity in the gesture and the utterances: she is talking about a perpendicular line, but she is moving the pen on a line parallel to AB. GP_2 is incoherent because if we consider a position of C within the drawn line (see *Drawing 1a*) and we imagine to trace AC and BC, then the triangle obtained is no longer right. So, the GP_2 maintains the constraint "fixed hypotenuse", but leaves out the property of being a right triangle at C. The corresponding new configuration (*Drawing 1a*) seems to be quite fuzzy: it does not show where the solver intends to place the point C in order to construct a new right-angled triangle. It is also evident at time 03:06, looking at the gesture. In addition, she seems a little bit uncertain at time 02:39, indeed she says "if I am not mistaken".

At time 03:06 a new GP process is undertaken and it leads the solver to yet another incoherent product: GP_3. Ilaria engages a sort of dialogue with the drawing. She is pointing to a possible new position for C on the line, but she does not seem really sure of her claim: she uses the modal verb “should”. Then, looking at the drawing she says that not all positions on the line are such that they maintain the triangle right. She says that if C is too far, the triangle is not right. So, she draws the two imagined extreme positions for the point C (*Drawing 1b*). Finally, she communicates her prediction:

GP_3: *C is on a line parallel to AB and between two projections of A and B on this line*.

The GP_3 is incoherent for the same reason why GP_2 is incoherent: if C is moving along the line and between the two projections, the constraint on ABC of being a right triangle at C is no longer maintained for any position of C. Probably this incoherent GP could be explained looking at the last drawing. In this picture there are two possible right-angled triangles: one is right at A and the other at B. We redrew (in red) the first one in Figure 27. If this were the case, we could claim that this GP is still influenced by the initial mistake on the hypotenuse. Moreover, it seems that in both cases the solver changes the constraints as follows:

- “C is the vertex with the right angle” turns into “ABC has a right angle”;
- the length of AB is maintained fixed, but AB is not ever the hypothenuse.

Figure 27 An instance of a right triangle recognized into the solver’s drawing

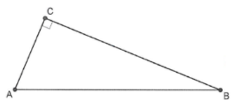
We can notice that, in this case, the incoherent products of GP are probably produced because of a mistaken interpretation of the given configuration. The interviewer’s effort to fix the mistake was not successful.

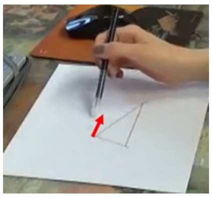
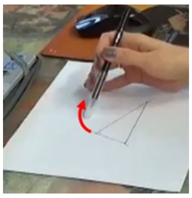
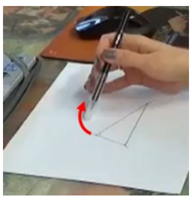
The three products of GP are incoherent because, within the reference Theory, the given constraints are not preserved. More specifically, in all cases the given constraints are modified by the solver as follows:

- GP_1 is produced changing the constraint "hypotenuse is fixed" into the constraint "CB is fixed";
- GP_2 and GP_3 are produced considering ABC as a generic right triangle that could be right indifferently at A, B or C.

6.1.2 Case (b): the solver adds constraints

In other cases, the product of GP is incoherent because, even if the given constraints are preserved, the communicated new configuration maintains other constraints that are arbitrarily added by the solver. We show an example of this kind of incoherent product of GP. The excerpt belongs to Laura's interview: *Laura_G10_T4_P1_(00:00 – 01:16)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:00.22	Int	Consider the right triangle in the figure with the hypotenuse of a fixed length.	 *Given drawing*	
00:06.10	Stud	Ok.		
00:06.25	Int	A and B are fixed and the length of AB has to always remain the same.		
00:12.05	Stud	Yes.		
00:13.00	Int	What can you say about the vertex with the right angle?		
00:16.29	Stud	Mmm...	She rotates the sheet of paper. Now the smaller side is the base of the triangle: 	It seems that she orients the triangle in a more suitable way for her.
00:26.13	Stud	If...I mean does the vertex C not have to always remain with a right angle can it change size of the	She is pointing at C	

		angle or does it have to always stay right?		
00:34.00	Int	What you can say about the vertex WITH the right angle.	She puts emphasis on the "with".	
00:38.06	Stud	That if AB has to remain...fixed, **the vertex C can move** ...well **like it wants**, in the end...[I don't think] there are many limitations....for the vertex C.	She is pointing to AB. She is pointing to C. She rotates the sheet of paper and she restores the original orientation.	
00:52.12	Stud	And obviously **if it moves it will no longer be right**.		
00:56.01	Stud	Instead **if maybe you move it in a certain way**, it could also...yes, **it could also be right**, if maybe...it moves a bit upwards so that ...it becomes **an equilateral triangle**...with also, I mean, well, **all sides...of 90 degrees**.	She points to C and moves the pen, tracing once a straight trajectory (a small segment) and repeatedly a curved path (a small arc of a circle):  	**GP_1_(0) (gestural – discursive): C is the vertex of a right and equilateral triangle** **Window gesture** Incorrect wording: side of 90 degrees.
01:16.08	Stud	It could become **again a right angle**, but you would have to do some movements, I mean do a movement that is...precise.	She repeats the curve gesture: 	**Window gesture**

In this short excerpt, we can see the process that leads Laura to communicate an incoherent product of GP. We see her first impression about the geometrical configuration at time (00:38) and (00:52). She says that is possible to move C within the plane, but if C is moved the triangle is no longer right. So, she takes into account that there exist some positions where C is no longer the vertex of a right triangle. Then at time (00:56), it seems that she shifts the focus onto possible positions of C that maintain the constraint "being a right triangle". The two window gestures suggest that, in order to find other positions for C, she is trying to move C on a line and on an arc of a circle. In particular, the second path seems to be suitable for her intention, because she repeats twice the gesture and then she claims that she obtains an equilateral triangle. Here she communicates her incoherent product of GP:

GP_1: *C is the vertex of a right and equilateral triangle*

It contains two mismatched constraints on a triangle:

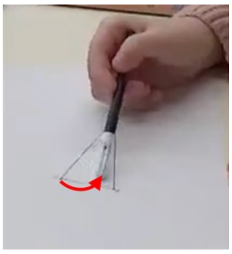
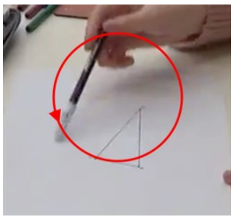
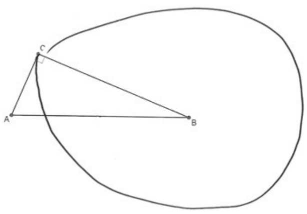
- being a right triangle;
- being an equilateral triangle.

The first constraint is given by the task, the second is added by the solver. Moreover, using a mistaken wording (i.e. "sides" in place of "angles"), she stresses that the imagined new configuration is a triangle with three right angles. So, for the solver the two mismatched properties have the consequence that the triangle has three right angles. As is well known, an equilateral triangle with three right angles is an inconsistent geometrical object within the TEG. Probably, it exists only in the solver's figural domain and reveals a lack of harmony between figural and conceptual components. So, we can easily see why the GP_1 is incoherent.

6.1.3 Case (c): the solver removes one or more constraints

The third case of incoherent GP occurs when the solver discards (eventually unconsciously) one or more properties within the set of the given constraints. Here an example of this kind of incoherent product of GP: *Sabrina_G9_T4_P1_(00:15 – 01:00)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:15.22	Int	What can you say about the vertex with the right angle?		
00:22.05	Stud	That…it can be moved, in such a way that CB is…the length of CB is		She regards CB as the hypotenuse.

		fixed…that is it is always the same.		
00:34.25	Int	How?		
00:36.23	Stud	I imagine by…lowering. That is, I imagine that it is moving along a circle.	She is pointing at C and she moves on a curve trajectory:  She moves the pen above the drawing, tracing a circle.	**GP_1_(2) (discursive – gestural): C on a circle [centered at B and with radius BC]** **Window gesture**
00:44.12	Int	Mm. How?		
00:47.24	Stud	Like this.	She is pointing at C and traces a circular trajectory that intersects AB and ends at C: 	**Window gesture**
00:49.12	Stud	Depending on where I move it.		
00:56.26	Int	Ok, show me this circle you imagine.		
01:00.24	Stud		She draws the circle:  *Drawing 1a*	

The excerpt begins when the interviewer asks the first question and the solver has already seen the given drawing, as in the excerpts previously reported.

Sabrina answers that the vertex with the right angle can be moved in order to maintain CB equal in length. The utterance reveals that she conceives CB as the hypotenuse of the triangle ABC and that she wants to maintain its length fixed.

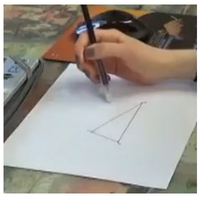
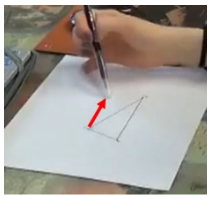
When the interviewer asks her to explain how she intends to move the point, she replies she imagines to move C on a circle. The two subsequent window gestures allow the researcher to infer the center and the radius of the circle. So, we can formulate Sabrina's product of GP:

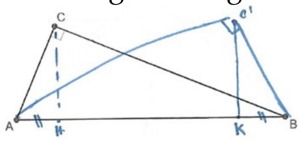
GP_1: *C on a circle [centered at B and with radius BC]*

Finally, the drawing (see *Drawing 1a*) confirms the researcher's inference about the center and the radius of the circle. The product of GP is incoherent because if C is moving on that circle, the triangle is no longer right for any position of C. In particular, it seems that the solver removes the constraint "C is a right angle" and she takes into account only the constraint upon the length of CB, which is regarded as the given hypotenuse.

6.1.4 Coherent product of GP

We show an example of a *coherent* product of GP looking further forward in Laura's interview. The excerpt *Laura_G10_T4_P1_(01:24-01:52)* comes after the previous one.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:24.16	Stud	Both maybe towards the center, towards...towards more to the right.	She is pointing to another point on the plane: 	**Window gesture**
01:31.15	Stud	If it moves... **in a parallel** way till it reaches …can I draw a figure?	She is tracing a straight trajectory: 	**GP_2_(2) (gestural – discursive): C at a symmetric position [with respect to the axis of AB]** **Window gesture**
01:38.01	Int		She is nodding	
01:38.13	Stud	Ok, if maybe here [AH]...there is...I don't know H. **If the length of AH were equal to**...ehm...**BK**...	She draws a segment through C and perpendicular to AB; she names the point of intersection H.	She starts constructing the new predicted point: C'.

			Then, she puts a sign on AH. Finally, chooses a point K on AB and puts the same mark on BK.	
01:52.10	Stud	...I could move **C so that it was perpendicular AB and it followed the trajectory of CH** and, in this case, I mean **to go back to a triangle**...ehm... a **right triangle,** more or less, yes.	She repeats the gesture at time (01:31) and draws a point. She connects K and this point with a segment; she connects the point to A and to B with two segments. She obtains the following drawing:  *Drawing 1a*	GP_2 (gestural)

As previously reported, the excerpt starts when the first question was already asked by the interviewer and the solver had produced a first GP.

In the first line of the table, we can observe a window gesture that shows a new position for C found by the solver in a figural way: she only points to the new position without adding any theoretical elements. At time 01:31 she explains how she intends to move C in order to find such a position. Here we find a first instance of a new product of GP, which is communicated in a discursive and gestural way:

GP_2: *C at a symmetric position [with respect to the axis of AB]*

The GP_2 is coherent with the given constraints: in that position, C is still a vertex of a right triangle; the length of AB does not change.

The product needs a certain amount of interpretation because the solver only describes and performs her movements. She does not use theoretical elements for describing the product of GP, so we cannot infer exactly how she intends to find the new point geometrically.

The subsequent instances of GP will clarify what the solver intends to do. Indeed, at time 01:38 the solver starts to construct geometrically the new position for the point: C′. First of all, she draws the height of the triangle passing through C. In that way she finds H and the length AH; she uses the length AH to find the segment BK in a figural way; then she repeats the window gesture and stops moving when the pen is at a position corresponding to K; in this way she draws the point C′.

Now, she connects C′ and K using a perpendicular segment. The perpendicularity is explicitly mentioned by the solver. Connecting C′ with A and B, she obtains the new configuration (*Drawing 1a*).

So, Laura's second product of GP is coherent with the given constraints within the TEG. Moreover, the new configuration is in line with the GP_2 and shows in a very clear way how the solver plans to find C′.

6.2 Detailed and fuzzy products of GP

When solvers communicate a product of GP, their utterances and their gestures can contain different amounts of details. These details can be of conceptual or figural nature and they can help the researcher understand and they may or may not be properly reported in a statement the product of GP. So, a product of GP can be communicated in a precise and detailed form or not. We have already highlighted that the fuzziness is connected with the degree of interpretation (0-1-2) that the researcher has to use in order to describe the product of GP in detail.

According to this perspective, we can distinguish the products of GP into:

- *detailed* product of GP;
- *fuzzy* or *vague* product of GP.

Moreover, our data reveal that this dichotomy is reflected in a different interplay between solvers and drawings during the GP processes.

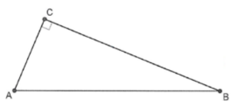
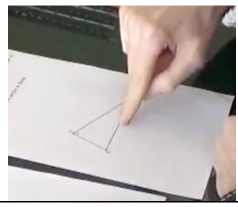
In the next sections, we provide examples of several detailed and fuzzy products of GP and we analyze the solvers' different interactions with the drawings. For the sake of clarity, we chose excerpts which belong to the same open problem: Task 4.

6.2.1 Detailed products of GP and top-down processes

In this section, we report two examples of detailed products of GP and we highlight how the solvers' interaction with the drawing is mainly a top-down process.

The first excerpt is from Filippo's interview and it constitutes the whole first part of his interview: *Filippo_PhD_T4_P1*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:00.28	Int	Consider the right triangle in the figure.		
00:04.06	Stud	Yes...with the hypotenuse of a	He is reading the task.	

		given length. A and B are fixed, the length of AB has to always be the same.		
00:12.19	Stud	Ok. So I fixed two points, I construct the right triangle on them above which the right angle is opposite.	 *Given drawing*	
00:21.01	Int	What can you say about the vertex with the right angle?		
00:24.15	Stud	What can I say about the vertex with the right angle?		
00:27.14	Stud	So, I can say that it is forced, though it doesn't want to, to **vary on a circle. With diameter AB**.	He keeps talking without producing gestures or drawings	**GP_1_(0) (discursive): the locus of C is a circle with diameter AB**
00:38.17	Stud	What can I say about this vertex?		
00:47.13	Stud	Yes, it has to stay there, it can't come too close, it can't go too far, **its distance from the midpoint of AB is fixed.**	He points at C and moves the finger bringing it closer or taking it away from AB. He is pointing at a point within AB: 	GP_1 is reformulated in terms of the locus of the points that are equidistant to a given point.
01:00.22	Stud	What else can I say...about this here?		

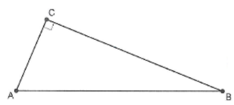
This excerpt shows the very beginning of Filippo's interview. He is a very *expert solver*: a solver who was exposed for a long time to the mathematical knowledge and, by virtue of this, is supposed to be expert. He answers to the interviewer's question very quickly. His answer at time 00:27 contains a discursive product of GP:

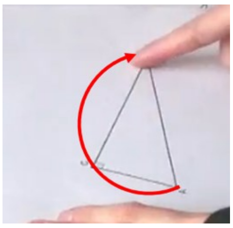
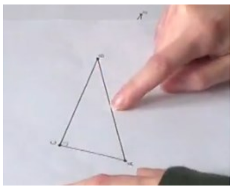
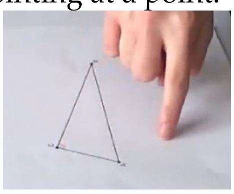
GP_1: *the locus of C is a circle with diameter AB*

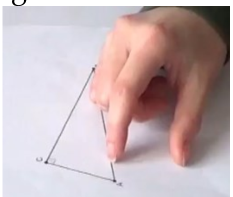
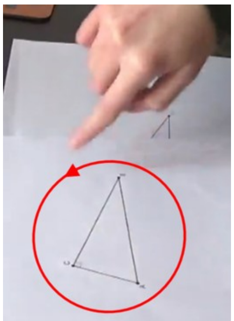
This is a detailed product of GP, indeed the locus is described as a circle and the solver makes the diameter explicit. After a few seconds he also declares and shows through a gesture the center of the circle: the midpoint of AB.

This is an extreme example of the interplay between the solver and the drawing: Filippo interacts with the drawing only in order to impose upon particular properties on the figure. To do this he only describes or shows through gestures the figural elements he was predicting; he does not draw anything; it seems that he controls very carefully both conceptual and figural components of the predicted configuration. This is an example of a top-down process because it seems that the solver does not need to get confirmation from the drawing.

For the sake of completeness, we provide another example of a detailed product of GP produced during an interview that was longer than Filippo's. The following excerpt is taken from Fiorella's interview and it constitutes the whole first part of her interview: *Fiorella_MD_T4_P1*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:00.14	Int	This is the next question.		
00:05.25	Int	Consider the right triangle in the figure, with a hypotenuse of fixed length. A and B are fixed.		
00:14.11	Stud	Mm mm.		
00:15.09	Int	The length of AB has to always be the same.		
00:18.20	Stud	Ok.	 *Given drawing*	
00:19.10	Int	What can you say about the vertex with the right angle?		
00:31.01	Stud	That I can move C so that it makes...**a half circle**.	She is pointing at point C. She traces a semicircle using her finger: she starts from A, passes through C and ends at B. During	She quickly communicates the product of GP: **GP_1_(0) (discursive – gestural): the locus of C is a semicircle**

			the process she does not talk:  She repeats the gesture at the end of the utterance.	**Window gesture**
00:40.19	Stud	Where does this **semicircle** have its center...at the **midpoint of AB**.	She is pointing at a point on AB: 	**Window gesture GP_2_(0) (discursive – gestural): the locus of C is a semicircle centered at the midpoint of AB**
00:47.04	Stud	There, the hypotenuse **AB always has a fixed length**, because I am only moving C.		
00:53.14	Stud	Since C is...the **vertex of a triangle that lies on a semicircle circumscribed to the triangle, it is always right**, so the triangle always stays... right. In any place I move C.		**Theorem** She uses the Theorem in order to explain and support her GP.
01:14.08	Int	Are there other positions for point C?		
01:16.17	Stud	So that it maintains this configuration?		
01:21.12	Stud	If I draw...the...point **transformed from C with respect to AB** [according to line symmetry], so I send the **perpendicular** from	Suddenly, she is pointing at a point:  She is pointing at C and then at AB. She	**GP_3_(0) (gestural – discursive):** **C symmetric point with respect to AB**

		C to AB, that is I draw a segment that is **always perpendicular to AB of the same length** of the one that I drew before. So, C can…can also stay on the **opposite side**.	traces with her finger a segment perpendicular to AB and through C. She is pointing at the segment "of the same length":  She repeats the first gesture.	
01:53.18	Stud	And so, in the end C can be on the...can be, yes, **lies on a circle with center at the midpoint of AB**.	She traces a circular path:  She is pointing at the midpoint of AB.	**GP_4_(0) (discursive – gestural): C on a circle centered at the midpoint of AB** **Window gesture**

During the first part of the interview, Fiorella communicates four products of GP. Each product is carefully described. Let's look at them in detail.

Right after the first question, Fiorella communicates her first GP: GP_1. She explains verbally that C is moving on a semicircle and she produces a gesture that is consistent with her utterance: she traces a curvilinear path with her finger starting from A, passing through C, and ending at B. She performs this gesture twice.

At time 00:40 she communicates a product of GP that adds some details to the first one (GP_2). Indeed, she explains that the semicircle is centered at the midpoint of AB. Moreover, she points to the drawing and in the center. At time 00:47 she stresses that the given constraints are maintained: if C is moving on this path, AB is of a fixed length and it is still a hypotenuse. She recalls a geometrical result, probably in order to provide evidence of her prediction: the angle C is right because it is a vertex of a triangle that is inscribed in a semicircle.

During her prediction process, she uses the drawing only for tracing the figural elements with her finger she was predicting; she does not draw anything. In this case, it seems that the prediction process is supported by top-down processes: the solver is imagining some properties or figural elements and she shows them to the interviewer on the drawing.

At time 01:14 the interviewer prompts Fiorella to explore further the configuration and at time 01:21 the solver communicates another GP: GP_3. She points to the drawing showing a new position for C and verbally describes how she intends to find it. Indeed, she explains she could use a geometrical transformation and she lists a sort of geometrical step-by-step construction. Finally, at time 01:53, she collects the first three products of prediction in another one:

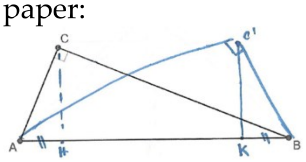
GP_4: *C on a circle centered at the midpoint of AB*

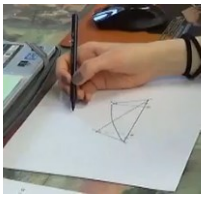
The product of GP is detailed. As we can see looking at the utterance and at the corresponding gestures, the locus is clearly named (a circle) and its center is made explicit.

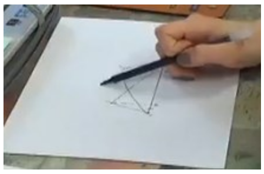
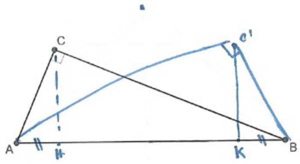
Even during these processes of GP, nothing is drawn; the solver seems to imagine the new configurations and she uses the drawing only for showing to the interviewer what she is imagining.

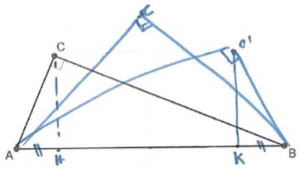
6.2.2 Fuzzy product of GP and bottom-up processes

This excerpt is taken from Laura's interview: *Laura_G10_T4_P1_(02:11 – 05:13)*. It shows an example of a fuzzy product of GP and the solver's interaction with the drawing.

Time	*Who*	*What is said*	*What is done*	*Comment*
02:11.28	Int	Make a prediction: do you think that point C can occupy other positions so that the angle stays right?		
02:18.20	Stud	Yes.		
02:19.06	Int	Which?		
02:20.15	Stud	Exactly, **this one here** that I just drew.	Figure on the sheet of paper:  *Drawing 1a*	GP_2

02:22.12	Int	Mm mm.		
02:22.25	Stud	Or ... ehm ... moving ... I don't know, ehm...		
02:31.26	Stud		She is pointing at a point on the plane: 	**Window gesture**
02:36.22	Stud	It could form a... triangle... ehm... **isosceles triangle**, I don't know if it's also **equilateral**. Yes, an **equilateral triangle** with also... ah, no... no, nothing.	She is moving the pen on AC: she starts from A and she goes towards C. Then she is points at the angle A.	GP_1 It seems that she considers other kinds of triangles: equilateral and isosceles. She seems to be guided by the figural elements. She changes her mind very quickly.
02:52.23	Int	What are you thinking?		
02:53.21	Stud	No, I was thinking that it has **all right angles** but that is impossible...or not?		It seems that she imagines that it is possible to obtain a triangle with three right angles. Contrast between figural and theoretical elements.
03:00.21	Stud	Is it possible that in a triangle there are all right angles?		
03:04.01	Int	You tell me.		
03:05.05	Stud	Ehm...no, it's not possible. No, then nothing.		She rejects the GP_1
03:08.06	Stud	I would say only...		
03:14.27	Stud	Or maybe there will be a point in any case a...**a point towards the center** such	She is pointing at a position on the sheet of paper:	**Window gesture** **GP_3_(1) (gestural – discursive):**

		that...C will be a right angle.		**C at a centered position [between C and C′]**
03:26.27	Stud	Yes, the initial point could be here, C towards the center and A towards here. But I would say that there is nothing else... I don't think there is anything else.	She is pointing at the found three positions of C:  *Drawing 1b*	GP_2 and GP_3
03:46.17	Int	What are you thinking?		
03:48.23	Stud	No, I was thinking about... yes, well the positions in the end I think I have only 3. But I was thinking that if maybe **moving it in an intermediate way** I could...	She is pointing at the three points C on the sheet of paper.	She is wondering about other positions between the three points she already found.
04:00.21	Int	Mm.		
04:01.21	Stud	...try finding another...another point with... so that C were a…right angle.		
04:08.28	Int	Mm.		
04:09.21	Stud	But... I mean this one and this one I'm sure, the I also think a point…at the center? There should be a position in which... C...I mean, stays right.	She is pointing at two of the three positions of C: the given one and C′.	GP_3
04:20.18	Int	Mm mm.		
04:22.08	Stud	And...or maybe not, I don't know.		
04:35.00	Int	What are you thinking?		
04:36.11	Stud	Eh, I was thinking in what position I had		GP_3

		to put C towards the center so that it…it is a right angle.		
04:50.03	Int	What do you imagine?		
04:53.07	Stud	Mmm... wait if I maybe use a piece of paper to do another right angle.	She turns the sheet of paper. She uses the corner of another sheet of paper in order to trace the right angle: 	
05:03.11	Stud	Yes, if it were here I could...another right angle could be formed.	She connects the new point C with A and B:  *Drawing 1c*	GP_3
05:13.13	Stud	So, I think there are 3 positions. Yes, necessarily.		

As mentioned previously, when the excerpt begins Laura has already communicated two products of GP: GP_1 "*C is the vertex of a triangle right-angled and equilateral*"; GP_2 "*C at a symmetric position [with respect to the axis of AB]*". Moreover, she has made explicit a new position for the point C (*Drawing 1a*).

At time 02:36 she seems a little bit uncertain about her first GP and she starts evaluating other triangles. It seems that she is imagining a position for point C such that the triangle will be isosceles or equilateral. At time 02:53 she says that, in the latter case, the triangle has three right angles, but she seems to be uncertain. Probably, the conflict between the figural elements of her prediction and the theoretical elements "three right angles" cause her uncertainty to the extent that she asks the interviewer if it is possible for a triangle to have three right angles. Finally, at time 03:05 she rejects her first prediction.

Although she did not succeed in describing the triangle obtained by placing C at the imagined position, she seems to be confident that such a position exists and consequently she communicates her product of GP:

GP_3 : *C at a centered position [between C and C′]*

We can notice the fuzziness of GP_3, looking at its instances:

- the utterance does not explain where she intends to place C (how far from AB);
- the drawing (*Drawing 1b*) shows the new position of C, but it does not show either the right angle or the new triangle;
- the gesture also does not show how she intends to construct the triangle.

The fuzziness is confirmed by the utterance at time 04:09. Laura states that she is quite sure about two positions: the given one and the one named C′. Moreover, she conjectures the existence of the position at the center, but she is uncertain: her uncertainty becomes manifest at time 04:22.

At 03:36 it seems that she is wondering where she can place point C in order to obtain a right angle. Its existence is no longer in doubt. She is engaging in a dialogue with the drawing: from time 03:26 to time 04:53 she is only looking at the drawing. We can infer that a bottom-up process is taking place and, in the end, Laura seems quite sure that another position exists for C, even if she is not able to draw the new triangle.

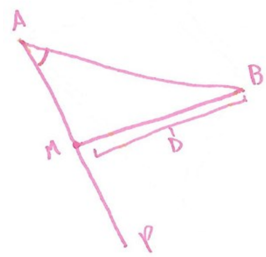
Finally, at time 04:53, in order to cope with this difficulty, she uses an improvised tool: the corner of the sheet of paper. We infer that, because of the fuzziness of her GP, she is not able to dominate the figural elements of the configuration and she resorts to a physical tool. Only after she makes use of this tool, she writes the letter "C" near the point at the center. We can interpret this last action as the end of the bottom-up process: only now that she draws out the right triangle, she seems more confident that by placing C in her selected position the triangle is really still right.

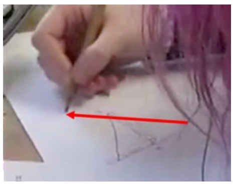
Nevertheless, even if she is tracing another triangle, the fuzziness of her GP still persists: at time 05:13 she states that there are three positions, but she uses "it seems". So, she is still not sure and she only interprets what is shown in the drawing.

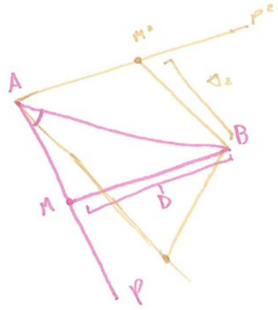
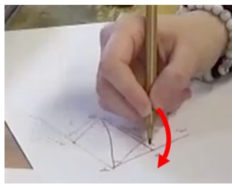
In excerpts like this, we can notice that the products of GP appear to be fuzzy: the solver heavily relies on the drawing to confirm, refine and talk about these products. Moreover, solvers, like in the case of Laura, do not seem to be confident about their predictions until the drawing confirms them in a figural way.

6.2.3 Fuzzy product of GP: an example from Task 2

For the sake of completeness, here is an example of a fuzzy product of GP produced during the resolution of another task: Task_2. The excerpt come from an interview with a 10th grade student: *Flavia_G10_T2_P1_(03:06 – 06:07)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:06.22	Int	Ok. Do you think that there are other positions?		
03:11.10	Stud	Ehm...there are other...	She is looking at the drawing produced at time 02:45:  *Drawing 1b*	
03:13.27	Int	For point P.		
03:15.05	Stud	Yes…		
03:21.22	Stud	It could be, because maybe **making it longer and changing the inclination,** I could find length d, say it is 10 and if I make it even longer, and therefor make the angle BAP bigger, d should get longer.	She points at P and traces a straight trajectory on AP moving the pen away from A. She points at M and traces a straight trajectory on MB.	She is talking about possible movements of P and M on two straight lines. She explains that these movements change some features of the geometrical figure. She seems to control the figural components of the geometrical figure. **GP_2_(2) (discursive – gestural): moving P or M, MB changes its length**
03:46.20	Stud	Buti f I decrease P, also the length should decrease, so it is likely that there are ...**more than two**	She points at P and traces a straight trajectory on AP bring the pen closer to A.	**GP_3_(2) (discursive): P can occupy more than two positions in order for the length of MB is the same**

		points to make d of this length.		
04:00.23	Int	Do you want to show me some?		
04:02.26	Stud	Ehm...what I said before, I think, it is ...		
04:06.06	Stud	Possibly with a color, so you can see the steps I did after and [otherwise] I myself forget the main figure.	She adds figural elements to the Drawing 1b.	
04:13.03	Int	Yes.	She is laughing	
04:16.13	Stud	I'll get...	She points at P, moves the pen on a straight line, and stops, showing where she intends to place P: 	**Window gesture**
04:20.04	Stud	...about this. If I take this here M2 and P2, ... theoretically.	She draws the symmetric segment of AP with respect to AB; she traces the midpoint M': she traces M'B.	
04:34.22	Stud	Very much so… the length we find here should be d2.	She writes "D2" on M'B.	
04:46.08	Stud	And another possibility could be, as I said before, one...		She is looking for another position.
04:54.01	Stud	No, but if I make it bigger, it stays bigger in any case. Because...		
05:02.02	Stud	Or I could...I could also do it here. So d, being here, would obviously be shorter.	She draws a new segment [AM]	
05:12.15	Stud	And...I bring it up to here.	She extends the drawn segment and connects it with B.	

			She obtains the following drawing:  *Drawing 1c*	
05:17.10	Stud	Or also higher, so that after d it goes to coincide, **BM goes to coincide with our measure of d.**	She points at the new point M and rotates the pen towards the first position of M: 	GP_3 **Window gesture**
05:27.08	Stud	But...I don't know, I should try having exactly the measures set up and being able to move them.		
05:34.04	Int	How do you imagine it?		
05:36.02	Stud	Mmm...		
05:37.07	Int	Do, like...ehm...you said probably there is another position of P. How do you imagine it beyond the drawing?		
05:44.09	Stud	Ehm... I imagine it with an inclination of the angle that is formed BAP either greater or less than it, and based on that, the length of P changes.		
05:58.06	Stud	I think that it would be enough to have it **at any angle** and then make P **longer or**		GP_3 She describes the method by which other positions of P

		shorter till you get the measure you want.		can be found. It involves only the figural components of the geometrical figure. **GP_4_(2) (discursive): other positions of P could be found combining a translation and a rotation**
06:07.18	Int	Ok.		

Previously during the interview, the solver had already communicated a first detailed product of GP:

GP_1: *P at a symmetric position [with respect to AB]*

The excerpt begins with the interviewer asking for possible other positions of the point P and Flavia says that there exist other positions. After 11 seconds of silence while she was looking at the drawing, she claims that P can occupy more than two positions because P could be moved back and forth along a line through AP, and M could be moved in order to change the angle at A. These two consecutive movements allow her to obtain a new position for P. The GP_2 and the GP_3 reported in the transcription table are quite fuzzy and, indeed, the solver is not able to show or describe the actual position of P. In addition, we notice some uncertainty in her utterance: she uses "*it could*" and "*maybe*". Nevertheless, she shows good figural control over the figure: she seems to be aware that the movement of P and M changes the length of MB, and that the described changing is coherent with the showed movement.

At time 04:00 the interviewer asks the solver to show on the drawing some of these new positions for P. Flavia starts drawing the symmetric position of P communicated in GP_1: she shows approximately where P will be placed; then, she directly draws AP and its midpoint M'; she connects M' and B. Then at time 04:46 she starts drawing a new position for M. We are more interested in the latter process, because she proceeds differently. It seems that she finds the point through "trial and error": at the beginning (time 04:54) she decides to increase the angle at A and she sketches the corresponding segment; she is looking at the drawing and she estimates that the segment AM is too short for obtaining the fixed length of MB; so, she extends the segment and, at a certain point, she stops drawing. At time

05:17 it becomes clear why: she stops because the new segment MB is such that it could overlap the first one through a rotation. After this dialogue with the drawing, Flavia explains how she intends to find the new positions of P and we find a new fuzzy product of GP:

GP_4: *other positions of P could be found combining a translation and a rotation*

In this example, we can also see that the same solver could communicate both detailed and fuzzy products of GP. Each kind of product corresponds to different usage of the drawing. We can claim that if the product is quite detailed the solver has used mainly top-down processes; on the contrary, if the product is fuzzy the solver has heavily used the drawing in order to support the prediction process. In the latter case, the solvers seem to mainly use a bottom-up process to the extent that they resort to physical tools.

6.2.4 Concluding remarks on detailed and fuzzy products of GP

The analyses of gestures, utterances and drawings allow the researcher to interpret the solvers' products of GP and report them in a sentence. Depending on the easiness of the interpretation we distinguish the products into detailed and fuzzy.

The following table summarizes our findings about this dichotomy.

Detailed product of GP	**Fuzzy product of GP**
The researcher can easily restate the product.	The formulation of the product in a statement requires a lot of effort.
The statement refers mostly to the theoretical components of the figure.	The statement refers mostly to the figural components of the figure.
The process that leads to the product is mainly a top-down process.	The process that leads to the product is mainly a bottom-up process.
During the process that leads to a detailed product of GP, the solver shows theoretical control over the figure.	During the process that leads to a fuzzy product of GP, the solver mainly shows a lack of theoretical control over the figure.
Usually during the process, the solver does not make other drawings.	The solver can produce several drawings or can add other details on the same drawing.

The drawings are used only for making an already communicated product of GP explicit.	The drawings play a key role, guiding the process.

6.3 The role of motion within GP processes

In order to describe the features of the GP process, in this section we will explore the relationship between motion and GP processes.

We start from the following working hypothesis:

- when the solvers are engaged in a GP process, they could generate figural expectations about the given configuration;
- the solvers can consider several figural expectations about the same configuration, changing some figural elements and eventually making use of motion;
- the request for possible other positions of a part of the configuration can, but not necessarily, support the use of motion during the GP processes.

So, the solvers could explore alternative arrangements of the configuration making use of motion or not. Depending on how motion is integrated into the process of GP, a process of GP may or may not have a dynamic dimension. So now we can make the distinction between products of GP that are produced:

- *dynamically* or *continuously*, using a continuous motion;
- *statically* or *discretely*, reconstructing the configuration starting from a new position of one of its parts.

Indeed, solvers can imagine, perform or mimic a continuous movement of one or more parts of the configuration (i.e. points, segments); otherwise, they can locate these parts at a specific position on the plane and reconstruct the corresponding configuration.

The first and the second point of the list represent two different approaches to the task, which reveal two different ways of looking at the geometrical object. In the former case, the configuration seems to be considered as a continuously changing object and the interaction with diagrams seems to be similar to the interaction that a solver could perform in a DGE when a point is dragged. In the latter case, the configuration is considered as a series of examples of the same object as if the solver were taking several snapshots of it. Moreover, these examples can be overlapped and appear in the same drawing, or they can be drawn in several places on the sheet of paper. We can investigate these different approaches by observing the presence of motion in solvers' productions.

Moreover, since the dynamic dimension can affect the GP processes, we will discuss this point in this chapter and, in particular, we will stress how the dynamic dimension can intervene into a process of GP that leads the solver to a coherent product of GP which support a complete solution to the problem.

As an example, here are two solvers' final productions that were accomplished during their resolution of Task 4.

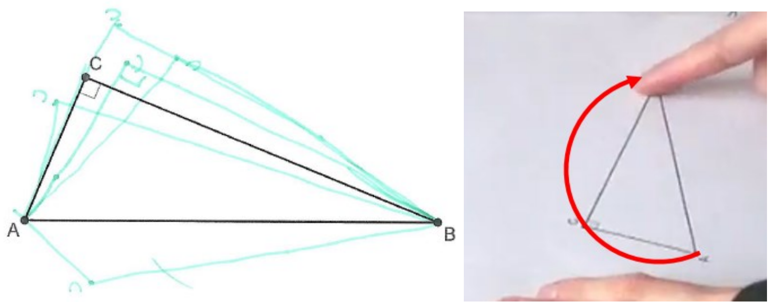


Figure 28 Instances of static (on the left) and dynamic (on the right) solvers' productions

The two productions are clearly different in nature: one is a drawing and the other one is a gesture. Nevertheless, as we will clarify shortly, we can compare these productions because our focus is on how they were produced. In both cases the solver is looking for other positions of C. In the first case (Figure 28 on the left), the solver tries to find the solution in a static way: she points at a position where she intends to place C; then she traces CA and CB. Her utterance confirms our inference about the process. While she is pointing at a specific position, she states:

Sofia: I was thinking of...I mean **if I put it** like **here...**I was thinking whether there were other points.

Then she connects the point with A and B; she repeats this procedure many times. The utterance confirms a static view of the configuration. Only when she obtains the drawing in Figure 28 with several positions for C which respect the given constraints she tries to identify the locus of C. Looking at the interview, we know that she does not succeed in reaching the locus.

In the second case (Figure 28 on the right), the solver continuously traces the locus of C and reveals that she intends to construct such locus dynamically. Her utterance, as well, contains dynamic elements:

Fiorella: **I can move C** so that it forms...a half circle.

We note that the interviewer only asks what Fiorella can say about the point C. So, Fiorella spontaneously mentions motion in her answer.

In the following section, we collected some illustrative instances of solvers' productions which show dynamic or static components.

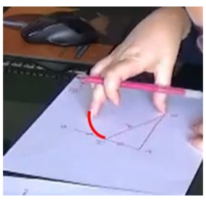
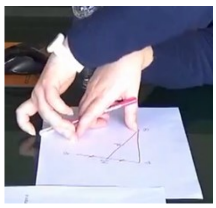
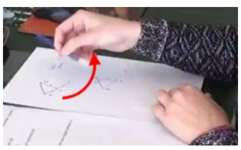
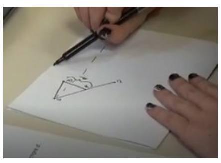
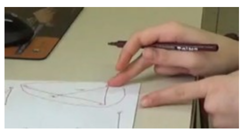
6.3.1 Observing dynamism

We can recognize several instances of dynamism in gestures, utterances, drawings, that are listed in the transcription table. In some cases, the dynamic component can be easily caught only looking at one solver's production.

In Table 5 we collect instances of dynamic and static gestures produced during the resolution of each task; it is not a complete list of all the gestures found in our data, because we only wish to show, for each task, some possible kinds of gestures. We divided the table into two categories. We will discuss in greater depth the role of gestures in another section (Section 6.4).

Looking at the table we notice that the gestures can have a dynamic component that can be captured. Dynamic components are used during the prediction process with several aims:

- to trace particular loci: (e), (i), (j), (n);
- to mimic the use of tools (i.e. the compass): (a), (b), (f);
- to trace imagined paths during their construction: (q), (r);
- to manipulate a given configuration: (m).

Dynamic		**Static**	
Task 2			
			
(a)	(b)	(c)	(d)
			
(e)	(f)	(g)	(h)

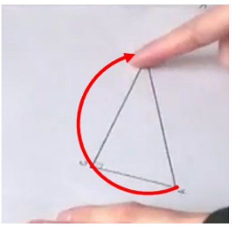
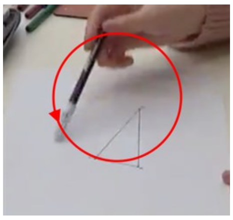
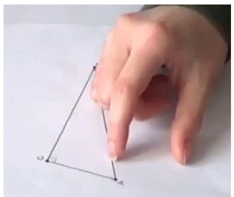
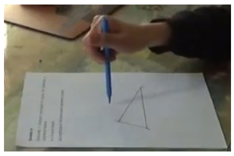
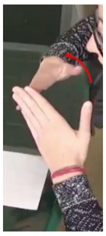
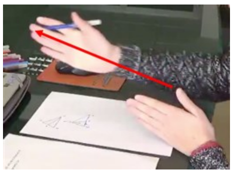
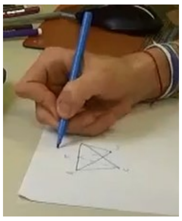
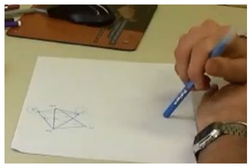
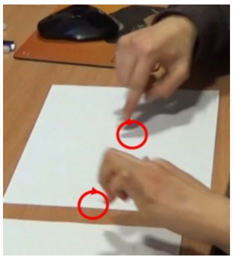
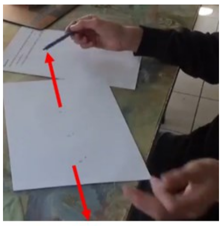
Task 4			
			
(i)	(j)	(k)	(l)
Task 5			
			
(m)	(n)	(o)	(p)
Task 6			
			
(q)	(r)	(s)	(t)

Table 5 Some instances of dynamic and static gestures that were performed during the resolution of the given tasks

The static gestures are used during the prediction process with the following aims:

- to mimic the use of tools (i.e. the ruler): (c), (d), (k);
- to point at a particular position of one part of the configuration: (g), (h), (l), (o), (p), (t);
- to show paths: (s).

We can also recognize dynamism in the utterances. In Table 6 there are some examples of utterances with dynamic and static components taken from the interviews. In Table 6 we use bolt type to highlight the dynamic and static components. In our data, we identify dynamism in the following cases: when the solvers use verbs that communicate motions; when they talk about events that will take place in the future. Otherwise, we find position, location and existence verbs.

Utterances with dynamic components	Utterances with static components
I can rotate MB.	*If the length has to always be d, M* ***is on the other side***.
If ***I move*** *point M to maintain the length d, a circle* ***will be created***.	*Point M* ***is always at the same distance*** *from A and from P.*
If ***I put it closer***, *we could have CM congruent to CB.*	*It's enough* ***to put*** *C on the perpendicular bisector of the segment.*
I move *C perpendicularly to AB.*	***If I draw*** *the image of C with respect to AB, C* ***can be*** *on the other side.*

Table 6 Some instances of utterances with dynamic and static components inspired by the actual solvers

In Task 6, we have few examples of utterances with static elements, because right from the beginning the questions themselves ask for possible movements of the points within the configuration. Indeed, the interviewer explicitly asks what happens if one of the points is moving along particular loci and, inevitably, the solvers mostly use dynamic elements to answer, frequently using the same verbs used by the interviewer.

Finally, we can also observe dynamism within drawings. The "paper and pencil" is a static environment: drawings cannot be moved, and points cannot be dragged continuously as a solver can in a DGE. Nevertheless, we can find an intention of motion also in drawing. A feature that communicates dynamism is the use of arrows (see Figure 29).

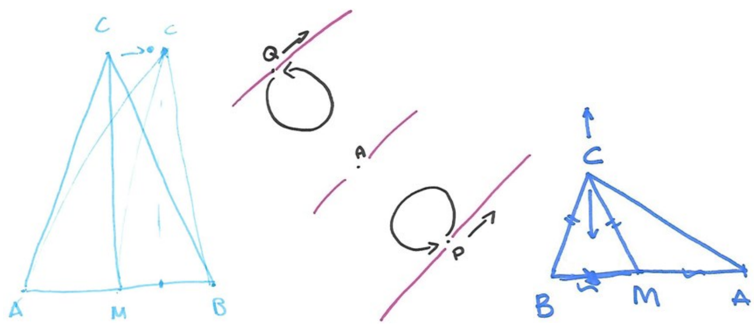


Figure 29 Examples of drawings which communicate a dynamic dimension

However, in our sample, this is not the only marker of dynamism in drawings. Indeed, in some cases, looking only at one solvers' production (i.e. drawing, gesture, utterance) is not enough to capture dynamism; moreover, an isolate product could be misleading. Instead, we can capture dynamism by looking at the interaction between the solver and the drawing. Since the drawing has a static nature (it is actually a picture), the final drawing alone could not reveal how it was constructed. Instead, we can conceive the figure as being made up of dynamic components or of static components. For example, looking only at the picture in Figure 28 (on the left) an observer could think that it is a drawing with a dynamic component, because we see many triangles in the same picture and this is a different situation with respect to drawing a lot of sketches of the same figure in several positions on the plane. However, analyzing the whole interview we discover that the solver was trying to find a locus by investigating many discrete positions of C. So dynamism can be captured also by analyzing drawings together with the gestures and the utterances that accompanied the construction of it. This point will be further clarified in the following section.

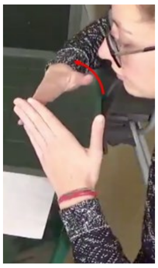
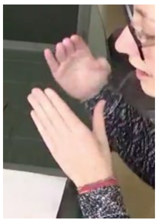
In summary, the production and communication of new possible arrangements seem to be a complex process, where gestures, words, and drawings interact step by step. So, in order to study the dynamic components of GPs, it is more useful to conduct both synchronic and diachronic analyses. We did this by using the transcription table, where we can see at a glance what is said and done; moment by moment, dynamism could be recognized in all solvers' productions at the same time, only by looking at a line of the table. Indeed, in each column we can see how utterances and gestures are developed by the solver during the interview.

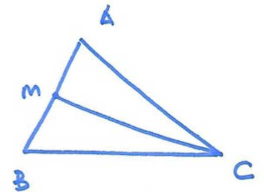
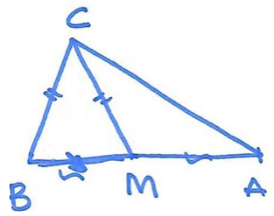
In the following section, we will describe in greater depth how we recognize dynamism looking at the interaction between solvers' productions. Moreover, we will provide examples of both processes of GP with and without a dynamic dimension. More specifically, our intention is to analyze the possible role of the dynamic dimension within a GP process, including also how dynamism could affect the solvers' reaching of a complete or coherent solution to the problem.

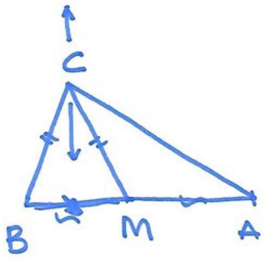
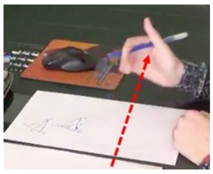
6.3.2 Dynamic and static approach: examples from Task 5

The first two examples come from solvers' resolutions of Task 5. Unlike the other tasks, this task requires the solver to undertake, for as long as possible, prediction processes without a given or sketched drawing. In this case it seems that dynamism plays an important role. Indeed, it seems that the ability of the solvers to imagine continuous transformations of the figural elements leads them to express a product of GP without using any drawings.

The following excerpt provides examples of GP processes with a strong dynamic dimension. The first is from Marta's interview: *Marta_MS_T5_P1_(00:23 – 02:44)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:23.16	Int	Make a prediction: is it possible that CM is congruent to CB?		
00:31.04	Stud	Is it possible that CM is congruent to CB?		
00:34.23	Int	Mm mm.		
00:35.15	Stud	So...I have a triangle...**M midpoint**, perfect. **CM**...great, so **it is a median**. Ok.	She is looking at the step-by-step construction.	She recalls some theoretical elements (the triangle, the midpoint) and introduces a new one (the median).
00:45.06	Stud	CM can be congruent to CB...I think so, I mean that…ehm...	She is looking up and ahead.	
00:52.27	Stud	**If CM is congruent to CB, I have a triangle BCM that is isosceles** and, just a bit big, I have **another triangle ABC** that was my initial triangle, I mean. So…eh…	While she is talking about the triangle ABC, she rotates the right hand:  While she is talking about the initial triangle: 	She reconstructs the configuration starting from an isosceles triangle (CMB) and manipulating one of its sides in order to obtain the triangle ABC. **Window gesture** **GP_1_(0) (discursive): BCM is an isosceles triangle**
01:11.27	Stud	Yes, I think that it is possible.		
01:15.07	Int	Ok. Make a drawing of what you imagined.		
01:20.27	Int	So, I have a triangle...	She draws a triangle	

01:29.09	Int	So I have a midpoint M and now I need to draw CM.	 *Drawing 1*	
01:36.22	Stud	Perfect. And...no, I don't like it like this. I'll try more...		
01:40.21	Stud		She starts a new drawing	
01:43.01	Stud	So let's say I start with CB and CM, like, the same.	First of all, she draws the triangle BCM	It seems that she repeats the construction previously described.
01:48.20	Stud	I made a mistake, this is equal to this one.	She marks two hash marks on CB and CM	
01:51.23	Stud	To this one extended...by another segment...and this one here. Makes a triangle.	She extends BM	
02:01.01	Int	Mm mm.		
02:01.23	Stud	This is what I imagined.	 *Drawing 2a*	She marks on the drawing the property "M is the midpoint of AB": she adds the same mark on BM and MA in order to highlight the congruence.
02:03.22	Int	Ok.		
02:12.11	Int	Are there other ways in which CM can be congruent to CB?		
02:17.02	Stud	CM can be congruent to CB? So...		
02:23.03	Stud	In what sense? I don't understand.		
02:24.18	Int	Imagine moving point C.		
02:26.16	Stud	Ok.		

02:27.26	Int	Do you think that there are other positions for point C so that CM is congruent to CB?		
02:33.29	Stud	Ok, so any translation of C **downwards or upwards**, **parallel** to BA, maintains this property. So...	She draws two arrows:  *Drawing 2b* "downwards or upwards":  "parallel": 	She says "parallel", but the gesture communicates the property of perpendicularity. **Window gesture** The gesture is more coherent than the words. **GP_2_(1) (discursive – gestural): C on a line perpendicular to AB** She does not mention the point where the perpendicular line intersects the segment BM. She does not seem to conceive the perpendicular line as the axis of BM.
02:44.20	Stud	CB will always be congruent to CM, therefore yes!		

The excerpt is from the very beginning of Marta's interview and it shows what happens right after the first interviewer's question.

At time 00:52 we observe that the solver uses the theoretical element "CM equal to CB" for communicating a conditional statement about the triangle BCM, using an expression that is quite similar to an "*if…then*" statement. She deduces that if CM needs to be congruent to CB, then BCM is an isosceles triangle. It seems that the theoretical element given in the problem leads the solver to think of a specific configuration. Here we identify a product of GP, GP_1: at time 00:45 the solver says that it is possible to have CB congruent to CM and then she explains why and how it is possible. So, our inference is that the GP process has started before, and it ends when the solver communicates the detailed product. Finally at time 01:11 the product of GP is confirmed. It seems that the dynamic component plays a role in this process: starting from the specific configuration of the triangle CBM (isosceles triangle), Marta dynamically constructs a new configuration for the given triangle ABC. She joins her hands, mimicking the two equal sides; she leaves the left hand fixed and moves the right hand continuously, maintaining the fingers of the two hands connected. So, ABC is obtained by a dynamic transformation of an isosceles triangle.

This particular construction is reflected in the second drawing she approaches at time 01:40. First of all, she draws an isosceles triangle and uses a sign to highlight that the sides are equal; then she extends the segment BM, until she constructs a segment with the same length; she uses another sign to stress that AM is equal to BM; finally, she connects C and A with a segment. So, the initial configuration is the isosceles triangle CBM.

When the interviewer asks for other possible positions of C, she communicates a new product of GP (GP_2) and her productions show a large use of dynamism. Indeed, in her utterance she explains that she intends to move C using a translation along a straight path; she also stresses the use of motion within the drawing, sketching two arrows. Finally, the first gesture shows the motion that she is talking about. The second gesture does not seem to be coherent with the corresponding utterance, but we will discuss this topic in another section (Section 6.4).

So, at time 02:33, the dynamic dimension is evident in Marta's utterances, gestures, and even in her drawing. Then, at time 02:44, she eliminates the dynamic dimension to move to a conditional statement. Paraphrasing what she said, we can formulate it like this: if C is on the described straight line, CB will be congruent to CM. We can see that in the last utterance there is no mention of movements, the verb does not reveal uncertainty, and she does not produce gestures.

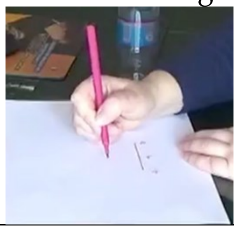
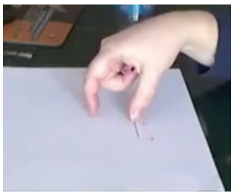
This excerpt shows the steps of a process of prediction which has a dynamic dimension:

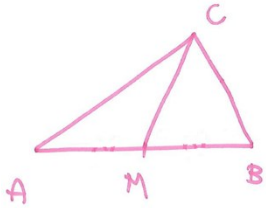
- the solver imagines a specific position for C in order to respect the given constraints;
- then she predicts and describes an entire locus such that, moving C on it, the configuration maintains the constraints;
- finally, she crystallizes the property into a statement.

Moreover, it shows how we can find instances of dynamism both in gestures and in utterances. In this case, the drawing also reveals the presence of a dynamic dimension.

Agnese uses a quite different approach to the task. During her resolution process, she produces several instances of the configuration which respect the given constraints. Her prediction processes are undertaken without an evident dynamic dimension. Let us analyze this in detail.

All the following excerpts are taken from the first part of Agnese's interview. Since it is quite long, we analyze only some relevant sequences. The first one is *Agnese_MS_T5_P1_(03:12 – 04:56)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:12.01	Stud	AB and the midpoint.	She starts drawing C: 	
03:12.23	Int	Then imagine... imagine only the third vertex.		
03:17.26	Stud	Ok. I want that CM is...		
03:21.17	Int	If it is possible that CM is congruent to CB.		
03:29.03	Stud	**CB yes**, because no.		**GP_1_(2) (discursive): CM could be congruent to CB**
03:32.24	Int	How?		
03:36.04	Stud	CM CB. So!		
03:40.24	Stud	Can I draw?		
03:42.01	Int	Of course.		
03:46.04	Stud	Or maybe...CM...CB		
03:47.19	Int	Mm.		
03:50.00	Stud	No, no no.		
03:53.24	Stud	This is CB.	She draws CB	
03:55.26	Int	Mm mm.		
03:56.22	Stud	So I have this triangle.	She connects C and A	
04:02.01	Stud	And CM.	She connects C and M. She obtains the following drawing:	

			 *Drawing 1a*	
04:04.14	Int	Mm mm.		
04:05.28	Stud	I want that CM is congruent to CB.	She writes the mathematical relationship of congruence between CM and CB.	
04:13.22	Stud	So...		
04:19.16	Stud	It goes...one side is always shorter than the sum of the other two.		Theoretical element
04:29.20	Stud	For sure. I mean...		
04:35.25	Stud	...**CB is always less than CM plus MB**.		The theoretical element applied to the configuration.
04:39.09	Int	Mm mm.		
04:40.13	Stud	Right away!		
04:44.01	Int	Yes, I was asking you to predict whether it is possible for CM to be congruent to CB.		
04:49.17	Stud		She writes the mathematical relationship: CB<CM+MB	
04:56.17	Stud	CB plus MB. That is you would like for this **triangle here to be isosceles**.		**GP_2_(0) (discursive): CMB is an isosceles triangle** She does not seem to be aware that GP_2 is a coherent answer to the question.

Before the excerpt, Agnese has attempted twice to describe the configuration that she was thinking about, but she was having trouble and asked if she could draw something. The excerpt starts when the interviewer allows her to draw.

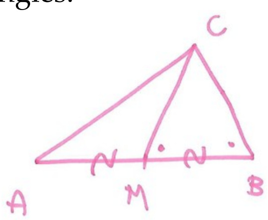
Looking at Agnese's first two gestures we can notice the absence of a dynamic component. At time 03:12 and 03:17 she is pointing at a possible position of C and she is mimicking the corresponding side of the triangle. Then she says that CM could be congruent to CB and we recognize a product of GP: GP_1. Probably the position of C that she pointed to leads her to the solution, coupled with the fact that, as she said, she does not see any reason why the two segments cannot be equal.

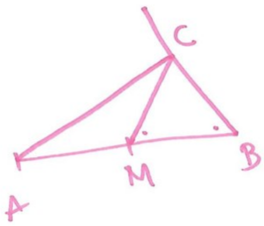
She completes the drawing starting from CB and then constructing CM. In *Drawing 1a* CM and CB seem to be congruent, but this does not seem to satisfy her. She is wondering why the two segments could be congruent and, indeed, she introduces two mathematically advanced theoretical elements at time 04:19 and 04:35.

Finally, at time 04:56 she communicates a product of GP:

GP_2: *CMB is an isosceles triangle*

This is a coherent answer to the task, but she does not seem to be aware of it: it seems to be more like a rephrasing of the given question because in the utterance she stresses that is the interviewer who wants the triangle to be isosceles. What happens in the following excerpt confirms this inference: *Agnese_MS_T5_P1_(05:53 – 07:37).*

Time	*Who*	*What is said*	*What is done*	*Comment*
05:53.15	Stud	So, **I should construct an isosceles triangle**.		GP_2
05:57.10	Int	Ok.		
05:59.07	Stud	Here and so...or have the...I mean I should show, I mean that I want the sides. If I consider the triangle CMB.	She marks the congruence of the angles.  *Drawing 1b*	
06:13.24	Stud	And...I want it to be isosceles.		GP_2
06:16.23	Int	Mm mm.		
06:17.01	Stud	I have to necessarily prove that in this case, the base angles are congruent. So,		She is looking for a mathematical proof of GP_2.

		that CMB is congruent to CB... eh so that the angles, eh!		
06:35.16	Stud	CBA. Can I?		
06:38.14	Int	Yes.		
06:41.05	Stud	A, B and I get M.	She starts a new drawing.	
06:57.17	Stud	I want that the angles at the base are congruent. I mean...	She is pointing at a position for the point C: 	
07:02.29	Stud	Eh, I want that the angles at the base...so this.	She draws a segment starting from B.	
07:12.10	Stud	This [angle at M] is equal to this [angle at B]. So considering that it is a free-hand drawing, so it is quite limited.	She connects C to M and B; she marks the angles at M and B within the triangle CMB.	She only constructs the triangle CMB and she focuses on it.
07:16.17	Int	Yes.		
07:20.15	Stud	I drew...the two...**the two congruent angles**.		
07:29.22	Stud	I mean the angle CBA and the angle BMC I drew them congruent.		
07:37.13	Stud	Ehm...the intersection of the ...of C... that is the intersection, the intersection, yes, I called the point of intersection of the sides and so... **I constructed the triangle ABC in which CB is congruent to CM**.	She connects A and C:  *Drawing 2*	

At times 05:53 and 06:13 she repeats the GP_2, respectively as a goal and to rephrase a constraint upon the figure. Then she focuses on the angles and she tries to obtain a position of C such that the two angles of the triangle CMB are congruent. At time 06:57, she places C at a specific position and draws CB, forming a specific angle with AB; she constructs CM as a side such that the angle CMB is congruent to the angle CBM; finally, she connects C and A.

In this excerpt we do not find any dynamic dimensions. After few seconds Agnese sketches another drawing using the same approach (see Figure 30): starting from AB and the midpoint, she draws CB with a fixed angle; she traces the corresponding side CM; she connects A and C with a segment.

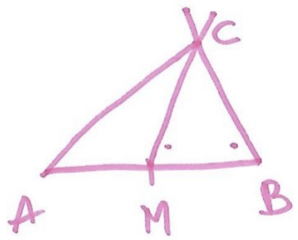
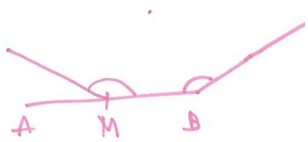


Figure 30 A picture of *Drawing* 2 sketched by Agnese during the resolution of Task 5

The lack of dynamism leads the solver to investigate some extreme configurations as a different case of the problem, as we can see in the following excerpt: *Agnese_MS_T5_P1_(08:56 – 11:24)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
08:56.10	Stud	Let's try another... obtuse.	She starts a new drawing.	The procedure is the same as before: first of all, she draws CB; then she tries to draw CM in such a way that it forms the same angle with AB.
09:01.08	Stud	With this...I only need to see if it fits.		
09:05.03	Stud	This is the point M, I want this angle.	She draws an angle at B greater than 90°	
09:12.14	Stud	Wow!		
09:18.15	Stud	This [B] is an obtuse angle, so...I need to put some restrictions		Theoretical elements

		on the...that has to be between 0 and 90.		
09:32.11	Stud	I mean it cannot be an obtuse angle because the angle in... BMC is not...I mean has to... then I couldn't construct the triangle that has to have the sum of its internal angles to be 180.		Theoretical elements
09:53.28	Int	Mm mm.		
09:55.05	Stud	I mean if I wanted to construct an obtuse angle...		
10:01.27	Stud	...congruent to this [B].	She is pointing at B.	
10:09.16	Stud	And that has a side in common with MBC. I mean that the side MB is in common.	She is pointing at M.	
10:18.13	Stud	I mean this **has to be like this** [segment AB]**, and for this one to be obtuse** ... I mean for it to be equal to this [angle at B] it should have been this [MC].	She draws MC starting from BC: 	
10:35.16	Stud	So in this way, I mean here I did not construct the triangle, I mean I do not have points of intersection, because...	 *Drawing 4*	
11:02.20	Int	Ok. Where would you say it is...say, based on... where would you say... ehm...		
11:08.05	Int	What positions can C take in order for CM to be congruent to CB?		
11:14.22	Stud	So...CM...		

11:24.17	Stud	C can be from A, I mean...**C has to be such that the angle ABC can be between 0 and 90**.		She restricts the exploration to the half plane above AB. She recalls the theoretical element introduced at time 09:18.

She starts exploring a configuration where the angle at B is greater than 90°. She draws the segment AB, the midpoint M and a segment that forms with AB such an angle. Then she starts wondering how the configuration can be completed in order to obtain a triangle and she draws a figure which maintains the rephrased constraint "two equal angles". Looking at *Drawing 4*, she concludes that the angle at B must to be smaller than 90°. We observe that she restricts her exploration to the half plane above AB, and that she does not seem to be aware that *Drawing 4* contains a possible new arrangement of the configuration: the one obtained by a line symmetry of C. We believe that possibly a stereotyped image of the triangle is influencing Agnese's exploration.

In Agnese's interview, we are not able to find any dynamic dimension: the utterances do not contain any words which refer to movement or motion; she produces few gestures and only for pointing at particular positions of a point or of a segment. Even the interaction with drawings shows a static dimension. Indeed, whenever she explores the problem, she produces a new drawing (see Figure 31).

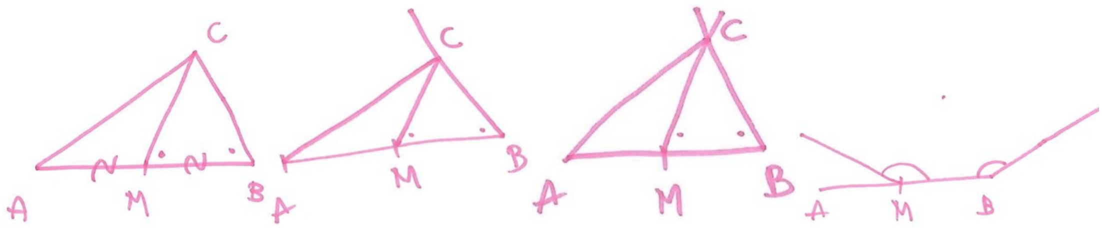


Figure 31 Pictures of the several drawings produced by Agnese during the first part of the interview

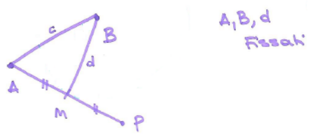
Possibly, the use of different drawings does not allow her to reach a more complete solution. After GP_2, she does not make explicit other products of GP. It seems that she sees the configurations shown in her drawings as several disconnected instances of the figure and the lack of dynamism restricts the possibilities that the solver is willing to explore.

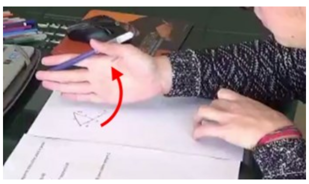
Moreover, we can notice that the two solvers find that the isosceles triangle is one of the useful properties for reaching a complete solution. So, we expected that Agnese would find the locus of C, but she did not. So, what made the difference could be actually the different dynamic dimension of the undertaken processes. Indeed, in both cases, the configuration is considered to be composed of a triangle ABC that contains an isosceles triangle CMB. However, Marta seems to conceive ABC as a triangle obtained through a dynamic transformation of CMB; instead, in Agnese's interview, we only recognize a static approach to the task. Even the locus of C is dynamically described by Marta and it seems to be conceived as a dynamic transformation of the same vertex of the triangle CMB; instead, Agnese explores discretely the configuration, sketching several drawings which seem to impede the solver to find the locus.

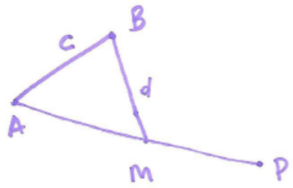
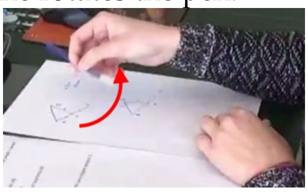
6.3.3 Dynamic and static approach: example from Task 2

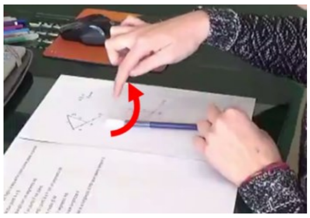
The second two examples come from the resolutions of Task 2, a complex task where the two loci must be constructed carefully to reach a complete solution. In this task also the dynamic component seems to play a role in leading quite quickly the solver to a first impression of the solution.

The following excerpt provides examples of GP processes with a strong dynamic dimension. It is from Marta's interview: *Marta_MS_T2_P1_(01:04 – 03:13)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:04.19	Int	What can you say about the point P?		
01:08.06	Stud	What can I say about P, ehm...		
01:16.10	Stud	Ok. So if A and B are fixed, it means that also this distance here [AB] will be fixed.	She adds "c" on the drawing:  *Drawing 1b*	Theoretical element
01:25.24	Stud	So.		
01:27.29	Stud	So this distance... So this distance. **I mean, I want to understand if point P can move or not**.		She spontaneously decides to explore the configuration dynamically, considering whether the point P is fixed or not.

01:34.20	Stud	So, the distance d is fixed, A and B are fixed, so...		Theoretical element
01:41.23	Stud	Their **distance is fixed**, perfect, but obviously **d** could, although it's fixed, **it could... move** let's say.	She fixes her left hand and moves the right hand, moving away and tracing an arc of a circle:  Then she only moves her right hand, tracing an arc of a circle: 	**Window gesture** Using two different gestures, she mimics the movement of M on an arc of a circle. **GP_1_(2) (gestural – discursive): several positions of MB** **Anticipatory Intuition**
01:49.12	Stud	And so AM can change length, therefore also MP and so also AP, so **P can move** within certain limits obviously.		**GP_2_(2) (discursive): several positions of P**
02:05.13	Stud	I guess I need to be precise. So ehm...		
02:11.15	Stud		She puts her right hand open on the drawing: 	
02:15.07	Stud		She moves her right hand as before: 	**Window gesture** (GP_2)

02:18.03	Stud		She starts a new drawing, following these steps: segment AB, labeled "c"; segment with an endpoint at B, labelled "d"; segment AP; point M.	
02:22.02	Stud	B, like this. So here we have a d, I'll draw it like this.	 *Drawing 2*	
02:32.10	Stud	So.		
02:35.25	Stud	This could be...M here.		
02:47.29	Stud		She puts the pen on the drawing at a position parallel to MB.	She still uses the first drawing.
02:49.18	Stud	Ok, **basically d can move**.	She rotates the pen. 	**Window gesture** It seems that the pen (or MB) represents the radius of a circle with center at B.
02:54.02	Stud	It is as if **B were the center of a circle and d can move**...	She leaves the pen and rotates her hand as at time (02:15).	**GP_3_(0) (discursive – gestural): [M on a] circle C(B,d)** **Window gesture**
02:58.10	Stud	It is a kind of radius for this circle, and so it can move.		GP_3
03:02.13	Stud	Since A, B are fixed and we leave them there therefore the segment AM and that is **the segment AP will follow that... the movement of d** I mean.	She is pointing at a point on the drawing with her index finger of the left-hand and at P with the index finger of her right-hand. While the index finger of the left-hand is fixed, she is moving the other index finger tracing a curvilinear path:	**Window gesture**

				
03:13.14	Stud	**So also** P actually will be able to move, **it will move along a circle**.	She moves her finger along a curvilinear path: 	**GP_4_(1) (discursive – gestural): P on a circle** **Window gesture**

The excerpt is from the beginning of Marta's interview. At time 01:27 the dynamic dimension is spontaneously introduced by the solver. After the interviewer's question about the point P, she decides to investigate whether P could move. We stress that the interviewer's question does not contain any reference to motion. At time 01:41, the solver focuses on MB and communicates her first product of GP: GP_1. Here we find two instances of dynamism: the utterance contains verbs that express motion and the gesture is dynamic. The solver mimics the rotation of the segment MB, which is used as a radius of a circle. Initially, the two hands are close as if they are the same segment MB; then, while the fingers of the two hands are touching, the left-hand stays fixed and the right-hand rotates, describing a small arc of a circle.

The second GP is produced as a consequence of the first one. The logical connection is evident if we consider the very beginning of the utterance at time 01:49: the solver uses "*and so*" and "*therefore*". This utterance also contains a reference to motion.

In order to reach a more complete solution, the solver keeps on moving her hand on the drawing, mimicking a rotation. The gesture shows a dynamic component.

At time 02:49, a rich sequence of utterances and gestures starts, ending with two consecutive products of GP. Initially, the solver says that MB could be moved, and we infer from her gesture that she intends to rotate the segment. The gesture is continuous, and we infer that she conceives MB as a radius. The following utterance, at time 02:54, contains a product of GP that confirms our inference:

GP_3: *[M on a] circle centered at B and with radius d*

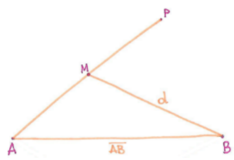
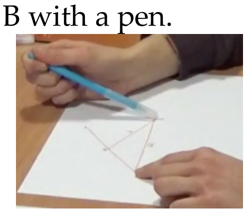
Then she also communicates a new product of GP, GP_4, addressing the positions of P, which are found as a consequence of the movement of M. We find evidence of this connection looking at what happens at times 03:02 and 03:13. The solver says that the segment AP will follow the movement of MB and she shows the motion using a gesture. Both the utterance and the gesture have a dynamic component. Then, she says that also P could be moved on a circle; looking at the words at the beginning of the utterance, we infer that this is considered a consequence of the movement of MP. In this case, as well, the utterance and the gesture have dynamic features.

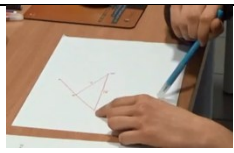
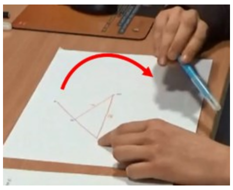
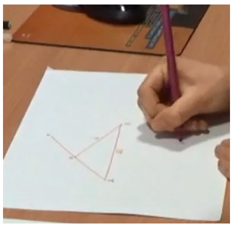
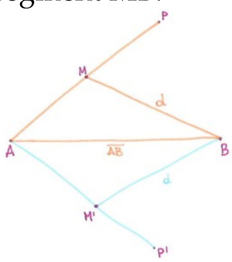
Marta undertakes several processes of GP which lead her to communicate four products. Each process has a strong dynamic component, which is evident in her utterances and mostly in her gestures. We notice that the possibility of motion is not introduced or pushed by the interviewer, but it seems to be a spontaneous direction of investigation undertaken by the solver. Moreover, in the entire excerpt, we do not find any hints by the interviewer.

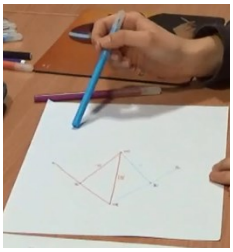
Dynamism seems to play a role in helping reach a first idea of the solution. Indeed, the supposed movement of M induces Marta to investigate the possible motion of P, which leads her to infer the locus of P. Obviously, this is only a first impression of the solution. Indeed, GP_4 is not very detailed, but it is coherent enough to become a starting point for further investigations.

The following excerpt is different in that it provides an example of a prediction process that does not contain any evident dynamic dimension. We will see that neither gestures nor utterances contain dynamic elements, and this seems to lead the solver to a very rigid final configuration. The excerpt is from the first part of Margherita's interviews: *Margherita_G13_T2_P1_(02:16 – 06:05)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
02:16.29	Int	So I will read the last sentence over again. A and B are fixed and the length MB has to always be d. What can you say about P?		
02:29.20	Stud	So... point P is as distant from M as A is distant from M. Ehm...		Theoretical element: equidistance

02:54.05	Stud	And it is also always distant from B. No?! Because...		
03:01.24	Stud	Because if **d** is always ...**is constant**. No, it can't be moved...and A and B are fixed. If d can't change, **M cannot move along AP**...so is the distance of P from B always the same?	The figure was already drawn and is the following:  *Drawing 1a*	
03:28.00	Int	Is it a question?		
03:29.08	Stud	I don't know. Ehm... I can only think of this.		
03:34.17	Int	Ok.		
03:36.11	Stud	Even though…also all of the segment AB is always the same.		
03:39.27	Int	Show me how.		
03:43.24	Stud	Mm...what do you mean?		
03:47.05	Int	Don't worry. So...Make a prediction.		
03:50.27	Stud	Yes.		
03:52.10	Int	Do you think that point P can occupy other positions?		
04:01.11	Stud	I don't think so!		
04:03.00	Int	Why?		
04:04.03	Stud	Because if A and B are...have to always be here...and distance d is fixed...	She is pointing at A with a finger and at B with a pen.  She is pointing at A with a finger and at M with a pen.	
04:17.18	Stud	Point M has to always be this.	She is pointing at A with a finger and at M with a pen.	
04:22.03	Stud	Well, it could be **on the other side**, it	She is pointing at a point on the plane:	**GP_1_(1) (discursive – gestural):**

		could be **the mirror image.**	 She is pointing at B. The she is turning over the pen with a circular movement towards herself: 	**P symmetric point [with respect to AB]** Initially she points at a position where she expected to find the point P. Then she produces a **Window gesture:** she performs line symmetry of P as a rotation around AB. **Anticipatory Intuition**
04:26.21	Int	What?		
04:28.05	Stud	For example...I don't do it very well, but more or less...If I put P here.	She draws a second point P:  She draws AP	GP1 (gestural) **Window gesture** She draws the point P as the symmetric point with respect to AB.
04:44.26	Stud	And then there will be point M...here.	She draws the point M	
04:54.11	Stud	And here the distance d will always be the same.	She draws the segment MB:  *Drawing 1b*	
05:02.29	Stud	Otherwise...it could also be...		
05:15.23	Stud	It could also be **over here**! No, well, absolutely not! Mmm...Also because it needed to start form A, right? So it can't...does it have to start from A? Yes.	She points at B and moves along a path from B to a certain position (of M):	**Window gesture** (about the possible position of M), but it is quickly rejected.

		No, I think this is the only one.		
05:36.12	Stud	Because if **d has to always be that**. I think...		It seems that she interprets "the same length" as "the same position on the plane".
05:40.27	Int	Make a prediction: imagine...mmm...do you think that P can occupy other positions so that MB stays of the same length d?		
06:00.06	Stud	I think it is this one. This is the only solution.		
06:05.11	Int	Ok.		

This excerpt starts with the same prompt from the interviewer about the point P. Nevertheless, the two solvers' approaches are different. Margherita answers with several utterances (at times 02:39, 02:54, 03:01, 03:36) which do not contain any dynamic features. They only show some crystallized properties, communicated through static verbs (i.e. is distant, is constant) and adverbs (i.e. always). Verbs that communicate motion are used only for denying the possibility of movement. We can notice that also the gestures communicate the absence of motion: she keeps on talking without performing any gestures.

At time 03:52, when it seems that her solution process is finished, the interviewer tries to trigger another one. She asks if there exist other possible positions for P. The solver says that she does not think so and she explains why. Utterances at time 04:04 and 04:17 show the same usage of verbs and adverbs of the previous utterances. In this case, she produces gestures, but only for pointing at specific points of the configuration.

So, until now the whole configuration, as well as each part of it, is fixed for Margherita. Suddenly, at time 04:22 a new solution arises and the solver communicates a product of GP: P could be at a symmetric position with respect to AB. Looking at the gesture, we can see that Margherita first points at the new position of P and then she shows how she can reproduce the configuration starting from this position. The gesture provides a window onto her process: it seems that she is able to find a new configuration by imagining a rigid transformation of the whole drawing. At time 04:28 she starts drawing a new sketch: she traces the new P and, starting from it, she completes the drawing tracing out the other segments. Similarly to Agnese, the solver places a part of the configuration (point P) in a position such that the given constraints are maintained and then she reconstructs the whole configuration.

At time 05:15 she tries to find another solution and she places P at a particular position on the plane. She quickly rejects this possibility and at time 05:36 she explains that the position communicated in GP_1 is the only one, right because "*d* is fixed". Even in this sequence we can notice a strong static dimension.

Finally, at time 06:00 Margherita repeats that the position included in GP_1 is the only one possible. One could conclude that Margherita makes a mistake on the constraints and she changes the invariance of the length of MB into the fixedness of the segment. However, this does not seem to be the case, because otherwise she would not have been able to find the symmetric position of MB, which she did. More likely, she conceives the configuration as very static and is not able to communicate other possible positions for P (her attempt in this direction fails). Possibly, she feels the necessity to give the interviewer an answer because she pushed her in this direction. So, she chooses a symmetric position for P.

This excerpt provides an example of processes of prediction, undertaken during the resolution of another task, without an evident dynamic dimension. The utterances make use of static verbs and communicate some general properties of the figure; the gestures are static and rare. Even the drawing does not reveal any dynamic dimension. Margherita's drawings appear to be different from Agnese's: Margherita uses the same drawing to show the solution, on the contrary Agnese uses several drawings of the same geometrical problem. Nevertheless, the two solvers show the same interaction with drawings: they start from a particular position of one part of the configuration (respectively a point and a segment) and reconstruct the configuration starting from there.

In particular during the resolution of this task, it seems that the dynamic dimension plays a very important role. The first solver reaches a solution really because she is able to conceive MB as a turning segment and, consequently, to become aware of the effect of this movement on P. The second solver is fixed in her static view of the configuration and is not able to further explore the situation.

6.3.4 Concluding remarks on the role of motion

Data analyses revealed that the solvers of our sample show two different approaches to the tasks, which are connected to the presence or lack of a dynamic dimension. Motion can be or can be not integrated into the GP processes.

We have analyzed the dynamic dimension looking both at:

- the singular solvers' productions;
- the interaction between the solvers' productions.

The dynamic dimension seems to play a role in reaching a product of GP that can lead the solver to a complete solution to the problem, rather than in communicating only a coherent product of GP. Indeed, also a static approach can lead to a coherent product of GP, but it cannot be the most complete.

We notice that considering the possibility of motion becomes crucial when the solver wants or is asked to undertake prediction processes without a given or sketched drawing. It seems that the solvers' ability to imagine continuous transformations of the figural elements leads them to express a product of GP without using any drawings.

The dynamic component plays a role also when the process is carried out with the support of a sketched drawing. In the reported excerpts, we have seen that, even when the solvers have reached an effective configuration, the dynamic dimension made the difference between a process that actually leads to a complete solution or a process that leads only to a partial solution.

The lack of dynamism is reflected also in the use of drawings. Some solvers who show a static approach tend to perform several drawings of the same problem. This seems to impede them to reach a complete solution. Generally speaking, the lack of dynamism seems to restrict the possibilities that the solvers are willing to explore.

Moreover, we stress that dynamism plays a role *during* the process of prediction. Indeed, when the process is over the solvers can crystallize the predicted property into a statement.

For the sake of clarity, we stress that a dynamic approach can be more effective or fundamental in particular cases that depend also on the task. For example, to be completely solved, Task 2 requires more than other tasks considering dynamically the locus of M and, consequently, the locus of P. Otherwise, the solvers can reach only some part of the solution (for example the locus of M).

6.4 Gestures as windows onto the prediction processes

During the resolution of the given prediction open problems, we observed that the solvers spontaneously perform gestures of different types. The solvers seem to use gestures for two aims:

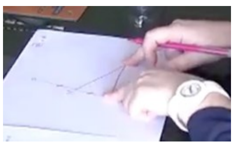
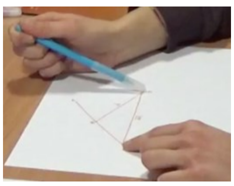
- as a complement of speech, in order to communicate a product of GP;
- as a tool that supports thinking, in order to reach a solution of the problem.

In this section we will describe: the kinds of gestures performed by solvers; the role of gestures within the prediction process; what we can infer about solvers' predictions looking at gestures.

6.4.1 Kinds of gestures performed by the solvers

Referring to the dimensions of gestures described in Chapter 3, in our data we found different numbers of occurrences of each of them. We infer the prevalent dimension looking at the corresponding utterances or considering the context. In this section, we include five tables that describe utterances and gestures. These tables are not intended to be exhaustive. Below we collected some typical gestures performed by the solvers during the resolution of each task. We use bold type in the presence of long sentences in order to stress exactly when the gesture is performed.

Among the most frequent gestures we identified are the *deictic* or *pointing gestures*, generally used by the solvers to locate objects and events. In our data, the solvers use gestures deictically for different purposes. The first aim is to focus on particular properties of the geometrical objects. Here are some examples.

Utterance	Gesture	Task	Comment
A and B are fixed.		2	She focuses on A and B and repeats the given theoretical element. The gesture emphasizes the fixedness of the points.
Because if ***A and B*** *are...****have to always be here****...*		2	She focuses on A and B and rephrases the given theoretical element. The gesture communicates the fixedness of the points.

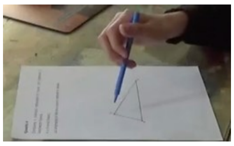
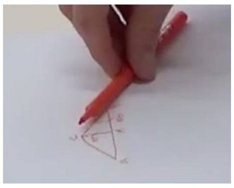
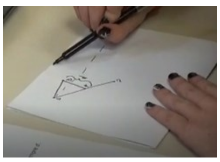
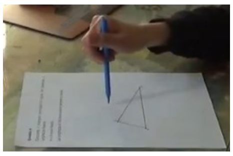
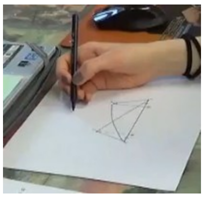
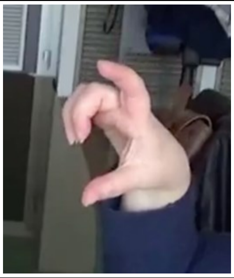
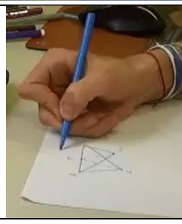
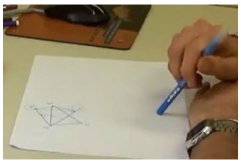
*Only that point... only that point...can...**C is a right angle**. Then the degrees change.*		4	He repeats the given theoretical element which refers to the angle at C.
I think...well it will be a bit longer.		5	He focuses on CB. We infer from the context that he compares the length of CB with the length of CM.
*On the contrary if the diagonal is...**like this**.*		6	She shows the orientation of a line. The gesture clarifies what she intends with "*diagonal*". It is not an instance of GP, because the interviewer explicitly asks to consider a line, so it is a given figural element.

Table 7 Instances of *deictic* gestures used to communicate properties and their corresponding utterances

These deictic gestures reveal the figural components of the geometrical objects in focus and the corresponding utterances explain their supposed properties or theoretical components. In our sample, these kinds of deictic gestures are not interpreted as instances of GP, because they are used only to repeat or stress an already known theoretical element.

Deictic gestures are also used to show objects or locations that are not present at that moment and that are predicted by the solver. Here some examples.

Utterance	Gesture	Task	Comment
But in this triangle...and indeed I imagine also with my hands I drew the triangle ...I mean side PB, but...		2	The utterance comes after the gesture. She uses a deictic gesture to complete the drawing of the triangle PMA.
[No utterance]		2	The gesture is performed without any utterances. We infer from the context that the solver is referring to a position

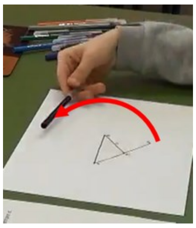
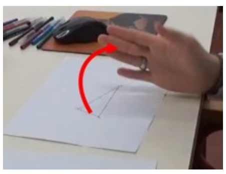
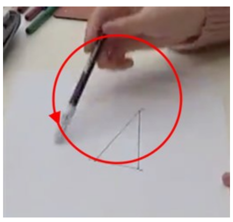
			of P. It is an instance of GP.
Yes, let's say another point M′, would that work?		2	The solver is referring to a possible position of the point M. It is an instance of GP.
Because if we move it ***here*** *it becomes an isosceles triangle. So...*		4	The solver performs the gesture right after she says "*here*". It is an instance of GP.
[No utterance]		4	The gesture is performed without any utterances. We infer from the context that the solver is referring to a position of C. It is an instance of GP.
And… CM.		5	The gesture mimics the segment CM. In this case the gesture reveals what happens during the resolution of a task without the use of drawings.
Well, obviously on the other side.		5	The gesture shows where the solver intends to place C. It is an instance of GP.
Ehm...also if I move it much closer always at this height...we could always have CM and CB be equidistant, so, yes I could do it, actually, also ***here****.*		5	The gesture is performed when the solver says "*here*", and it reveals where he intends to place C. It is an instance of GP.
[No utterance]		6	The solver uses the gesture to show a path.

That if I move it...that if I move P up Q stays, the distance QA stays QA or PA stays the same.		6	The solver uses the gesture to show a new position of P. It is an instance of GP.

Table 8 Instances of *deictic* gestures used to communicate a new figural element and their corresponding utterances

In these cases, the gestures are mostly instances of GP and, in particular, they are part of the communication of a product of GP. In some cases, the solvers do not show the necessity to add speech to their gestures. All deictic gestures performed by solvers are particularly helpful when they are performed without speech because they allow the researcher to infer the figural elements in focus.

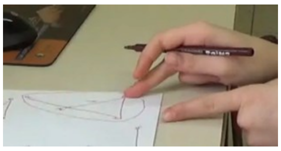
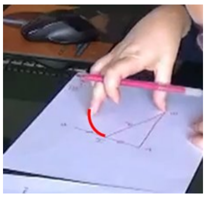
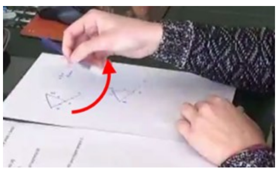
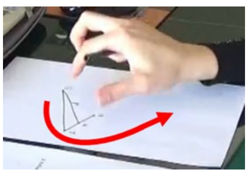
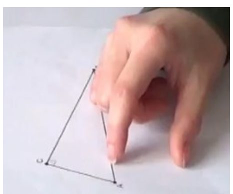
Iconicity is a very frequent dimension of gestures found in our data. In particular, *iconic-symbolic* gestures are used to describe loci or geometric transformations. We stress how these gestures also show a prominent dynamic dimension.

Utterance	Gesture	Task	Comment
*But **another circle will be created with point P**, because, that's it, it is not fixed, so...*		2	The gesture shows the predicted locus.
*I mean I take P and I put it at the same...let's say **like a mirror**… like this.*		2	The gesture shows that the solver intends to find a position of C by a symmetric transformation.
The one mirroring C where it is now.		4	The gesture shows that the solver imagines constructing a new configuration through a symmetric transformation. The gesture is performed before the utterance.
Like this.		4	The gesture shows the predicted circle.

...I only want to move C ***along, say, a line that is perpendicular to BA*** *and this way it maintains the same property.*		5	The gesture shows the locus of C.
Here, too, I think you need to see ***a diagonal*** *and...based on whether P moves up, Q moves down and vice versa.*		6	The solver uses the gesture to communicate the symmetric position of the new locus of Q.

Table 9 Instances of *iconic-symbolic* gestures

Iconic-physical gestures are mostly used during the process of GP. It seems that they allow the solver to reach a prediction. In our data, these gestures are used to mimic tools like the ruler and compass.

Utterance	Gesture	Task	Comment
Well, it could be projected again, I mean...to the other side of A, so I'll do it on this side, MB takes on the same distance only that it takes it from, well, here it would be its distance, ***it should be like this****.*		2	She mimics the use of a ruler for drawing a segment with the same length of MB.
*Wait, I...***can rotate d***.*		2	She uses her fingers as a compass.
Ok, basically d can move.		2	She uses the pen for fixing the distance and she constructs the locus of M by the radius rotation.
What am I thinking? The problem could be this: starting from…from this ***MB*** *and creating others within the plane, I mean…*		2	She fixes the length with two fingers and she mimics the use of a compass.
If I draw...the...point transformed from C with respect to AB [according to line symmetry], so I send the perpendicular from C to AB, ***that is I draw a segment that***		4	She mimics the use of a ruler that is placed at a specific point on AB, is perpendicular to AB, and replaces a specific length.

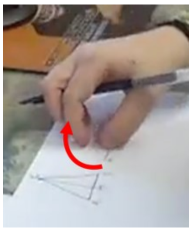
is always perpendicular to AB of the same length of the one that I drew before*. So, C can…can also stay on the opposite side.*			
So I was thinking: at this moment CM and AB are not congruent, but if: AB, midpoint.		5	She uses her fingers as a compass in order to construct the height with the same length of the base.
[No utterance]		6	We infer from the context that he uses his fingers as a ruler with a fixed length to carry over the distance AP and find the point Q.

Table 10 Instances of *iconic-physical* gestures

In our sample, the gestures that mimic the use of a compass have a dynamic dimension; on the contrary, the gestures that mimic the use of a rule have a more static dimension[7]. Moreover, we notice several gestures used for mimicking the use of a compass: they seem to reveal a different construction of a circle. Solvers who use the pen as a radius seem to see the circle constructed as an effect of the physical rotation of the radius. Instead, the solvers who use their fingers seem to construct the circle following the definition: the locus of points that are equidistant from a fixed one. So, in some sense, iconic-physical gestures do not only show us how a solver mimics the use of a tool, but they also allow us to infer how the solver was constructing the product of GP.

Finally, *metaphoric* gestures are the last type that we identified in our data. These gestures are used to construct several instances of the same configuration, to manipulate a geometric figure, to perform a geometric transformation. Also in this case, the dynamic dimension is dominant.

[7] This is a particular feature of the data we have collected, and it could be provoked by the kind of tasks that have been proposed to the solvers. Generally speaking, using a ruler presumes the act of tracing a line, thus a movement; in retrospect, a line can be conceived both a process and a product, that is as a trajectory (Bartolini Bussi & Mariotti, 2008, p. 747).

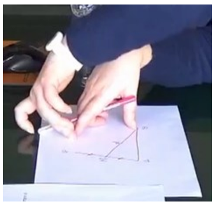
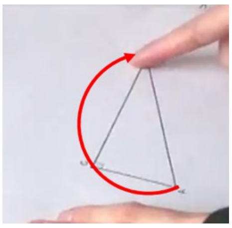
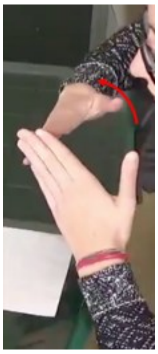
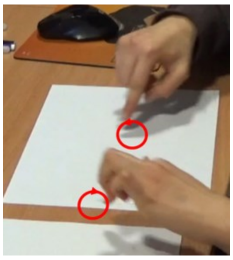
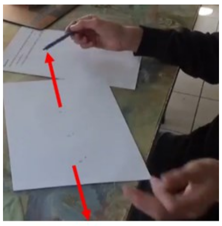
Utterance	Gesture	Task	Comment
*So I always have M defined and **the distance AM...MP**.*		2	The underlying metaphor is the equidistance.
Ok. As P varies...		2	The gesture reveals how the solver intends to move the point P. It is an instance of GP.
*That I can move C so that it makes...**a half circle**.*		4	The solver shows how she intends to dynamically construct several instances of the triangle moving C on a semi-circle.
*If CM is congruent to CB, I have a triangle BCM that is isosceles and, **just a bit big**, I have another triangle ABC that was my initial triangle, I mean. So…eh…*		5	She mimics the reconstruction of the configuration starting from an isosceles triangle and manipulating one of its sides in order to obtain the initial triangle.
So they make... a...a circle that... P does it in one direction and Q does it in the other.		6	The underlying metaphor is the symmetry, which is dynamically and synchronically performed.
If I move it here it moves on the other side.		6	The underlying metaphor is the symmetry, which is dynamically and synchronically performed.

Table 11 Instances of *metaphoric* gestures

6.4.2 The role of gestures within the process of GP

The solvers' use of gestures seems to be useful both for solvers and for researchers. From the solvers' point of view, the gestures:

- support the process of GP;
- allow them to carry out the process without adding other details to the drawing;
- help them to explain in a more effective form the products of GP.

From the researchers' point of view, the gestures:

- reveal the features of the figure that the solver is reasoning upon;
- reveal some details of the products of GP in a more accurate or complete way than the speech;
- anticipate an upcoming product of GP.

In the following sections, we provide examples from the transcription tables which show evidence of the listed claims about gestures.

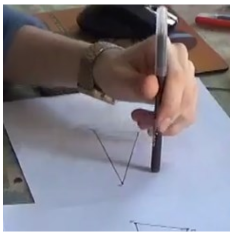
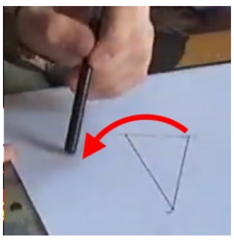
6.4.3 The use of gestures for communicating the products of GP

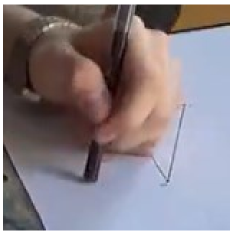
Looking jointly at gesture and speech we can discover the features of the products of GP. In many cases, the gestures communicate more effectively to the researcher what the solver has predicted. In particular, it can happen that:

- the utterances are quite vague or unclear and the gestures clarify the meaning;
- the utterances seem to communicate a coherent product of GP, but the gestures clarify that the product is incoherent and vice versa.

We find an example of the first case in Sergio's resolution of Task 4: *Sergio_G10_T4_P1_(00:27 – 01:50).*

Time	*Who*	*What is said*	*What is done*	*Comment*
00:27.06	Int	What can you say about the vertex with the right angle?		
00:46.18	Stud	That...since A and B are fixed and the length of AB has to always be the same, in order for the triangle to remain	Long silence. He is pointing at C. He is pointing at a new position for C:	**GP_1_(1) (discursive – gestural):** **C symmetric point with respect to [the axis of] AB**

		right **I cannot move the angle** **C**...I mean I can move it, but only putting it **at the opposite point with respect to AB**, moving P [C] to the other side and putting it in the same...at the same inclination with respect to A. Putting it **at the same inclination bit with respect to B**.	 He is pointing at the angle at B.	**Window gesture** Speech and gesture have mismatched features. For example, he uses "opposite point with respect to AB", but the gesture suggests that he does not intend a line symmetry with respect to AB.
01:14.13	Int	Make a prediction: imagine moving C, do you think it can occupy other positions so that this angle remains right?		
01:25.01	Stud	Yes		
01:25.22	Int	Which?		
01:27.21	Stud	Eh... that is had with... with respect to angle A, putting the same inclination with respect to angle A from angle B and putting the point there.	He is pointing at the angle at B. He is pointing at the new position of C: 	GP_1 We infer that he intends to perform a new construction of the triangle: the angle at A and at B are switched.
01:42.21	Int	Do you think that there are other positions?		
01:50.19	Stud	There are two other positions if we calculate that we can move C **on the opposite side** and therefore put it on the opposite side **with respect to A** and then do the same thing first **with**	He points at two positions of C: 	**GP_2_(1) (discursive – gestural):** **C symmetric point [with respect to AB]** **Window gestures** It is a new GP because the gesture is not the same one used for

		respect to B, that is with the same inclination.		communicating the GP_1. In the case of GP_1 it seems that he intends to reconstruct the figure; now he obtains C using a transformation within the plane. **Window gesture** He combines the GP_1 and the GP_2 into: **GP_3_(1) (gestural): C symmetric point [with respect to AB and the axis of AB]**

In the first utterance of the solver, at time 00:46, he talks about an opposite position of the point C "*with respect to AB*". So, looking only at the speech, one might think that he was talking about a line symmetry of C such that AB represents the axis of symmetry. However, the gesture clarifies that he is considering a different geometrical transformation. We cannot be sure that he is aware of the geometrical features of such a transformation, nevertheless we infer through the gesture that he is performing a line symmetry where the axis is a line perpendicular to AB. The utterances are useful to discover that he intends to construct the new configuration reproducing the two angles in opposite positions.

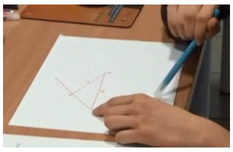
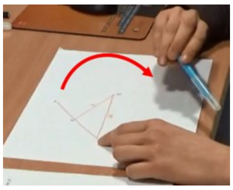
After the interviewer's hint, at time 01:50, the solver uses a similar word expression to communicate another product of GP. Now he talks about the opposite position of C with respect to a point. Following the utterance, one can expect that the solver refers to a point symmetry centered at A, which would be an incoherent product of GP. However, looking at the gesture we discover that he was referring to a position of C that can be found through symmetry about the line AB.

Finally, he says that he will apply the same procedure used before with respect to B in order to obtain the fourth position of C. The utterance alone is quite vague, but the gesture clarifies its meaning: the solver wants to use the same procedure used to find GP_1 to obtain the final predicted position.

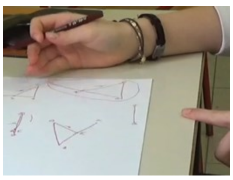
In all cases, the gestures play a fundamental role in clarifying the solver's products of GP and in making them more communicable. Also the solver seems quite satisfied with his answers, to the extent that he does not add any other explanatory utterances.

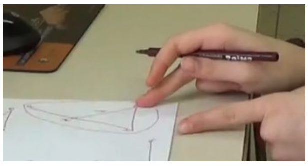
Moreover, from the researcher's point of view, the gestures coupled with speech are useful to make a distinction between two different GPs: GP_1 and GP_2. Looking at the gestures corresponding to the two products of prediction, we can identify two different approaches. In the first case, the solver intends to reconstruct the figure shifting the role of the angles at A and at B; in the latter, he seems to perform a geometrical transformation of C.

In some cases, different solvers use similar utterances to communicate products of GP, and these reveal their coherence with the given constraints only if we look at them jointly with gestures. Here we show two solvers' answers to a question about the possible position of P within the resolution of Task 2. The first is from Margherita's interview (also presented in Section 6.3.3).

Time	*Who*	*What is said*	*What is done*	*Comment*
04:22.03	Stud	Well, it could be **on the other side**, it could be **the mirror image.**	She is pointing at a point on the plane:  She is pointing at P and turns the pen, bringing it close to her with a circular movement (rotation around AB): 	**GP_1_(1) (discursive – gestural): P symmetric point [with respect to AB]** **Window gesture**

The second is from Carolina's interview: *Carolina_9G_T2_P1_(08:49).*

Time	*Who*	*What is said*	*What is done*	*Comment*
08:49.20	Stud	Well, it could be **projected** again …that is... **on the other side** of A, so I'll do it on this side, MB takes on the same distance only it takes it on from I	She points at a position for P: 	**GP_4_(1) (discursive – gestural): P symmetric point with respect to A**

		mean here would be its distance, it should be like this.	Then she points at the new position of MB: 	

As we can notice, the solvers use the same verbal expression to describe their products of GP: "*on the other side*". This description is quite vague; the researcher can interpret it in several ways (for example, a line symmetry, a point symmetry, a rotation). The gesture is what better clarifies the content of the two products and their coherence with the given constraints. In the first case, we can see that the "*other side*" refers to the segment AB and the solver intends to construct the point using a line symmetry. This is a coherent product of GP. In the second case, the same vague utterance leads us to infer that the solver intends to do the same thing. However, looking at the gesture, we can see that she intends to construct the symmetric point of P through a point symmetry centered at A; this is an incoherent product of GP. Considering the gestures in addition to the utterances, the researcher is able to make a distinction between the two products of GP (reported in the transcription table) and re-formulate them.

As a special occurrence of this case, we identify the *mismatch* phenomenon: the conveying of different (eventually opposite) information in speech and in gesture. This phenomenon is also evident during the GP process; we find that the meaning communicated through gesture becomes dominant within the process. Let us consider the following two examples of mismatch found in our data.

The first example has already been presented in the previous section (see Section 6.3.2) and it is a part of the Marta's resolution of Task 5: *Marta_MS_T5_P1_(00:23 – 02:44)*. The excerpt shows a gesture that communicates a coherent product of GP coupled with an utterance that contains an incoherent description. At time 02:33 she says that she intends to move C up and down on a path parallel to BA. This is an utterance that is incoherent with respect to the given constraints, but also with respect to the drawing: BA is actually a horizontal segment. While she is talking, she produces a gesture that, instead, is coherent: she moves one hand forward from her position. Looking at the drawing (*Drawing2b*) we can claim that the dominant meaning is that expressed through the gesture.

The second example has already been presented in another section (see Section 6.1.1) and it is part of Ilaria's resolution of Task 4: *Ilaria_G9_T4_P1.* At time 02:30, Ilaria talks about the locus of C as a line perpendicular to AB. However, looking at the gesture, we notice that she mimics twice a line parallel to AB. Moreover, as we highlighted in the transcription table, the gesture comes before the description of the locus. So, the solver performs a gesture accompanied later by a mismatched utterance. Also in this case the meaning communicated through the gesture becomes dominant within the prediction process, to the extent that she reproduced the parallel line on the drawing.

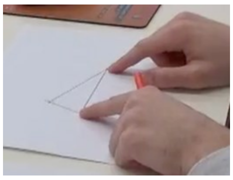
6.4.4 The use of gesture for undertaking a GP

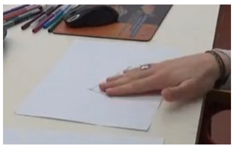
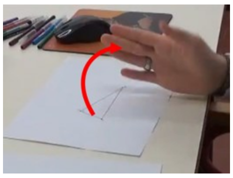
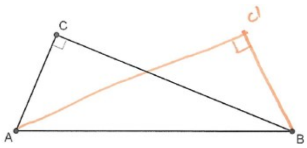
Another role of the gestures is to help the solver undertake a GP. This is quite evident in Marta's resolution of Task 2 presented in the previous section. We can notice how the dynamic gestures, performed in order to control the possible positions of MB, lead the solver to several GPs.

Moreover, gestures are so fundamental within the process that they can anticipate or replace the utterances. In the five tables below, we mark several gestures performed without any utterances. In many cases they communicate by themselves a product of GP, but generally the solver later communicates by speech the corresponding product of GP.

Here we provide examples of gestures that come before the corresponding utterance. The two examples are part of the resolution of Task 4. The fist is already reported in another section (see Section 6.1.4) and is from Laura's interview: *Laura_G10_T4_P1_(01:24-01:52).* At time 01:24 the solver seems to be a bit uncertain about the description of another position for C. We infer the uncertainty looking at the pause in the utterances, stressed by the ellipsis. During the pause Laura performs the deictic gesture that communicates a possible position of C. The gesture allows the researcher to infer that the question actually triggers a prediction process: only after she has performed the gesture, she seems to be able to describe how to move the point C in order to find another right angle. In this case, it seems that the process follows these phases: firstly, the solver finds a possible configuration that could be an answer to the problem; meanwhile, she checks the position using gestures; then, she communicates the product of GP through speech. So, when the sentence is pronounced, the process seems to be over and speech is useful only for communicating its product. Instead, the gesture constitutes a genuine step of the process.

We find another example in the first part of Giorgio's interview: *Giorgio_G13_T4_P1_(00:46 – 02:38).*

Time	*Who*	*What is said*	*What is done*	*Comment*
00:46.29	Int	Ok. What can you say about the vertex with the right angle?		
00:53.26	Stud	With the right angle. This [C] can be moved, right?	He points at C.	
00:58.02	Int	You tell me.		
00:59.04	Stud	I mean, I think this can, **it can be moved** because in any case this segment here, AB, that can't be moved, then moving the segment AC at most...the side CB or the side CA change.	He is pointing at A and B:  He points at C.	
01:14.04	Stud	So... yes, CB can be moved... it can make different configurations and **it can become a… an isosceles triangle** like in this case or…no, what is this? Yes, isosceles, right? Yes. **Equilateral** and so on.		He seems to have some difficulties in classifying the triangles.
01:31.08	Int	Ok. Make a prediction: do you think that point C can occupy other positions so that the angle stays right?		
01:45.10	Stud	M m m m.	He is looking at the given drawing.	
01:53.21	Stud	That C...that C remains always right, right?		
01:56.17	Int	The angle in C, yes, so that it stays right. I'm asking you if C can occupy other positions so that the angle in C remains right.		
02:04.16	Stud	I think so. But only one position.		
02:08.11	Int	Mm. Which?		

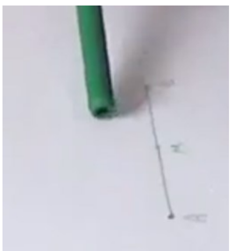
02:10.05	Stud		He rotates the palm of his hand:  	**Window gesture** The gesture comes right before the utterance.
02:11.00	Stud	The one that **mirrors** where C dis now.		**GP_1_(1) (gestural – discursive): C symmetric [with respect to the axis of AB]**
02:13.09	Int	Show me.		
02:14.13	Stud	In the sense that this AB does not...does not move, it does not get smaller, they are fixed…		
02:19.20	Int	Yes.		
02:21.04	Stud	...and this CB...	He draws a new segment CB. It seems to have the same length of AB. Then he connects the new point C with A.	
02:30.07	Stud	I take this here and I move it over here...C1, this will be right.	He points at C and then on C':  *Drawing 1*	
02:38.10	Int	Ok.		

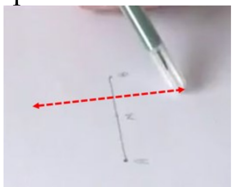
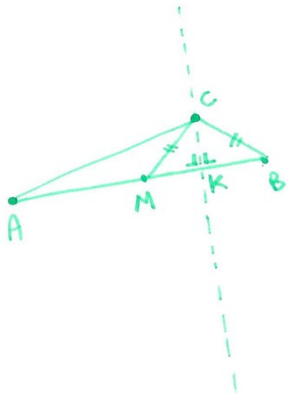
The excerpt begins right after the interviewer has asked the first question about C. At times 00:59 and 01:14, the solver repeats some given properties and talks about several static arrangements of the given configuration, like an isosceles triangle and an equilateral triangle. We cannot be sure that in these arrangements he maintains all the given constraints. Probably he forgets the constraint on the right angle. At time 01:31, the interviewer seems to perceive the lack of a constraint and, in order to be sure that this is not the case, she asks another question. Giorgio rephrases the constraint, making explicit which is the vertex with the right angle.

At time 02:04, we find a first instance of GP: the solver states that there exists a single position for C in order for the triangle to be right. Almost simultaneously with the interviewer's question at time 02:08, the solver starts performing a gesture in order to communicate what he has predicted. We infer that he finds the new position of C by a geometrical transformation of the initial point. Only after making the gesture, he communicates verbally the product of GP. Gestures and utterances are coherent with a final position of C that is symmetric with respect to the axis of AB. The drawing performed at time 02:30 confirms our inference.

As we observed, in this case the gesture comes before the utterance and, probably, it would replace the verbal expression if the interviewer had not asked for further explanations. The sequence of utterances and gestures makes the phases of the process more explicit. It seems that the questions activate the prediction process, a first instance of which we find at time 02:04. The gesture confirms to the solver his prediction and is used to communicate the product in a first formulation. Finally, the utterance is used to describe in greater detail the product of GP. At this point, looking at the transcription table, the drawing could be perceived almost as a redundant explanation.

The last example is from Fiorella's resolution of Task 5: *Fiorella_MD_T5_P1_(03:41 – 05:07)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:41.18	Stud	C…CB...		
03:45.00	Stud			**Window gesture**
03:47.15	Stud	Maybe if I try to draw C...	She is resolutely pointing at a position for C: 	**Window gesture**
03:55.14	Stud	...on the... **perpendicular line to MB at the midpoint**.	She is moving the pen on a trajectory perpendicular to AB	**GP_1_(0) (gestural – discursive): C on a line perpendicular to**

			and passing through the midpoint of MB: 	**MB, which passes through the midpoint [of MB]** **Window gesture**
04:06.28	Stud	Can I try?		
04:07.26	Int	Mm mm.		
04:10.04	Stud		She draws: a dotted line, the segments CA and CB, two right angles.	
04:35.06	Stud	I mean, if this is the initial triangle ACB.		
04:41.17	Int	Yes.		
04:44.17	Stud		She draws the segments CM and a hashmark on CM and CB. She moves the pen tracing a curvilinear trajectory.	
04:59.17	Int	What are you thinking?		
05:07.05	Stud	I mean that if **I move C along this perpendicular line through the midpoint K of the segment MB**...this way I construct an isosceles triangle and so CM and CB will always be of the same length.	She labels "K" the midpoint of MB and obtains the following drawing:  *Drawing 1b*	GP_1 (discursive) "I move": the dynamic component is spontaneously introduced, it is not pushed by the interviewer.

When the excerpt starts the solver has drawn the segment AB and its midpoint, and she is wondering if there exist some positions for C so that the median of the given triangle is congruent to the side CB.

At the beginning of the excerpt, the utterance reveals that she focuses on CB. Then, at time 03:45, she points at a particular position on the sheet of paper and she stays in that position for a while. The gesture without utterance reveals an instance of GP: the solver seems to be considering the pointed-to position as a possible solution.

At time 03:47, the solver decisively repeats the gesture and says that she could draw C in that position. Finally, at time 03:55, she performs a dynamic gesture and explains it: she intends to move C on a line that is perpendicular to AB and that passes through the midpoint of MB. The gesture coupled with the utterance communicates in a detailed form the product of solver's prediction. At the end of the excerpt, Fiorella shows her product of GP on the drawing.

In this case, the first gesture not only comes before the verbal explanation but, in some sense, it anticipates a well-structured product of GP: GP_1.

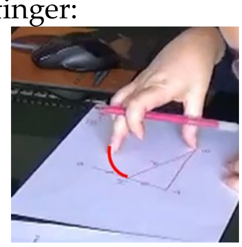
In the examples below we recognize a common approach:

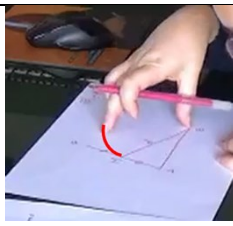
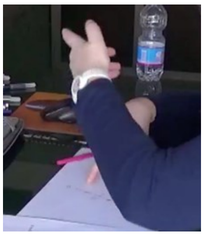
- the solvers communicate by a deictic or iconic gesture a possible configuration;
- they communicate the product of GP by speech;
- they reproduce it on the drawing.

In some sense, it seems that in these cases the gesture is more useful during the process of GP, while the speech is used only for communicating the products.

Gestures can anticipate an incoming product of GP, but they can also help the solver shape it during the process. More specifically, this is the case of iconic-physical gestures. One of the best examples is provided by an excerpt from Agnese's interview during the resolution of Task 2: *Agnese_MS_T2_P1_(03:55 – 05:14)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:55.17	Stud	Can P take on other positions? And I always want that A and B are these, that this is d.	She points at A and B with two fingers. She points at MB with one finger.	
04:03.09	Stud	I am looking for a theorem, something...		
04:09.25	Int	Yes, you want that A and B are fixed and that the length of MB remains d.		

04:15.01	Stud	Wait, I... **can rotate d**.	She puts two fingers of the same hand at M and B:  Starting from that position, she leaves the thumb fixed at B and rotates clockwise the other finger: 	**Window gesture** **Window gesture**
04:21.21	Stud	I mean, so **rotating d**, I mean...no instead...	She performs the same gesture, occupying a greater space.	**GP_3_(1) (gestural – discursive): MB rotates** **Window gesture**
04:29.00	Stud	Drawing, I mean do you want me to formalize it or...?		
04:32.03	Int	However it comes to you.		
04:33.13	Stud	Ok. So...		
04:34.03	Int	Say it however you, however you are thinking about it.		
04:36.10	Stud	Ok. Ehm...		
04:39.20	Stud	So **if** I want that ehm...		
04:43.21	Stud	I want to **keep the distance** MB **fixed**, that it stays d.		Theoretical element
04:47.02	Stud	**So** all the points on the...**circle**...		Theoretical element
04:56.00	Stud	...of ra...of **center B** and of **radius MB** they are all points that...	She points at B. She repeats the same circular movement.	Theoretical element
05:08.20	Stud	Ta ta ta.	She repeats the same circular motion.	**Window gesture**

				
05:10.22	Stud	They are all points with distance d, bon*.	* This has the meaning of "good, I'm done".	**GP_4_(0) (discursive): points of the circle C(B, MB) are distant d from B**
05:14.15	Stud	And so P has to be...ehm...has to have ...ehm...has to be such that MA is equal to MP, so I can choose a different P, **yes I can change P**.		She changes her mind about GP_2. The gesture is quite fuzzy. **GP_5_(0) (discursive): P in a position such that MA=MP**

The excerpt starts when the solver repeats the interviewer's question: she is wondering if the point P can occupy other positions. At the beginning, she tries to find a theoretical element (a theorem) to answer the question, but she does not succeed. At time 04:15 she resorts to figural elements. She notices that a rigid rotation of the segment MB allows her to maintain the given constraints. Even if she keeps talking about "*d*", we infer from the gesture that she is focusing on MB. She performs two connected gestures: a first deictic gesture, pointing at MB; a second, iconic-physical gesture, mimicking with two fingers of the same hand the use of a compass. She also repeats the gesture at time 04:21. We interpret the gesture as an instance of the process of GP that leads the solver to GP_3, but it also provides notice of a more detailed product of GP. Indeed, GP_3 is quite vague; on the contrary, GP_4 is very detailed.

There is a short pause; then at time 04:39, the prediction process is resumed when the student recalls the theoretical element "*MB must be constant*". According to the aim of this section, the most interesting sequence starts at time 04:47 and ends at time 05:08: we find an interplay between gestures and utterances, where gestures help the solver to shape a product of GP. Indeed, probably inspired by the first gesture (time 04:15), she introduces the theoretical element "circle"; she starts communicating through speech the center and the radius, but she hesitates and repeats the circular gesture; at time 05:08, she performs again the same gesture and we infer that she does so without speech because the utterance does not have really a meaningful content. Finally, at time 05:10, she is able to communicate a discursive product of GP without performing any gestures. In particular, a word in the utterance reveals that the process is over: she uses the word "*bon*" that we can translate as "*good*" or "*great*". This particular slang is used in the North of Italy to say that something works, is fine or is over.

It seems that, in this case, the first gesture announces a new process of GP. Since the solver probably has not produced all the details of the product of GP immediately, she uses the same gesture for finding and better communicating these details. We observe a strong interplay between gestures and speech within this process.

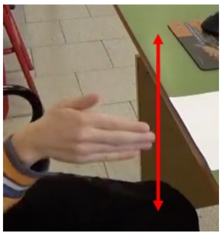
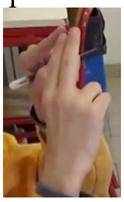
6.4.5 Gestures that replace or do not use the drawing

As elsewhere highlighted, we finally want to observe how a process of GP could be carried out when the solver is not allowed to produce or use any physical supports, like drawings or dynamic figures in a DGE. This is why, by design, the very beginning of Task 5 pushes the solver to solve the problem only imagining the configuration. In this case, much more than in others, the gestures can clarify what happens.

Generally speaking, for the solvers of our sample it was not so easy to make predictions in this particular situation. Many solvers seemed to hesitate for a long time, and in this case the interviewer allows them to draw something. However, when the solver starts to undertake a prediction process without the use of drawing, the gestures allow the researcher to identify the details of the product of GP.

An excerpt of Andrea's interview shows how the gestures help in inferring the products of GP: *Andrea_G9_T5_P1_(01:07 – 02:03)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:07.10	Int	Is it possible that CM has the same length of CB? This is the question.		
01:13.24	Stud	Mmm...I don't think so.		
01:17.19	Int	Why?		
01:18.22	Stud	Because I imagined the...**the segment CM as the height of the**	He moves his hand up and down:	**Window gesture**

		triangle I imagined it, to make life easier.		
01:29.26	Stud	In this sense: I imagined the triangle, I imagined **point C as the tip** that...that **the segment CM was the height of the triangle**. So I don’t think that it is equal or congruent. But the question is "can" it be congruent?	“triangle”:  “the tip”:  When he talks about the height, he moves one finger from the top: 	He imagines CM as the height of the triangle ABC. In this case CM cannot be congruent to CB. He seems to be aware that this is only one instance and the question asks for possible configurations where CM is congruent to CB. **Window gesture** **GP_1_(1) (discursive – gestural): the height CM of the triangle [ABC] is not congruent to the side CB**
01:47.24	Stud	In theory it can because if, I can change the sides, then **I could also make them the same**. My prediction is that if I changed the shape like we did before for the quadrilaterals I could have made them the same.		Theoretical answer: changing the initial data, the segments could be congruent.
02:00.13	Int	In what sense to they change shape?		
02:03.03	Stud	If I had...for example made longer...ah, no it’s true A and B are fixed. If I could **have**	He moves two connected fingers in front of him:	**Window gesture** **GP_2_(1) (discursive – gestural): moving**

		made C longer, that is make it **put it further up or down** I could have modified the shape, **I could have made them the same**.		**C [along the height of the triangle] CM becomes congruent to CB**

The excerpt starts with the interviewer's question. She needs to rephrase the usual first question because the solver asks for an explanation about the meaning of "*CM congruent to CB*". The solver answers that CM is not equal to CB. After the interviewer's hint, at time 01:18, he explains that CM is not congruent to CB because he imagines CM as the height of the triangle. The verb "*to imagine*" is actually used by the solver. The gesture makes explicit that he was reasoning about a triangle with a vertical height and, probably, with AB as a horizontal base.

At time 01:29 the gestures that accompanied the utterance clarify the figural components of the imagined figure: he uses two hands for mimicking the whole triangle; then, he leaves the right hand fixed and uses one finger for pointing at the supposed vertex C; he moves the finger up and down to show the height of the triangle. So, we infer that he is considering a triangle: the base is AB and one height is a vertical segment CM. Even if we cannot infer whether the triangle ABC has other properties, the theoretical and figural elements expressed by the solver are detailed enough to formulate the first product of GP:

GP_1: *the height CM of the triangle [ABC] is not congruent to the side CB*

At time 01:47 he seems to be aware that the answer is not general: he stresses that he can reconstruct the configuration with different lengths, and in such a way he could obtain two congruent segments. However, the answer is quite vague, and we do not recognize instances of GP. So, the interviewer asks the solver to clarify his last utterance. At time 02:03, the gesture shows how he intends to manipulate the figure in order to obtain two congruent segments: we infer that Andrea wants to move C on a vertical path. We notice that the corresponding product of GP is incoherent: if at the beginning CM and CB are not congruent, moving C in such a way they maintain this property.

In this excerpt, we can see the fundamental role of gesture in inferring the solver's products of GP. In particular, without any other support, gestures make the figural elements of the considered figure more communicable for the solver and understandable for the researcher.

Another example of this specific role of gesture is provided by Marta's interview: *Marta_MS_T5_P1_(00:23 – 02:44)*. In particular, we look at what happens at time 00:52 of the excerpt previously presented (see Section 6.3.2). Also in this case, the gesture reveals the figural elements of the figure the solver was reasoning upon.

At the end of this analysis of gestures within our data, we want to stress one last finding: in some cases the use of gestures can replace the use of drawing. In this sense, gestures diminish the usefulness of the act of drawing. This phenomenon is related to two of the characteristics of GP already discussed: the dynamic dimension and the degree of detail.

As shown in the excerpt reported in Section 6.3.3, during the resolution of Task 2, Marta carries on her prediction process and communicates the corresponding products without adding other graphical elements on the drawing. She talks about loci and movements of some figural elements but using only gestures for embodying them.

As shown in the excerpt reported in Section 6.2.1, during the resolution of Task 4, Filippo and Fiorella produce detailed products of GP. Moreover, they use the given drawing only for pointing at particular positions or for highlighting figural elements in an iconic or metaphoric way.

These examples share a common feature: the solvers use gestures upon an already sketched out or given drawing without adding anything to the sketch.

6.4.6 Concluding remarks on gestures

At the end of our analyses on gestures performed by the solvers of our sample, we can claim that the prediction process is accompanied by the production of several gestures that support the process of GP and clarify the content of solvers' productions.

In particular, during the process of prediction, solvers can perform several kinds of gestures, but not all the gestures are instances of GP. For example, *deictic gestures* that refer to an already drawn element are used mostly for focusing on particular figural elements or for stressing their corresponding theoretical elements. Instead, when the solver introduces a new figural element that is not drawn, we interpret the deictic gesture as an instance of GP: actually, a product of GP.

Iconic-symbolic, *iconic-physical* and *metaphoric* gestures are mostly used during the prediction process and present a prominent dynamic dimension.

We identify two major roles of the gestures:

- at the end of the process, the gestures clarify the details of a product of GP;
- during the prediction process, the gestures shape the products of GP.

In particular, our findings on gesture within the prediction process are listed below:

- gestures clarify the meaning of solvers utterances;
- gestures are useful to make a distinction between similar utterances;
- in the case of mismatch between gesture and utterance, the gesture reveals what the solver intends to communicate;
- gestures coupled with speech are useful to make a distinction between two processes of GP with two different products;
- gestures can anticipate or replace an utterance that contains an instance of GP;
- gestures can announce the beginning of a new process of GP;
- a gesture can help the solver in constructing a more accurate product of GP.

6.5 Immediacy

As highlighted in Chapter 5, the interviewer makes use of a list of questions to get the solver more involved in the prediction process. Usually, the first question is quite general, and then questions ask for particular configurations suggested by the interviewer. In this way, we can make inferences about the extent to which the prediction process is undertaken naturally by the solver.

Analyzing the transcription table, we can notice that sometimes the solvers answer the interviewer's questions very quickly and that the answers contain instances of GP that are more or less detailed and well-structured. In particular, we look at:

- how much time elapses between the question and the first evidence of GP;
- how many questions the interviewer has to ask before the communication of a product of GP.

When the solver answers right after the first question with an evidence of GP, we can say that the GP process was undertaken in an immediate way.

We refer to *immediacy* as the quality of a result or reaction that is provided without any delay. In our perspective, immediacy is not only a synonym of speed, but it is also the quality of a process of GP that is undertaken without a strong intervention of the interviewer.

So, immediacy can be a characteristic of GP processes and we consider it as an indication of the naturalness of the process. More specifically, our data reveals that:

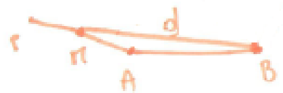
- a process of GP can be carried out in an immediate way, but not all the processes of GP are immediate;
- immediacy seems to be a feature of the GP processes that are undertaken by expert solvers.

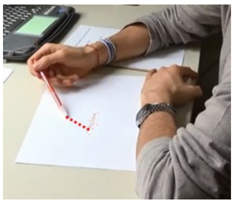
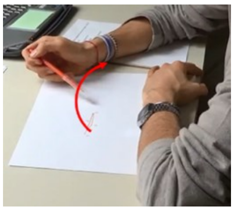
With the term *expert solver* we intend a solver who was exposed for a long time to the mathematical knowledge and, by virtue of this, is supposed to be expert.

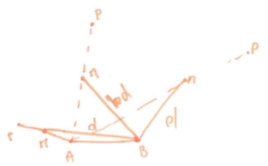
In the following section we provide four examples and one non-example of immediate processes of GP. The last example is provided for the sake of clarity, just to show when immediacy is lacking.

6.5.1 Immediacy: an example from the resolution of Task 2

The best example within the high school students' population in our sample is provided by the Emilio's interview. He is a 13th-degree student and was interviewed in the middle of his last year of high school. So, we can say that he had been exposed to mathematical knowledge for as much as possible within the school system. The excerpt is part of the resolution of Task 2 and it starts when the interviewer asks the first question: *Emilio_G13_T2_P1_(01:44 – 03:32)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:35.05	Int	The question is: what can you say about the point P?		
01:38.25	Stud	P??	The drawing was already sketched out:  *Drawing 1a*	
01:39.26	Int	Yes!		
01:41.03	Stud	Ah ok, then no.		
01:44.02	Elia	Um…so…A and B are fixed. The length of MB must always be d.		Theoretical elements
01:50.22	Elia	Eh…		

01:54.06	Elia	Well…**If we move the point M** to maintain the distance d, **a circle around the point B will be created**…	He points at M and moves the pencil on a segment:  Still pointing at M, he describes an arc: 	**GP_1_(0) (discursive – gestural): the locus of M is a circle centered at B** **Window gesture** Gestures and verbs express the construction process of a circle. They both have strong dynamic features: M is moving; the circle is being created; B is a fixed point compared to M which is moving around it.
02:07.10	Elia	But **another circle will be created with point P**, because, that's it, it is not fixed, so...	He points at P and describes a circumference: 	**GP_2_(1) (discursive – gestural): the locus of P is a circle** **Window gesture**
02:21.16	Stud	...it should **follow**...Ah no! That's not true! Because...		Dynamism
02:27.18	Stud	Point P **has to remain always equidistant** from A ...with respect to M, so if we...	He points at: P, A and M.	Theoretical element: equidistance
02:35.08	Stud	can I draw?		
02:35.23	Int	Of course!		
02:37.04	Stud	If we put, for example, a distance... we put here more or less...	He draws a new position of M within the same drawing: 	He starts drawing M and then he finds the corresponding position of P. It seems that the position of P depends on the position of M.

02:44.29	Stud	So d is fixed...		
02:50.16	Stud	M...		
02:52.16	Stud	Instead here this distance here...	He connects A and with the new point M.	
03:00.03	Stud	...has to be...		
03:05.09	Stud	...here.	He draws the point P	
03:09.29	Stud	So? So? So? So? So?		
03:12.24	Stud	So...then if...		
03:18.04	Stud	...instead we accept a situation like this... M… d…and this has to always be...	He draws a new position of M.	This construction confirm that he finds P depending on M.
03:30.04	Stud	Ok, yes!	 *Drawing 1b*	From the left to the right we can see the three points he has constructed.
03:32.27	Stud	**It comes to create another circle** also P ...maybe...		GP2 (discursive) Dynamism. He talks about a circle, which is not drawn.

The interviewer asks for whatever the solver is able to or wants to observe about P. Emilio starts recalling some of the given theoretical elements: A and B are fixed points; MB always has the same length. At time 01:54 the solver starts communicating the first product of GP: GP_1. It takes approximately 20 seconds from the question (time 01:35) to the discursive instance of GP (01:54) and 10 seconds from the utterance which refers to the given theoretical elements. Moreover, GP_1 is detailed and dynamically communicated. Looking at the utterances and at the gesture, we infer that the solver dynamically constructs a circle centered at B and with radius *d*.

Right after GP_1 and smoothly, the solver communicates another product of GP: GP_2. This is less detailed then GP_1: we cannot infer the center and the radius of the circle. However, it is very immediate: it seems to be constructed almost at the same time as the first product of GP. Moreover, the two products of GP share the same genesis: they are communicated starting from the recalled theoretical elements at time 01:44.

As we stressed, GP_2 is not very detailed and Emilio also seems to be aware of this. Indeed, at time 02:21, he uses the modal verb "*should*" which stresses that he is not so sure about his prediction. Probably in order to clarify and to provide evidence of GP_2, at time 02:27 he tries to introduce a theoretical component (the equidistance) and at time 02:35 he restores to the figural components. Starting from this moment, he performs several sketches of the points M and P within the same drawing. The technique is always the same: he points at a specific position of M; he connects the new M with A; he finds the corresponding position of P. Looking at the geometrical construction, we can make two inferences. First, the drawing reveals that the solver conceives P and M as connected points; in particular, P depends on M. Then, the construction seems to mirror the theoretical control over the figure: probably, the solver is able to control the locus of M, and GP_1 is actually detailed, but he is not so confident about the locus of P; for this reason, he first constructs the positions that he can easily control and then the corresponding positions of P.

At time 03:30, the solver has obtained *Drawing 1b* and he seems to be quite satisfied. Moreover, at time 03:32, he repeats more assertively that the locus of P is a circle. The uncertainty still remains at the end of the utterance, probably because he was not able to provide a better description of the circle. However, we notice that, while he is looking at the drawing, he keeps talking about a circle even though it is not drawn. So, the GP_2 is quite evident for the solver to the extent that he talks about a circle that is not present on the sheet of paper.

The excerpt provides two examples of coherent products of GP characterized by immediacy. We recognize the immediate quality of the process looking at the time, but also at the number of hints of the interviewer. Indeed, except for the first question, she does not need to intervene during the process to help or push the solver towards a prediction process. In this case, it seems that the products of GP give the solver a first idea of the solution, which has to be followed by a more analytical one.

6.5.2 Immediacy: an example from the resolution of Task 4

The best example within the part of the sample composed by undergraduate and graduate students is provided by Filippo's interview. He is a mathematician and, when the interview was videotaped, he was soon to defend his Ph.D. thesis in Algebraic Geometry. So, we can undoubtedly say that he is an expert solver. The excerpt reports on the whole first part of the resolution of Task 4 and has already been presented in Section 6.2.1 (*Filippo_PhD_T4_P1).*

First of all, we can notice that the whole first part of the interview lasts only one minute: the solver goes straight to the solution. Let's see in detail.

At time 00:21, the solver rephrases the task:

Filippo: So, I fixed two points, I construct the right triangle on them above which the right angle is opposite.

We notice that he recalls some of the given theoretical elements, but he does not mention that ABC is a triangle. Actually, it may not be necessarily useful for reaching a solution.

At time 00:21 the interviewer asks about the vertex with the right angle. Right after, the solver considers the question and repeats it as if he is talking with himself. Only after 3 seconds he answers:

Filippo: So, I can say that it is forced, though it doesn't want to, to vary on a circle. With diameter AB.

The answer is immediate and contains a product of GP that is: detailed, complete and completely verbally communicated.

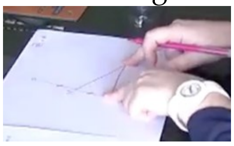
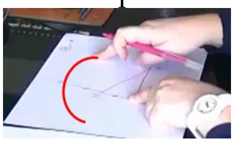
Moreover, at time 00.47 the solver shows the center of the circle. This is the only information that was not explicitly communicated before but that could be easily inferred through the description of GP_1.

Comparing Filippo's utterances with Emilio's, we can observe that Filippo speaks in a more assertive form. This reveals to what extent the solver is sure of his product of GP.

This excerpt and, in general, all of Filippo's interviews show immediate processes that leave the researcher out from what happens during the time frame that goes from the question to the answer. Indeed, the process is undertaken so quickly that the solver is not able or does not need to use gestures, utterances or drawings to communicate its phases.

6.5.3 Predictions without any explicit hints

In some cases, the process of GP is undertaken spontaneously, without any question from the interviewer. Here we show two examples. The first example is from the initial part of Agnese's interview during the resolution of Task 2: *Agnese_MS_T2_P1_(01:54 – 02:08)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:54.09	Stud	A and B are fixed, and the length of MB has to always be d.		
02:01.23	Stud	A and B are fixed.	She uses two fingers for pointing at A and B on the drawing: 	
02:08.01	Stud	Ok. As P **varies**…	She points at A with a finger and at P with another. Then, she starts moving the finger that is pointed at P back and forth on a curvilinear path: 	**GP_1_(2) (gestural): P is moving on an arc of a circle** **Window gesture**

At the beginning of the interview, Agnese performs the drawing and the interviewer intervenes only for giving reading out the task and for supporting the solver during the drawing, for example bringing another sheet of paper. The excerpt starts when the solver is recalling the given constraints. At time 02:08 we find a gestural instance of GP naturally performed by the solver. The reported product is strongly inferred, but it reveals that the solver is wondering about possible positions for the point P. It is important to stress that the interviewer does not ask anything. So, we guess that the prediction process, that shows a strong dynamic dimension, is undertaken spontaneously by the solver.

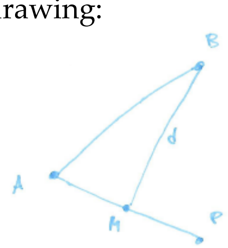
Another example is provided by one excerpt from Ilaria's interview during the resolution of Task 4, which was already presented in Section 6.1.1 (*Ilaria_G9_T4_P1*).

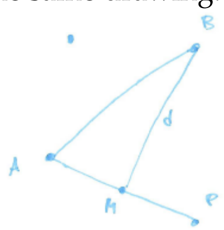
The excerpt from the whole first part of the interview shows that the first question about C is introduced only at time 01:22. Nevertheless, even before this moment, we find some instances and a product of GP. At time 00:23, right after Ilaria has finished reading the list of constraints, we recognize in her utterance an instance of GP: she verbally communicates that the points cannot be moved. At time 00:41 we find a product of GP. Although it is incoherent, probably because of an imprecise interpretation of the constraints, it is communicated without any hint from the interviewer. At time 01:02, we find a second discursive instance of GP which stresses that each part of the configuration is fixed.

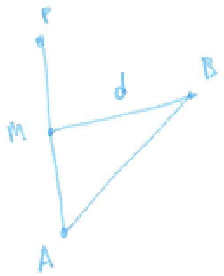
We are not discussing here the coherence of the products of GP communicated by these two solvers, but the spontaneous way in which they undertake the prediction process. In both cases, the exploration of other arrangements of the configuration starts naturally, without any explicit prompts. This is noteworthy because it reveals that in certain cases and for certain solvers the prediction process could be undertaken in such a spontaneous way that it can take place even when it is not explicitly requested.

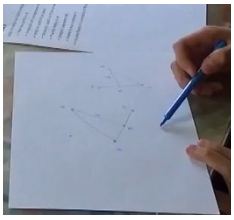
6.5.4 Non-example of immediacy

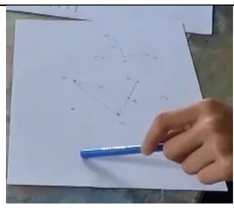
For sake of clarity, we want to provide an example of a quite long process of prediction that is neither spontaneous nor immediate. The excerpt is part of the interview with a 10th-grade student: *Carlo_G10_T2_P1_(01:22 – 04:52)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:22.26	Int	What can you say about the point P?		
01:26.17	Stud	Mmm, point P?	The solver has already performed a drawing:  *Drawing 1a*	
01:28.05	Int	Mm mm.		
01:33.17	Stud	Maybe...I don't know if I need this, ehm...		
01:35.23	Int	Mm.		
01:38.06	Stud	This was used to find the center of AP and to connect it with M and make the right triangle.	He points at M.	Figural elements. He refers only to the figural elements that are present in the drawing. He focuses on the triangle AMB, which is considered a right triangle.

				The segment MP seems not to be relevant.
01:45.05	Int	Mm mm.		
01:47.19	Stud	Eh...at least it is right, **I got a right one.**	He is laughing.	He guessed that the *Drawing 1a* is a particular arrangement of the given configuration.
01:51.14	Stud	and...it is any extension of AM.		The triangle and the segment MP seem again to be disconnected.
01:57.25	Stud	Ah no! it is the extension of AM and it is...AP is twice AM, so...		He changes his mind and recognizes that AP is twice AM.
02:06.02	Stud	It is the extension of AM of the same length of AM, so AM is congruent to MP.		He seems to be aware that AM is equal to MP just now. It is a discovery for the solver.
02:16.10	Stud	Ok.		
02:17.05	Int	Ok. Make a prediction: do you think that point P can occupy other positions?		
02:24.02	Stud	Uhm!	He looks at the step-by-step construction.	
02:32.22	Stud	Yes, any because "choose a point P on the plane".		Theoretical element
02:36.18	Int	Make a prediction…		
02:38.01	Stud	Let's say...		
02:38.16	Int	Mm mm.		
02:39.03	Stud	Let's put P here...I'm coming.	He adds a point on the same drawing:  *Drawing 1b*	

02:44.17	Stud	I'll try to quickly do it over.	He starts sketching a new drawing.	
02:45.29	Int	Yes.		
02:48.13	Stud	P. So it says...		
02:51.11	Stud	If I choose a point P on the plane and I connect it with A.		
02:59.17	Stud	Midpoint M.	 *Drawing 2*	
03:09.19	Int	Mm mm.		
03:12.01	Stud	I always get a right one.	He looks at the last drawing.	Figural elements. The figural elements seem to drive his exploration. He does not seem to control theoretically the configuration.
03:16.27	Stud	In any case yes, it could occupy any region of the plane, because it says so.	He moves the finger on the whole sheet of paper.	Recalled theoretical element.
03:21.11	Int	Ok. make a prediction: do you think that P can occupy other positions in the... other positions so that MB remains of length d?		
03:33.11	Stud	Yes.		
03:34.06	Int	Which?		
03:37.19	Stud	Do I have to make it so that MD corresponds to d?	He points at M, then at B and then on d.	He does not seem to make a distinction between MB and d.
03:41.17	Int	Mm mm.		
03:42.05	Stud	In any case, it's always like this, because... d is its length, of MB and they are connected,		He does not seem to distinguish between MB and its length.

		therefore...it's like, it's saying the same thing with different words I think.		The length and the segment are not distinct.
03:57.01	Int	Ok, concentrate on the first drawing, for example, that you made.	She refers to *Drawing 1a*.	
03:59.02	Stud	Mm mm.		
04:02.14	Int	In this configuration it is possible... mmm... let's say, suppose that you have measured MB. You found that this distance is 4 centimeters for example.		
04:12.29	Stud	Yes.		
04:13.22	Int	Is it possible that P occupies other positions so that d remains of length 4?		
04:21.04	Stud	Eh, no! Because **say that we had put P** here, at this point...	He traces a point on the sheet of paper and labels it P: 	
04:30.02	Stud	...and we connected ...do you say "we connected"?	He moves the pen from A to the new point P. He hesitates on the tense of the verb.	
04:34.00	Stud	And **BM would have become longer**.	He moves the pen back and forth from A to the new point P.	
04:37.10	Int	Mm mm.		
04:38.21	Stud	And so...the lengths would, would change.		
04:42.29	Stud	So only...one other way in which **B is**	He points at a new position of P:	**GP_1_(0) (discursive – gestural): P at a**

		congruent to d I think it is the **opposite**.		**symmetric position with respect to A** **Window gesture**
04:52.06	Stud	That P would be on this side.		

The excerpt starts with the question about P. Looking at what happens until time 02:16, we notice that the question does not activate a process of GP. The solver simply describes the configuration sketched in *Drawing 1a*. We notice that the characteristics of the configuration are driven by the drawing or in general by the figural components. Indeed, the solver speaks about P as a point used to find M, which is described as the "*center*" of the segment AP; he focuses on the shape that is made up of the segments and he describes it as a right triangle. We stress that the triangle is right not as a logical consequence of the particular step-by-step construction but only because it seems so in the solver's drawing. Moreover, at the beginning the solver describes AP as any extension of AM; then, he recognizes that AP is twice AM and therefore that AM is congruent to MP. The tone of the last utterance (time 02:06) reveals that the solver perceives the property as a discovery as if he did not know before. In fact, we know that it is only an immediate consequence of the theoretical element "*M is the midpoint*".

So, we notice that the most natural reaction of the solver to the question is to describe the configuration obtained following the step-by-step construction. The description is carried out without any dynamic dimension and we cannot recognize any product of GP.

At time 02:17 the interviewer introduces the second question, asking explicitly for a prediction on the possible positions of P. The solver's answer recalls a theoretical element given at the beginning: P is a point on the plane. So, he does not seem to be aware of the theoretical constraints that the configuration must maintain. The theoretical element leads the solver to consider another position for P, which is pointed to, at time 02:39. The solver does not seem able to add other information about this position. So, he restores to a figural approach: at time 02:44, he starts making a new drawing. At time 03:13, in order to describe the configuration, the solver again refers to a right triangle. The figural elements seem to drive the exploration of the problem. Moreover, at time 03:16, he tries again to introduce a theoretical element that justifies his answer.

Within this sequence we do not find any instances of GP.

At time 03:21, in order to trigger a GP process and to recall the given constraints, the interviewer asks the third question about P. The solver says that it is possible to find other positions of P, but still the solver does not succeed to describe them. The utterance at time 03:42 reveals that the solver does not distinguish between the segment MB and its length d. Moreover, probably "*d*" is only a label for him and not the indication of a fixed length.

Then, the interviewer makes the last attempt to trigger a GP process. She asks the solver to focus on the first drawing and to suppose that MB is 4 centimeters long. At time 04:13, she asks for possible positions of P that maintain this length. Since the interviewer makes explicit one of the crucial constraints, the solver says that if he chose another position for P, the lengths would change. In order to provide this answer, the solver focuses on a particular P that he points at time 04:21.

Finally, at time 04:42 we find the first product of GP verbally and gesturally communicated: GP_1. We infer that Carlo wants to construct the point P as the symmetric point with respect to A. This is a detailed but incoherent product of GP. Indeed, placing P in that position the configuration does not maintain the given constraint on the length of MB.

Summarizing, the first question is asked at time 01:22 and we find the first product of GP at time 04:42. So, more than 3 minutes pass until we recognize an evidence of GP. Moreover, the first question induces the solver only to describe the configuration that he sketched before; one process of GP is undertaken only after three explicitly interventions of the interviewer, the last of which is strongly oriented making a prediction. The process that leads the solver to communicate GP_1 is not immediate at all. We would add that GP_1 is incoherent and is the only product of GP communicated by the solver during the first part of the interview.

6.5.5 Concluding remarks about immediacy

Our tasks are designed in order to also catch spontaneous features of prediction processes. The first question is generally open also for this reason. The data reveals that immediacy can actually be a characteristic of GP processes.

In some cases, the process is so natural for the solver that it is triggered without any support from the interviewer. In other cases, the first question is sufficient to induce a prediction process.

In particular, immediacy seems to be mostly a quality of the process of GP undertaken by a solver who was exposed for a long time to the mathematical knowledge and, by virtue of this, is supposed to be expert. In a certain sense, communicating a coherent and detailed product of GP in an immediate way seems to be a *habit of mind* of the expert solvers. We infer that, if the solver can produce an evidence of GP in an immediate way, the product is quite evident for the solver and, therefore, well communicable. For expert solvers, the product of GP seems to constitute a first idea of the solution that needs to be followed by a more analytical discussion in order to provide evidence of it.

However, not all the prediction processes are immediate. There are several cases of GP processes undertaken only after several explicit and strong interviewer's requests for a prediction.

Obviously, between the non-immediate and the immediate processes there is a range of occurrences in which the process can be gently supported by the interviewer and the solvers construct their products of GP step-by-step.

6.6 Intuition

Immediacy is directly connected with a theoretical construct that is often involved in the mathematical activity of *intuition*.

A priori we have conjectured that intuition can have a role within the prediction process. Our data reveals that in some cases the GP process is actually accompanied by evidences of intuition. As previously highlighted (see Chapter 3), we are particularly interested in *anticipatory intuition,* a kind of intuition specifically involved in problem-solving activities.

We have found several evidences that an anticipatory intuition occurred right before or combined with the communication of a product of GP. We can recognize evidences of anticipatory intuition when the solver communicates a new piece of mathematical knowledge about the solution:

- suddenly;
- without an explicit or recognizable connection with the processes previously undertaken;
- after a long silence.

An evidence of anticipatory intuition can have one or more of the features listed above. The context, the tone of the voice, the way the solver performs the utterances are additional markers of anticipatory intuition.

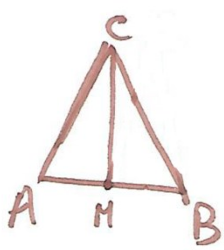
Emilio's interview previously analyzed offers a good example of a product of GP reached through an anticipatory intuition. Indeed, GP_1 and GP_2 are communicated suddenly after a descriptive sentence (time 01:44). The subsequent meaningless utterance (time 01:50) denotes a pause while we can infer that he is thinking about the configuration. Looking at the subsequent utterance, and more specifically at its tone and incipit ("*well*"), we identify evidences of anticipatory intuition. In particular, we recognize that GP_1 is communicated suddenly; it reveals a global view of the solution; it is quite evident for the solver to the extent that it is not questioned later; it seems not to be directly connected with the content of the previous utterances. So, it has several features that also characterize an intuition.

Even if Filippo's process is immediate, we cannot recognize instances of intuition within the excerpt. The process is carried out so quickly that we did not find any elements that suggest that there are instances of intuition. The solution is more likely from an analytical process than from a global view or a guess at the solution.

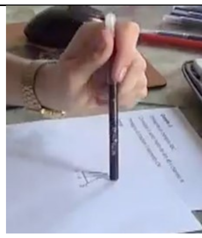
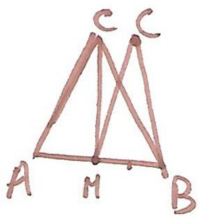
In the next sections we provide examples of GP processes during which we can recognize instances of anticipatory intuition.

6.6.1 Anticipatory intuition within the resolution of Task 5: an example.

The first example is a part of Sergio's interview and shows how an anticipatory intuition can lead to a detailed product of GP: *Sergio_G10_T5_P1_(01:44 – 03:35)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:44.21	Int	So make a drawing of what you imagined.		
01:48.25	Stud		He draws: the triangle ABC; the midpoint of AB; the segment CM.  *Drawing 1a*	
02:13.20	Stud	Ok.		
02:14.10	Int	Ok. Is it possible that CM is congruent to CB?		

02:19.18	Stud	With...A and B that are fixed points no.		
02:25.14	Stud	No, no in any case, no it is impossible.		GP_1 (discursive)
02:27.09	Int	Why?		
02:28.11	Stud	Because the side CM, that represents the height, is in any case... since it is perpendicular to the sides AB that are fixed points, it is shorter than the side CB since it is slanted instead with respect to the side AB.	He points at CM and AB. He is points at CB while he is saying: "slanted"	CM is considered a height of the triangle. This recalls the properties of the height.
02:45.15	Int	Ok. Do you think you could move C so that CM is congruent to CB?		
02:51.18	Stud	No.		GP_1
02:52.13	Int	Why?		
02:54.15	Stud	Because **moving point C, the length of CM would increase**, but consequently also the length of CB would increase and since they are not congruent in this case any position I could choose ...		
03:09.00	Stud		Suddenly, after the last sentence, he stops talking.	
03:17.12	Stud	No! No, no, there would be one point.	After 8 seconds of silence, he claims: "*No!*"	He suddenly seems to discover a new configuration: **Anticipatory Intuition**
03:21.03	Stud	I should....**I should move C perpendicularly to the midpoint of segment MB** and therefore these two,	He points at a specific position:	**GP_2_(0) (discursive – gestural): C on a perpendicular line through the midpoint of MB**

		these two sides would be identical.	 He points at a position between M and B.	
03:31.26	Int	Show me how.		
03:35.15	Stud	If here...if I moved **C to the midpoint of MB**, so here, this side MC would become like this and M...MB...CB would become like this. So consequently they would be...identical.	First, he marks the midpoint of MB; then he traces another point placed above it; finally, he draws the sides.  *Drawing 1b*	

For the sake of completeness, we briefly report what happens before the excerpt. The solver was asked to solve the task without using any supports; he communicates the first product of GP, highlighting that CM cannot be congruent to CB; he explains that he thinks so because CM is a height of the triangle. The excerpt starts when the solver is asked to draw what he has imagined. Looking at *Drawing 1a*, we can see that actually CM is drawn as a height of the triangle and CM is not congruent to CB in this particular arrangement.

Now the solver has the support of a drawing, so the context is different, and the interviewer can ask again if it is possible for CM to be congruent to CB. He repeats that it is impossible, and after the interviewer's prompt he explains why: he thinks so because CM is the height of the triangle. The theoretical element "height" also activates the properties or the definition of the height that the solver uses to justify its first prediction. The solver uses a very assertive tone in which we do not find any uncertainty.

At time (02:45) the interviewer changes the question a little and asks if he thinks that C could be moved in order to have CM congruent to CB. He repeats the same answer and starts explaining why. He supposes to place C in another position and starts describing what he will obtain, but suddenly he stops. After a few seconds of silence, suddenly and with an exciting tone, he claims that there exists a position. The pause, the tone and the meaning of the sentence lead us to recognize an anticipatory intuition and, at time 03:21, a new product of GP: GP_2. This is communicated through gesture and discourse and is detailed: the solver clearly described the new position of C as located on a perpendicular line passing through the midpoint of MB. After the interviewer's prompt, the solver draws the new configuration. The details contained in the utterance and the clarity of the drawing reveal that for the solver this solution is quite evident.

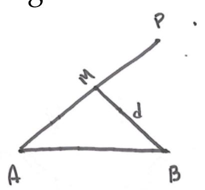
In this excerpt, we notice how an anticipatory intuition can lead to a coherent product of GP that is communicated in a detailed way and that contains a good number of theoretical elements introduced by the solver for the first time, like the perpendicularity and the midpoint. Until time 03:13, the solver seems quite convinced about his first prediction but suddenly he changes his mind. Looking at what happens before, we cannot be sure about what triggers the GP process that leads to GP_2. A plausible explanation is the sudden intervention of an intuition with an anticipatory role. Moreover, what happens at time 03:17 and 03:21 has the features of an anticipatory intuition.

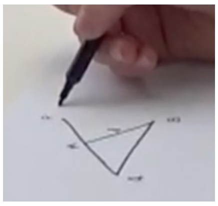
- Suddenness: all at once, the solver stops talking and silently keeps looking at the drawing.
- Evidence: the solver confidently points at the new position of C and draws the new configuration.
- The new insight into the solution comes after other attempts.
- We cannot find an explicit connection with what happens before. In fact, GP_2 seems to be in sharp contrast with the fixedness of C and of the whole configuration, previously supposed by the solver.

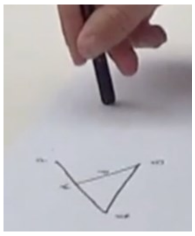
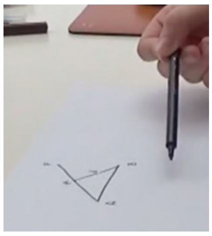
6.6.2 Anticipatory intuition within the resolution of Task 2: an example.

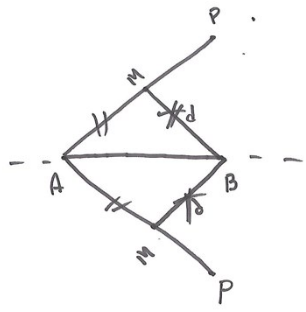
As elsewhere highlighted, the resolution of Task 2 can be quite long and require the generation of several products of GP in order for a complete and detailed solution to be reached. So, this task, more than others, gives us the opportunity to observe several instances of anticipatory intuition within the interviews. In this section, we provide two examples of GP processes accompanied by an evidence of anticipatory intuition: the first one leads to a coherent product of GP and the second one to an incoherent product.

The first example is from Giacomo's interview: *Giacomo_G9_T2_P2_(01:38 – 05:55)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:38.09	Int	Are you with me, great. So the question is: what can you say about point P?		
01:47.04	Stud	That it is external to the figure ABM.	He moves the pen over the figure:  *Drawing 1a*	Figural elements
01:55.00	Stud	It is half of AM.		Incorrect wording: the length of a point. We infer that he refers to MP.
01:57.15	Stud	No, of...it is equal to AM.		
02:02.20	Stud	and it is half of AP.	After the utterance he remains silent.	
02:27.07	Int	Ok, can I ask you something else?		
02:28.11	Stud	Yes.		
02:29.05	Int	Ok. Make a prediction: do you think that point P can occupy other positions?		
02:37.23	Stud	Well, yes, it can, it could occupy **infinitely many positions** on the plane.		**GP_1_(2) (discursive): countless position of P** He does not seem to have a precise idea of the positions of P.
02:44.29	Int	Which?		
02:48.13	Stud	I mean considering that it is there still on the figure or that before making the whole figure, having	He points at another point on the plane:	GP_1 shows a lack of the given constraints.

		only side A and B and then be able to construct [by pointing a compass] the segment P?		
03:02.28	Int	Do you think that the point P can occupy other positions so that MB remains of length d?		
03:11.00	Stud		He looks in silence at the drawing.	
03:21.15	Int	When you told me "infinitely many positions"...		
03:23.13	Stud	Yes.		
03:23.24	Int	...what were you thinking about?		
03:24.28	Stud	I was thinking about when I did not yet have the figure...	He points at AB	GP_1 refers to the construction of the configuration, rather than to the maintaining of the given properties.
03:27.07	Int	Mm mm.		
03:27.27	Stud	...AM...ABM with the...with d like MB and so if I had only the point...		
03:35.05	Int	Mm mm.		
03:35.23	Stud	...I could have positioned it **anywhere**.		Theoretical element
03:37.19	Int	Of course. Now imagine that MB has to remain of length d.		
03:43.07	Stud	Ok.		
03:44.07	Int	Make a prediction: do you think that point P can occupy other positions?		
03:51.25	Stud		He points at a position with certainty:	**Window gesture** We infer that he is considering a new configuration

			 He stays in silence at that position for a while.	starting from this position of P.
04:00.24	Stud	No, **it cannot occupy other positions**.		**GP_2_(1) (discursive): P is fixed**
04:05.19	Stud	Because **it has to always remain fixed in its points** so that MB always has distance d. I mean always has length d.		
04:22.24	Int	Why?		
04:35.17	Stud	Or, **if I mirrored the figure across AB...MB would always be the same**. But it would be mirrored on the other side.		**Window gesture Anticipatory Intuition** **GP_3_(0) (gestural – discursive): P at a symmetric position with respect to AB**
04:45.21	Int	Show me how.		
04:47.18	Stud	So I should start…AB...		
04:51.22	Stud	...a point P...more or less...	He traces out a new point P	
04:53.28	Int	Of course.		
04:57.12	Stud		He traces out AP, the new M and MB	
05:10.15	Stud	**If I mirrored it**...in this point P I would have...I mean MB would have the same distance that it has on...P...on the other side. I mean if I drew this thing here.	He traces out two dotted lines at the extremity of AB.	Theoretical element inferred by the drawing: axis of symmetry.

05:23.28	Stud	The two figures would be mirrored: this and this distance here are congruent, like this one and this one.	He makes two marks on the segments:  *Drawing 1b*	Several theoretical elements are inerrable from the drawing.
05:31.27	Int	Ok. Do you think that there are other positions for point P? that leave MB of length d.		
05:53.10	Stud	No, there are no more.		
05:55.24	Int	Ok.		

The excerpt starts with the interviewer's question about P. From time 01:47 till time 02:02, the solver describes various figural components of the figure as if he were describing the drawing. He focuses not only on P, but also on MP. Looking at the utterances, we guess that he is considering the configuration as composed of a triangle ("*the figure*") and a point that is connected with the figure through a segment.

The sequence ends at time 02:02 with a long silence (25 seconds). So, the interviewer guesses that the solver will not add any more information and decides to ask the second question. At time 02:37, Giacomo says that P can occupy infinitely many positions within the plane, and we find his first product of GP: GP_1. This is communicated only in a discursive way and is quite fuzzy: the solver does not perform any gesture or drawing that could clarify where he intends to place P. The answer seems to be quite vague.

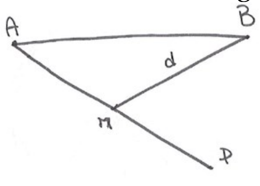
The utterance at time 02:48 reveals that, while he was talking about the positions of P, he did not consider all the constraints. So, at time 03:02, the interviewer recalls in the question the constraint over the length of MB but Giacomo does not answer. At time 03:24, we find evidence of our inference: the GP_1 concerns a possible configuration that does not maintain all the given constraints, and Giacomo refers to a completely new configuration. At time 03:37, the interviewer rephrases the question and the solver replies with a new product of GP: GP_2. If MB must always have the same length, P must be fixed.

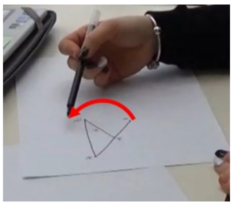
When the solver is asked for a further explanation, he remains silent for a while (13 seconds) and all at once he communicates a coherent product of GP that is completely disconnected with the previous one: GP_3. The utterance reveals that Giacomo intends to place P at a symmetric position with respect to AB. He does not explicitly mention the line symmetry, but he performs a gesture that shows the new position of P. The utterance combined with the gesture shows that GP_3 is quite evident for the solver. In this sequence, we find an evidence of anticipatory intuition: the answer comes out suddenly, without an explicit hint for a prediction and after some solver's attempt to find a solution, and it is in sharp contrast with the previously supposed fixedness of P.

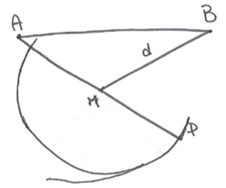
At time 05:23, the solver performs a drawing of the new configuration. Since after this moment he does not communicate other products of GP even if the interviewer asks for other predictions, we infer that he does not undertake other GP processes. This could happen because of a lack of dynamism within the Giacomo's exploration or, most likely, because the solver adds another constraint to the configuration: the length of AM must be fixed. This is evident looking at the *Drawing 1b*: the solver marks the two segments labeled AM.

In this excerpt, we see that the request for a further explanation triggers an anticipatory intuition that leads to a product of GP. Even if GP_3 is quite rigid, it is coherent and detailed.

Nevertheless, an anticipatory intuition does not necessarily lead to a coherent product of GP. The following excerpt provides an example of this particular case. The excerpt is from Isabella's interview: *Isabella_G13_T2_P1_(01:37 – 04:08)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:37.29	Int	What can you say about the point P?		
01:42.29	Stud	Eh I can say that it is a random point on the plane.	The drawing which is already sketched out is the following:  *Drawing 1a*	She recalls theoretical elements
01:48.11	Stud	In this case it is also the endpoint of a segment.		Theoretical elements
01:52.01	Stud	Eee...I don't know what else I could say.		

02:01.08	Stud	Enough. I mean I don't know what else to say.		
02:05.05	Int	Ok. Make a prediction: do you think that point P can occupy other positions?		
02:13.26	Stud	Do you mean on the plane? Or?		
02:15.11	Int	Yes.		
02:17.22	Stud	Yes, I think it could.		
02:20.23	Int	Which?		
02:23.28	Stud	You mean in the sense that I could move it? Or that it could be another **endpoint for segments**?		
02:33.14	Int	So, imagine... The question is: do you think that point P can occupy other positions on the plane so that MB remains of length d?		
02:52.06	Stud	Mmm.	She points the pen at P and moves to the top along a trajectory perpendicular to AB.	**Window gesture**
03:04.02	Stud	So, no wait, eh!		
03:08.27	Stud	Yes. Yes.	Suddenly, she answers.	**Anticipatory Intuition**
03:09.21	Int	Which?		
03:12.16	Stud	Eh...if for example we placed the **midpoint as the center of a circle** making **P rotate**, that is making **MP become the radius of the circle**, then the distance, I mean **the length MB would not change**.		**GP_1_(0) (discursive – gestural): P on a circle centered at M and with radius MP** **Window gesture** MP and the triangle are independent figural elements.

03:34.13	Int	Show me what you imagine.		
03:36.24	Stud	I think!		
03:39.03	Stud	I mean considering that. Should I draw it?		
03:41.03	Int	Yes, yes.		
03:42.17	Stud	So considering that **M is the central point**– I am losing my voice – of the circle and making the circle, then moving P to one side or the other, well OK I made a very bad drawing but anyway.	She starts drawing the predicted circle.	
04:02.14	Stud	**M would not change its position, but P would** and so.		
04:08.21	Stud	The length does not change, I mean I think, **it would move around the circle**.	She draws a semicircle:  *Drawing 1b*	

The excerpt starts with the first interviewer's question about P, right after the solver has finished sketching *Drawing 1a*. From time 01:42 till time 02:01, the solver talks about the figural elements of the figure, as she is describing the drawing: P is a point on the plane; P is the endpoint of a segment.

She does not add any other information, so the interviewer decides to explicitly ask for a prediction. Isabella seems to be confused about the question and asks for a further explanation. The interviewer decides to ask the most explicit question about P. It seems that this last hint triggers a process of GP. Indeed, at time 02:52 we find a window gesture that suggests that a process of GP is undertaken: the solver seems to evaluate a possible position for P, without adding any utterances. Suddenly, at time 03:04, the solver claims that there exist other positions for P. Here we recognize an evidence that an anticipatory intuition occurred, because of the tone of the voice and the suddenness of the answer. This intuition leads the solver to a product of GP: GP_1. Isabella says that, in order to maintain the given constraints, P can rotate around a circle centered at M and with radius MP. The utterance reveals that she is quite sure about her prediction: the sentences are quite assertive. We notice that, although GP_1 is detailed, it is incoherent. The solver seems to consider the configuration as composed of a fixed triangle and an independent segment. Moreover, the configuration described in GP_1 has new constraints and loses some of the given ones. This reveals a lack of theoretical control of the solver over the figure. The new constraint is: "*the segment MB is fixed*" or "*M is fixed within the plane*", which leads to the fixedness of the whole triangle. The lost constraint is: "*M is the midpoint of AP*". At time 03:42, Isabella repeats that M is the center of the circle and then she performs a drawing that shows what she has predicted.

Intuition seems to have a role within the prediction process, but it is not necessarily connected with the coherence of the product of GP. The excerpt shows an evidence of anticipatory intuition that leads the solver to an incoherent but convincing product of GP. The suddenness and the tone of her voice suggest that it is actually an intuition that supports the communication of GP_1. The solver seems quite convinced of her prediction: after the communication of GP_1, she repeats the features of the configuration and is able to sketch it out in a drawing. Moreover, GP_1 is the only product of GP that the solver communicates.

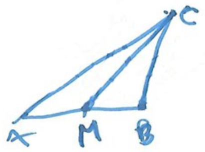
So, it seems that it is not the immediacy or the evidence of the intuition that leads the solver to a coherent product of GP. In this excerpt, we see that the intuition only leads to a solution, not necessarily to the most correct one. Nevertheless, it leads to something that is very evident for the solver. Probably the dominance of figural elements and the lack of theoretical control over the figure could more suitably explain why the solver did not reach a coherent product of GP.

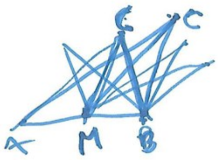
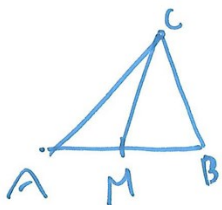
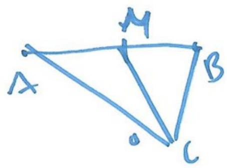
6.6.3 A product of GP is not ever accompanied by an intuition

Although in our data we find several evidences that an anticipatory intuition occurred, it is not ever a feature of the GP process. Not every process of GP is accompanied by or coupled with an anticipatory intuition; there are also interviews during which a product of GP is communicated without any observable or recognizable evidences of intuition.

One example is provided by the previously reported interview of Carlo. The process that leads to a product of GP proceeds very slowly and leads to an incoherent product of GP. Looking back at the previous sections, we find several examples of products of GP (coherent and incoherent) that are communicated without a recognizable intervention of an anticipatory intuition.

Here we show one last example of a product of GP produced without an anticipatory intuition, but that is strongly supported by figural elements. The excerpt is from Silvia's interview: *Silvia_G10_T5_P1_(05:21 – 08:15).*

Time	*Who*	*What is said*	*What is done*	*Comment*
05:21.18	Stud		She starts a new drawing:  *Drawing 3a*	
05:29.20	Stud	No, it could not be the same.		**GP_1_(2) (discursive): CM cannot be equal to CB**
05:32.01	Int	Why?		
05:39.14	Stud	I mean graphically you can see it, but...ehm...		Figural elements
05:51.18	Stud	Because...		
06:04.25	Stud	Because...		
06:11.15	Stud	I mean I don't know how to explain it.		
06:24.29	Stud	Because in any case the midpoint is not the same as the point... point B is not the same as M, they are different points in the space of the plane.	She points at M and B	Figural elements
06:38.09	Stud	And always connecting them to P, **the length is always different**, it's impossible for them to be the same.	She points at CM e CB	GP_1 Figural elements
06:45.25	Int	Concentrate on point C.		
06:47.19	Stud	Mm mm.		
06:48.11	Int	Do you think it could occupy other positions so that CM is congruent to CB?		
06:53.22	Stud		She is looking at *Drawing 3a* and	

			starts drawing others points C.	
07:06.08	Stud	Aaah! Ah well then, in that case yes.	The claim coincides with the drawing of C as the vertex of an isosceles triangle:  *Drawing 3b*	She considers the possibility CM=CB only because she sees it on the drawing. It is not a GP. It is only an interpretation of the drawing. Bottom-up process.
07:11.16	Stud	When... yes, when, when you connect M with C and B with C they would form, in this way, **they would form an isosceles angle and so the two sides would be the same**, so there are... basically like before, so there is more that one possibility that they be ...		The figural elements dominate the process.
07:35.20	Int	Which?		
07:37.19	Stud	This one, so...	She starts a new drawing:  *Drawing 4*	
07:51.17	Stud	One.		
08:00.12	Stud	This is practically the same.	She starts a new drawing:  *Drawing 5*	**GP_2_(0) (gestural): C is the vertex of an isosceles triangle and is located at a symmetric position with respect to AB**
08:06.28	Stud	Two. **But that is always the same one**. One, only one.		The position of C and its symmetric position are the same.
08:15.03	Stud	**I think only one**.		Last solution: only one position.

The excerpt starts right after the interviewer has proposed the question again. Until now the solver has erroneously reasoned upon the possibility that CM is congruent to AB and she did not communicate any product of GP. The solver starts performing a new drawing and at time 05:29 she communicates a first incoherent product of GP: GP_1. When the interviewer asks for an explanation of her answer, she resorts to figural elements. Indeed, from time 05:39 to time 06:24, the solver talks about the particular drawing she has accomplished and admits that she is not able to provide a better explanation. At time 06:53, after an interviewer's hint, she starts drawing again. She draws several instances of a triangle on the same drawing. At a certain point, she draws something that seems like an isosceles triangle and stops drawing. The utterance at time 07:06 coincides with this moment. The discovery that there exists a position for C so that CM appears equal to CB triggers to an utterance that contains more theoretical elements. Indeed, at time 07:11, Silvia describes the configuration as an isosceles triangle that, for this reason, has two equal sides.

The answer arises quite suddenly during the process of drawing. It seems a bottom-up process that was triggered by seeing a particular configuration of the figure, rather than an anticipatory intuition. It is neither a product of GP because it seems more like an observation of some figural elements discovered almost by accident.

At time 07:37, Silvia draws the new configuration and at time 08:00 another one. Here we find the second product of GP: GP_2. This is communicated by performing *Drawing 5*. The solver is not aware that this is a possible solution to the problem, and she explains that she thinks of the two configurations as the same situation.

Even if the insight on the solution comes suddenly, we do not see it as an evidence of anticipatory intuition: the solution is discovered trying to construct on the drawing several cases.

6.6.4 Concluding remarks about *anticipatory intuition*

As conjectured a priori, our data reveals that the theoretical construct of intuition interacts with the prediction processes. In particular, we find several products of GP that are preceded or accompanied by an evidence of *anticipatory intuition*.

After some efforts to reach a solution, it could happen that suddenly, the solver claims that there exists a solution. At a moment like this, we identify an evidence that an anticipatory intuition occurred. The subsequent product of GP communicated shares some features with an intuition:

- suddenness;
- evidence;
- it reveals new insight on the solution;
- it is quite disconnected from the previous attempts.

Moreover, the intervention of an anticipatory intuition can explain what happens when, after a period of silence, the solver provides a solution to the problem.

An anticipatory intuition can precede or accompany the communication of a product of GP. When a GP is coupled with an anticipatory intuition, its products seem quite evident for the solvers.

Moreover, anticipatory intuition is not connected with the coherence of the product of GP. Indeed, it could lead to both coherent or incoherent products of GP.

Nevertheless, not every process of GP is accompanied by or coupled with an anticipatory intuition; there are examples of products of GP that are communicated without any observable or recognizable evidences of anticipatory intuition.

Supported by our data, we can claim that a product of GP is not an intuition, for the reasons listed below.

- Intuitions are characterized by self-evidence and vivid. If a product of GP is an intuition, we expect all of the products of GP to be detailed. Instead, we have both detailed and fuzzy products of GP.
- Intuitions are characterized by intrinsic certainty. Instead, we find several *hedges* within the solver utterances that reveal the solver's feeling of uncertainty.
- Intuitions are immediate; instead, we also find a product of GP at the end of a long process.
- Intuitions are resistant to change; instead, often during the resolution process, the solvers change their idea about a product of GP.

In summary, the products of GP can share features of anticipatory intuitions, but not all of them are actually intuitions. We prefer to claim that anticipatory intuition can support the process of GP, leading to products that share some features of the intuitions.

7. The role of theoretical elements: findings from the funnels

A key issue in the resolution of the open problems in geometry is how the solvers use their mathematical knowledge and how they are able to impose upon the figure and to maintain particular geometrical properties. Indeed, as discussed in Section 5.2.2, all the solvers of our sample are supposed to have been exposed to the same geometrical knowledge, even if in different levels of depth.

The mathematical theory (the TEG in our case) and in particular the theoretical elements recalled by the solvers are strongly involved in the process of GP. We will describe such an interaction in the next sections.

7.1 Findings from the funnels

The funnels are the best tools for analyzing the intervention of theoretical elements within the GP process and, in particular, how they affect the characteristics of the products of GP. More specifically, the funnels allow us to see at a glance:

- the theoretical elements recalled by the solver;
- how many theoretical elements are recognizable;
- whether the solver introduces new theoretical elements;
- what kinds of these theoretical elements (properties, theorems, …) there are;
- whether the theoretical elements are coherent or incoherent with the given geometrical problem.

Moreover, the funnels highlight:

- the figural elements focused on by the solver;
- eventually, the intervention of anticipatory intuition previously recognized.

From the transcriptions we can capture the kind of vocabulary used by the solver to recall the theoretical elements highlighted in the funnel. Indeed, for example, the wording could be more or less detailed or mathematically correct.

Looking at both these tools and comparing funnels, we can say that the theoretical components play a crucial role within the GP processes. In solvers' utterances or gestures we identify theoretical elements because of the nature of geometrical objects. This is an expected finding.

What is new is the role of the theoretical elements introduced for the first time by the solver. In Chapter 5, we highlighted the difference between a new theoretical element and an already known one. Referring to this dichotomy, in our sample we recognized four prototypical situations, which lead to different products of GP:

a) there are no new theoretical elements;
b) there are new theoretical elements, but they are incoherent with the given constraints;
c) there are few new theoretical elements and they are both coherent and incoherent;
d) there are new theoretical elements and they are proper and consistent with the given constraints.

It seems that the presence and the quality of the new theoretical elements are connected to the quality of the products of GP. In particular, the funnels reveal that:

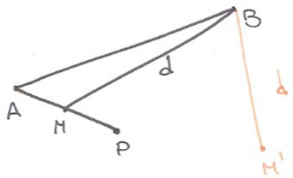
- if the solver does not introduce new theoretical elements, the products are very simple, almost obvious, and they do not give any new information on the problem;
- when there are new theoretical elements and they are incoherent, the products of GP are strongly incoherent, very connected to figural elements, and they seem to move the solver away from a correct solution;
- when the solvers communicate a product of GP passing through (eventually incoherent) theoretical elements introduced by themselves, they seem to be convinced of their findings even if the perceptual feedback appears to be inconsistent in the eyes of the interviewer;
- when there are new coherent theoretical elements, the products of GP are coherent and, depending on the number of new theoretical elements, the products of GP can be more or less detailed and well-described.

In the next sections, we will discuss the four cases and how they lead us to the findings listed above.

7.1.1 Case (a): there are no new theoretical elements

Solvers can communicate a product of GP recalling only already known theoretical elements. It is not so common, but we find some instances of this case. An example is contained in one of Tiziana’s funnels, which refer to the resolution of Task 2.

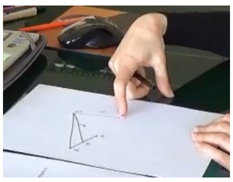
Configuration 5 (*Drawing 1b*):

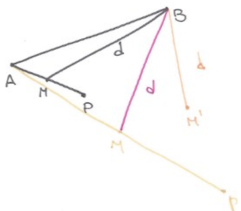


	Theoretical elements	4	Figural elements
(03:55)			M′
			AM′
	Congruent segments/Midpoint		AM′=M′P[′]
			Another P
	Constant length		BM[′′]=d
			AM[′′]
	Congruent segments/Midpoint		AM[′′]=MP[′′]
			B
	Constant length		BM[′′]=d
			B

GP_4_(2) (gestural – discursive): P on the plane constructed using AM=MP

Configuration 6 (Gesture and *Drawing 1c*):





Funnel 4 contains only theoretical elements that are already given in the step-by-step construction. Moreover, all the theoretical elements are correct and are used properly, indeed during the coding we did not put an "X" in the middle column. On the other hand, she introduces several new figural elements that are not actually present in the drawing.

Looking at the product of GP, we can see that it is correct, not very detailed, and almost obvious from the beginning. Indeed, the solver refers to a particular position of P that she finds only by applying one of the given constraints.

We see a connection between the lack of new theoretical elements and the quality of the product of GP: all the theoretical elements are correct, but none of them are new, so the solver tends to communicate only a trivial product of GP. The number of new figural elements suggests that the solver resorts to figural considerations in order to carry on the GP process. We will see that this also happens when the new theoretical elements are incoherent.

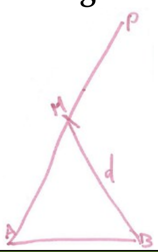
7.1.2 Case (b): incoherent new theoretical elements

When there are theoretical elements introduced by the solver, but they are inconsistent with the given constraints, the products of GP are strongly inconsistent as well. Moreover, the solvers seem to resort to figural elements in order to reach a product of GP.

Here we show two examples of funnels that show a product of GP that is quite disconnected with the given constraints.

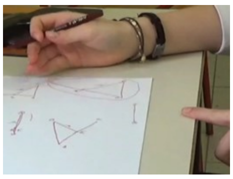
The first example is from the excerpt *Carolina_G9_T2_P1_(01:27 – 09:52)* zooming in on what happens from time 08:33 to time 09:02 of the interview.

Configuration 3 (*Drawing 3*):



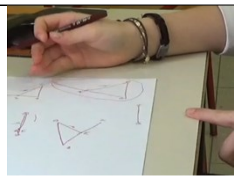
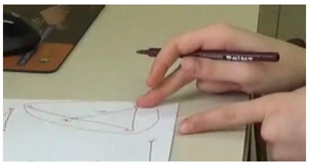
	Theoretical elements	**4**		**Figural elements**
(08:33)				B
	Projection			[P]
	[Point symmetry]	X		P, A
	Fixed/Constant length	X		MB
	[Point symmetry]	X	X	[MB]
	GP4_(0) (discursive – gestural): P symmetric with respect to A			

Configuration 5 (Gesture):



Carolina introduces new elements: the projection of a point and the point symmetry. Unfortunately, these are not consistent with the constraint on the length of MB. As we can see in *Configuration 5*, she obtains a position of P that does not maintain the length of MB. However, she is not aware of this inconsistency and she talks about the new position of P as a possible correct solution. The corresponding excerpt is the following.

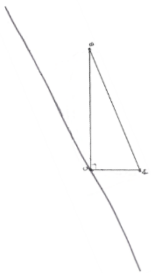
Time	*Who*	*What is said*	*What is done*	*Comment*
08:49.20	Stud	Well, it could be **projected** again …that is... **on the other side** of A, so	She points at a position for P:	**GP_4_(1) (discursive – gestural):**

		I'll do it on this side, MB takes on the same distance only it takes it on from I mean here would be its distance, it should be like this.	 Then she points at the new position of MB: 	**P symmetric point with respect to A**
09:02.18	Stud	It is probably turned upside down.	She mimics a rotation: 	GP_4 **Window gesture**

The excerpt shows that at time 09:02 she performs another instance of GP_4 using a gesture and a describing utterance. Moreover, she refers to GP_4 again at the end of the first part of the interview. These observations reveal that the solver is quite convinced of her product of GP to the extent that she does not communicate any other products of GP.

The second example belongs to the excerpt *Ilaria_G9_T4_P1* looking at what happens from time 03:06 to time 03:37 of the interview. The transcription table is reported within Section 6.1.1.

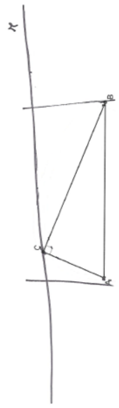
Configuration 4 (Gesture):



	Theoretical elements	**3**		**Figural elements**
(03:06)	Point on a [parallel] line	X		C
				A[on the line], B[on the line]
	Perpendicular	X		[Straight line through C]
	Point on a [parallel] line	X		C
	Point on a [parallel] line	X		C
	Projections on a line	X		A[on the line], B[on the line]
	Right angle	X		C

GP_3_(0) (discursive): C on a line parallel to AB and between two projections of A and B on the line

Configuration 5 (*Drawing 1b*):



Ilaria introduces a long sequence of theoretical elements: she sees C as a point on a line; the line is parallel to AB; on the line she fixes two points that are obtained by an orthogonal projection of the points A and B. Moving C along the line and between these two new points, Ilaria predicts that the angle ACB is still a right angle. We notice that all the theoretical elements (new and already known) are incoherent.

The product of GP is incoherent: putting C on one of the projection points is sufficient to observe that the angle is no longer right. Nevertheless, Ilaria does not seem to be aware of this inconsistency. The utterance connected with GP_3 is the following:

Ilaria: *r...ehm...C...I can move on any point of this line r...between the projections of...of B and of A on this line...and it should remain...C...and the angle C always right.*

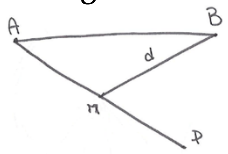
We can only recognize a slight uncertainty in her words: there are a lot of pauses and she uses modal verbs as "*should*".

So, when the theoretical elements introduced for the first time by the solver are incoherent, the funnel ends with a product of GP that is strongly inconsistent with the given constraints. In this case, the solver seems to be blind to the contradictions that the obtained configurations (i.e. drawings or gestures) reveal to the interviewer.

Moreover, when the theoretical elements are few and incoherent, the products of GP can be strongly influenced by the figural elements of the figure that the solver has drawn. Here we show two examples from Task 2.

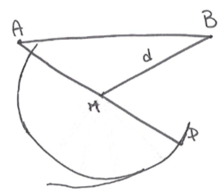
The first example is a funnel which is drawn from the excerpt *Isabella_T2_P1_(01:42 – 04:08)* (see Section 6.6.2) zooming in on what happens from time 01:23 to time 02:16 of the interview.

Configuration 1 (*Drawing 1a*):



	Theoretical elements	1		Figural elements
(01:42)	General point on the plane			[P]
	Endpoint of a segment			[P]
	Anticipatory Intuition			
	Midpoint			[M]
	Center of a circle	X		[M]
	Rotation			P
	Radius	X		MP
	Fixed/Constant length			MB
	Center	X		M
	GP_1_(0) (discursive – gestural): P on a circle C(M, MP)			

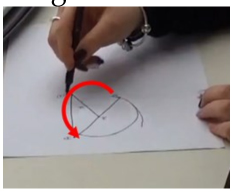
Configuration 2 (*Drawing 1b*):



Isabella's funnel starts with two very simple properties: P is a point on the plane, as reported in the step-by-step construction; P is the endpoint of a segment, which is a piece of information that she gathers from the *Drawing 1a*.

Suddenly, she reaches a solution and recalls some theoretical elements which are already known, like the midpoint and the fixed length. Moreover, she refers to new properties: the circle center and the radius. These theoretical elements show their inconsistency with the given constraints. They are not incoherent in an absolute way, indeed we have other funnels where these elements have not been labeled as incoherent, but here they are listed as properties of incoherent figural elements for the following reasons. Looking at the funnel, we can see that Isabella is talking about a circle centered at M and with radius MP. In general, this is a possible locus for P, but here it is incoherent with the given constraints: it is impossible to move P along such a circle, without consequently also moving M. Following this locus, the length of MB changes and the solver loses a constraint. As highlighted in Section 6.6.2, the solver seems to both lose a constraint and to add another one. Thus, the product of GP is incoherent with the TEG.

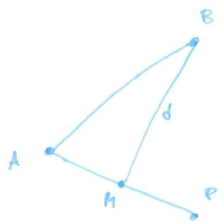
Looking at the transcription table of the excerpt that follows the first one, we find an explanation of the GP_1: *Isabella_T2_P1_(04:22 – 05:15)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
04:22.03	Int	The question is about P: what can you say about the point P?		
04:28.13	Stud	This, I mean I think, eh then I don't know.		
04:31.01	Int	Mm mm.		
04:32.18	Stud	And indeed **translating P along the whole circle**, M does not vary because it is the center of the circle, and so the length [d] does not change either.	She is moving the pen in order to complete the circle she is talking about: 	**Window gesture** (GP_1) She explains the GP_1
04:47.10	Int	How do you imagine this circle? Describe it to me.		
04:52.22	Stud	I imagine MP as a radius of the circle, therefore.		Theoretical element
04:59.01	Stud	I mean in what sense do you mean how do I imagine? With respect to the dimensions?		
05:02.24	Int	Like, tell me what you think you need to describe this circle to me.		
05:09.24	Stud	Mmm, oh well, mmm.		
05:15.08	Stud	That M, no, that MP is indeed the radius of the circle and that therefore **translating the segment MP** we can obtain the circle so that indeed the length stays unvaried.		

We can say that the figural components of the figure are dominant in Isabella's solution because it seems that, after having performed the drawing, she forgets the properties given in the step-by-step construction. She also shows a lack of theoretical control over the figure. Indeed, she explains that it is possible to move P while M is fixed.

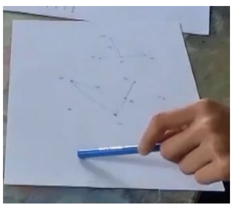
Another example is provided by Carlo's funnel. The related excerpt is *Carlo_G10_T2_P1_(01:22 – 04:52)* and is analyzed in Section 6.5.4. Here is the funnel.

Configuration 1 (*Drawing 1a*):



	Theoretical elements		Figural elements
(01:38)			P
			AP
	Center [Midpoint]		M
	Right triangle	X	AMB
	Right triangle	X	AMB
	Extension of a segment		[MP]
	Double length		AP = 2 AM
	Extension of a segment		[MP]
	Congruence		AM, [MP]
	Congruence		AM = MP
	[…]		
			P
			BM
	Fixed length		MB or d
	[Point] symmetry	X	P
	GP_1_(0): P at a symmetric position with respect to A		

Configuration 2 (Gesture):

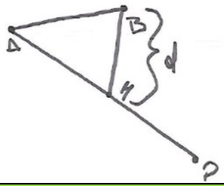


As highlighted before, the figural elements drive Carlo's exploration of the configuration and he shows a lack of theoretical control over the figure. The funnel, too, confirms this inference: there are a small number of new theoretical elements and only one is coherent. Both new and already known theoretical elements refer to the figural description of the configuration: the triangle seems to be right; he discovers that the segments are congruent only looking at the drawing; MP is part of a longer segment. Also in this case, the incoherence of new theoretical elements leads the solver to communicate an incoherent product of GP.

Apparently, like Isabella, after having performed the drawing, Carlo forgets the given properties and starts conceiving the configuration as composed of a triangle and an independent segment.

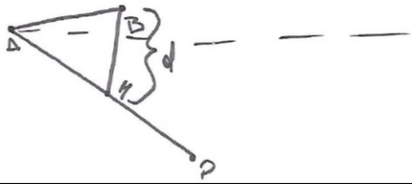
The last example is taken from the excerpt *Alessia_T2_P1_(01:27-03:09)*.

Configuration 1 (*Drawing 1a*):



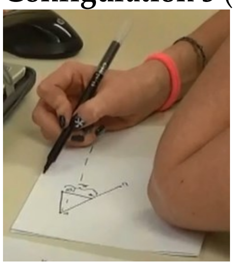
Theoretical elements		1		Figural elements
(01:27)	General point on the plane			P
	Fixed point	X		P
	GP_1_(0) (discursive): P is a fixed point			

Configuration 2 (*Drawing 1b*):



Theoretical elements		2		Figural elements
(02:37)	Anticipatory Intuition			
	[Line] symmetry	X		P
	[Axis of symmetry]	X		Half-line (start point A)
	Start point of the half-line			A
	Fixed/Constant length	X		d [o MB]
	GP_2_(1) (discursive – gestural): P is the symmetric point with respect to a [horizontal] line through A			

Configuration 3 (Gesture):



The two funnels end with two incoherent products of GP. In the first funnel, there are only two theoretical elements, which lead to the impossibility of placing P in another position. Suddenly, looking at the drawing, Alessia undertakes another GP. She introduces new theoretical elements probably inspired by the particular drawing. Indeed, she draws a horizontal axis, as shown in *Configuration 2*, and she points at a new position for the point P.

As we know, the line symmetry of P could be a theoretical element that leads to a coherent product of GP. However, in this case, the chosen axis is not the correct one. We infer that probably the half-line through A has appeared to the solver the most suitable according to the particular drawing she has performed.

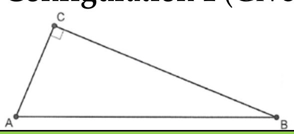
Moreover, there are few theoretical elements and they are incoherent with the given constraints. This supports the conjecture that the solution is driven by the figural elements of the figure sketched out in *Drawing 1a*. This seems to be confirmed by the *Configuration 3*: the new position of P is suitable graphically, but it does not maintain the given constraints. Here we recognize a lack of harmony between figural and conceptual components of the given figure. Nevertheless, Alessia does not seem to be aware of the inconsistency between the figural and the conceptual component of the new configuration.

7.1.3 Case (c): few new theoretical elements

When the solver introduces a small number of theoretical elements, the products of GP are very simple. However, they are not similar to the ones described in the first case. Although they are simple, they are connected with the new theoretical elements and they contain new information about the geometrical configuration. Below are examples from the resolution of two tasks.

The first is from the excerpt *Giorgio_G13_T4_P1_(00:46 – 02:38)* (see Section 6.4.4), zooming in on what happens from time 00:53 to time 02:30.

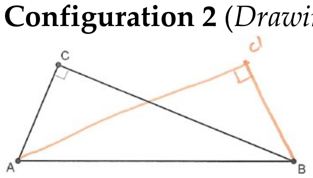
Configuration 1 (Given drawing):



	Theoretical elements		Figural elements
(00:53)			C
			C
	Fixed segment		AB
	Not fixed		AC or C
	Not fixed		CB and CA
			CB
	Isosceles triangle		
	Equilateral triangle	X	
	Right angle		C
	[Line] symmetry		C

GP_1_(1) (gestural – discursive): C symmetric point [with respect to the axis of AB]

Configuration 2 (*Drawing 1*):



Giorgio starts with some given theoretical information. Then he focuses on the side CB and he lists the kind of triangle that the given triangle should become moving the segment CB: isosceles and equilateral. "*Equilateral triangle*" and "*isosceles triangle*" are marked in the funnel as pure theoretical elements, because the solver does not point to anything on the sheet of paper and he does not refer to a particular object on the drawing: he is talking in a general way. It seems that the exploration of other configurations helps him to remember one of the given constraints: the triangle must be right. Indeed, he recalls the property "*ACB is a right angle*". Finally, using line symmetry, he finds a correct position of point C such that the angle right is maintained.

The geometrical transformation is inferred from the utterance coupled with the following window gesture (Figure 32) and it is confirmed by *Drawing 1*.

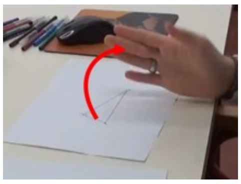
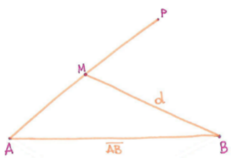


Figure 32 Giorgio's window gesture of a line symmetry of the point C

The product of GP is directly connected with the last new theoretical element. Nevertheless, it is simple, and it remains the only product of GP communicated during the first part of the interview. The new theoretical elements introduced by the solver are not enough to find other or more detailed products of GP.

The second example is drawn from the excerpt *Margherita_G13_T2_P1_(02:16 – 06:05)* (see Section 6.3.3), looking at what happens from time 02:29 to time 04:54.

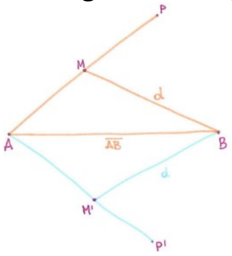
Configuration 1 (*Drawing 1a*):



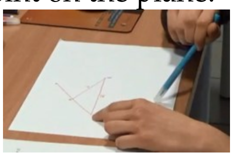
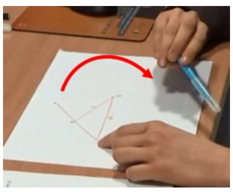
	Theoretical elements			Figural elements
(02:29)	Equidistance			PM=AM
	Fixed/constant distance	X		PB
	Fixed/constant distance			d [o MB]
	Fixed points			A, B
	Fixed/constant distance			d [o MB]
	[Fixed point]			M
	Fixed/constant distance	X		PB
	Fixed/constant distance	X		d(P,AB)
	Fixed points			A, B

Fixed/constant distance			d [o MB]
Fixed point			M
Anticipatory Intuition			
[Line] symmetry			[P]
GP_1_(1) (discursive – gestural): P symmetric point [with respect to AB]			

Configuration 2 (*Drawing 1b*):



In the beginning, Margherita recalls the same already known theoretical elements. These elements are: explicitly given in the step-by-step construction, like the fixedness of points and segments; rephrased by the solver, like the equidistance directly deduced from the constraint on M. They do not help her to find a solution of the task. Suddenly, she talks about a new position of the point P, introducing a new theoretical element: line symmetry. The utterances and the gestures which communicate the product of GP are reported below.

Time	*Who*	*What is said*	*What is done*	*Comment*
04:22.03	Stud	Well, it could be **on the other side**, it could be **the mirror image.**	She is pointing at a point on the plane:  She is pointing at B. The she is overturning the pen with a circular motion toward herself: 	**GP_1_(1) (discursive – gestural):** **P symmetric point [with respect to AB]** Initially, she points at a position where she expected to find the point P. Then she products a **Window gesture**: she performs line symmetry of P as a rotation around AB. **Anticipatory Intuition**

The solver does not describe the prediction in greater depth and seems a bit uncertain: she uses the modal verb "*could*". However, her pointing gesture to the new position of the point P is quite precise and GP_1 contains new information about the configuration.

Also in this case, the new theoretical elements do not suffice to communicate a detailed product of GP. Indeed, we need to infer the axis of symmetry and GP_1 is expressed in a simple way. Moreover, there is only one product of GP during the first part of the interview.

Comparing this example with Tiziana's product of GP reported in the Case (a), we can see that, despite the simplicity of both of them, Margherita's introduces new information about the solution. This information is strictly related to the line symmetry.

7.1.4 Case (d): coherent new theoretical elements

The most coherent and detailed predictions are produced when the funnels contain a good number of new theoretical elements which are consistent with the given constraints.

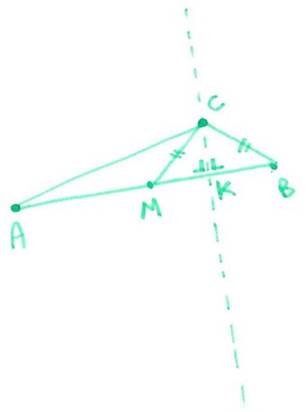
Here an example of this kind of funnel. It is from the excerpt *Fiorella_MD_T5_P1_(02:05-07:07)*.

Configuration 1 (*Drawing 1a*):

	Theoretical elements	**Figural elements**
(02:05)		C
	Congruency	CM=CB
		C
	Angle	C
		C
	Angle	C [or ACB]
	Right-angled triangle	[ACB]
		CM, CB
	It is not a height	CM
	Vertex of an isosceles triangle	C
	Congruency**	CA=CB
	Height*	CM
	Median*	CM
	Bisector*	CM, C
		CB
	Point on a perpendicular line	C, MB
	Midpoint	[Midpoint of MB]
	Point on a perpendicular line	C
	Midpoint	K, MB
	Isosceles triangle	[MCB]
	Same length	CM=CB

GP_1_(0) (gestural – discursive): C on a line perpendicular to MB, which passes through the midpoint of MB

Configuration 2 (*Drawing 1b*):



* She uses a Theorem about the height of an isosceles triangle.
** Congruency as a consequence of having an isosceles triangle.

We see at a glance the most evident feature of this funnel: there is a long sequence of new theoretical elements which are consistent with the given constraints.

The funnel starts with already known and very simple theoretical elements. Fiorella recalls the constraint on the sides requested in the question: CM equal to CB. Then she focuses on particular figural elements: the point C, which is regarded as a point and as an angle; the segment CM.

The funnel clearly shows when Fiorella starts communicating a new product of GP. She looks at point C as a vertex of an isosceles triangle and then it starts a long sequence of new theoretical elements which leads her to a correct solution. The sequence begins with very powerful and well-structured theoretical elements, highlighted in the funnel with a star (*). Indeed, the solver recalls one of the definitions of an isosceles triangle and a theorem on its height. The theorem is the following:

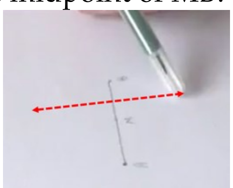
Theorem: *In an isosceles triangle (where the base is the side which is not equal to any other side) the height drawn to the base is the median and the angle bisector.*

She does not formally recall the theorem, but she only uses it to clarify the inferred properties of CM. The transcription table below shows how she recalls these theoretical elements.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:01.09	Stud	If I instead try to draw **C so that it is an isosceles triangle**, [so that] CA...CA is equal to CB. CM is	She points at a position for C:	She explores the case: ABC is an isosceles triangle. She recalls a lot of theoretical elements:

		height, median and bisector of the angle at C. But in any case, it is not of the same length.		the median, the height and the bisector. She uses a Theorem: the height of an isosceles triangle is also median and bisector.

Even if the isosceles triangle she was talking about is not CMB as we expected, the recalled theoretical elements help her find not only a new position of the point C, but also to recognize an entire locus for the point C.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:47.15	Stud	Maybe if I try to draw C...	She is resolutely pointing a position for C: 	**Window gesture**
03:55.14	Stud	...on the... **perpendicular line to MB at the midpoint**.	She is moving the pen on a trajectory perpendicular to AB and passing through the midpoint of MB: 	**GP_1_(0) (gestural - discursive): C on a line perpendicular to MB, which passes through the midpoint of MB** **Window gesture**

Even if the recalled theorem is related to the triangle ABC, these theoretical elements seem to open the way to the solution. Probably, the solver transfers the imagined properties of the triangle ABC onto the triangle CMB. Indeed, in the funnel we find a second set of new theoretical elements related to the triangle CMB or to some parts of it. She finds a locus for the point C and starts describing its properties: she traces the midpoint of MB; she imposes that the line passes through the midpoint; in this way she obtains an isosceles triangle and, therefore, CM equal to CB.

As we can see looking at the corresponding utterance, the solver stresses a conditional connection between the location of C and the property of CM and CB having the same length.

Fiorella: *I mean that **if** I move C on this perpendicular line through the midpoint K of segment AB…I construct this way an isosceles triangle and **so** CM and CB will always be of the same length.*

She uses an "if…then" form to express that if she makes C a vertex of an isosceles triangle with base on MB, then the sides CM and CB have the same length.

The product of GP is well described by the solver: she finds the shape of the locus (a line); she constructs the midpoint of MB that the line must intersect perpendicularly; she properly draws the line, its properties and the properties of the triangle CMB.

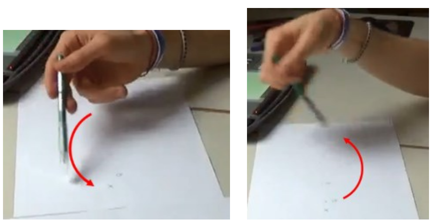
Funnels of this kind are not necessarily as rich in mathematical theory as the one shown in the previous example. What we want to stress is that they share a common structure. Indeed, they have a sequence of new theoretical elements which the researcher can capture by just glancing at the funnel. In these cases, the products of GP are coherent, detailed, well-described. Moreover, they add new information about the solution of the problem.

This feature is also evident during the resolution of simple tasks like Task 6. The following funnel is constructed from Emilio's interview.

Configuration 1 (*Drawing 1a*):

	Theoretical elements	**7**	**Figural elements**
(05:55)	Center of the circle		A
	Point on a circle		Q
	Semicircle		Arc PQ
	Semicircle		Arc QP
	Circle		
	GP_8_(0) (discursive – gestural): P and A on a circle centered at A and with diameter PQ		

Configuration 8 (Gestures):



The funnel shows the elements introduced by the solver for answering the question about what happen to the configuration if the point P is moving on a circle.

We can see at a glance that the solver introduces 5 new theoretical elements organized in a dense sequence. All of them are coherent with the given constraint: P and Q must be symmetric points with respect to A (the center of the symmetry).

Moreover, the product of GP is detailed and mathematically rich: we can easily recognize the locus (a circle), its center, and its diameter.

Finally, the solver shows good theoretical control over the figure. The gesture reveals that, while he is moving P along a semicircle, he imagines a consequent movement of Q along another semicircle; the two movements are coherent with point symmetry.

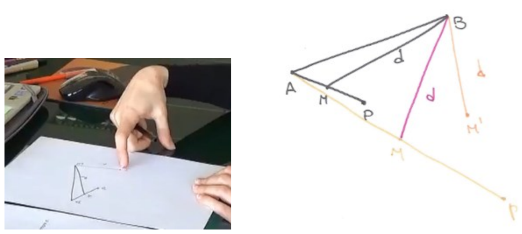
7.1.5 Additional examples of Case (c) and Case (d)

We want to further stress the features of the funnels that lead to coherent and detailed products of GP. In particular, it seems that the more theoretical elements are recalled, the more details the products of GP contain. The number is not to be considered in an absolute way, but in relation to the total number of theoretical elements. So, we can look at the ratio between the new theoretical elements and the total number of entries in the first column.

Let's compare two different solvers' approaches to solving Task 2 that lead to a similar product of GP: Tiziana's and Laura's.

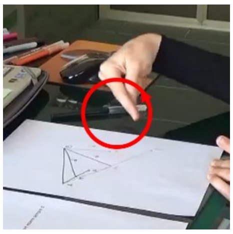
The following funnel is constructed from the analysis of Tiziana's interview.

Configuration 6 (Gesture and *Drawing 1c*):



Theoretical elements	5	Figural elements
(05:15)	Anticipatory Intuition	
Radius		d [or MB]
Circle (as a locus)		
Center		B
Radius		d [or MB]
Point on a circle		M
GP_5_(0) (gestural – discursive): M on a circle C(B, d)		

Configuration 7 (Gesture):



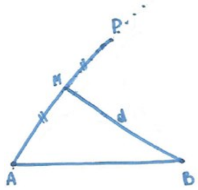
As in Fiorella's funnel, we notice a sequence of new and coherent theoretical elements. What is more, all the entries of the first column are new coherent theoretical elements. Seemingly, this large set of theoretical elements allows the solver to reach a clear and detailed product of GP:

GP_5: *M is on a circle centered at B and with radius d*

The product is very detailed: the solver makes explicit the kind of locus (a circle), its center and its radius.

The following funnel was constructed from the analysis of Laura's interview.

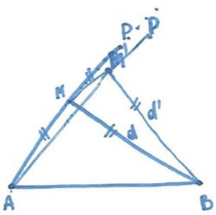
Configuration 2 (*Drawing 1b*):



	Theoretical elements	**2**	**Figural elements**
(01:56)	Fixed point		B
	[Point on an arc]		M
			M, B
	Congruence		MB and another MB
	[Fixed length]		d
	Not fixed		P
	Fixed length		d
	Fixed [segment or length]		MB
	Fixed length		d
	Rotation		d [or MB]
	Vertex		B
	Fixed point		B

GP_2_(1) (discursive – gestural): M turns around the fixed vertex B

Configuration 3 (*Drawing 1d*):



Laura communicates a product of GP that refers to the same figural element of Tiziana's: point M. Nevertheless, Laura's GP_2 is not very detailed. We can only infer that M could be generically moved around B. We cannot be sure that she intends to move P along a circle.

Laura's funnels contain many theoretical elements: some are given at the beginning and recalled by the solver at a particular moment; others are introduced by the solver for the first time. However, only 3 out of 11 are new theoretical elements.

So, it seems that the number of theoretical elements introduced for the first time by the solver is related to the level of detail of the product of GP.

7.2 A general overview of the resolution of Task 2 and concluding remarks

We now show the data collected from the funnels analyzing the resolution of Task 2 to give a general overview of the role of theoretical elements within the GP process.

For each funnel, Table 12 shows:

- the name of the solver;
- the total number of theoretical elements (column "*Tot*"), the number of new theoretical elements (column "*New*"), the number of incoherent already known theoretical elements (column "*Inc. – K*"), and the number of incoherent new theoretical elements (column "*Inc. – N*");
- the numerical label of the product of GP as reported at the end of the funnel (column "*#*");
- the statement that contains the product of GP itself;
- the level of fuzziness of the product (column "*Fuz.*");
- the possible incoherence of the product of GP, present if the cell is marked with an "X" (column "*Inc.*").

In the last two columns, we show the ratio between:

- the number of new theoretical elements and the total number of theoretical elements;
- the number of incoherent new theoretical elements and the number of new ones.

Solver	Theoretical elements				Products of GP				Ratio	
	Tot	New	Inc. K	Inc. N	#	GP	Fuz.	Inc.	$\frac{New}{Tot}$	$\frac{(N)}{New}$
Agnese	5	1			1	P on an arc of a circle	2		1/5	0
	6	2			2	P cannot occupy other positions	2	X	1/3	0
	1	1			3	MB rotates	1		1	0
	5	4			4	points of the circle C(B,MB) are distant d from B	0		4/5	0
					5	P in a position such that MA=MP	0			
	5	4		2	6	M on a circle [with radius MB]	0		4/5	2/4
Alessia	2	1		1	1	P is a fixed point	0	X	1/2	1
	4	3	1	2	2	P is the symmetric point with respect to a horizontal line through A	1	X	3/4	2/3
Carolina	0	0			1	several positions of MB	2		ND	ND
	5	3			2	P rotates	2		3/5	0
	14	2	1	2	3	P belongs to the line through AB	2	X	2/14	1
	4	3	1	2	4	P symmetric with respect to A	0	X	3/4	2/3
	7	4			5	if P moves along AP, the length of MB grows	1		4/7	0
Emilio	5	2			1	the locus of M is a circle centered at B	0		2/5	0
	0	0			2	the locus of P is a circle	1		ND	0
Flavia	10	3			1	P at a symmetric position [with respect to AB]	0		3/10	0
	5	3			2	moving P or M, MB changes its length	2		3/5	0

	3	0			3	P can occupy more than two positions in order for the length of MB is the same	2		0	ND
	0	0			4	other positions of P could be found combining a translation and a rotation	2		ND	ND

Giacomo	6	1		1	1	countless number of positions for P	2	X	1/6	1
	3	2		2	2	P is fixed	1	X	2/3	1
	4	3			3	P at the symmetric position with respect to AB	0		3/4	0

Ilaria	9	3	2	1	1	P on a symmetric position with respect to AB	2		3/9	1/3
	2	2		1	2	the locus of P is a half-line from M	0	X	1	1/2

Isabella	8	4		3	1	P on a circle C(M, MP)	0	X	4/8	3/4

Carlo	10	4		3	1	P at a symmetric position with respect to A	0	X	4/10	3/4

Laura	8	4		4	1	P on a half-line starting from M	2	X	4/8	1
	11	3			2	M turns around the fixed vertex B	1		3/11	0

Margherita	12	4		3	1	P symmetric point [with respect to AB]	1		4/12	¾

Marta	7	1			1	several positions of MB	2		1/7	0
	4	1			2	several positions of P	2		1/4	0
	5	5			3	[M on a] circle C(B,d)	0		1	0
	5	4			4	P on a circle	1		4/5	0

Sergio	8	5		2	1	if P is moved, the length of MB changes	2		5/8	2/5
	4	3		3	2	P is fixed	2	X	3/4	1

Silvana	10	3	2		1	P symmetric point with respect to AB	1		3/10	0

Stefano	8	3		1	1	P symmetric point with respect to AB	1		3/8	1/3
	4	2		2	2	P on a line [through AM]	2	X	1/2	1

Tiziana	4	1			1	the symmetric point of P with respect to AB	0		1/4	0
	6	5			2	the symmetric point of M with respect to AB	0		5/6	0
	1	1			3	M on an arc of a circle	1		1	0
	4	0			4	P on the plane constructed using AM=MP	2		0	0
	5	5			5	M on a circle C(B, d)	0		1	0
	3	3		1	6	the locus of P is a circle passing through B	0	X	1	1/3
	2	2			7	P on the maximum distance from B	0		1	0
	9	6		2	8	the locus of P is a circle centered at M	0	X	6/9	2/6
	7	7		2	9	the locus of P is a circle	0	X	1	2/7

						C(a particular M, (AB+d))				
	9	8			10	the center of the circle (of P) is the midpoint of P'P''	0		8/9	0
	4	4			11	the locus of P is a circle C(C,(AB/2)+d)	0	X	1	0

Table 12 Data from funnels of Task 2: an overview of the theoretical elements recalled and introduced by the solvers within the resolution of the task.

We now comment on the findings listed at the beginning of this chapter by reading the content of Table 12.

- Seemingly, if there are not any new theoretical elements, the products of GP are almost obvious, and they do not add other information on the solution. This is not a very common behavior in our solvers. Indeed, it happens only in the two reported cases: Flavia (GP_3) and Tiziana (GP_4). This is an expected finding: because of the dual nature of geometrical objects, when the solvers talk about a configuration, they (possibly unconsciously) refer to the theoretical counterpart of the figural elements that are part of the configuration.

  In other cases, the product of GP comes spontaneously at the beginning of the interview or as a consequence of other products of GP, we find a "0" in the column labeled "*New*": see Carolina (GP_1), Emilio (GP_2), Flavia (GP_4). This is a different case because we also have an empty entry within the column "*Tot*", so the ratio is not determinable (N/D).

- We claim that if the new theoretical elements are incoherent, the products of GP are incoherent or related to figural elements. Looking at the column named "*Inc.*", we find several cases of incoherent products of GP where all or a great number of the new theoretical elements are incoherent: Alessia, Carolina (GP_3, GP_4), Giacomo (GP_1, GP_2), Ilaria (GP_2), Isabella, Carlo, Laura (GP_1), Sergio (GP_2), Stefano (GP_2). Looking at the last column, we can see that in these cases the ratio is close to 1. In this case, the products of GP are completely incoherent, and they constitute the only solution provided by the solver

  Tiziana's is a special case of incoherent products of GP, different from the other cases. She has found that the locus of P could be a circle. Then she is engaged in a dialogue with the drawing, trying to refine the products of GP by making explicit new details. The products of GP are not completely incoherent: they contain the coherent information that the locus is a circle. Despite the incoherence of some products of GP, the solver introduces a huge number of new theoretical elements, and this is the reason why the ratio is not so high.

Also Agnese's is a rather special case: the incoherence of GP_2 is not caused by the incoherence of new theoretical elements, but seemingly it is due to the impossibility to conceive other arrangements of the configuration. Moreover, we notice that frequently the incoherence of the products of GP is related to an overabundance of theoretical constraints imposed on the figure. Indeed, one of the most frequent incoherent products of GP is "*P is fixed*".

- We had conjectured that the new incoherent theoretical elements seem to be connected with a sort of solvers' blindness of such incoherence and of the inconsistency between figural and theoretical elements. We have provided examples of this phenomenon and, now, looking at the table, we notice that very often there are no other products of GP after an incoherent one. This suggests that the solver is not able to find any other solution or that she is quite convinced about her prediction.

- Looking only at the coherent products of GP, we notice that when there is a relatively large number of new theoretical elements, the products of GP are quite detailed, and they provide new insights into the problem. Let us look at the following excerpts from some solvers' interviews: Agnese's (GP_4, GP_5, GP_6), Carolina's (GP_5), Giacomo's (GP_3), Marta's (GP_3, GP_4), Tiziana's (many cases). As previously mentioned, the funnels relative to these excerpts usually contain a dense sequence of new theoretical elements introduced before a new product of GP.

  Instead, when there are few new theoretical elements, even if the products of GP provide new insights into the problem, they are quite simple. This is shown in excerpts from the following solvers' interviews: Flavia's (GP_1), Ilaria's (GP_1), Laura's (GP_2), Margherita's, Marta's (GP_1, GP_2), Sergio's (GP_1), Silvana's, Stefano's, Tiziana's (GP_1).

Moreover, Table 12 reveals that the same solver can communicate both coherent and incoherent products of GP as well as both detailed and fuzzy ones.

So, the funnels confirm our conjecture that the presence or absence of the new theoretical elements strongly influences the products of GP.

More specifically, if the solver does not introduce new elements the products are very simple, almost obvious, and they do not give any new information on the problem. When there are theoretical elements, but these are incoherent, the products of GP are strongly incoherent and they seem to move the solver away from a correct solution.

Moreover, when the solvers communicate a product of GP passing through (potentially incoherent) theoretical elements introduced by themselves, they seem to be convinced of their findings even if the perceptual feedback appears inconsistent to the interviewer.

By only looking at the funnel and in particular at the "X" collocated in the central column, we can recognize prediction processes which will produce an incoherent product of GP. Consequently, we can advance the hypothesis that if the solvers are theoretically poorly equipped and they recall incoherent theoretical elements, they resort to using figural considerations that are disconnected with the theoretical constraints.

Moreover, we see a connection between the lack of new theoretical elements and the quality of the products of GP: all the theoretical elements are correct, but none of them are new, so the solver can communicate only a trivial product of GP.

Instead, when there are new theoretical elements and they are coherent, the products of GP are consistent with the given constraints and strictly connected with the new elements. Moreover, the products of GP contain original information about the configuration. Depending on the number of new theoretical elements, the products of GP can be more or less detailed and well-described.

Finally, in our sample, a sequence of new coherent theoretical elements indicates the production of a more accurate and detailed product of GP: it does not reflect the length of the interview. In the funnel we highlight each property of the different figural elements in focus; so, we can have a single sentence which takes up more than one box of the funnel. Fiorella's coding of the theorem in the funnel is a good example: she utters a single sentence which is coded in the funnel with 5 boxes.

The funnel was a priori designed to explore the involvement of theoretical and figural elements within the GP process. Funnels reveal an intricate intertwining between predictions and theoretical elements. However, a posteriori, funnels also seem to be a powerful tool for predicting the quality (in terms of coherence and richness in theoretical details) of the products of GP.

8. A second level of findings: the structures of GP processes and possible obstacles

In this chapter we will present some additional characteristics of the processes of GP. More specifically, in Section 8.1 we will describe how several GP processes, or their products, can interact within the resolution of the given tasks; in Section 8.2 we will describe the general and local obstacles that could inhibit the solvers in accomplishing processes of leading to coherent products.

8.1 Structure of GP processes

In the previous sections, we have highlighted several features of GP processes, stressing that we cannot talk about only one process of GP, since several processes seem to lead the solvers to make and consequently to communicate a prediction. For example, there are processes of prediction with a strong dynamic dimension, but other processes are supported by a static approach; the processes can be immediate, but they can also come after long reasoning about the problem; intuitions play a role in some processes of GP but not in others.

In this section, we describe how several GP processes, or their products, can interact within the resolution of the given tasks. Indeed, during the resolution process – and mostly if this takes a long time – the solvers can explicitly recall a previously communicated product of GP or they can use some pieces of knowledge made explicit during another GP process; otherwise they can communicate several seemingly independent products of GP.

The possible connections are determined according to the rules described within Section 5.4 and they are marked using a blue arrow to indicate that they appear in funnels describing the process. At the end of the analyses of each interview, we can summarize the connections between funnels with a diagram as we will show below.

So, looking at the funnels and the diagrams, we observe that the products of GP arise and can be organized in several structures:

a) a chain of configurations with a single product of GP;

b) a chain of configurations with several disconnected products of GP;

c) one or more chains of configurations with several connected products of GP.

We say that "*GP_n is connected to GP_m*" if the funnel which leads to *GP_n* is connected with the funnel which leads to *GP_m*. Otherwise, we say that the two funnels, and consequently the two products of GP, are *disconnected*. So, in the following, we will indifferently refer to the connection between funnels or between products of GP.

The diagram of each case has a prototypical shape or some common features that will be shown and discussed in the next sections.

8.1.1 Case (a): the solver communicates only one product of GP

During the resolution of the given tasks within the paper and pencil environment the solvers of our sample may communicate only one product of GP. The prototypical shape of this kind of diagram is shown in Figure 33.

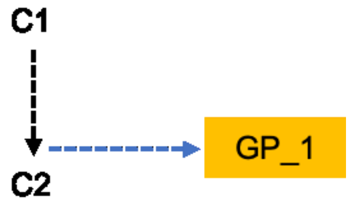


Figure 33 Prototypical diagram of Case (a): the solver communicates only one product of GP

These isolated products can be coherent or incoherent; we did not find a connection between the coherence of the products of GP and their number. Frequently the isolated product is not the most complete one that a solver can reach within the resolution of that specific task. Only, in two cases we find a different behavior: the product of GP is very detailed and rich in new theoretical elements. In these two cases, the solver was an expert.

So, we recognize two alternative characteristics of the products of GP:

1) the isolated product of GP is not the most complete, it shows a rigid use of the configuration, and it comes after prolonged reasoning upon the task; the solver imposes on the figure additional constraints;

2) the isolated product of GP is the most complete, it is rich in theoretical elements and details.

We infer that the solvers do not communicate other products of GP respectively because:

- the additional constraints imposed upon the figure or the rigid view of the configuration do not allow them to find any other arrangements that would lead to another solution;
- their theoretical control over the figure suggests that the product of GP is actually the most general and complete that they could reach.

In the following, we provide examples of these two different cases.

The first subcase is very common in our sample. We provide three examples: two are taken from the resolution of Task 2 and one from the resolution of Task 4. In all cases the solvers communicate only one product of GP; the funnels show few or incoherent theoretical elements; the products of GP are very connected with figural elements.

The first example is Isabella's funnel, already shown in Section 7.1.2. The corresponding excerpt is *Isabella_T2_P1_(01:42 – 04:08)* (see Section 6.6.2). This is the case of an incoherent product of GP produced adding a constraint ("*the segment MB is fixed*" or "*M is fixed within the plane*") and losing another one ("*M is the midpoint of AP*"). These additional and missed constraints lead to the fixedness of the whole triangle and to the possibility of only a rigid movement of the segment MP. In such a way, the point P can trace a circle, but a circle that is incoherent with the given constraints: the center is at M and the radius is MP. Looking at the excerpt *Isabella_T2_P1_(04:22 – 05:15)* analyzed in Section 7.1.2, we know that the solver does not feel the necessity to renegotiate or complete her prediction: it seems that she has the feeling that the answer is complete.

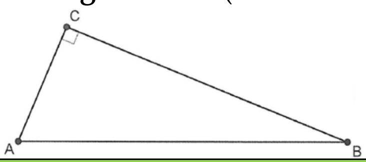
The second example is Margherita's funnel, already analyzed in Section 7.1.3. The corresponding excerpt is *Margherita_G13_T2_P1_(02:16 – 06:05)* (see Section 6.3.3). This is the case of a coherent product of GP that remains the only one communicated. As we have already highlighted the process that leads the solver to GP_1 manifests a lack of dynamic features. The motion is considered only for denying the possibility of the movements. Seemingly, the rigidness of the configuration does not allow the solver to undertake other GP processes. At time 06:00, Margherita explicitly says that the symmetric position is the only one that she is willing to consider.

The third example is Giorgio's funnel, which has been analyzed in Section 7.1.3. The corresponding excerpt is *Giorgio_G13_T4_P1_(00:46 – 02:38)* (see Section 6.4.4). Also the diagram of this funnel looks like the diagram in Figure 33. The funnel reveals the presence of a few theoretical elements and suggests a lack of theoretical control: the solver says that he is imagining to obtain an equilateral triangle, but this contrasts with the constraint on the right angle. After an exploration of the problem, Giorgio succeeds in communicating only one product of GP that is very connected with figural elements: C at the symmetric position.

The second subcase (the isolated product of GP is the most complete, it is rich in theoretical elements and details) is well represented by the funnels of two expert solvers: Filippo and Fiorella.

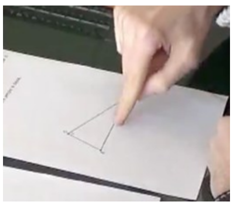
The first funnel is from the excerpt *Filippo_PhD_T4_P1* (see Section 6.2.1), looking at what happens from time 00:27 to time 01:00 during the resolution of Task 4.

Configuration 1 (Given drawing):



	Theoretical elements	Figural elements
(00:27)	Point on a circle	C
	Diameter	AB
	Midpoint	Point on AB
	Fixed distance	C, midpoint of AB
	GP_1_(0) (discursive): the locus of C is a circle with diameter AB	

Configuration 2 (Gesture):



The funnel shows that Filippo immediately talks about C as a point on a circle. The subsequent theoretical elements are useful for specifying the features of the locus: the diameter of the circle, the center of the circle, the reason why the locus is actually a circle. Moreover, all the theoretical elements are coherent and introduced for the first time by the solver. The product of GP is very detailed, complete, and theoretically solid. So, we can easily grasp why the solver does not further investigate the situation.

The second example is taken from Fiorella's interview during the resolution of Task 5. The funnel is already reported in Section 7.1.4 as an example of a theoretically rich funnel and the corresponding excerpt is *Fiorella_MD_T5_P1_(02:05-07:07)*. Fiorella starts exploring the configuration considering several positions of C: looking at the funnel we see that she focuses longer on C. A long sequence of new theoretical elements begins when she focuses on a specific position of C: at the vertex of an isosceles triangle. Starting from this theoretical element she introduces several mathematically advanced theoretical elements in which we recognize a Theorem. The product of GP is detailed, well described and theoretically solid. Indeed, as highlighted before (see Fiorella's highlighted utterance in Section 7.1.4), at the end of the interview Fiorella uses a conditional statement to justify the product of GP. Since it is justified, well described and properly sketched out in the drawing, there is no necessity for further investigations: GP_1 remains the only product of GP communicated during the resolution of the task.

Overall, we can claim that the solvers communicate only one product of GP in the two cases listed below:

- Case 1: the solver did not succeed to discover further interesting arrangements of the first configuration. This could happen because of a lack of theoretical control, because of superimposing the theoretical constraints upon the figure, or because of restoring to figural components.
- Case 2: the solver finds a complete and detailed product of GP that is theoretically founded. This happens when the solver manifests theoretical control over the figure and is able to properly introduce theoretical elements.

8.1.2 Case (b): a chain of configurations with several disconnected products of GP

During the resolution of the tasks, the solvers can produce several configurations starting only from the first one. The funnels highlight the chains of configurations performed by the solvers. Each chain is coupled with several products of GP. In some cases, the solvers only produce one chain of configurations with disconnected products of GP. The prototypical shape of this kind of diagram is sketched out in Figure 34.

In all cases of our sample, this kind of diagram contains one or more incoherent products of GP. We find a connection between disconnected products of GP and their coherence with the given constraints, for two reasons:

- the solvers change their mind about a product of GP (coherent or incoherent) and they find a new solution to the problem;

- the solvers communicate several products of GP referring to situations that they seem to perceive as independent, as if they were different problems.

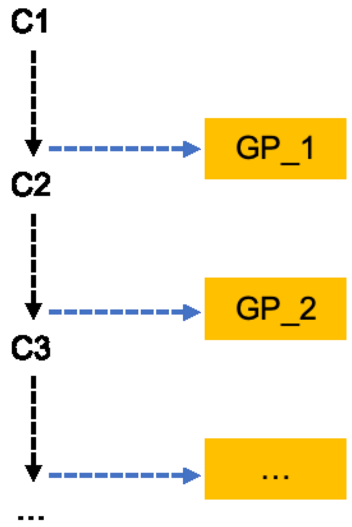
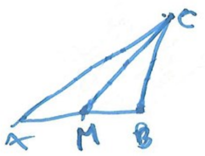


Figure 34 Prototypical diagram of Case (b): one chain of configurations with several disconnected products of GP

Here we provide two examples of the processes that can be sketched out in a diagram like that in Figure 34.

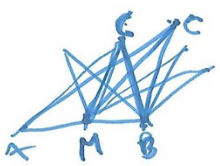
The first example is already presented and discussed in Section 6.6.3. It is taken from Silvia's interview: *Silvia_G10_T5_P1_(05:21 – 08:15)*. Here are the corresponding funnels.

Configuration 1 (*Drawing 3a*):



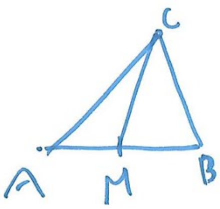
	Theoretical elements	**1**		**Figural elements**
(05:29)	Not coincident	X		B, M
	Points on the plane			B, M
	Always not congruent	X		CM, CB
	Not equal	X		CM, CB
	GP_1_(2) (discursive): CM cannot be equal to CB			

Configuration 2 (*Drawing 3b*):

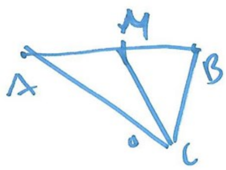


	Theoretical elements	2	Figural elements
(07:11)			CM
			CB
	Isosceles [tri]angle		[MCB]
	Sides of a triangle		CM, CB
	Congruent sides		CM, CB
			C (new position)
	[Line symmetry]		C
	GP_2_(0) (gestural): C is the vertex of an isosceles triangle and is located at the symmetric position with respect to AB		

Configuration 3 (*Drawing 4* and *Drawing 5*):



and



Silvia communicates two products of GP: the first is a quite fuzzy and incoherent product and it concerns the impossibility of having two equal segments within the configuration obtained by the step-by-step construction; the second product is detailed and coherent. As highlighted, the process that leads to GP_2 is supported by figural elements: the discovery within the drawing that there exists a position for C so that CM appears equal to CB. The obtained configuration (see *Drawing 3b*) supports the introduction of a theoretical element: Silvia describes the configuration (CMB) as an isosceles triangle. This theoretical element is recognizable into the funnel and is followed by two other ones: "*congruent sides*", which is directly deduced from the first new theoretical element; "*line symmetry*", which arises as an instance of GP.

The configurations and the funnels can be sketched out in a diagram that contains two disconnected products of GP (Figure 35). We do not add any blue arrows between the two funnels because the solver does not recall either theoretical or figural elements previously used to communicate GP_1 nor the product of GP itself.

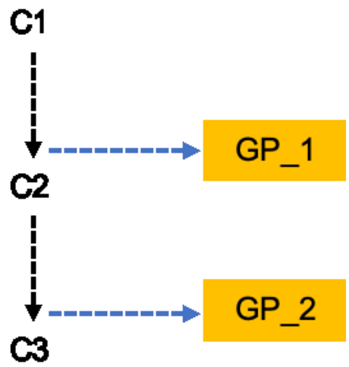
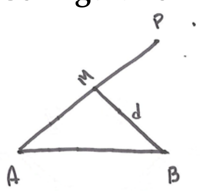


Figure 35 The diagram summarizes the configurations and the products of GP produced by Silvia during the resolution of Task 5

In this example, the first product of GP is rejected because a new solution arises. The solver seems to change her mind about GP_1 because of the support of figural elements and interviewer's prompts. Indeed, at time 06:48 and after GP_1 was communicated, the interviewer asks if there exist some positions for C so that CM is equal to CB. This prompt seems to induce the solver to investigate further the situation.

The second example is from Giacomo's interview during the resolution of Task 2 and it is already contained in Section 6.6.2: *Giacomo_G9_T2_P2_(01:38 – 05:55)*. Here are the corresponding funnels.

Configuration 1 (*Drawing 1a*):



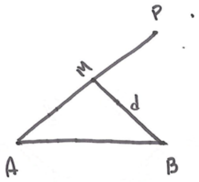
	Theoretical elements	**1**		**Figural elements**
(01:47)	Point outside a figure			[P]
	Geometrical figure			ABM
	Half length	X		MP, AM
	Equal length			MP, AM
	Half length			MP, AP
	Point on the plane			P
	GP_1_(2) (discursive): countless number of positions for P			

No configuration

The solver does not perform any gesture or drawing.

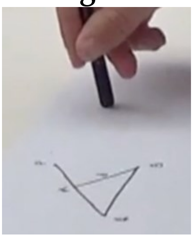
New sequence

Configuration 1 (*Drawing 1a*):



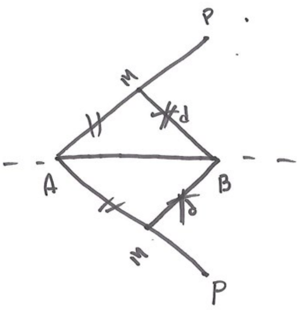
	Theoretical elements	2		Figural elements
(03:51)	Fixed point	X		P
	Fixed point	X		P
	Fixed length			MB
	GP_2_(1) (discursive): P is fixed			

Configuration 2 (Gesture):



	Theoretical elements	3		Figural elements
(04:35)	Anticipatory Intuition			
	[Line] Symmetry			[ABM]
	[Axes of symmetry]			AB
	Constant length			MB
	[Line] Symmetry			MB
	GP_3_(0) (gestural – discursive): P at the symmetric position with respect to AB			

Configuration 3 (*Drawing 1b*):



Giacomo undertakes three GP processes; he communicates two incoherent products of GP and a final coherent product. As previously highlighted, GP_1 is quite fuzzy: the solver does clarify where he intends to place P; the funnel ends without a new configuration and it does not contain new coherent theoretical elements. At times 03:24 and 03:35 Giacomo explains that GP_1 refers to the situation where the step-by-step construction is not already performed, and the first configuration is not drawn.

At time 03:51, Giacomo focuses again on the first drawing and he starts a new GP process. The starting point is identified in the window gesture which addresses a possible position of P. It seems that the impossibility of reaching a new position for P that is coherent with the constraint upon the length of MB leads the solver to conceive the configuration statically.

The question for explanation induces a new GP process that finishes with the communication of GP_3. It comes as an evidence of anticipatory intuition. As usual, looking at the funnel we find a sequence of new coherent theoretical elements before GP_3.

The chain of configurations and the sequence of products is reported in Figure 36.

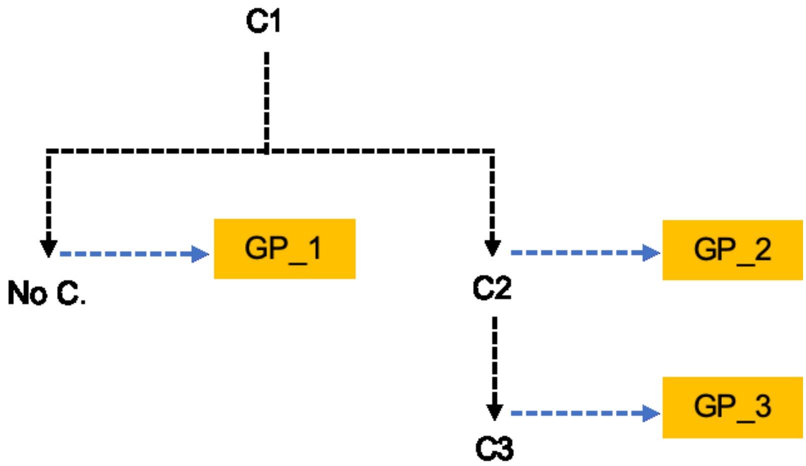


Figure 36 The diagram summarizes the configurations and the products of GP produced by Giacomo during the resolution of Task 2

Within the diagram we recognize two sub-diagrams: the first finishes without a configuration and the second with *Configuration 3*. The products of GP are not connected. Indeed, GP_1 is disconnected with GP_2 and GP_3 because the solver explicitly claims that he was not reasoning about the same configuration. So, the answer to the second question seems to be perceived as a solution to a problem different from that one proposed thought the first question. GP_1 is quickly left. GP_2 and GP_3 are disconnected because we were not able to find any elements of continuity between the two funnels. Probably this is the case because GP_3 comes suddenly as an anticipatory intuition and it is in sharp contrast with GP_2. Moreover, we notice that the third funnel is very different from the others: it contains a good number of new coherent theoretical elements that support a coherent product of GP. After GP_3 has been communicated, GP_2 is left.

After GP_3 the solver does not express further products of GP. Probably because of the fixedness of the configuration, the solver cannot reach another position for P. As previously highlighted, the fixedness could be caused by the solver's superimposing of theoretical constraints. Indeed, we infer from the drawing (see *Configuration 3*) that Giacomo adds another constraint: "*the length of AM must be fixed*". With this additional constraint, GP_3 is a complete solution to the problem.

8.1.3 Case (c): chains of configurations with connected products of GP

This is the most heterogeneous case, but also the most interesting in terms of unveiling how a product of GP can arise, evolve or be refined during the solution process.

The diagrams do not have a prototypical structure, as in the previous cases. Nevertheless, we recognize some common features:

- there is one recognizable initial product of GP, which is connected with one or more of the others;
- there is one recognizable final product of GP, in which other products of GP converge.

Looking jointly at the transcription table and the funnels, we can distinguish two subcases:

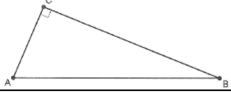
1) several connected products of GP add details to an initial rough one;
2) several (potentially independent) products of GP contribute to express a new product.

In the following we will provide examples of the two subcases and discuss in greater depth the underlying processes.

We will provide two examples of the first subcase: one from the resolution of Task 4 and another from the resolution of Task 2.

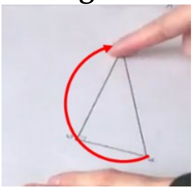
The first example is taken from Fiorella's interview during the resolution of Task 4 and it constitutes the whole first part of her interview: *Fiorella_MD_T4_P1*. The transcription table has been entirely reported in Section 6.2.1 as an example of several detailed products of GP produced through a top-down process. Here are the funnels.

Configuration 1 (Given drawing):



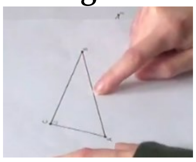
GP_1_(0) (discursive – gestural): the locus of C is a semicircle

Configuration 2 (Gesture):



GP_2_(0) (discursive – gestural): the locus of C is a semicircle centered at the midpoint of AB

Configuration 3 (Gesture):



Theoretical elements		3	**Figural elements**
(00:47)	Hypotenuse		AB
	Fixed/Constant length		AB
			C
	Theorem*		C
	Right		C
	Right-angled triangle		[ABC]
	[Line symmetry]		C[′], AB
	Perpendicular		Segment C[-point on AB]
			Segment [point on AB-C′]
	[Line symmetry]		C

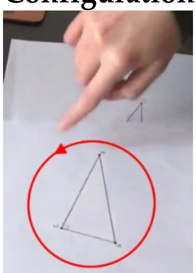
GP_3_(0) (gestural - discursive): C symmetric point with respect to AB

Theorem*: *"Since C is...the vertex of a triangle that lies on a semicircle circumscribed to the triangle, it is always right."*

Configuration 4 (Gesture):

GP_4_(0) (discursive – gestural): C on a circle centered at the midpoint of AB

Configuration 5 (Gesture):



As we can see, Fiorella communicates four detailed products of GP and each funnel is connected to one another. Moreover, focusing on the content of the orange boxes, we notice that Fiorella communicates the first product of GP that is refined and completed by the others. Fiorella first says that the locus of C is a half circle (GP_1); the second product of GP specifies the center of the half circle and Fiorella carefully and coherently refers to it as the midpoint of AB. Indeed, she does not only point at the center (see *Configuration 3*) but she also geometrically explains how to find the point. So, GP_2 refines GP_1 and introduces additional details.

In the third funnel, the solver recalls a theorem for supporting her previous products of GP; we reported at the end of the funnel the solver's utterance which expresses the theorem. GP_3 is supported by several new coherent theoretical elements that lead the solver to communicate the existence of another position for C so that the given constraints are maintained invariant. Looking at the transcription table, we observe that GP_3 comes as the answer to the interviewer's prompt about other possible positions of C. In particular, the solver first points at the new position and then she explains how to geometrically find it. Fiorella says that she intends to construct the new point C as a "*point transformed from C*" according to a line symmetry; to do this, she would use a line through C perpendicular to AB and mark on the line the distance between C and AB. She actually talks about a perpendicular segment, but we can refer to a line without changing the geometric construction she would follow.

GP_3 quickly leads to another product of GP, without adding any other theoretical or figural elements, as if it were a direct consequence of the previous exploration. GP_4 expresses that the locus of C is a circle centered at the midpoint of AB. In this sentence we can recognize the contribution of the processes that have led to GP_2 and GP_3. The last product of GP is a refined and more complete version of GP_1. The corresponding diagram allows us to see this finding at a glance (Figure 37).

In this example the first product of GP is communicated very soon in an initial formulation. The process that leads to GP_1 is very condensed. The subsequent process of GP allows the solver to better communicate the features of the predicted locus. The process that ends with the communication of GP_3 is less condensed and it allow us to see how the theoretical elements play a role within such a process. Finally, GP_4 seems to be supported by the previously communicated products of GP.

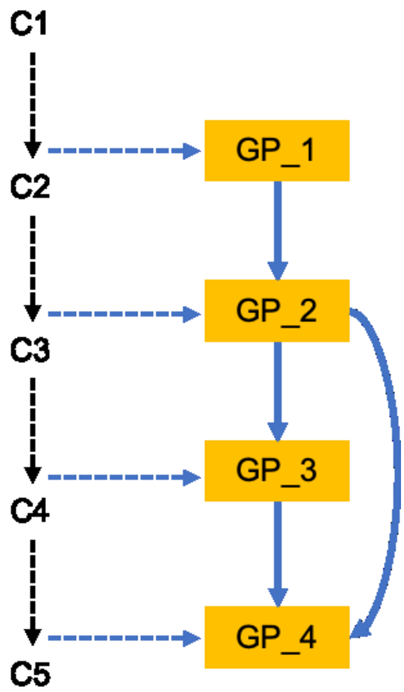


Figure 37 The diagram summarizes the configurations and the products of GP produced by Fiorella during the resolution of Task 4

The second example is from Agnese's interview during the resolution of Task 2. An overview of her products of GP is provided by the diagram (Figure 38), which is composed of three directed sub-diagrams. The first on the left (from C1 to C2) ends with a fuzzy product of GP; the process has already been discussed within Section 6.5.3. The second (from C3 to C4) ends with an incoherent and quite vague product of GP, which communicates the impossibility of finding another position for P so that the given constraints are maintained invariant. GP_1 and GP_2 address the problem of the possible positions of P and are quickly abandoned by the solver. We can consider these sub-diagrams as additional examples of Case (a).

Then Agnese focuses again on C3 and starts exploring the problem from a different point of view, wondering about the possible positions of M. The last sub-diagram on the right (from C3 to C7) sketched what happens during this part of the interview. The transcription table is reported in Section 6.4.4 as an example of a strong interplay between gestures and speech within the prediction process: *Agnese_MS_T2_P1_(03:55 – 05:14)*.

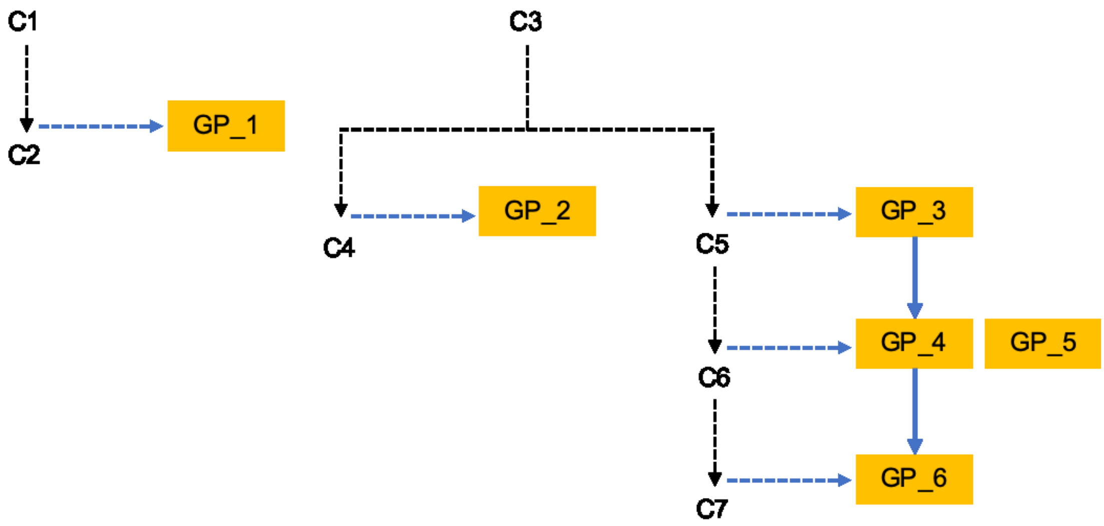
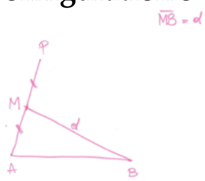


Figure 38 The diagram summarizes the configurations and the products of GP produced by Agnese during the resolution of Task 2

Since in this subsection we are not interested in analyzing the processes that lead to GP_1 and GP_2, here we include only the sequence of funnels that corresponds to the sub-diagram from C3 to C7.

Configuration 3 (*Drawing 2b*):

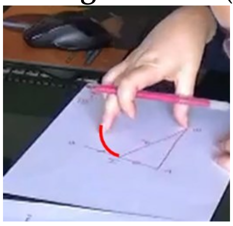


	Theoretical elements	**3**	**Figural elements**
(03:55)			A and B
			d [or MB]
	Rotation		d [or MB]
	GP_3_(1) (gestural – discursive): MB rotates		

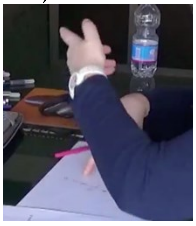
Configuration 5 (Gesture):

	Theoretical elements	**4**	**Figural elements**
(04:39)	Constant/fixed distance		MB=d
	Points on a circle		
	Circle (as a locus)		
	Center		B
	Radius		MB
			Points of the circle
	GP_4_(0) (discursive): points of the circle C(B, MB) are distant d from B		
	GP_5_(0) (discursive): P in a position such that MA=MP		

Configuration 6 (Gestures):

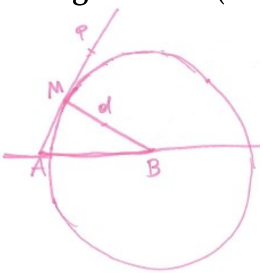


and



	Theoretical elements	5		Figural elements
(06:35)	Circle with fixed radius			MB
				[Arc of a circle through M]
				M
	Congruence			AM and MP
				Arc of a circle through M
	Point on a circle			M
	Tangent	X		
	[Point on a] perpendicular line	X		M
	GP_6_(0) (discursive): M on a circle [with radius MB]			

Configuration 7 (*Drawing 3*):



The first product of GP is coherent and quite vague, but it is connected with an iconic-physical gesture (see *Configuration 5*) that will support the subsequent prediction processes.

At time 04:39, Agnese undertakes another exploration of the problem. Looking at the funnel, we see that she recalls a given theoretical element (“*constant/fixed distance of MB*”) and introduces new ones. In particular, the circle and its points are first introduced as pure theoretical elements (see time 04:47). Then the radius and the center of the circle are made explicit also using their figural components. At the end of the funnel there is a very theoretically rich product of GP: GP_4. During the process the solver repeats many times the iconic-physical gesture which she first performs at time 04:15.

The same gesture is also repeated during the last process of GP. An instance of the gesture is recognizable in the second inferred figural element of Funnel 5. Another evidence of the funnels’ connection is provided by the presence of the theoretical element “*point on a circle*” referred to M, which was already used for communicating GP_4. The process starts after the interviewer's request for possible other positions of P; an imagined movement of P is asked for explicitly. Even if Funnel 5 contains two incoherent new theoretical elements, the product of GP is coherent and detailed. The elements added in *Drawing 3* confirm the coherence of GP_6.

Also in this example, we can recognize three products of GP which refer to the same figural element: the point M. GP_3 is recognizable as an initial product and it is quite vague. Instead, GP_6 is recognizable as a final and more detailed product. The previously accomplished processes of GP support the last one and the connected products refine the first product of GP.

For the second subcase, we will provide several examples from the resolution of different tasks.

The first example is from Marta's interview during the resolution of Task 2: *Marta_MS_T2_P1_(01:04 – 03:13)*. The transcription table was already reported within Section 6.3.3 and the excerpt has been discussed as an example of GP processes with a strong dynamic dimension. The diagram has the following structure (Figure 39).

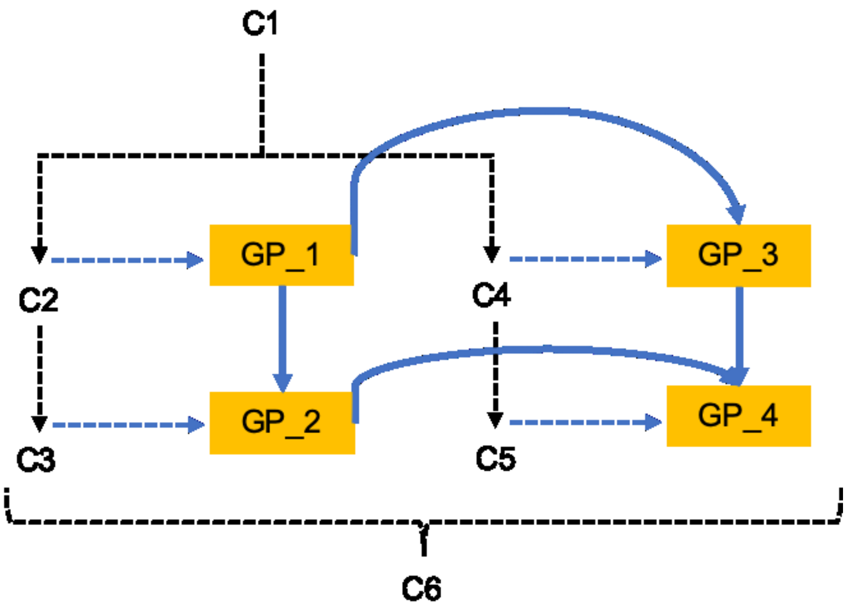
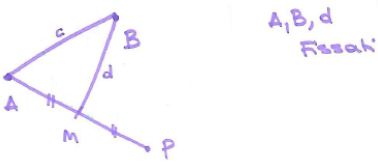


Figure 39 The diagram summarizes the configurations and the products of GP produced by Marta during the resolution of Task 2

We recognize two sub-diagrams and four products of GP, each of which is part of connected funnels. The figure shows that two parallel processes converge into a final product of GP: GP_4. Moreover, the four GP processes flow into a final configuration: C6. Below we list the corresponding funnels.

Configuration 1 (*Drawing 1b*):

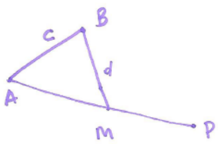


Theoretical elements		1	Figural elements
(01:08)	Fixed point		A and B
	Fixed/Constant distance		[AB]
	Distance		[AB]
			P
	Fixed/Constant distance		d [or MB]
	Fixed point		A and B
	Anticipatory Intuition		
	Fixed/Constant distance		[AB]
	[Rotation]		d [or MB]
	GP_1_(2) (discursive– gestural): several positions of MB		

Configuration 2 (Gesture):

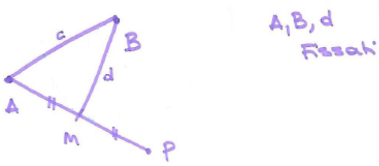
Theoretical elements		2	Figural elements
(01:49)	Variable length		AM
	[Variable length]		MP
	[Variable length]		AP
			P
	[Rotation]		[MB]
	GP_2_(2) (discursive): several positions of P		

Configuration 3 (*Drawing 2*):



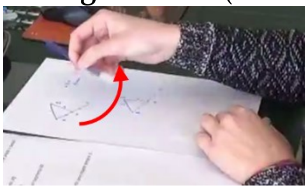
New sequence

Configuration 1 (*Drawing 1b*):



	Theoretical elements	3	Figural elements
(02:47)			[MB]
	[Rotation]		[MB]
	Circle center		B
	[Rotation]		d [or MB]
	Radius of a circle		d [or MB]
	[Rotation]		d [or MB]
	GP_3_(0) (discursive – gestural): [M on a] circle C(B,d)		

Configuration 4 (Gesture):

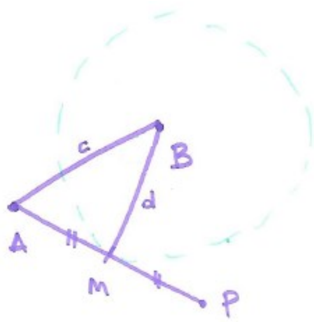


	Theoretical elements	4	Figural elements
(03:02)	Fixed point		A and B
			AM
	[Rotation]		AP
	[Rotation]		d [or MB]
	[Rotation]		P
	Point on a circle		P
	GP_4_(1) (discursive – gestural): P on a circle		

Configuration 5 (Gesture):

At the end, the solver produces the following additional configuration.

Configuration 6 (*Drawing 1c*)



The first GP process proceeds slowly and it involves an anticipatory intuition. Only by looking at the funnel, we can notice a long sequence of already-known theoretical elements and a final crucial new theoretical element ("*rotation*") which helps the solver reach a first idea about the possible positions of M. Indeed, the solver grasps that, even if MB must always have the same length, it could be placed in several positions.

GP_2 emerges as a consequence of GP_1. As elsewhere highlighted, the solver's utterances reveal a logical connection between the two products. At the end of the first path within the diagram, the solver has communicated that MB can occupy other positions and that *consequently* P can be placed in different positions as well.

Then, even if Marta has made another drawing, she does not focus any further on it and she comes back to her first drawing. Looking at and interacting with *Drawing 1b*, Marta undertakes another GP process and communicates GP_3. The funnel is very rich in new theoretical elements and ends with a detailed product of GP. Within the funnel we recognize a previously used theoretical element ("*rotation*"); this is the reason why we can define a connection between Funnel 1 and Funnel 3: the solver uses the same theoretical element which is supported by a similar window gesture for communicating two products of GP that are distant in time.

GP_4 comes as a consequence of GP_3. More specifically, the movement of M on the described circle induces the solver to conceive P as a point of another circle. The connection is stressed at times 03:02 and 03:13 by the solver's use of logic connections: "*therefore*" and "*so also*".

Funnel 4 seems to also be connected with Funnel 2. At the beginning, the solver seems to grasp that P can occupy more than one position, but she does not know which ones; after the accomplishment of the processes of GP which lead Marta to find the locus of M, she is more confident in communicating a possible locus for P. Actually, it seems to come as an analogy with the locus of M.

At the end of the prediction processes, Marta collected and reported some of her products of prediction in *Configuration 6*. The drawing is only used for making the already communicated products of GP explicit, as in the other cases of detailed products.

So, during Marta's resolution process, the communicated products of GP converge into a final one (GP_4). GP_4 is a refined version of GP_2 and its communication is supported by GP_3. Moreover, GP_3 plays a role in suggesting the locus of P through analogy. Instead, deductive reasoning seems to play a role in connecting GP_1 to GP_2. In this example, we can observe how the processes of prediction which address M and P proceed along parallel but interacting trajectories.

The second example is from Laura's interview during the resolution of Task 4. Three excerpts of this interview have been reported in two other sections (see Section 6.1.2, Section 6.1.4, Section 6.2.2). As previously discussed, Laura undertakes three GP processes and communicates three products: GP_1 is incoherent, the other two are coherent but quite fuzzy. Below is the summarizing diagram (Figure 40).

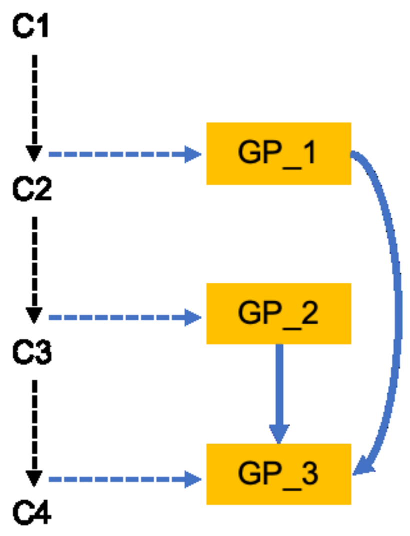
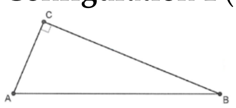


Figure 40 The diagram summarizes the configurations and the products of GP produced by Laura during the resolution of Task 4

Also in this case, the last product seems to be the outcome of three GP process. In particular, we will show that GP_3 gathers specific elements which belong to the previous processes. Below are the corresponding funnels.

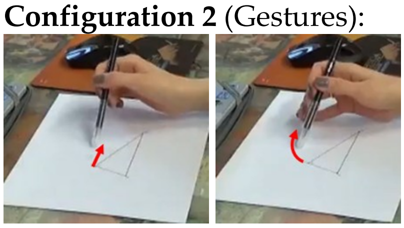
Configuration 1 (Given drawing):



	Theoretical elements	1		**Figural elements**
(00:38)	Fixed segment			AB
	Vertex			C
	Point on the plane			C
	No longer right			[C] or [triangle ACB]
	Point at a particular position			C
	Equilateral triangle	X		[triangle ACB]
	Right	X		Sides [AB, BC, CA]
	Right angle			C

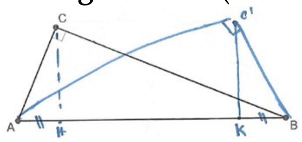
GP_1_(0) (gestural – discursive): C is the vertex of a right and equilateral triangle

Configuration 2 (Gestures):



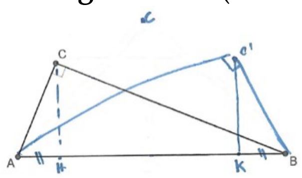
	Theoretical elements	2	Figural elements
(01:24)	Point on a parallel line		[C′]
			H
	Congruence		AH=BK
	Point on a perpendicular line		C
	[Parallelism]		CH [and C′K]
	Right triangle		AC′B
	GP_2_(2) (gestural – discursive): C at a symmetric position [with respect to the axis of AB]		

Configuration 3 (*Drawing 1a*):



	Theoretical elements	3	Figural elements
(02:18)	Isosceles triangle		
	Equilateral triangle	X	
	Equilateral triangle	X	
	Right angle	X	A, B, C
	[Midpoint]		C, CC′
	Right angle		C
	GP_3_(1) (gestural – discursive): C at a centered position [between C and C′]		

Configuration 4 (*Drawing 1b*):



Looking at Funnel 1 we can anticipate the incoherence of GP_1, because the only two new theoretical elements introduced by the solver are incoherent. As already discussed, the reason for such an inconsistency actually lies in the mismatching of two constraints: "*being a right triangle*" and "*being an equilateral triangle*". Nevertheless, the latter theoretical element is quite important within the resolution process, because it will play a role in the last process of GP.

Funnel 2 contains a sequence of new theoretical elements. Indeed, Laura explains how she intends to geometrically find the new predicted position of C. This process seems to be independent from the first one.

After the interviewer's request for possible other positions of C, Laura undertakes the last process of GP. At the beginning she refers to GP_2 and also to GP_1. In particular, she recalls the theoretical element "*equilateral triangle*". Although Laura quickly rejects GP_1, she keeps a part of the figural component of GP_1: the position of the vertex "*toward the center*". We can reasonably set up a connection between Funnel 1 and Funnel 3. Funnel 2 and Funnel 3 are connected as well. Indeed, Laura recalls GP_2 and makes use of the two positions of the vertex (C and C’) in order to find the third one. It seems that, at the beginning of the interview, Laura guesses the central position of C, but she does not exactly reach it. After having drawn C’, she seems to be more confident about the existence of such a position to the extent that she can communicate the product of GP in a statement.

This is an example of three processes of GP, two of which are independent. They lead the solver to a final product which appears as a collection of several interacting theoretical and figural elements previously expressed. We notice that when the resolution process does not proceed too quickly and when the solver is willing to further explore the situation, even the incoherent products of GP or some part of the process that leads to them can be useful for reaching a coherent product of GP.

The last example is taken from Emilio’s interview during the resolution of Task 5. Emilio communicates four products of GP. The processes that lead the solver to a solution are mainly top-down and Emilio uses the drawings only for making the products of GP explicit. Here is the summarizing diagram (Figure 41).

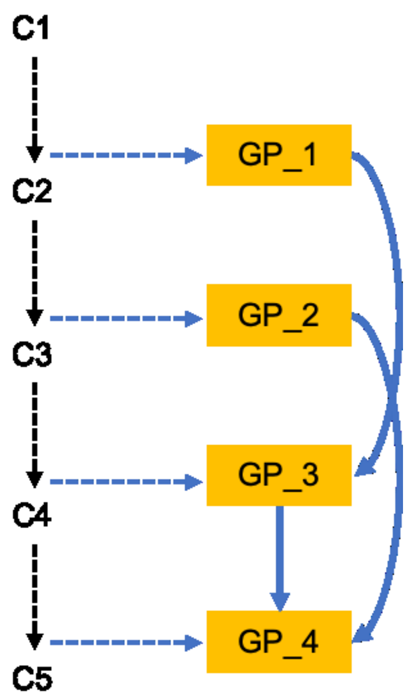


Figure 41 The diagram summarizes the configurations and the products of GP produced by Emilio during the resolution of Task 5

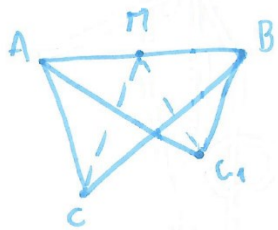
The diagram shows a first and a final product of GP (GP_1 and GP_4), and two products that are disconnected from the first one (GP_2 and GP_3). The previous processes contribute to the construction and communication of GP_4. Looking at the following funnels we can see the processes in detail.

Configuration 1 (*Drawing 1*):

Theoretical elements	1	Figural elements
(01:09)		C
Fixed points		A, B
Congruency		CB=CM
Fixed points		A, B
Equidistance		[C] such that CB=CM
		CM
Variable length		CM
Equidistance		[C] such that CB=CM
Triangle		
Triangle		[ABC]
Isosceles triangle		[MBC]
Two equal sides*		

GP_1_(0) (discursive): C in such a position that an isosceles triangle is formed

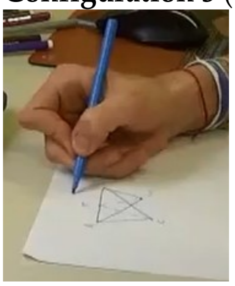
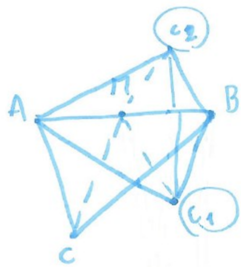
Configuration 2 (*Drawing 3*):



*"Two equal sides" is the definition of the isosceles triangle.
The solver recalls this property.

GP_2_(0) (discursive – gestural): C symmetric with respect to AB

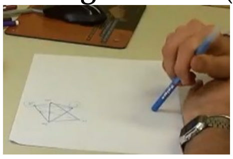
Configuration 3 (Gesture - *Drawing 4*):


and


Theoretical elements	3	Figural elements
(04:00)		[C]
Point on the height		C and height [of MC_1B isosceles]
Equidistance/Congruency		CM=CB

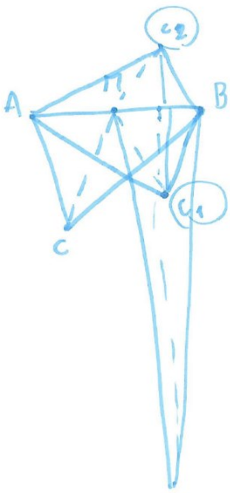
GP_3_(1) (discursive – gestural): C on the height [of the isosceles triangle MBC]

Configuration 4 (Gesture):



Theoretical elements	4	Figural elements
(04:11)		[C]
Point on the height		C and height [of MC_1B isosceles]
Midpoint		Point between M and B
Segment through a point		Height [of MC_1B isosceles]
[Line symmetry]		[C]
Point on the height		[C]
Midpoint		Point between M and B
GP_4_(0) (discursive): the locus of C is the height through the midpoint of MB		

Configuration 5 (Gesture and *Drawing 5*):



Emilio reaches a possible position for C, considering MCB as an isosceles triangle. The congruency between CM and CB is later introduced as a pure theoretical element. Below is the corresponding utterance.

Emilio: *Because I would make another triangle the triangle and so [with] the other triangle I could make an isosceles triangle if it is the one with two equal sides, if I am not wrong.*

The interviewer's question about the existence of possible other positions for C triggers a new GP process: Emilio reaches the symmetric position of C, through a line symmetry. We do not recognize any explicit connections between the first and the second funnel, so we have considered the two products to be independent.

The drawing that is part of *Configuration 4* suggests that the solver was considering the heights of the triangles. So, the "*point on the height*" is introduced as a theoretical element into the prediction process and it leads the solver to GP_3. Funnel 3 is connected with Funnel 1 because the isosceles triangle, first considered for the production of GP_1, is a fundamental component of GP_3.

Funnel 4 is the richest funnel in new theoretical and figural elements. Emilio predicts an entire locus for C so that the given constraints are maintained invariant. Moreover, the solver explains in further detail GP_3, making explicit that the height passes through the midpoint of MB. GP_4 is carefully described and it is very detailed.

Furthermore, GP_4 emerges as a collection of theoretical information previously made explicit: the height of an isosceles triangle (Funnel 3), the line symmetry (Funnel 2), and the isosceles triangle MC_1B (Funnel 1 and Funnel 3). So, we consider GP_4 connected with and derived from GP_3 and GP_2.

In this example, we can see how two seemingly independent GP processes (sketched out in Funnel 1 and Funnel 2) and a connected one (sketched in Funnel 3) converge into a final process. This chain of funnels allows Emilio to reach a coherent product of GP that constitutes a complete solution to the problem.

8.1.4 Analyzing connections: a further example from the resolution of Task 6

Looking at the diagrams that summarize the outcomes of solvers' GP processes during the resolution of Task 6, we have drawn an unexpected finding. We recall how Task 6 is composed of four questions that could be considered as four independent open problems, each of which expresses an additional constraint on the given configuration. The solver can answer each question communicating one or more products of GP. Analyzing the interviews, we expected to obtain a diagram with disconnected funnels and products of GP. Indeed, the point symmetry is the only common element of the four questions. So, Filippo's diagram is one of the expected outcomes (Figure 42).

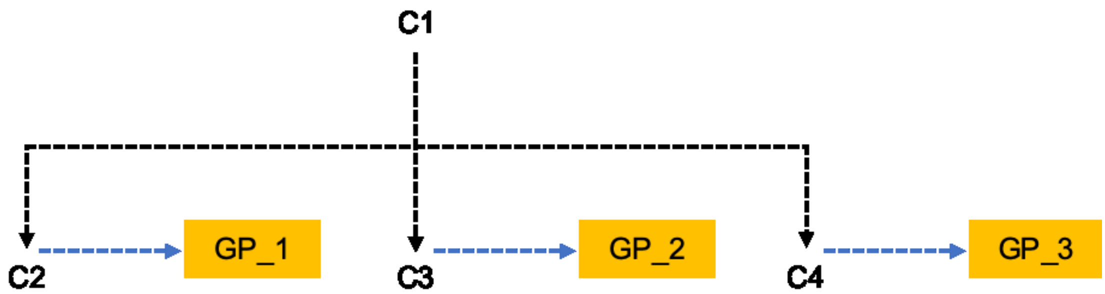


Figure 42 The diagram summarizes the configurations and the products of GP produced by Filippo during the resolution of Task 6

There are only three vertical paths because Filippo has spontaneously considered the second case (P on a line) within the answer to the first question.

However, we have found another solvers' approach to the task: it is closer to Case (c), more than to Filippo's.

The best example is provided by Emilio's interview. Below is the corresponding diagram (Figure 43).

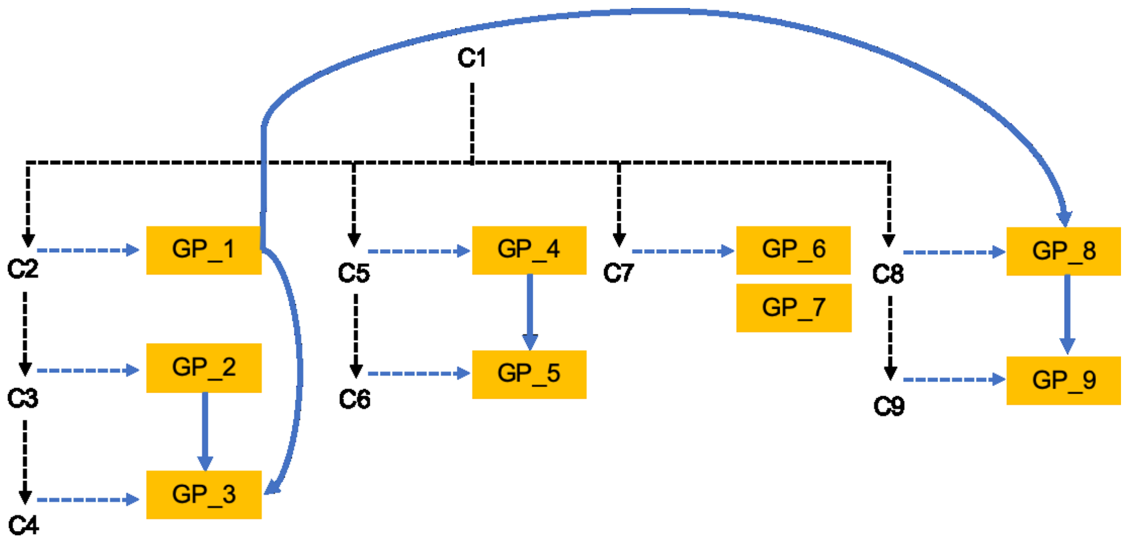


Figure 43 The diagram summarizes the configurations and the products of GP produced by Emilio during the resolution of Task 6

As expected, in the diagram we recognize four vertical paths and each of them starts from the first configuration (C1). The diagram is quite heterogeneous, containing both connected and disconnected funnels and in particular revealing a far connection between the initial and the final processes of GP.

Below are funnels describing in greater depth Emilio's processes.

Configuration 1 (*Drawing 1a*):

	Theoretical elements	1	Figural elements
(01:09)	Symmetry		
	Point [on the plane]		Q
	Fixed point		A
	Diameter		QP
	Circle		

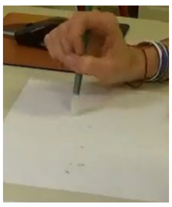
GP_1_(0) (discursive - gestural): P and Q on a circle with diameter PQ

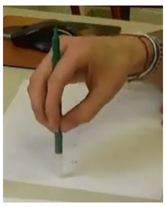
Configuration 2 (Gesture):

	Theoretical elements	2	Figural elements
(01:28)	Point [on the plane]		P
	[Point symmetry]		Q

GP_2_(1) (gestural): Q at the symmetric position with respect to A

Configuration 3 (Gestures):



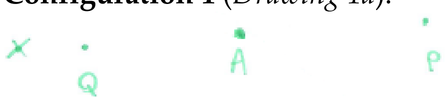


GP_3_(1) (discursive - gestural): [P and Q on] concentric circles

Configuration 4 (Gesture): he repeats the same gesture of *Configuration 2*

New question (P on a line)

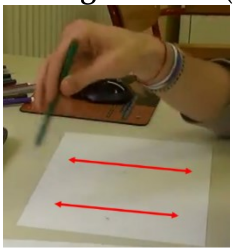
Configuration 1 (*Drawing 1a*):



	Theoretical elements	4	Figural elements
(02:05)	Opposite points		P and Q
	Point on a parallel line		Q
			Line through P
	Triangles		

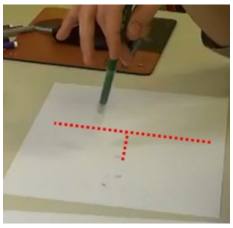
GP_4_(1) (discursive – gestural): P and Q on parallel lines

Configuration 5 (Gesture):

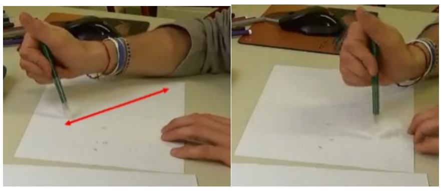


	Theoretical elements	5	Figural elements
(02:26)	Parallel line		
	Line		AP
	Perpendicular line		Line through P
	Right-angled triangles		[A, B and …]
	Triangles		
			Line [through P]
	[Point symmetry]		Q
	GP_5_(2) (discursive – gestural): P is the vertex of several triangles		

Configuration 6 (Gestures):



and

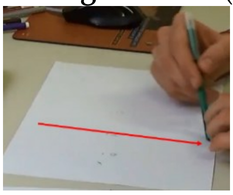


New question (A on a line)

Configuration 1 (*Drawing 1a*):

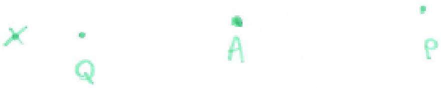
	Theoretical elements	6	Figural elements
(03:02)	Symmetric point		Q
	Fixed point		P
	[Point symmetry]		Q
	[Symmetry]		Q, A
	Equidistance		[AP=AQ]
			P, Q
	Parallel line	X	Line through P
	Parallel line		Line through A
	Parallel line		Line through Q
	Line		Line through A
	Parallel line		Line through Q
			Line through A
	GP_6_(1) (discursive – gestural): Q, A, P on three parallel lines		
	GP_7_(1) (discursive – gestural): Q on a line parallel to the line through A		

Configuration 7 (Gesture):



New question (P on a circle)

Configuration 1 (*Drawing 1a*):



GP1 "*as I said before*"

	Theoretical elements	**7**	**Figural elements**
(05:55)	Center of the circle		A
	Point on a circle		Q
	Semicircle		Arc PQ
	Semicircle		Arc QP
	Circle		

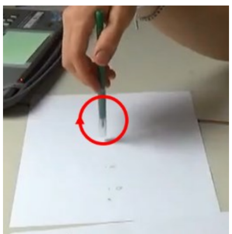
GP_8_(0) (discursive – gestural): P and A on a circle centered at A and with diameter PQ

Configuration 8 (Gestures):

	Theoretical elements	**8**	**Figural elements**
(06:15)	Circle		
	External circle through a point		[Circle through P]
	Point on a circle		P
	Point on a circle		Q
	Symmetry		Symmetric circle

GP_9_(0) (gestural – discursive): Q on a symmetric circle [with respect to A]

Configuration 9 (Gestures):



The first product of prediction comes as the answer to the first question about the configuration while the point P is imagined to be (randomly) moved. As the funnel reveals, GP_1 is supported by the theoretical element "*symmetry*", explicitly mentioned by the solver at time 01:09.

Emilio: *So if this principle of symmetry remains ...also point Q will move and point A will remain fixed. And so we could consider for example... Q and P as a hypothetical diameter of a circle that...ehm...we could go draw.*

The already-known theoretical element leads the solver to introduce three additional theoretical elements that describe a locus. This is further shown through the gesture.

GP_2 is quite simple, as we expected by only looking at the theoretical elements in the funnel, and it only recalls the effect of the point symmetry on Q.

GP_3 comes right after GP_2 and it is connected with it. The logical connector "*so*" suggests such a relation as well. GP_3 is also connected with GP_1. Indeed, it seems to be a more refined version of GP_1 and it reveals that the solver conceives the circle through P and Q as two circles with the same center. The adjective "*concentric*" could be considered as a synonym of "*coincident*" or "*overlapping*". Moreover, the catchment observed in gesture (see Configuration 2 and Configuration 4) suggests a connection.

A new sequence of prediction processes starts when the interviewer introduces the second situation (P is moving on a line) and asks for a new prediction about the whole configuration. The first product is GP_4: it is detailed and the window gesture reveals a strong theoretical control of the point symmetry over the configuration. Emilio mimics a line through Q moving the pen from the top down on the sheet of paper; immediately after, he mimics a line through P moving the pen from below upwards on the sheet of paper. The last new theoretical element is quite interesting because it is recalled during the subsequent process and it supports the description of GP_5. Although it remains an isolated speculation of the solver, it is an attempt to describe a prediction with a strong figural nature. Indeed, the solver seems to crystallize the configuration, considering the line through P as a static figural element. He reviews the line as a part of a triangle composed by a horizontal line, a perpendicular segment through P, A, Q and probably the several positions of P on the line. The product of GP is quite fuzzy but it communicates well what the solver is predicting.

A new sequence of prediction processes starts when the interviewer introduces the third situation: A is moving along a line. The point symmetry seems to also support this process: it is the first theoretical element of the funnel and it appears again in a new formulation as the equidistance between the three points. Although moving A along a line can give rise only to two parallel lines, Emilio talks about three parallel lines because he also considers the case when P is moving on a line. He specifies that he is considering two cases: if P is fixed, or if P is moved. Nevertheless, the dependency of Q from A and from P, and the point symmetry are not questioned. This reflects an awareness of the given constraints and induced, in which we recognize a strong theoretical control over the figure.

We notice that GP_6 and GP_7 are supported by several new theoretical elements, but also by new figural elements: several lines to which he refers as they were already drawn on the sheet of paper.

Finally, the last sequence starts when the interviewer introduces the fourth situation (P is moving on a circle) and asks for a new prediction about the whole configuration. The funnel is the richest in theoretical elements and at the beginning the solver recalls GP_1. Below is the corresponding utterance (time 05:55).

Emilio: *Mmm...well...**as I said before** indeed if...A becomes the center of the circle, Q will complete the circle with...*

The two funnels are actually connected because the solver explicitly talks about the connection. Moreover, we find an evolution of a theoretical element. Initially the point A is a "fixed point" (Funnel 1) and now it is a specific fixed point: the center of the circle. We guessed that Emilio has conceived A as the center, but now we find evidence of our inference and the property of A is unveiled also for the solver.

Looking at Funnel 8, we can see several theoretical elements that suggest the connection with the previous funnel. This is also stressed by the solver who refers to GP_9 as another case of GP_8.

We can consider Emilio's as an additional example of Case (c). Even though the vertical connection between funnels that address the same question is expected, it is not so for the horizontal connection. In particular, we have found: a product (GP_8) that is reviewed as a refined version of GP_1; a product (GP_9) that constitutes another answer to the first question. Emilio's resolution of Task 6 shows that the processes of GP interact, even when this is not expected. This finding stresses once again the complexity of GP processes.

8.1.5 Concluding remarks about the structure of GP processes

Looking at the funnels and at the diagrams, we observe the interaction between the processes of GP, even when they were undertaken and accomplished in different time frames. As a consequence, the products of GP arise and can be organized in several structures listed below.

a) A chain of configurations with a single product of GP.

   Only in two cases the product of GP is the most complete and rich in theoretical elements. In those cases, the solvers have a long mathematically successful experience (i.e. they are mathematicians). Moreover, they show good theoretical control over the figure, which probably suggests to them that the product of GP is actually the most general and complete that they could reach.

   In all the other cases, the product of GP does not provide a complete solution to the problem and it shows a rather rigid use of the configuration. The product of GP arises as a consequence of the large number of constraints that the solvers have imposed on the figure. Moreover, the products of GP are very much connected with figural elements. All these factors seem to inhibit the solvers' reaching another solution or prediction.

b) A chain of configurations with several disconnected products of GP.

   This case is strongly connected with the presence of one or more incoherent products of GP. We have found two reasons why the funnels and the products of GP are not connected: the solvers change their mind about a product of GP and they find a new solution to the problem; the solvers provide answers to problems that they seem to perceive as independent, as if they were different tasks.

   In these cases, more than in others, the interviewer's prompts play a crucial role in guiding the solver within the resolution process.

c) One or more chains of configurations with several connected products of GP.

   There is not a prototypical diagram that sketched out this case. Nevertheless, within several diagrams, we can recognize a final product of GP in which the previous processes and products converge.

   The final product of GP can be a more refined version of an initial rough one. If this is the case, the previously accomplished processes of GP support the last one; the final product of GP is incrementally constructed adding new details that geometrically specify it.

Otherwise, the final product of GP looks like a composition of several other products. Interacting with theoretical and figural elements previously expressed, it comes as the outcome of several previously accomplished prediction processes.

In both cases, the solvers seem to conceive the question as part of the same problem, and the interviewer plays a very marginal role within the resolution process.

Moreover, the data suggest that even the incoherent information considered by the solver at the beginning can be later used for shaping a coherent GP.

The overview of the different structures of GP processes shown in this section provides evidence of the complexity of this topic. In some cases, the process can be very simple, linear, and easily grasped; in other cases, it can be very complex and composed of several interacting components of different natures. The latter are the most interesting cases according to our research purposes because they unveil the multifaced nature of the processes of prediction where the theoretical components play a role as do the figural ones.

8.2 Critical elements that could hinder the prediction processes

In the previous sections, we only touch upon some elements or approaches that, during the resolution of the given prediction open problems, could hinder the exploration or, in some cases, even the processes of GP.

Among them, there are an incoherent interpretation of the theoretical constraints given in the step-by-step construction; or a lack of awareness of the theoretical constraints induced by the given ones. So, the solver could be induced to consider additional constraints and to forget or modify some others. These issues are strongly connected to the solvers' lack of theoretical control over the figure.

Moreover, some solvers have shown a static or discrete approach to the tasks. The static use of the figural elements of the given configurations seems to inhibit the consideration of alternative arrangements for the solvers of our sample.

Another crucial element is the lack of harmony between the theoretical and the figural components of a geometric figure. In some cases, this is so prominent that the stereotyped image of the geometrical figure in focus strongly affects the exploration.

In this section, we will describe in greater depth the components of the solvers' approaches that could hinder the prediction processes in general and the communication of a coherent product of GP in particular. Furthermore, we wish to draw attention on two additional issues:

- the prevalence of bottom-up processes accomplished by the solvers could lead to a sort of "*devolution to the drawing*" that induces them to consider incoherent theoretical elements;
- the good theoretical control over the figure could be in contradiction with the solver's choice of theoretical elements.

In the following, we will discuss some of the claims listed above and also better describe and clarify these last two phenomena. Moreover, we will provide examples from the resolution of each task. Indeed, even if we can draw some common approaches which impede the solver in reaching a coherent product of GP, each task can show specific features and obstacles. So, for the sake of clarity, we intend to also stress these local findings.

8.2.1 Task 2

There are several solvers of our sample who show a common approach to the configuration obtained through the step-by-step construction given at the beginning of Task 2. Indeed, they seem to consider the configuration as composed by a triangle AMB and an independent segment MP. We have found various evidence of this way of seeing the configuration in solvers' utterances.

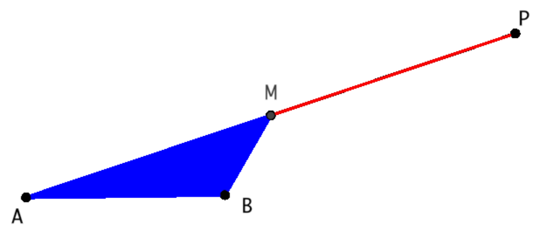


Figure 44 An instance of a possible drawing of the given configuration where the triangle (blue) and the independent segment (red) are highlighted

In our opinion, this approach is interesting within the analyses of prediction processes because apparently all the solvers who consider the configuration in this way had trouble reaching a product of GP or a coherent one.

An example is provided by Carlo's interview of which an excerpt is reported in Section 6.5.4 as an example of a process of GP that is undertaken very late during the interview: *Carlo_G10_T2_P1_(01:22 – 04:52)*. We have already highlighted that at the beginning the figural components of the figure seem to guide Carlo's answer. He simply describes the configuration sketched out in *Drawing 1a* and stresses that the point M is part of a right triangle. Actually, the figural components drive the answer because the right angle is not a consequence of the given theoretical constraints: the angle with vertex at M seems right only within the figural domain of the drawing.

The interviewer's prompt for triggering a GP process has the effect of restoring a figural approach for the solver. Carlo starts a new drawing (*Drawing 1b*) and continues referring to AMB as a right triangle. Moreover, he stresses that the triangle seems to be "*always*" right.

After a last effort of the interviewer to trigger a GP process, the solver communicates a product of GP:

GP_1: *P at a symmetric position with respect to A*

Nevertheless, the figural components continue guiding the exploration. Indeed, in a subsequent excerpt, when the interviewer asks the solver to show the imagined position of P, Carlo performs two additional drawings, here reported (Figure 45).

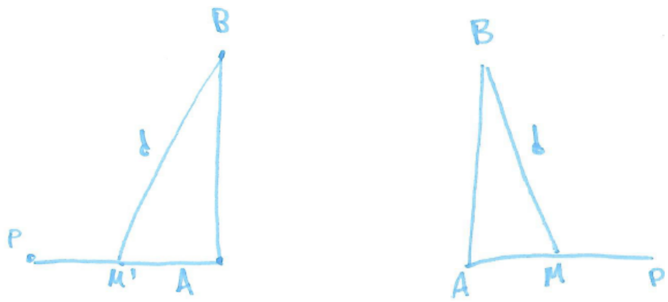


Figure 45 Two drawings consecutively performed by Carlo during the resolution of Task 2: *Drawing 3* (on the left) and *Drawing 4* (on the right)

Even though, referring to *Drawing 1b* and in general, GP_1 is incoherent, we notice that in those particular drawings (Figure 45) the new position of P works. This highlights again that the figural components drive the exploration and in particular suggest the solver to maintain the whole triangle.

At the times 06:35 and 06:46 we can see the crucial role that the triangle AMB plays in the resolution process. The corresponding utterances are:

Carlo: […] *if this triangle* [triangle AMB in *Drawing 1b*] *we move it over to this side* [triangle AMB in *Drawing* 2], *we transfer it…in any case it is moved in a different way*.

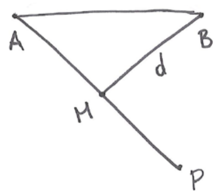
Carlo: *I mean I can copy the triangle wherever I want.*

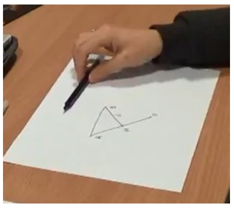
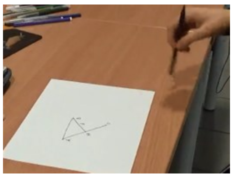
GP_1 remains the only product of GP communicated by the solver. The utterances reveal the two elements that impede the solver to reach another and eventually more coherent product of GP:

- the configuration seems to be considered as composed of a triangle and an independent segment;
- the triangle is rigidly maintained in order to show another position for P.

So, the domain of figural elements and argumentations coupled with the fixedness of the configuration does not allow the solver to explore further the situation.

Another example is provided by an excerpt from Ilaria's interview: *Ilaria_G9_T2_P1_(01:48 – 03:26)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
01:48.12	Int	What can you say about the point P?		
01:52.19	Stud	The point P…meanwhile is part… of the straight line in which…on which there is also… AM and is outside the triangle AMB.	On the sheet of paper there is the following picture:  *Drawing 1a*	She seems to consider the configuration composed of a triangle and an "external" segment. Theoretical elements: triangle, line.
02:13.27	Stud	In this case – but I think it is a particular case – perpendicular to… not the point P! It is a straight line on which there is the point P that is perpendicular to MB, but it is a particular case, I think!		She seems to be uncertain about the perpendicularity between MB and the line.
02:31.09	Int	Ok. Make a prediction. Do you think that the point P		

		can have other positions?		
02:39.20	Stud	If I draw again…if **the length of MB must always be d**, it could be **like a mirror**, so it could take the place…the same position, only **on the other side of the segment**.	She points to the step-by-step construction. She places the pen on AB and moves it upwards. 	**GP_1_(2) (discursive – gestural):** **P on a symmetric position with respect to AB** **Window gesture**
03:00.14	Stud	Ehm and then...there are no other ...other positions that it could occupy if the length has to always be the same d as the segment MB.		
03:13.20	Int	Ok...Imagine... Make a prediction: imagine moving point P. Do you think it could occupy other positions so that MB remains of length d?		
03:26.09	Stud	Yes, **it** [point P] **moves along** the segment M…along **the half-line MP** and then etcetera, etcetera… **along the half-line that would continue**, because it does not interfere with the triangle.	She points at P and starts moving the pen on a straight line: 	**GP_2_(0) (discursive – gestural):** **the locus of P is a half-line from M** **Window gesture** She seems to look at the configuration as composed by a triangle and an independent segment. She seems to imagine moving P without changing the triangle. The drawing is independent from the geometrical construction.

The excerpt starts right after the solver has accomplished the step-by-step construction and when the interviewer asks the first question about P. In her answer Ilaria collects the observable figural elements of the configuration: P is on a line, AM is part of the line, AMB is a triangle. Moreover, she stresses a topological property: P is "*outside the triangle AMB*". The description of *Drawing 1a* suggests that the solver is considering the configuration as composed of a triangle and a point P, which eventually is connected to the triangle with a segment. She adds that AP is perpendicular to MB, but she seems to be aware that this could be a special case whose rationale lays on the figural domain. We will see in the following that the perpendicularity does not affect the subsequent exploration; instead, the triangle AMB plays a crucial role.

At time 02:32, the interviewer asks for other possible positions of P. The first utterance suggests a discrete approach to the configuration: the solver predicts a symmetric position of P with respect to AB and she intends to find it "drawing again" the figure. The product of GP is coherent, quite fuzzy and it maintains the length not only of MB but also of the sides of the whole triangle. At time 03:00, Ilaria stresses that to maintain the length of MB, P cannot occupy other positions.

The fixedness of the configuration, and in particular of the triangle AMB, affects the second product of GP. The process of GP seems to be triggered by the interviewer's question about the possible positions of P such that MB has the same length; it is explicitly required to consider the motion of P. Ilaria says that P can be moved on a half line from M; she explains that it is so "*because*" the movement does not infer with the triangle. We can notice the extent to which the triangle plays a role within the prediction process: even though the solver has previously communicated a coherent product of GP, GP_2 does not consider all the given constraints. Indeed, the described movement of P changes the length of the segment AP and consequently of the segment AM, but the solver seems to forget that M is the midpoint of AP and she considers AM and MP as independent segments.

GP_1 maintains all of the given theoretical constraints and some additional ones. However, during the process that leads Ilaria to GP_2, she does not show an awareness of these constraints. In general, the figural components of the configuration dominate the process. It looks like, after the step-by-step construction was accomplished, the properties addressed to the obtained configuration are different from the given ones and mainly relies on the figural elements. This is what we have called a sort of "*devolution to the drawing*".

Ilaria's excerpt is an example of a bottom-up process and of the focusing on figural elements which affect the GP process to the extent that the solver reaches an incoherent product of GP even though she has previously communicated a coherent one.

The same phenomenon can be observed in Isabella's interview (also presented in Section 6.6.2 and Section 7.1.2). We have already noticed that also Isabella seems to consider the configuration as composed of a fixed triangle and an independent segment. Moreover, the configuration described in GP_1 (P on a circle centered at M and with radius MP) has new constraints and loses a given one (i.e. M is the midpoint of AP). At times 04:52 and 05:15, she describes in greater depth the locus she has imagined: she intends to translate the segment MP that she considers as the radius of the circle. Looking at the second part of the interview, we can see a drawing that confirms our inference about the configuration (Figure 46).

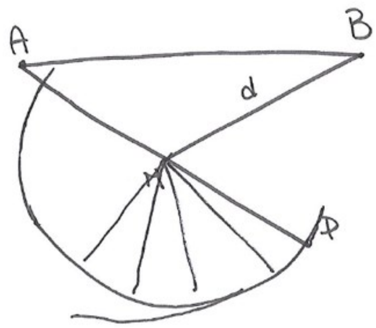


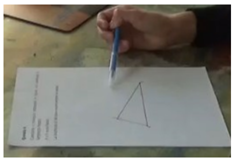
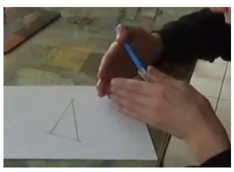
Figure 46 ***Drawing 1c*** **performed by Isabella during the second part (time 09:46) of the resolution of Task 2**

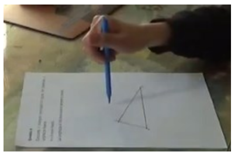
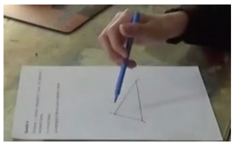
In Isabella's resolution the figural components are dominant to the extent that, after she has performed the drawing, the solver seems to forget the properties given in the step-by-step construction.

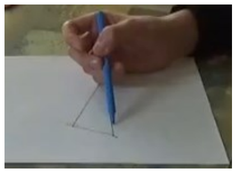
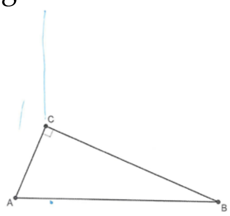
8.2.2 Task 4

The task does not present particular common obstacles and most of the solvers who approach the task reach at least a new position for C. Nevertheless, also the resolution of this task provides examples that support the role that a rigid use of the configuration plays within the prediction processes. Furthermore, some excerpts give us the opportunity to highlight that the prototype effect negatively influences reaching a coherent product of GP.

The first example is from Carlo's interview: *Carlo_G10_T4_P1_(00:58 – 03:47)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:58.21	Int	What can you say about the vertex with the right angle?		
01:07.08	Stud	That is can move wherever it wants.		**GP_1_(0) (discursive): C can occupy any position within the plane**
01:12.16	Int	How?		
01:13.06	Stud	In any point that it wants ... wants, because... no, but if it moves over to that side the angle changes.	He looks up. He randomly moves the pen above the drawing. He looks up.	He seems to consider some positions for C.
01:21.06	Stud	If we move C to this side...can I?	He is pointing at C and then at another position: 	**Window gesture**
01:27.12	Int	Yes.		
01:29.19	Stud		He draws two points on the sheet of paper.	
01:30.09	Stud	They are fixed.		
01:36.03	Stud	**I cannot imagine how** C can move here if it has to maintain a right angle.	He points at the same position of 01:21. Then he uses two hands for mimicking the motion of the right angle:  He moves the two hands together from the left to the right.	**Window gesture**
01:47.10	Stud	Ah no, the triangle would move in any case.	The utterance comes after a long silence.	

01:50.11	Stud	Because if we move it here **it becomes an isosceles triangle**. So...	He points at another position: 	**Window gesture**
01:58.04	Stud	What was the question?		
01:59.08	Int	What can you say about the vertex with the right angle?		
02:02.04	Stud	**Only that point**...only that point...can...C is a right angle. Then the degrees change.	he points to the initial position of C on the drawing: 	**GP_2_(0) (discursive – gestural): C is fixed**
02:13.25	Int	Ok. So if I ask you to make a prediction and to tell me...		
02:19.23	Stud	But, but!		**Anticipatory Intuition**
02:20.12	Int	Mm!		
02:22.05	Stud	I think that there is always the **vertical line** that...because...	He points at C and moves the pen on a strainght trajectory, perpendicular to AB: 	**GP_3_(1) (gestural – discursive): C on a vertical line [perpendicular to AB]** **Window gesture**
02:31.22	Stud	Yes, I think that there are actually **infinitely many points** in which it can become a right triangle.		
02:38.29	Int	Which?		
02:41.00	Stud	These.	He starts drawing a segment through C and perpendicular to AB.	

02:43.16	Stud	Ah no, but if it goes here it becomes a straight angle.	He points at a point on AB: 	GP_3 (discursive – gestural)
02:46.09	Int	Mm.		
02:47.16	Stud		He is laughing.	
03:00.03	Stud	If it goes down there for sure it decreases until it becomes a straight angle.		
03:07.13	Int	Ok, tell me there. Wait, this one here later.		
03:15.06	Stud	Maybe it can only go up. If it goes up...	He draws a vertical segment:  *Drawing 1*	
03:24.10	Stud	No.		
03:26.18	Stud	It is that...I think that **only this is possible**.		GP_2
03:29.10	Int	Mm mm.		
03:29.29	Stud	But like if one makes **a straightedge.**		
03:31.25	Int	Mm mm.		
03:34.03	Stud	Ah...**one can make it bigger** and it is always a right angle. Eh.		
03:40.01	Int	In what sense bigger? How do you imagine it?		
03:43.14	Stud	**But the hypothenuse would have to get bigger I think.**		
03:47.03	Stud	So if the hypothenuse stays fixed maybe only...**there is only one possibility**.	He points at the initial position of C.	GP_2 (discursive)

The given drawing shows a right triangle which is sketched so that the hypotenuse AB is horizontal on the sheet of paper. Before the excerpt, Carlo has rotated the paper to see the triangle with a leg horizontal and the other leg vertical in front of him.

The excerpt starts with the interviewer's question about the vertex with the right angle. Even if the answer is quite vague, it communicates that C can occupy any position within the plane. However, we cannot infer where the solver intends to place C. At time 01:21, the window gesture suggests that another GP process is undertaken. Indeed, Carlo seems to investigate another configuration, which he would obtain placing C at a particular position. At time 01:36, he mimics the motion of the right angle. This is quite rigid; indeed Carlo tries to translate the whole angle. The impossibility of obtaining another right triangle with AB as the hypotenuse, explicitly expressed at times 01:36 and 01:50, leads the solver to say that C is fixed (GP_2).

GP_3 comes after an evidence of anticipatory intuition. Suddenly, Carlo communicates an entire locus for C: a line through C perpendicular to AB. GP_3 is sketched in *Drawing 1* and contains an incoherent product of GP: moving C along the line the angle is not maintained right. Probably the solver imposes over the figure so many theoretical constraints that impede to find other arrangements of the configuration. However, the locus of C seems very strange and it is difficult to properly guess what has driven the solver's process of GP. From time 03:39 we find some revealing utterances. We grasp that Carlo considers the right triangle as a whole fixed and rigidly given figure. The metaphor of the set square suggests this interpretation. Moreover, it seems that he has imagined several enlarged versions of the given right triangle or set square. Overlapping these figures, he could see the point C occupying many positions on a straight line. An attempt to reproduce the imagined situation is reported in Figure 47.

Nevertheless, at time 03:43, Carlo realizes that, if C is on the line reported in *Drawing 1,* one of the given constraints is not maintained invariant: he grasps that the hypotenuse could change its length. So, he restores GP_2.

The excerpt provides instances of two elements that could impede the solver reaching a coherent product of GP: the fixedness of the configuration and the consequent impossibility of conceiving other possible arrangements.

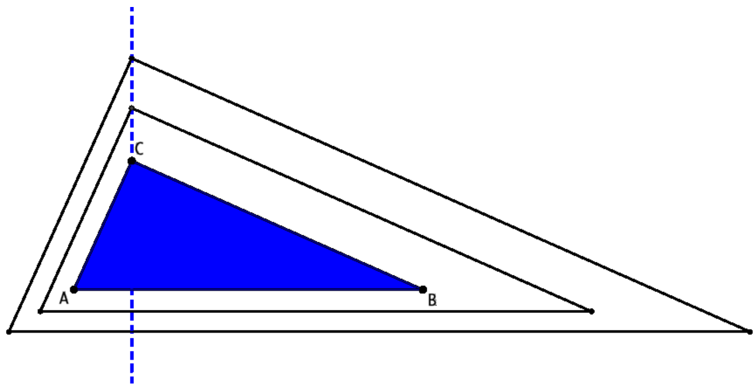
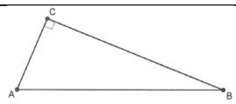


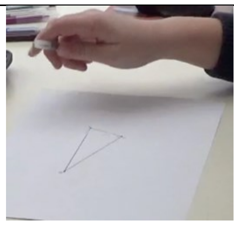
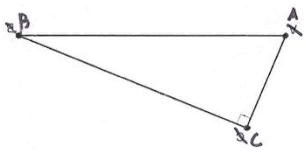
Figure 47 A possible interpretation of the configurations imagined by Carlo

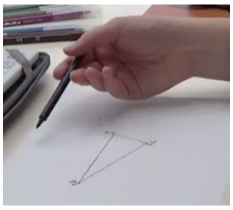
The solver does not show a lack of theoretical control over the figure (simply look at times 01:50 and 02:43) and he dynamically interacts with the drawing (see time 01:36). Nevertheless, it seems that he interprets the given constraints too rigidly: the fixedness of the length of AB could have induced the fixedness of the whole triangle. The figural counterpart of this situation becomes an obstacle for undertaking another process of GP.

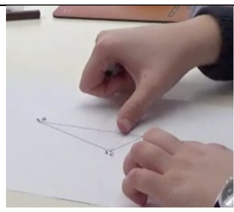
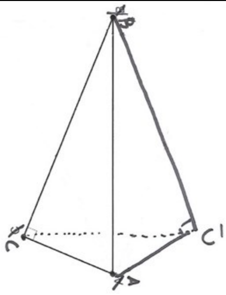
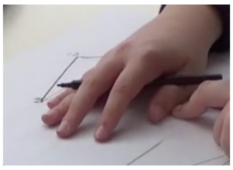
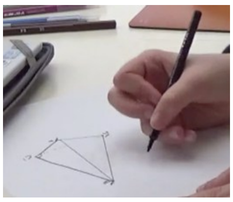
Another element that could inhibit an effective process of GP is the prototype effect. This is most evident in this task because it involves a well-known figure to which the interviewer refers explicitly: a right triangle. The awareness that the triangle must be right could induce a stereotyped image of it that could contrast with the particular drawing proposed to the solver. A first example of the prototype effect is provided by Ilaria's interview. It has been already discussed in Section 6.1.1.

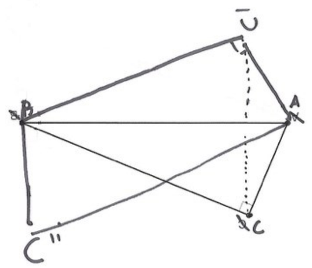
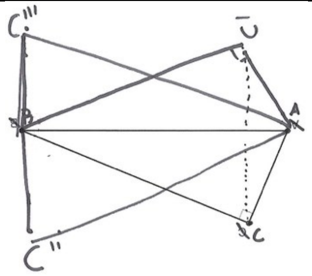
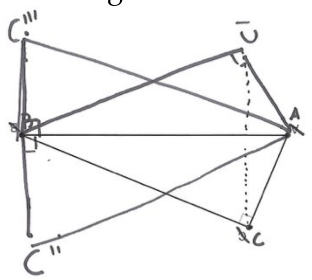
Another, and more telling, example is provided by Giacomo's interview: *Giacomo_G9_T4_P1_(00:03 – 03:54)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:03.12	Int	Consider the right triangle in the figure.	 *Given drawing*	
00:07.05	Stud	Ok.		
00:10.14	Int	With the hypotenuse of fixed length.		
00:13.02	Stud	Alright.	He rotates the sheet of paper:	

				
00:13.22	Int	A and B are fixed.		
00:16.08	Stud	Ok.		
00:17.03	Int	The length of AB has to always be the same.		
00:21.08	Int	Here there is everything written.	She points to the step-by-step construction.	
00:22.15	Stud	Ok.		
00:22.26	Int	Ok, this is your drawing.	She points to the sheet of paper with the given drawing.	
00:25.02	Stud	Ok, I drew it.		
00:27.29	Int	Ok, when you are ready, I will ask you the question.		
00:30.16	Stud		He rotates many times the sheet of paper.	
00:36.03	Stud	Can I...write on it?		
00:37.21	Int	Of course.		
00:39.05	Stud		He labels again the vertices:  *Drawing 1a*	
00:44.23	Stud	Ok, that is better.		
00:45.13	Int	Ah ok, you made the letters bigger?		
00:46.26	Stud	No no, I turned it because I was not at ease keeping it like this right angled.	He rotates the sheet of paper: first in the initial position and then in the new orientation (time 00:39).	
00:48.21	Int	Ah ok. Alright.		
00:51.03	Stud	This is better.	He takes the sheet of paper rotated so that the triangle appears lying on the longer leg BC.	

00:52.23	Int	What can you say about the vertex with the right angle?		
01:13.22	Stud		Long silence.	
01:19.05	Stud	What can I say about the vertex with the right angle?		
01:21.20	Int	Mm mm. Yes.		
01:23.01	Stud	I know that it is right. I mean a point with a right angle.		He recalls an already known property.
01:27.17	Stud	And then...	Then there is a long silence.	
01:50.29	Int	What are you thinking, Giacomo?		
01:52.17	Stud	I am thinking...I mean I know that it is right and that's it. **It is the only thing that I know about point C** I don't know anything else.		
01:59.01	Int	Ok. Make a prediction: do you think that it can occupy other positions?		
02:06.28	Stud	The point C?		
02:08.02	Int	Yes.		
02:08.14	Stud	So that AB is constant? They are fixed, ok.	He looks at the step-by-step construction.	The solver seems to recall some of the given constraints.
02:13.07	Stud	I hope it's not like the one before.		Probably he refers to Task 2.
02:16.09	Stud	Always **on the other side of segment AB, mirrored**.		**GP_1_(1) (gestural – discursive): C at a symmetric position with respect to AB Window gesture**
02:23.07	Int	Yes, show me how.		
02:24.20	Stud		He uses the thumb to have the same length:	He performs a sort of step-by-step construction for

			 Then he draws a new position for C and the adjacent segments.	drawing the exact position of C.
02:48.16	Stud	Like this.	 *Drawing 1b*	
02:48.17	Int	Ok.		
02:49.17	Stud	Like...in this way here.	He rotates many times the sheet of paper; he stops and covers half of the drawing: 	
02:52.25	Stud	And like this AB is always constant.		
02:54.15	Int	Mm mm.		
03:02.02	Stud	Then I can move this point C.		
03:06.14	Stud	In this position here, **moving all the figure**. And putting point C for instance here making it remain 90 degrees.	He points at a new position for C:  He draws the new position and the adjacent segments.	**GP_2_(2) (gestural – discursive): C at a position such that the triangle is right** **Window gesture**
03:14.04	Int	Mm mm.		

03:17.26	Stud	And like this AB is always the same, **but here I have C that is in this position here** and it is...AB is always constant.	 *Drawing 1c*	He maintains only some constraints: for example, AB is no longer a hypotenuse.
03:24.17	Stud	And so also in this case **on the other side**.	He points at a symmetric position of C with respect to AB: 	GP_1
03:33.16	Stud	And so I have another right triangle with A [and] B always constant.	 *Drawing 1d*	He maintains only some constraints.
03:44.04	Int	Right where?		
03:45.11	Stud	I mean it would be...	He points at C′′′ and looks at the drawing for a while.	He directly points at C′′′, but then he seems uncertain.
03:50.00	Int	Where is the right angle?		
03:51.13	Stud	Here, it is moved to B.	He signs the right angle at B. The drawing is the following:  *Drawing 1e*	Figural elements.
03:54.10	Stud	In these two cases.		

The excerpt starts at the very beginning of the interview. Right from the start it is evident that the solver feels uncomfortable with the particular arrangement of the figural elements of the triangle within the sheet of paper and he rotates it many times: see what happens at times 00:13, 00:30, 00:39. Finally, at time 00:46 he explicitly says that he has trouble with that particular drawing. So, he relabeled the vertices and, at time 00:51, he takes the drawing rotated so that the triangle looks lying on the longer leg (the side BC). We infer that the stereotyped image of a right triangle influences this initial phase of the resolution process.

The first interviewer's question does not trigger any prediction process. We cannot find any explicit clues that would suggest a possible reason.

At time 01:59 the interviewer asks the second question, that explicitly requests a prediction. The solver answers recalling one of the given constraints: A and B are fixed points. He does not mention either the constraint on the angle at C or the fixedness of the hypotenuse. The question triggers a GP process that finishes with the communication of GP_1: the gesture reveals where the solver intends to place C and the utterance suggests the geometrical transformation that allows him to find the new position of C. Probably the process is supported by the inferred similarity with the prediction process that was accomplished by the solver during the resolution of Task 2 (see time 02:13). GP_1 is carefully sketched on *Drawing 1b*; Giacomo performs a sort of step-by-step construction, starting from the distance between C and the segment AB. The configuration, which corresponds to the new position of C, rigidly maintains all the given constraints and additional ones. Indeed, we can notice that also the lengths of the legs are maintained invariant.

The solver seems to spontaneously[8] undertake another GP process that, at time 03:02, leads him to communicate GP_2. At time 03:06, the utterance suggests that the solver is trying to maintain the constraint on the right angle. However, the figure sketched on *Drawing 1c* reveals that he is maintaining an alternative version of the constraint: ABC is a right triangle. GP_2 is an incoherent product of GP. The given constraints are not maintained, indeed AB is no longer a hypotenuse and the right angle now is at B. Moreover, looking at the snapshot of the window gesture, we notice that the orientation of the drawing used to accomplish the process of GP follows one of the prototypical images of a right triangle (i.e. a right triangle that lies on a leg). So, it seems that the prototypical effect affects the process of GP. The solver does not seem to be aware of such a bias.

At time 03:24 the solver applies GP_1 to the new situation and he reaches another position for C. When the interviewer asks for an explanation about the right angle, Giacomo seems to be a bit confused. He points directly at C''' and probably

[8] We mean that the process is undertaken without a new explicit interviewer's request for a prediction.

realized that the angle is no longer right. Only looking at the drawing he can point at the actual vertex with the right angle: point B. So, it seems that the configurations which were obtained starting from C'' and C''' are so natural for the solver that he becomes aware of the actual location of the right angle only when the interviewer asks to show it.

Looking at the final drawing, we notice that also the positions of the labels of the vertices, and in particular their orientation with respect to the several triangles, suggest an influence of a stereotyped image of the right triangle.

We highlight that the solver seems to know and manage well the line symmetry. Indeed, he uses twice this transformation to reach two new positions of C. So, we would expect that he can reach the symmetric positions of C with respect to the axes of AB as other solvers did, but he did not. We advance the hypothesis that the prototype effect is so dominant within the prediction process that it strongly influences the possibility that the solver is willing to consider. Moreover, we stress that in the following the solver was not able to undertake other GP processes, despite the interviewer's requests.

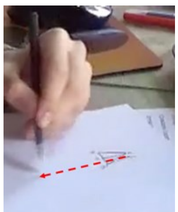
8.2.3 Task 5

At the very beginning, the task asks the solver to undertake a process of GP without the support of the drawing. As expected, this part of the task was quite difficult for some solvers. Probably, it depends on the number of information that the solver has to manage only imagining the situation. We have already discussed that when the solver undertakes a prediction process, the gestures are powerful windows onto the GP process (see Section 6.4).

To help the solvers who have trouble with not being allowed to draw, the interviewer can allow them to draw only the segment AB and the midpoint. In the following, we will focus our analyses on this part of the interviews.

The most common obstacle to undertake an effective GP process is the theoretical element that the solvers use for referring to CM. The segment CM is the median of the triangle ABC, even though it is not explicitly mentioned within the step-by-step construction. However, frequently the solvers of our sample have considered CM as the height of the triangle ABC. According to the aim of a data analysis that intends to unveil the possible obstacle for a prediction process, it is an interesting finding. Indeed, until the solvers consider CM as a height, they have trouble reaching a product of GP or a coherent one. Seven solvers explicitly refer to CM as "*the height*" of the triangle.

The best example is provided by Sergio's interview (also presented in Section 6.6.1). We have already discussed the role of the instance of anticipatory intuition within Sergio's resolution process. A coherent product of GP arises suddenly as an intuition quite late at time 03:17. Before this moment Sergio seems quite sure that CM cannot be congruent to CB because CM is the height of the triangle ABC. The solver explicitly makes use of this theoretical element (the height) to justify the impossibility of having the two segments congruent, that was his first product of GP. Suddenly, he changes his mind and communicates a product of GP that is in sharp contrast with the previous one. Now, we can grasp the element that supports the anticipatory intuition: dropping the idea that CM must be the height. This claim is supported by what happens in the following excerpt: *Sergio_G10_T5_P1_(03:52 – 04:17)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
03:52.05	Int	Are there other ways in which CM can be congruent to CB? Other positions for point C?		
04:01.17	Stud	Ehm... the other points would be **moving point C perpendicularly to MB**, I mean raising it...	He points at C and mimics a straight trajectory that is perpendicular to MB: 	GP_2 is better detailed by GP_3. **GP_3_(0) (gestural – discursive): the locus of C is a line perpendicular to MB that passes through the midpoint of MB** It is a locus because he moves the pen up and down along the perpendicular line: **Window gesture**
04:07.29	Int	Mm mm.		
04:08.18	Stud	...or lowering it and, bringing **to the other side**, raising it or lowering it.	He mimics a straight trajectory perpendicular to MB: 	**Window gesture**

04:15.14	Stud	Yes, moving point C only **perpendicularly to MB**.		GP_3
04:17.13	Int	Mm mm.		

The excerpt starts after the communication of GP_2 when the interviewer asks for other configurations where CM could be congruent to CB. Looking at the utterances, we notice that the solver focuses on the point C, more then on CM; the segment CM is no longer mentioned as a height of the triangle; even though the solver resorts to figural elements (see time 04:08), he introduces new theoretical elements (i.e. perpendicularity, line symmetry) which support a new process of GP. The process seems to be triggered by the interviewer's question, supported by the new theoretical elements and leads the solver to a detailed product: GP_3. It is a refined version of GP_2. Indeed, GP_2 addresses only a static position of C; instead, GP_3 depicts an entire locus using some elements of the previously communicated product of GP. The gestures are very detailed and reveal that the solver is strongly convinced about his prediction.

Sergio's interview provides an example of a way to look at a part of the configuration (i.e. the segment CM) that inhibits finding a coherent product of GP. This is also an example of a spontaneous breakthrough in the situation which allows the solver to undertake an effective GP process. The dynamic dimension coupled with the identification of a non-stereotyped position of C within the triangle seem to play an important role.

The identification of CM as a height seems to occur for two reasons:

- an unconscious addition of constraints, so that the solver tries to maintain CM as a height or ABC as an isosceles triangle;
- a stereotyped image that does not allow the solver to consider the most general kind of triangle.

In both cases, the drawing seems to have the effect of crystallizing the situation, making it hard to consider another configuration. Sergio was able to overcome this difficulty using a top-down approach to the configuration.

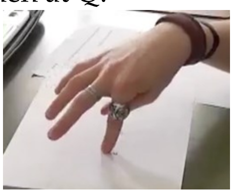
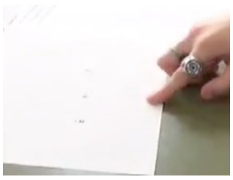
8.2.4 Task 6

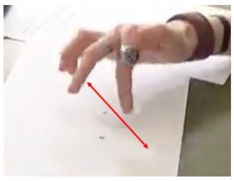
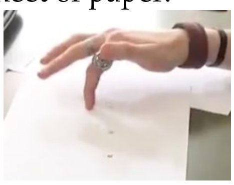
The task is composed of four sub-tasks, which refer to different new positions of P or A proposed by the interviewer. The tasks are quite simple if the solver knows and is able to control point symmetry. All the solvers of our sample seem to be quite at ease during the resolution of this task. However, several solvers communicate strongly incoherent or fuzzy products of GP.

This task gives us the opportunity to remark some findings already discussed, like the role of figural elements and of a static approach to the configuration within the GP processes. Moreover, during the processes of GP that accompanied the resolution of Task 6, a particular integration of theoretical elements and theoretical control over the figure emerge. Indeed, even if the solvers exhibit knowledge of the proper TEG, they can choose to apply to the configuration an incoherent theoretical element that is in sharp contrast with their good theoretical control over the figure.

Here are three examples that support our claims.

The first excerpt from Stefano's interview: *Stefano_G9_T6_P1_(00:58 – 02:11)* shows a lack of theoretical control, a static approach to the configuration, and a dominance of figural elements.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:58.05	Int	Ok. Imagine... make a prediction: imagine moving point P.		
01:06.13	Stud	Mm mm.		
01:06.28	Int	And make a prediction. What happens to the configuration?		
01:16.21	Stud	That if I move it...that **P if I move it on Q** it remains, the distance QA remains QA or **PA remains the same**.	He points at P and then at Q:  He points at Q and A.	**GP_1_(0) (discursive – gestural): if P coincides with Q, QA is equal to PA**
01:24.15	Stud	If instead I **move it under** the same distance of QA becomes...PA is equal to QA.	"Under":  He points at Q and A as they are in the drawing.	**GP_2_(0) (discursive – gestural): P at a position such that PA=QA, Q and A are fixed** He considers a position of P such that the distance between the new position of P

				and A is congruent to QA in the initial position. The segments are actually congruent, but the fixedness of Q is not coherent with the given point symmetry.
01:32.19	Int	Mm.		
01:33.02	Stud	**If I move P up** ...the distance QA is equal to PA. If I move it above it is known.		GP_2 P is at a position such that the segments are congruent, but Q is considered to be fixed.
01:41.14	Stud	If instead I move Q on P it becomes, it remains always equal to PA, QA equals the length, the distance.	He points at Q and P.	GP_1, but moving Q.
01:50.11	Stud	I think [this is the case] also if I move it to some side here...	He moves the finger on the drawing tracing several lines: 	
01:55.12	Int	Mm mm.		
01:56.21	Stud	Between here and here, and it remains equal.	He points to two positions on the sheet of paper:  	GP_2 applied to positions of P on a line neither horizontal nor vertical. The pointed positions of P return segments that are congruent to QA.
02:00.04	Stud	And also if I move it on top one on top.	He points again at the same two positions of (01:56)	

02:02.26	Int	That is if you move P what do you think happens to...A and to Q.		
02:08.14	Stud	Ah, **Q A stay fixed**.		Evidence of our inference about the fixedness of Q.
02:10.16	Int	Ok.		
02:11.00	Stud	If I change P, it can remain, it can, the length of di AP can vary or it can remain the same if I move it in certain positions.		GP_2

For the sake of completeness, we summarize what happens before the excerpt. Stefano has considered P as the center of the point symmetry. When the interviewer repeats that Q is the "*symmetric point of P with respect to A*", Stefano reconsiders his drawing and coherently traces Q (Figure 48). This could be a careless mistake, but more likely it seems to be due to a lack of control of the theoretical element "point symmetry". The following of the interview supports this interpretation.

Figure 48 The coherent drawing produced by Stefano during the resolution of Task 6 (*Drawing 1*)

The excerpt starts when the interviewer proposes to imagine a motion of P and consequently asks for a prediction on the configuration. We find a product of GP in the solver's utterance and gesture: if P is placed on Q, then QA is congruent to PA. The dynamic dimension of the utterance is only a repetition of the interviewer's prompt. Indeed, the rest of the utterance reveals a very static approach: Stefano repeats three times the verb "*remains*". Moreover, the gesture suggests that he conceives the configuration statically: Stefano points at A and Q at their initial positions. We infer that his prediction is the following: P is placed on Q, Q is fixed and A is as well; at that new position of P, PA is congruent to QA, because they are two overlapping segments. Moreover, Stefano does not mention another position for Q. In line with this interpretation, we must consider GP_1 to be an incoherent product of GP. We notice that, even if the dynamic dimension is strongly suggested by the interviewer, the static approach is dominant in Stefano's process of prediction.

At time 01:24 the solver undertakes another GP process: he imagines moving P to another position on the sheet of paper and he finds two congruent segments. The position of P actually returns two congruent segments in that particular drawing, but the fixedness of Q is not coherent with the point symmetry. So, we cannot consider GP_2 as a coherent product of GP. In the following we find several instances of GP_2; in all cases the point Q is considered to be fixed.

At time 02:08, Stefano's utterance confirms our inference: he explicitly claims that Q is fixed. Analogously, in the rest of the interview, whenever the interviewer asks what happens to the other two points (A and Q or P and Q), the solver says that they are fixed. Every time the tone of the voice suggests that the solver considers the fixedness of the points as an obvious finding. Moreover, at time 03:40, he stresses his point again:

Stefano: *If I move P only the distance between A and P can change.*

This first excerpt shows a static approach to the configuration, dominated only by the figural elements that are visible in the drawing. Indeed, after the solver has performed the drawing, A and Q become fixed elements of the configuration. The theoretical constraint that Stefano has used to obtain the configuration seems to be forgotten. The theoretical control over the three points is absent: the solver is not able to recognize the effect of the movement of P on Q.

The second excerpt from Stefano's interview presents the same characteristics: *Stefano_G9_T6_P1_(02:19 – 02:57).*

Time	*Who*	*What is said*	*What is done*	*Comment*
02:19.21	Int	Ok. Imagine moving P along a line and make a prediction: what happens to the configuration?		
02:28.24	Stud	That if I mo...**if I move P to...the right**, outwards, the length PA changes and it is no longer equal to QA.	He points at a position for P with the thumb: 	**GP_3_(0) (gestural – discursive): there are positions for P such that PA is not congruent to QA** Q is still considered fixed.

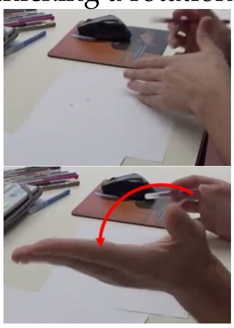
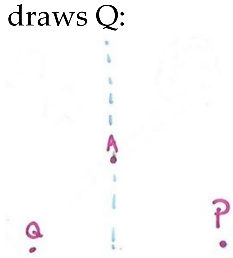
				The point symmetry is absent.
02:38.16	Int	Mm mm.		
02:39.04	Stud	If instead I move it towards Q and it goes on point Q, as I said before, QA always remains equal to PA. And they are **on top of one another**.	He points at Q.	GP_1: the talks about P coincident with Q.
02:47.13	Stud	If instead I move it **outwards here**, to the left...Q.	 Then he moves on the left the whole sheet of paper.	
02:52.03	Int	Mm mm.		
02:52.26	Stud	Anyway the distance of PA changes and becomes greater than Q.		GP_3
02:57.12	Stud	And if instead I move it towards the inside of Q, it becomes...PA becomes less than QA.	He points at a point between A and Q.	

The interviewer explicitly asks the solver to imagine moving P along a specific path. Stefano answers showing a position for P such that PA is not congruent to QA. He continues referring to the initial positions of A and Q as if they both are fixed points. At time 02:39, there is another instance of GP_1. Finally, at time 02:57, he concludes with another instance of GP_3.

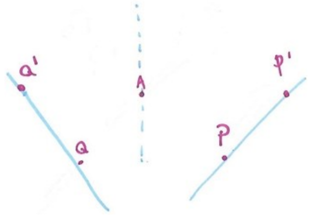
In this excerpt as well, Stefano shows a lack of theoretical control over the figure. The solver undertakes several GP processes. However, his products of GP are always incoherent and driven by the particular drawing he has performed. The incoherence of his products of GP seems to depend on the insufficient control of the point symmetry. Moreover, the lack of theoretical control does not allow Stefano to predict the effect on Q of the movement of P. So, he is induced to consider point Q fixed, as A in fact is.

The second and the third are examples of GP processes that show a paradox: a good and fine-grained theoretical control over the figure that leads the solver to communicate several incoherent products of GP. We will see that the reason resides in the initial choice of the theoretical constraints.

The second example is from Giulio's interview: *Giulio_G13_T6_P1_(00:37 – 02:34)*.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:37.28	Stud	I need to draw the symmetric point to P with respect to A, so I assume that **A is a bit like a line of symmetry**, I would say, so...		Incoherent wording: A is an axis. Theoretical element: line symmetry.
00:49.20	Stud	Let's draw in another color.	He draws a vertical segment through A.	
00:58.12	Stud	So if it is symmetric it needs to be reflected on the other side exactly as it is.	He moves the hand, mimicking a rotation: 	**Window gesture**
01:04.07	Stud	Theoretically. So more or less here. More or less.	He draws Q:  *Drawing 1a*	The drawing is coherent with a line symmetry with a vertical axis.
01:13.15	Int	Ok. Imagine moving point P and make a prediction.		
01:21.19	Stud	Ok.		
01:22.29	Int	What happens to the configuration?		

01:29.05	Stud	I move point P in any direction, then **in the case that I move it closer to the line of symmetry, then also point Q would get closer**.	While he is talking, he uses two fingers as if he is holding something and he moves the hand as follows: 	**GP_1_(0) (discursive – gestural): P is closer to the axis, Q is closer as well** GP_1 is coherent with the line symmetry.
01:38.06	Stud	If I **moved it upwards point Q would go along the same trajectory**, the same movement.	"upwards": 	**GP_2_(2) (discursive): P is moved upward, Q follows the same path**
01:44.07	Int	Mm mm.		
01:46.09	Int	Imagine moving point P along a line.		
01:49.27	Stud		He looks at the drawing.	
01:50.05	Int	And make a prediction. What happens to the configuration?		
01:59.00	Stud	Imagine moving it along a line...can I draw anyway?		
02:02.20	Int	Of course, whatever you think is useful.		
02:08.14	Stud	I imagine that this is the line, I translate, say, point P, with respect to the line I bring it here, I call it P prime.	He draws a line and a point P on this line: P'.	
02:21.27	Stud	I do the same procedure basically also with on this point Q.	He draws a line that is symmetric with respect to the sketched axis.	
02:34.22	Stud	I obtain the same movement. Let's say it could look like a trapezoid, but if I connect all the points.	He draws the symmetric point Q':	**GP_3_(0) (discursive): Q on a line that is symmetric to the axis through A**

			 *Drawing 1b*	

At the beginning of the excerpt, Giulio is performing the first drawing. We notice that he interprets the theoretical constraint “*Q is the symmetric point of P with respect to A*” as if A is on a line that plays the role of an axis of symmetry. The utterance (time 00:37) and the window gesture (time 00:58) suggest this inference, which is confirmed by *Drawing 1a.* We notice that the drawing is incoherent with respect to a point symmetry centered at A.

After the solver has sketched the situation, the interviewer asks for a prediction. The process of GP is quickly accomplished, and the solver communicates a product of GP that is coherent with the constraints that he is considering (the line symmetry), but incoherent with the given ones. GP_2 also shows the same features of GP_1. Even if the products of GP are more referred to the figural elements of the configuration than to the theoretical ones, they are carefully described by the solver.

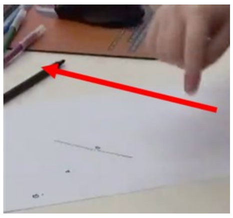
The second question of the interviewer triggers a new GP process, that finishes with the communication of GP_3. This is a detailed product of GP, but it is incoherent according to the point symmetry. The drawing is carefully sketched out as the previous ones and it reveals a good control over the figure.

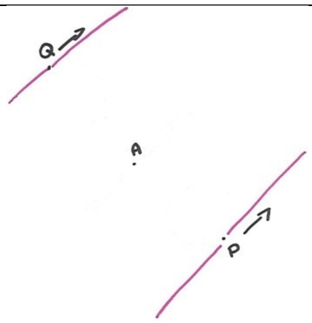
For the sake of brevity, we include only the excerpt that contains the first three processes of GP. The fourth one shows the same characteristics of the previous ones, so it is not so interesting for the aims of this section. All the prediction processes undertaken by Giulio are characterized by strong theoretical control over the figure; consequently, the products of GP are coherent with line symmetry but incoherent with respect to the given constraint (the point symmetry); the processes of GP have a dynamic dimension that is evident looking at all the solver’s productions.

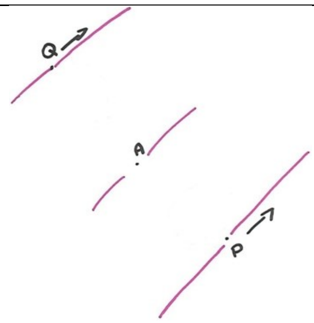
This is the case of strong theoretical control that is applied to the configuration using an incoherent theoretical constraint, which is induced by the solver and not by the step-by-step construction. In this case as well, the figural components of the particular configuration sketched out in the drawing seems to affect the prediction process: the axis of symmetry seems to dominate the process.

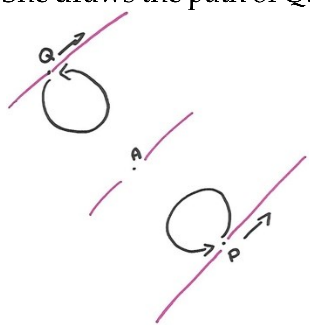
The third example is from Sabrina's interview: *Sabrina_G9_T6_P1_(00:00 – 03:21)*. Here we can observe another reason for the paradox: the interference between two different theoretical elements.

Time	*Who*	*What is said*	*What is done*	*Comment*
00:00.17	Int	Read and perform the following step-by-step construction.		
00:05.16	Stud	Mm...ok.	She is looking at the step-by-step construction.	
00:11.26	Stud	So...I draw **any two points**.	She draws two points.	
00:14.20	Int	Yes.		
00:15.00	Stud	On...the piece of paper. **Not necessarily horizontal or vertical**.		
00:20.17	Stud	And then I draw the symmetric point, so there are A...and P.	She labels the points A and P.	
00:26.19	Stud	I draw the symmetric point to P with respect to A and so I call it...with respect to A I take it like an **axis** let's say.	She moves the pen on a straight path, neither horizontal nor vertical: 	Incoherent wording: A is an axis. She seems to consider an axis of symmetry that is not reported in the step-by-step construction.
00:35.07	Stud	Ehm...symmetric, so I draw it **higher at the same distance** and I call it Q.	 *Drawing 1a*	
00:40.16	Int	Ok. Imagine moving point P and make a prediction.		
00:47.00	Stud	Ok.		
00:48.05	Int	What happens to the configuration?		

00:50.13	Stud	That...it will move like **a mirror.** So if I **move P upwards, also Q moves upwards**. And...the same if I move it downwards.	She points at P and moves the pen upwards on a vertical line: 	**GP_1_(2) (discursive): P on a vertical line, Q on a vertical line** GP_1 is not coherent neither with a point symmetry nor to an axis symmetry with the axis expressed at (00:26). It is coherent with a line symmetry with a vertical axis.
01:01.21	Int	Ok. Imagine moving point P along a line and make a prediction. What happens to the configuration?		
01:10.17	Stud	If I take like a line for example...one that goes like this.	She traces a line neither horizontal nor vertical through P.	
01:17.27	Stud	I imagine that **also point Q then would move in the same way on a same line**, but moved, indeed, to the other side with respect to A.		**Window gesture**
01:27.17	Int	How?		
01:28.25	Stud	In the same way. If I move P I move it upwards, so I **always have a line...parallel to the first**.	She draws a line through Q.	
01:37.27	Stud	If I **move P in this direction, also Q will move in this direction**.	She draws two arrows on the lines:	**GP_2_(0) (discursive): Q on a line that is parallel to the line through P and traced in the same direction** GP_2 is coherent with a line symmetry with an

			 *Drawing 1b*	axis parallel to the line through P.
01:44.25	Int	Ok. Imagine moving A along a line.		
01:49.01	Int	And make a prediction: what happens to the configuration?		
01:54.17	Stud	That depending on where I move it, also the other two points will move.		
01:59.27	Int	How?		
02:01.19	Stud	Ehm...if I move A towards P also the distance, so diminishing the distance, also the distance between A and Q will get shorter.		Theoretical answer
02:12.00	Int	Ok. If instead you move A along another type of line?		
02:18.05	Stud	Mmm...I think that **the other two points will not move**.		
02:22.01	Stud	If...		
02:22.09	Int	Mm mm.		
02:22.27	Stud	...the **line is parallel**, that is if the line is like like this.	She draws a line parallel to the other two:	**GP_3_(0) (discursive): A on a line, Q and P fixed** GP_3 is coherent with a line symmetry. The axis

			 *Drawing 1c*	is drawn in the picture.
02:26.21	Int	Mm mm.		
02:28.19	Stud	So I move it only horizontally, **I think that the other two points will not move**.		
	Int	Imagine moving P along a circle and make a prediction. What will happen to the configuration?		
02:44.26	Stud	That whenever I move it close A also... that is shortening therefore the distance, also Q will shorten its distance from A.		Theoretical answer
02:52.16	Stud	And...the **Q will move in the same way**.	She moves the finger tracing a circle under the desk: 	**Window gesture**
02:56.16	Int	Ok. P moves along a circle, how do you imagine moving it?		
03:01.00	Stud	In...the counterclockwise direction.	She draws a circle through P.	

03:05.11	Int	And what happens to the configuration?		
03:07.18	Stud	That...it changes, it changes. Also Q...		
03:10.18	Int	How. Mm.		
03:11.28	Stud	...**it moves in the same way**.		
03:14.04	Int	How?		
03:14.16	Stud	But...**always in the counterclockwise direction**, but downwards.	She draws the path of Q:  *Drawing 1d*	
03:19.04	Stud	This one upwards and the other downwards.		**GP_4_(0) (discursive): P on a circle, Q on a circle traced in the same direction** GP_4 is coherent with the point symmetry.
03:21.17	Int	Ok.		

The excerpt shows the whole first part of Sabrina's interview. From the beginning, the solver seems to introduce an implicit axis of symmetry. Indeed, at time 00:26, Sabrina exhibits the same interpretation of Giulio: she refers to A as an axis. We know that the wording is incoherent within the TEG, but the utterance suggests that Sabrina is considering a line through A as an axis of symmetry. Unlike Giulio's first drawing, Sabrina's *Drawing 1a* is not completely incoherent with point symmetry; on the contrary, it is potentially coherent with both the symmetries (the point and the line symmetry).

The first question of the interviewer triggers a process of GP whose product reveals another axis of symmetry: GP_1 is coherent with a line symmetry with a vertical axis.

At time 01:01, the interviewer asks another question that requires imagining a motion of P on a line. First, the solver considers and draws a line through P neither horizontal nor vertical. Then, the gesture and the utterance refer to a parallel line through Q. Actually, Sabrina does not explicitly talk about the parallelism, but the expression "*a same line*" and the window gesture suggest this interpretation. This is confirmed by the solver's answer to the interviewer's request for further explanation. At time 01:28, Sabrina explicitly refers to a parallel line. Moreover, at time 01:37, she explains that the two points move on the two lines following the same direction; *Drawing 1b* marks this property. In particular, this last detail reveals that the solver is considering a line symmetry and that the axis is the first line that she has performed through gestures at time 00:26. Nevertheless, GP_2 is incoherent with the given constraint: the line can be parallel, but the two points should move in opposite directions.

GP_3 is also coherent with a line symmetry and incoherent with respect to the given constraint. Indeed, while A is moving along a line, P and Q are considered to be fixed. Finally, the solver makes explicit the implicit axis, that we can see looking at *Drawing 1c*. We stress that Sabrina uses strong theoretical control over the figure, but addressed towards an incoherent theoretical element. We can say that Sabrina's theoretical control is coherent to the theory that she has chosen.

The last process of GP is in sharp contrast with the previous ones. Indeed, GP_4 is coherent only with point symmetry, because Sabrina says that P and Q are on two circles and they move in the same direction. The arrows within the drawing also stress this property.

As mentioned at the beginning of this section, Sabrina's interview provides an example of the interference between different theoretical elements that the solver tries to use: line symmetry with a vertical axis, line symmetry with a non-horizontal and non-vertical axis, point symmetry. The theoretical control is always coherent with a different theoretical constraint and consequently the products of GP are incoherent. During the production of GP_1, GP_2 and GP_3 it seems that the solver uses a different axis according to her aims. Only GP_4 is coherent, very detailed and carefully reported within the drawing. However, the solver does not seem to be aware of the inconsistency among her products of GP.

8.2.5 Concluding remarks about the possible obstacles

Analyzing the approaches of the solvers of our sample to the given open problems, we have found that both general and local obstacles inhibit the solvers in accomplishing processes of leading to coherent products.

Figural elements play an important role within GP processes. When the figural elements dominate the exploration, the configuration can be considered as composed of sub-configurations that do not mirror all the given or induced theoretical constraints. For example, this is the case of the triangle and the segment in Task 2. This can be interpreted as a lack of harmony between the theoretical and the figural elements of the given configuration.

Moreover, when the solvers mainly use bottom-up processes, the drawing can induce the solvers to consider theoretical elements that are not coherent with the constraints given in the step-by-step construction. In this case, we can observe a phenomenon that we have called "*devolution to the drawing*". After the step-by-step construction is performed, the solver induces on the configuration theoretical elements that are suggested by the particular drawing rather than deduced by the given theoretical constraints. Moreover, the theoretical elements inferred by the drawing can be in sharp contrast with the given constraints. In this case, the figural elements that are observed looking at the drawing can suggest to solvers incoherent theoretical elements that influence the prediction process.

We have widely discussed the role of theoretical elements within the prediction process, in particular if they are introduced by the solver for the first time. One example is provided by the incoherent right triangle considered by Carlo and Ilaria during the resolution of Task 2. Moreover, the resolution of Task 5 provides a good example of the consequence of considering an incoherent theoretical element applied to a specific figural element. As long as the solvers of our sample do not refer to CM as a height of the triangle, they have trouble in reaching a product of GP or a coherent one. Not all the solvers were able to overcome by themselves this difficulty, as Sergio did.

The use of coherent theoretical constraints seems to be directly connected to the solvers' theoretical control over the figure. The lack of theoretical control is a crucial obstacle to reach a coherent product of GP. One of the reasons lies in the difficulties in managing a particular theoretical element, without which the solver cannot reach a coherent prediction. However, we also observe a paradoxical situation: strong theoretical control over the figure coupled with several incoherent products of GP. We have seen that in this case the reason seems to lie in the initial choice of an incoherent theoretical constraint coupled with a lack of solver's awareness of the mistake; otherwise, it can be caused by a sort of interference between two theoretical elements that the solvers use independently.

During the resolution of simple tasks, like Task 6, the role of the theoretical control is reduced to a careful identification or interpretation of the theoretical elements given by the step-by-step construction. Instead, the resolution of a more articulated task, like Task 2, requires more advanced theoretical control. Indeed, the solver must not only consider the given theoretical elements coherently, but she must also infer the theoretical elements that are induced by the given constraints, and finally control their mutual interaction.

Another obstacle can be an extremely rigid manipulation of the configuration. The reason for such a fixedness could lie in an inaccurate interpretation of the given constraints or on an incoherent deduction of the constraints induced by the given ones. The effect is the impossibility of considering other possible arrangements of the configuration. Carlo’s resolution of Task 4 provides an extreme example of the effects of the fixedness.

The influence of a stereotyped image of a geometrical figure within the problem-solving process is widely documented in the literature. The prototype effect seems to also play a role during the process of GP: it seems to inhibit the solver's considering properly the given constraints and exploring different arrangements of the configuration. When the prototype effect is strongly dominant, it seems to influence the alternative arrangements that the solvers are willing to consider.

In some cases, the lack of dynamic dimension and the prototype effect seem to be a natural and persistent phenomenon. Indeed, also in cases like the resolution of Task 6 where motion is required for explicit, the solvers can manifest a static approach to the task; even when the given drawing does not propose, by design, a stereotyped image of a geometric figure (like that one used in Task 4) the solver can resort to a prototypical orientation.

9. Exploration within a DGE

As we approached this study initially, we had wished to analyze what happens when a solver who has undertaken GP processes in a paper-and-pencil environment moves to a Dynamic Geometry Environment (DGE). This choice was supported by several arguments that we list below.

- Since the dynamic figure is available on the screen to both the user and the interviewer, a DGE seems to be a good environment for observing more transparently how the solver intends to change the configuration. So, we could use the DGE to confirm or reject our inferences about the products of GP that was communicated during the first part of the interview.
- A dynamic figure maintains all the constraints that were used for constructing the figure on the screen, even when the solver drags a point. Now, let us consider the case when the solvers undertake a process of GP during the first part of the interview that lead us to an incoherent product of GP; if they try to drag a point of the dynamic figure according to their incoherent prediction, the feedback of the DGE may be in sharp contrast with the expected final figure. We advanced the hypothesis that this possible mismatch between the solver's figural expectations and the actual behavior of the dynamic figure may elicit some surprise. In turn, such a surprise can have a twofold effect: trigger new GP processes or a new resolution process; reveal to the researcher additional or different details of the previously communicated products of GP.
- Since literature has widely highlighted the important role of a DGE within the exploration tasks in geometry, we wanted to observe whether the interaction with a dynamic figure, which corresponds to the configuration that the solvers have reasoned upon during the first part of the interview, supports the resolution process, in particular of those solvers who had trouble in communicating a coherent product of GP or a product of GP at all. In other words, we wanted to gain insight into whether the use of a DGE leads to overcoming obstacles that the solvers may find in the paper and pencil resolution of the given task.

These arguments could potentially be additional lines of research. However, because of the limited amount of time that could be devoted to this doctoral study, we were not able to investigate them adequately.

9.1 Some general findings from the second part of the interviews

In this perspective the interaction between solvers and the dynamic figure was only used to support our inferences about the products of GP that was communicated during the first part of the interview, rather than for a separate discussion. Nevertheless, we have collected some general findings, each of which would need deeper analyses.

Overviews of the main findings are listed below:

- the products of GP communicated during the first part of the interview have an influence on the exploration of the dynamic figure;
- the surprise, generated by the unexpected behavior of the dynamic figure, can lead the solvers to a new insight into the problem and trigger a new resolution process.

9.1.1 The products of GP affect the dynamic exploration

When a solver, who had reasoned upon a given figure in a paper-and-pencil environment undertaking GP processes, moves to a DGE the products of GP previously communicated can drive the exploration supporting or inhibiting the reaching of new insight into the problem.

More specifically, the products of GP can drive the exploration, suggesting particular movements or positions of the points.

A first example is provided by Emilio's interview during the resolution of Task 2. As reported in Section 6.5.1, during the first part of the interview, Emilio suddenly communicates two products of GP:

GP_1: *the locus of M is a circle centered at B*

GP_2: *the locus of P is a circle*

Finally, he reported GP_1 on a drawing (Figure 49).

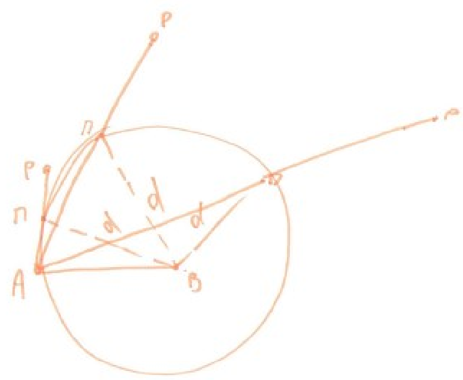


Figure 49 A picture of *Drawing* 3 performed by Emilio at the end of the resolution of the first part of Task 2

Then he moves to the resolution of the same task in a DGE. We can notice that, during the solver's interaction with the dynamic figure, he makes use of *maintaining dragging* (MD) in order to maintain *d=3* in different ways: at the beginning he moves by jumps; then his movements become more fluid. Moreover, he passes through positions that do not maintain the label "d=3" but only the general shape of the locus of P previously predicted: a circle. We interpret this change in movement as an intervention of GP_2: he starts with his prediction of the trajectory of P and drags P according to this prediction. Emilio continues the exploration, constructively interpreting the feedback provided by the DGE to refine the products of GP. In particular, GP_1 is no longer discussed, to the extent that he draws a circle on the dynamic figure (see the small circle in Figure 50); GP_2 is enriched by the investigation of the center and the radius of the circle. In the end, Emilio produces the following dynamic figure (Figure 50), where he also points at the center of the circle of P.

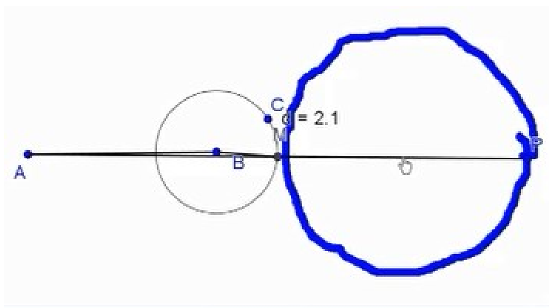


Figure 50 A screen capture of Emilio's exploration of the dynamic figure

Other examples like this show solvers' constructive interactions with the DGE, using the feedback to confirm and refine their products of GP. In all cases, the solver's use of MD for imposing certain properties suggests the influence of a previously communicated product of GP.

In some cases, the interaction with the dynamic figure leads the solvers not only to refine their product of GP but also to reach a new insight into the resolution of the task. Let us consider an example from the resolution of Task 5. It is from Agnese's interview. Excerpts from the first part are reported in Section 6.3.2. During the paper and pencil resolution of the problem, Agnese communicates two products of GP:

GP_1: *CM could be congruent to CB*

GP_2: *CMB is an isosceles triangle*

When she moves to the DGE exploration, on the screen she finds only the segment AB. Then, she directly constructs C as the vertex of an isosceles triangle (*CMB*) (see Figure 51). We interpret this as an influence of GP_2.

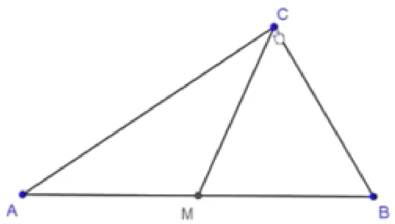
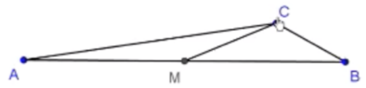


Figure 51 A screen capture of the dynamic figure constructed by Agnese during the resolution of Task 5

The following excerpt shows what happens in the subsequent exploration: *Agnese_MS_T5_P2_(13:19 – 13:35).*

Time	*Who*	*What is said*	*What is done*	*Comment*
13:19.15	Int	Are there other positions for point C so that CM is congruent to CB other than this one that I see?		
13:26.15	Stud	Mmm.		
13:29.20	Stud	Those **along** ...aaaah…	Starting from the initial position of C, she drags C on a vertical trajectory and then she stops:	Maintaining Dragging

				
13:35.16	Stud	The **height** of the triangle C...MB. **That is also median and bisector**.		**SGP_1: C is on the height of the triangle CMB** Theorem on the height of an isosceles triangle.

At time 13:29, Agnese moves C along a particular path, making use of MD, probably for maintaining the triangle isosceles. We recognize in her utterance at time 13:35 a product of GP suggested by the interaction with the dynamic figure. This product emerges from the interaction between GP processes and the feedback from the DGE, so we labeled this product *SGP*, intending a product of a "*scaffolded GP*"[9]. In the end, for making explicit the locus of C, she constructs a line perpendicular to MB and passing through the midpoint of MB.

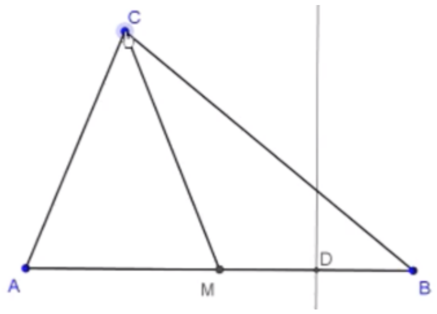


Figure 52 A n instance of the dynamic figure constructed by Agnese during the resolution of Task 5

In examples like this, we can notice how the feedback from the DGE constructively interpreted by the solvers can lead them to reach a new solution to the given problem.

More specifically, analyzing the first part of the interview, we advance the hypothesis that the lack of dynamism restricts the possibilities that Agnese is willing to explore, impeding her to undertake further GP processes. Instead, the exploration within the DGE seems to make Agnese accept to dynamically transform the figure and, consequently, discover additional theoretical and figural details.

[9] Because of the possibility of having an immediate figural feedback from the DGE, we cannot properly talk about products of GP in a DGE referring to the outcome of the same process that leads to a product of GP in a paper-and-pencil environment.

However, the positions or movements of the points suggested by the previously communicated products of GP can also inhibit reaching another or more detailed solution, and in general the dynamic exploration. Indeed, the solvers might be so convinced about their predictions that:

- they are not willing to explore other configurations;
- they seem to be "blinded" to the contradictory feedbacks provided by the DGE.

Let us consider two examples which respectively address the first and the second point.

The first example is from Lidia's interview during the resolution of Task 4. During the resolution of the first part of the task, Lidia explicitly refers to CM as a height of the triangle and states that CM cannot be congruent to CB. She supports her claim adding the following explanations:

Lidia: *No, because even if I ...made the segment CM longer, it would not be...equal to...*

Lidia: *So if I had a right triangle, any leg would be of a different size with respect to the hypothenuse.*

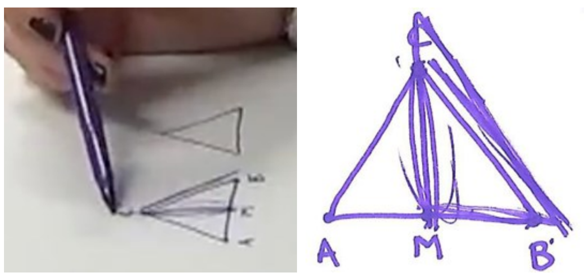


Figure 53 A window gesture and a dragging performed by Lidia during the resolution of Task 4 in a paper-and-pencil environment

In Section 8.2 we have already discussed how conceiving CM as a height could inhibit prediction processes that lead to a coherent product of GP. We infer from the gesture and the drawing (Figure 53) that Lidia is investigating a possible position of C on a line perpendicular to AB; then, she concludes that CM cannot be equal to CB. The impossibility of having CM congruent to CB and the conception of CM as a height seem so convincing for her that they also affect the exploration of the dynamic figure. Indeed, when Lidia moves to the DGE, she constructs a dynamic figure corresponding exactly to the figure she was reasoning upon in the paper-and-pencil environment (Figure 54).

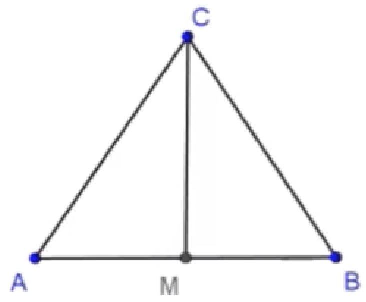


Figure 54 An instance of the dynamic figure constructed by Lidia during the resolution of Task 4

Starting from this configuration, she drags C only along a vertical trajectory. When she stops dragging, she repeats that CM cannot be congruent to CB and adds the following explanation:

Lidia: *Because in any case C...CM will never be equal to CB. So I could not,* ***I could not exchange them*** *in the sense that ...I could never put CB in place of CM, because the* ***height*** *remains the height and the* ***side*** *is always side.*

We can notice how an incoherent convincing prediction on a figure can affect the exploration in a DGE to the extent that the solver does not try to move the points following trajectories different from the one she has predicted: Lidia is so convinced about her prediction that continues moving C only on the height of the triangle.

The second example is from Valeria's interview during the resolution of Task 2. The example shows a phenomenon that we addressed in a paper (Miragliotta & Baccaglini-Frank, 2018). We had advanced the hypotheses that when an incoherent product of GP is very convincing for the solver, it does not allow her to generate and interpret constructively the feedbacks from the DGE to the extent that the solver appears to be "blinded" by her original prediction.

More specifically, during the first part of the interview, Valeria claims that P is fixed and adds the following explanations:

Valeria: *Because for...from what I remember, through three points only one line passes, so first of all I could not do it.*

Valeria: *But, even if I moved it, even a little bit, it would change in any case because the midpoint would be different, because the length would be different from my segment [AP] and therefore it could never be equal [to MB].*

When she moves to the DGE, the strength of her prediction seems to inhibit her ability to constructively interpret the feedback obtained from the DGE. Indeed, when Valeria is asked to explore the dynamic figure, at first, she drags P passing through position very close to the initial position of P (Figure 55); she states that her expectations are confirmed because moving P the length of MB changes.

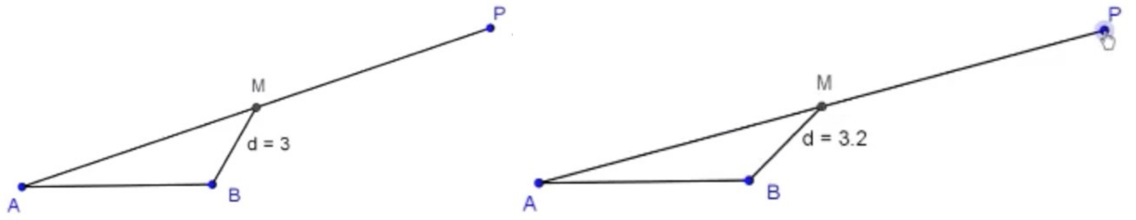


Figure 55 An instance of the initial dynamic figure (on the left) and of the dynamic figure after dragging (on the right)

Then the interviewer asks for further explorations. She starts moving P maintaining d=3 and passes through almost two positions of P that maintain the constraint on MB and are different from the given one (see Figure 56).

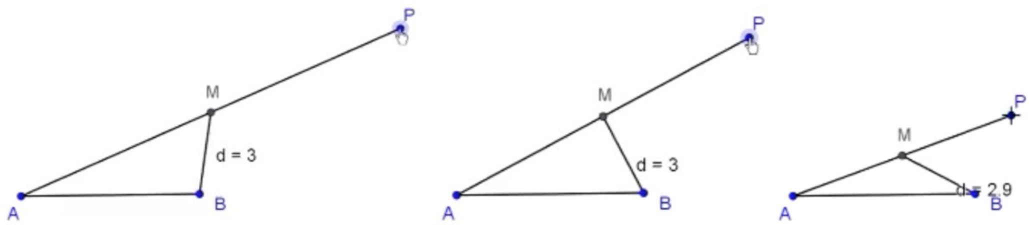


Figure 56 Two instances of the dynamic figure while point P is dragging (on the left and in the central position); an instance of the final dynamic figure after dragging (on the right)

Although these two positions are visible on the screen, Valeria ignores them and repeats that P has to be fixed at its initial position. Because of her original prediction, Valeria does not seem to be able to see any "good positions" for P other than the original configuration.

9.1.2 The role of surprise

In all the examples shown previously, we do not find any instances of surprise. When the products of GP are coherent it is quite obvious: when the solver drags a point, the figure behaves as she is predicted because actually the product of prediction is close to the proper behavior of the figure under dragging. When the products of GP are incoherent, the lack of solver's surprise is quite unexpected; nevertheless, it can be explained considering that the solvers add reasonable (at least for them) explanations that confer to the product of GP a sense of confidence and certainty.

This is confirmed by another observation: when the products of GP communicated during the first part of the interview are quite fuzzy and, moving to a DGE, the solvers see behaviors of the dynamic figure in contradiction with the expected one, they can be quite surprised. Utterances which communicate instances of surprise are the following:

- *<<Oh my Goodness! What had I thought?!>>*
- *<<What is this?!>>*
- <<I thought that…>>
- <<I take back everything I said>>
- <<It doesn't move…why not?>>
- <<*No!*>> or <<*Noooooo!*>>
- *<<Oh no!>>*
- <<*Ah!*>> or <<*Aaaaaah!*>>

The tone of the voice suggests that certain verbal expressions are instances of surprise. Moreover, often the solvers were laughing when they see behaviors of the dynamic figure in contradiction of their expectation: in this reaction we also recognize an instance of surprise.

For the sake of clarity, we will show two examples of reactions of surprise.

The first example is from Carolina's interview during the resolution of Task 2. As presented and discussed in Section 7.1.2, during the first part of the interview, Carolina communicates a product of GP on P:

GP_4: *P symmetric point with respect to A*

This is accompanied by a window gesture (Figure 57) that shows where she was intending to place P.

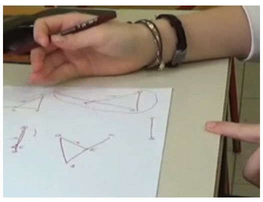
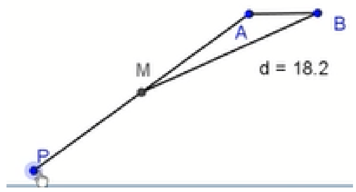


Figure 57 A window gesture of Carolina's performed during the resolution of Task 2 and connected to GP_4

When she moves to the DGE, she drags P to a particular position, where probably she was expecting that the configuration maintains the constrains on the length of MB. Here is an excerpt from the second part of the interview: *Carolina_9G_T2_P2*.

Time	Who	What is said	What is done	Comment
12:14.02	Stud	Nooo...	She moves P to the half-plane below AB and towards to A. She drags P using small movements for finding d=3.  Then she moves P along the lower edge of the screen; finally, she moves P on a line through AM.	Guided Dragging **Surprise** It seems that, when she drags P to a symmetric position with respect to A, she was expecting to see on the screen d=3.

Looking at the gesture and at the screen-capture of the dynamic figure, we can notice that Carolina tries reproducing the predicted configuration. When the configuration that was obtained dragging P in that position appears on the screen, Carolina seems surprised. From this moment on, Carolina undertakes a new exploration of the figure and a new resolution process. The feedback of the DGE allows her to communicate several positions of P which maintain the constrains on the length of MB and to conjecture that the locus of P could be a circle.

Another example is from Carlo's interview during the resolution of Task 4. The excerpt *Carlo_G10_T4_P1_(00:58 – 03:47)* was already shown in Section 8.2.2. As previously discussed, Carlo seems to interpret the given constraints too rigidly: the fixedness of the length of AB could have induced the fixedness of the whole triangle.

During the first part of the interview, Carlo communicates mainly these two products of GP:

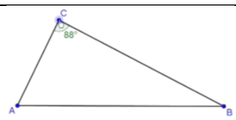
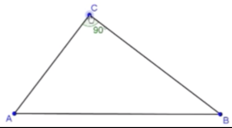
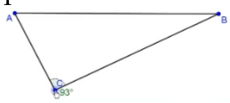
GP_2: *C is fixed*

GP_3: *C on a vertical line [perpendicular to AB]*

In the end, he restores to GP_2 which is regarded as the only possible prediction.

The following excerpt starts at the beginning of the exploration within the DGE: *Carlo_G10_T4_P2_(04:04 – 04:17).*

Time	Who	What is said	What is done	Comment
04:04.12	Stud	Aaah!	He drags C upwards passing through positions very close to the initial one.	**Surprise**: he sees new positions of C so that the angle is right.

				
04:07.24	Stud	There are two points.	He drags C passing through positions very close to the initial one. 	Guided Dragging
04:11.24	Stud		He quickly drags C on the half-plane below AB.	Guided Dragging
04:17.13	Stud	The opposite is true.	He drags C on the half-plane below AB and he stops when the following figure appears on the screen: 	Guided Dragging Although the label is different from "90°", he recognizes a "good position" of C. **Surprise:** he is observing new positions of C so that the angle is right.

Right after he has dragged C, exploring positions that are very close to the initial one, Carlo seems to be surprised because of the behavior of the dynamic figure. Looking at the rest of the excerpt, we can see that the surprise triggers a new resolution process: Carlo starts using *guided dragging* for obtaining new positions that leave the angle right. At time 04:17 he seems surprised again because he finds also a symmetric position of C with respect to AB.

Starting from this moment, he further explores the dynamic figure moving C on a circular trajectory and making use of MD for maintaining the angle right. After the exploration, he concludes that C is on a circle:

Carlo: *A circle will form...with all the triangle...with the inscribed segment.*

Moreover, he shows the circle performing a gesture on the screen (Figure 58).

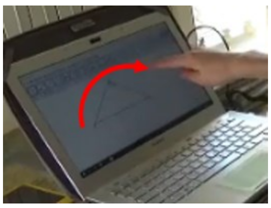


Figure 58 A gesture performed by Carlo for communicating the locus of C during the resolution of Task 4

At the end of the interview, when the researcher asks again "*What can you say about the vertex with the right angle?*", Carlo answers as follows:

Carlo: *They can move and make it so it stays...so that its angle is right infinitely many ...at infinitely many points...that make a circle. Whose diameter is AB.*

We can notice how many differences there are between Carlo's initial prediction (C is fixed or on a line) and his last answer. Moreover, the interaction with the DGE strongly influences the last answer. Finally, we can claim that what triggered a new resolution process was actually the surprise, possibly coupled with products of GP that were not so clear or convincing for the solver.

9.2 Concluding remarks

In sum, we can say that our hypothesis on the conditions that could induce surprise is not totally confirmed. We had advanced the hypothesis that when a solver, who has accomplished GP processes in a paper-and-pencil environment, moves to the exploration within a DGE, the behavior of the dynamic figure that is in contradiction with her products of GP should make her surprised. However, we have noticed that, even if a solver has communicated incoherent products of GP and the researcher sees on the screen a mismatch between the solvers' products of GP and the actual behavior of the dynamic figure under dragging, the solver may not perceive such a mismatch in the way the researcher does. Consequently, the solver might not be surprised. So, it seems that what triggers surprise is how convinced the solvers are about their product of GP, rather than the incoherence of the products themselves.

However, when the solvers are surprised because of the feedback from the DGE, such surprise triggers new exploration and resolution processes that can lead the solvers to reach a coherent solution to the problem.

Our general analyses of the second part of the interview reveal that the DGE can play an important a role in supporting the resolution process of those solvers who had trouble in reaching a coherent solution to the problem in the paper-and-pencil setting. In particular, it seems that the exploration of the task within the DGE helps solvers in considering motions in their investigations.

Nevertheless, from an educational point of view, it is interesting to notice that the products of the GP processes have a strong influence on the interaction with the DGE. These products can guide solvers along particular trajectories to be investigated and can influence the interpretation of the DGE's feedback. More specifically, if the products are coherent, the dynamic exploration allows the solvers to refine and add details to their products of GP; if the products are incoherent but quite convincing for the solver, the dynamic exploration can be ineffective for changing the solver's mind.

We referred to the latter phenomenon as "the power of GP" (Miragliotta & Baccaglini-Frank, 2018): the products of GP, and in particular in the case of incoherent products of GP, can drive the solver to see on the screen only what they have predicted and are, therefore, prepared to see.

This phenomenon is consistent with the *overconfidence* described by Fischbein (1987):

> As a matter of fact even after a certain decision is taken one frequently tends to be overconfident about the conclusion reached and to overlook possible counter-arguments. The need for verification usually is less honored than it should be. (ibid., p. 28)

The overconfidence leads the solvers to ignore or minimize the significance of possible counter arguments, manifesting "a bias to confirm" (ibid., p. 36).

On the contrary, "the power of GP" seems not to be coherent with the *transformational-saliency hypothesis* (Battista, 2007) and we suggest addressing it in future research.

10. Findings, discussion, conclusions

In this concluding chapter, we will explicitly describe the model of *geometric prediction processes* and explain how it leads to significant findings concerning the research questions we had set out to investigate.

The model provides a description of the prediction processes accomplished by a solver who engages in the resolution of *prediction open problems* proposed in this study. Moreover, it provides a lens through which it is possible to analyze solvers' productions and gain further insight into the prediction processes. More specifically, it sheds light onto the role of theoretical elements introduced by the solvers.

As mentioned in the description of the methodology, our findings have no statistical ambitions because of the limited number of cases analyzed. However, the fine-grained qualitative analyses that were carried out provided a richness in detail and depth which would not have otherwise been possible. Furthermore, many commonalities emerged during the analyses, outlining the features of the GP processes accomplished by a solver who was exposed for a long time to mathematical knowledge and, by virtue of this, is supposed to be *expert*. So, in a search for more general results, quantitative research can be fruitfully grounded upon our findings.

After answering the research questions highlighting the theoretical contributions that this study offers, we contextualize our findings within the existing literature, and then we describe possible implications and directions for further research.

10.1 Answers to the research questions

In the following sections, we will provide answers to each of the questions concerning our findings described in Chapter 6, Chapter 7, Chapter 8 and Chapter 9.

All together these chapters answer to the research questions, providing both the elements useful for reaching a model of the GP processes and the description of the complex interaction between these elements emerging from data analyses.

For the sake of brevity and completeness, we specify here that in the following we refer to findings that address our topic in the particular context we had constructed: geometric prediction processes accomplished by a solver who engages in the resolution of the given *prediction open problems* during a task-based interview.

10.1.1 How can these processes be modeled?

Our analyses seem to confirm that there is not only one single process of prediction with certain fixed characteristics. More likely, there are several prediction processes that have in common the production of figural expectations.

Nevertheless, our analyses (see Chapter 8) reveal that a GP process can be accomplished in coordination with other GP processes, giving rise to several structures of GPs. Indeed, during the resolution of a prediction open problem, the solvers can accomplish several processes of GP that can be explicitly connected or not, but all of them address the same issue: to give an answer to the interviewer's question which requires a prediction, more or less explicitly.

So, we can describe *micro-processes of GP* with general and stable components, but different features. The way in which the features are embedded into the model define a particular *micro-process of GP*. Although we have found instances of different GP processes, we can draw a common model of the micro-process. What makes the processes different are the ways through which the different components of the model interact (see the next sections), producing a particular micro-process of GP. We will show this below, with help of visual diagrams.

Our model of GP has been elaborated within the particular domain of plane Euclidean Geometry; it maintains a strong relationship with this mathematical domain of reference. Each micro-process of *geometric prediction* is a complex process within which different cognitive components (visuo-spatial abilities and the solver's knowledge of the TEG) intervene, mirroring the dual nature of the objects that come into play: geometrical objects.

Mirroring the dual nature of geometrical objects, we consider both the figural domain (Figure 59 on the right) and the theoretical domain (Figure 59 on the left).

On the background there are (Figure 59):

- the constructs developed by the Cognitive Psychology explaining interactions between an ideal solver and the spatial objects through her visuo-spatial abilities;
- the scholarly Theory of Euclidean Geometry, as a logical system made up of definitions and theorems (these are known to an ideal expert solver).

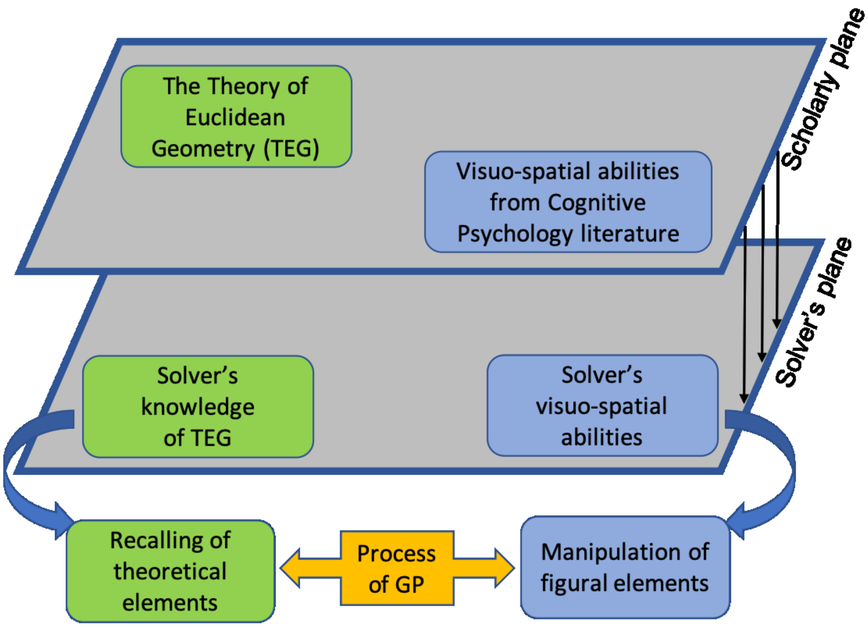


Figure 59 A general overview of the process of GP within the conceptual and figural domains

However, a process of GP is accomplished by an actual solver. The solver:

- makes use of *some* of her visuo-spatial abilities for manipulating the figural components of a geometrical object;
- recalls her interpretation and use of the TEG, and therefore her personal *figural concepts*.

So, in the foreground we find:

- the manipulation of the figural elements on which the solver focuses during the resolution of a prediction open problem;
- the specific theoretical elements recalled by the solver for exploring and solving the task.

The manipulation is accomplished within the figural domain and could be supported by visuo-spatial abilities. The theoretical elements introduced by the solver are supported by her personal knowledge of and, more importantly, mastery in using the TEG. The GP process is at the interplay between pure manipulation of figural elements and pure recalling of theoretical elements.

Below is a visual diagram summarizing the main components of the model of the micro-process of GP and their connections (Figure 60).

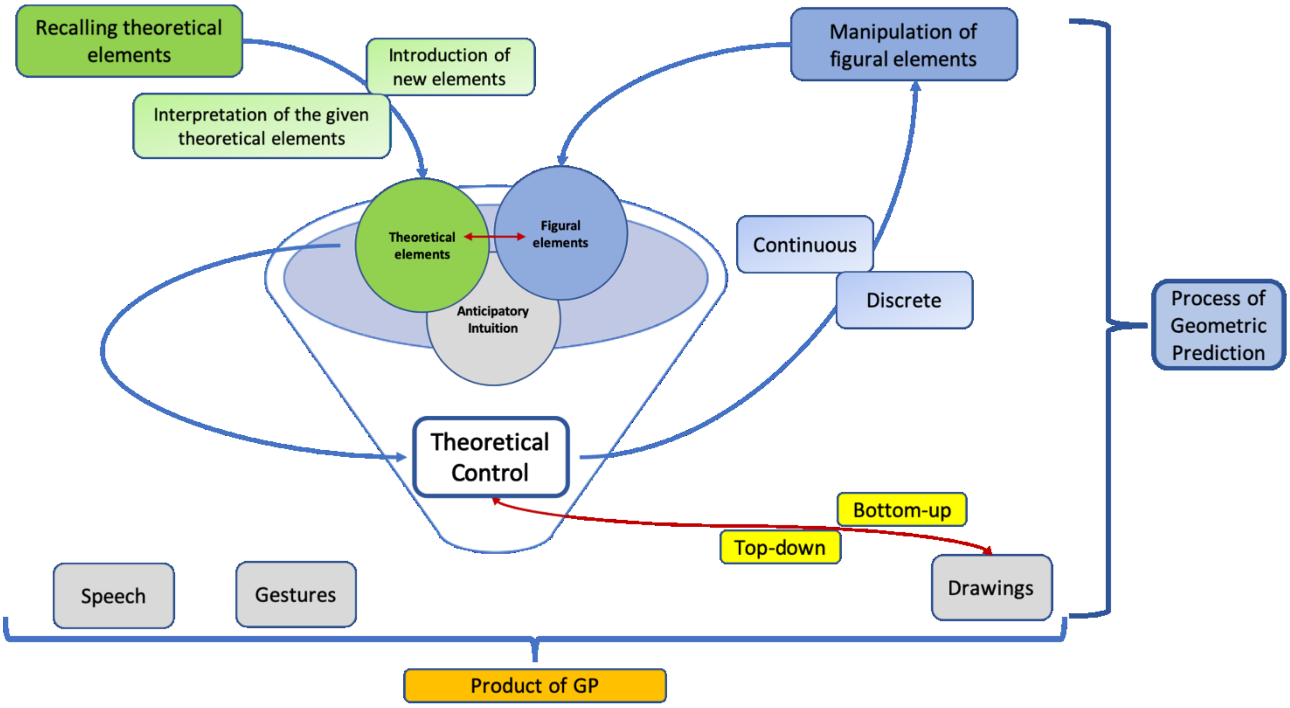


Figure 60 Visual diagram of the model of the micro-process of GP and its main elements

The model is composed of the following:

- *theoretical elements*, recalled by the solver through introduction of new elements or interpretation of the given ones;
- *figural elements*, on which the solver focused, and which can be manipulated;
- *theoretical control*, that supports figural manipulations that are coherent with respect to the solver's TEG.
- potentially but not necessarily, *anticipatory intuitions*.

The arrows make explicit the connections between the elements and their possible features. Starting from the upper left side and following the blue arrows, we will describe the model.

When the process starts, a geometrical configuration is interpreted by the solver, who recalls some theoretical elements that characterize certain figural elements. Through speech and gestures, the solver makes these elements explicit - we collect these in the *funnel*. The funnel is not properly a component of the model, but a research tool used to make explicit the elements that come into play. The funnel also serves the purpose of graphically showing the main elements in focus for the solver and within the model.

The solver can also introduce new theoretical elements that characterize certain figural elements. Figural and theoretical components are always intertwined (see the red arrow). *Anticipatory intuition* may or may not intervene, suggesting to the solver an interesting combination of figural and theoretical elements; for this reason it is sketched within the funnel as an intersecting set with the sets of the figural and the theoretical elements.

The solver can exercise upon the elements of the funnel (figural elements with theoretical properties) her *theoretical control* (see the blue arrow on the left). Using *theoretical control* the solver manipulates or decided to focus on particular figural elements (see the blue arrow on the right). In this way she obtains new figural elements that enter into the funnel or additional theoretical properties of the figural elements that enter into the funnel as theoretical elements. The cycle described can be repeated.

Manipulation can be accomplished *continuously* or *discretely*, according to the integration of motion into the manipulation (see the blue arrow on the right).

The process involves geometrical objects and figural concepts. Indeed, according to Fischbein (1993) the solvers can manipulate figural concepts:

> […] usually *in the process of mathematical invention* we try, we experiment, we resort to analogies and inductive processes by manipulating not crude images or pure, formal axiomatic constraints, but *figural concepts, images intrinsically controlled by concepts.*(ibid., p. 160, italics in the original)

Moreover, the author stresses that geometrical objects can be manipulated:

> the student has to learn to mentally manipulate geometrical objects by resorting simultaneously to operations with figures and to logical conditions and operations. (ibid., p. 158)

A GP process does not produce either a pure theoretical or a pure figural object, but an object that is a composition of the two. We refer to the outcome of a GP process as a "*product of GP*", intending in the Fischbein's perspective the outcome of a manipulation of a geometrical object or a figural concept, strictly controlled by theoretical constraints. Manipulation of figural elements that possess certain properties defined by the corresponding theoretical elements, coordinated by the solver's theoretical control, leads to the product of GP. A product of GP is actually a *new solver's geometrical object*, intending an object with figural and theoretical components that are coherent according to the solver's TEG. At the end of the prediction process, the solver has gained new insights into the initial geometric configuration and therefore she has constructed a geometrical object with new figural and theoretical elements. Such a geometrical object can be coherent or incoherent with respect to the given constraints within the TEG. This object can be drawn or only mentally controlled, according to the theoretical control that the solver can exercise.

The manipulation is always controlled by the solver's theoretical control, more or less coherently with the given constraints. During the process the solver can interact with drawings using a bottom-up or top-down approach or an alternative combination of them.

The coherence of the manipulations that a solver can accomplish on a geometrical configuration is determined according to the compliance with the theoretical constraints (i.e. the theoretical elements) that belong to the TEG. When the solver is not able to exercise a good theoretical control over the figure, the coordination between the figural and the theoretical elements fails, and the solver restores to only one of the two domains: the figural or the theoretical. In this case, the products of GP are vague, incoherent or not reachable at all.

Drawings, speech and gestures are heuristic aids for the solver and they can be considered actualizations of the process that communicate features of the process and of its product.

Speech and gestures are not connected with a particular component of the model because they intervene during all the process.

According to the interaction between the components of the model, we have several kinds of GP processes. For example, we can observe:

- GP processes with a dynamic approach, i.e. with a continuous manipulation of the figural elements;
- GP processes carried out through a static approach, i.e. with a discrete manipulation of the figural elements;
- GP processes in which anticipatory intuitions intervene;
- GP processes mainly guided by top-down processes;
- GP processes mainly guided by bottom-up processes.

Using our model as a tool for analysis, we can gain a deeper insight into the connections between the main components of the model. We will provide an example of how the model can be used as a tool for analyzing solver's processes of GP in Section 10.1.3.

When the process of GP has terminated, the product may or may not enter into another process of GP and the products may be connected.

We have widely discussed the connections between processes of GP (see Chapter 8). Now we can say that the solver may accomplish one single or more than one process of GP according to the complexity of the task, the control that the solver exercises on the coherence of the product of GP, the solver's awareness of the completeness of the products with respect to the question proposed into the task.

We have found that within the resolution of the same task, the processes of GP can be organized into more complex structures. So, it seems that the micro-processes of GP can be combined into a larger process: *Macro-process of GP*.

A Macro-process is composed of several *micro-processes of GP,* potentially with different features. So, the macro-process of GP appears to be a melting-pot of micro-processes.

We have provided several examples of connected and disconnected processes of GP. The reason why two process of GP can be connected are the following:

- the solver uses figural elements that she reached at the end of or introduced during another GP process;
- the solver recalls theoretical elements expressed or introduced during another GP process;
- the solver recalls the whole product of GP previously reached.

An explicit reference in the solver's discourse or a *catchment* (McNeill, 2005) can reveal these connections.

In general, it seems that the processes are connected when the solver uses the insight provided by the previously reached products of GP as starting point or helpful elements for a new GP. When this happens, the solvers seem to conceive the interviewer's questions as part of the same problem.

More specifically, we have found two cases:

- the final product of GP looks like a composition of several other products. Interacting with theoretical and figural elements previously expressed, it comes as the outcome of several previously accomplished prediction processes.
- the final product of GP can be a more refined version of an initial rough one. If this is the case, the previously accomplished processes of GP support the

last one; the final product of GP is incrementally constructed adding new details that specify it.

As shown in Chapter 8, in some cases the solver accomplishes only one process of GP. In this case, the macro-process coincides with the micro-process: this is the case of the shorter chain of GPs. This case encompasses either processes of GP that lead to simple products or very condensed processes of GP that lead to rich in theoretical elements and detailed products.

The macro-process of GP ends when the solver does not undertake others micro-processes of GP, even if the interviewer asks for a prediction. This can be the case, because the solver is not able to further explore the situation or because the last GP is actually the most complete. In this context, the *completeness* of a product of GP refers to a product that is reachable with the means that the solver has (paper and pencil without additional tools) and it represents an answer to the question proposed in the task.

We observed that during the macro-process of GP, some products of the micro-processes of GP can be rejected while others can be further explored and expressed in more detail. In retrospect, we can say that, during the macro-process, the solvers can exercise a sort of control over the products of GP that allow us to recognize a product of GP reach enough in theoretical details to be considered a solution to the problem.

The details of the several features of the (micro) processes of GP are addressed by the answers to RQ2 about the new insight into the solvers' actual process of GP, and RQ3 about the role of the main components of the model. In the following, unless otherwise specified, we will use the expression *process of GP* in reference to the micro-process modeled.

10.1.2 What insight into students' actual processes of GP can be gained when our model is used for analyzing solvers' figural expectations?

The model gives us insight into a very local process: the GP process of *that particular solver* at a *given moment* while she is engaging the resolution of *that specific prediction open problem*. Consequently the tools of the model show different components depending on the solvers' approaches and depending on the task. This is evident if we look at the composition of the funnels (Chapter 7) or of the structures of GPs (Chapter 8). Nevertheless, we were able to draw general findings on the processes of GP, as reported in Chapter 6, Chapter 7, and Chapter 8.

Since we have found several possible features of GP processes, we claim that different GP processes are possible. The model (Figure 60) shows the main elements of the process, but these can interact differently.

- *Motion* can be integrated into the GP process, giving the process a dynamic dimension. The new geometrical object can be produced *dynamically*, which means using continuous motion. Solvers can imagine, perform or mimic a continuous movement of one or more figural elements of the geometrical configuration. In this case, the configuration seems to be considered as a continuously changing geometrical object and the interaction with diagrams seems to be similar to the interaction that a solver could perform in a DGE when a point is dragged.

  The dynamic dimension was studied conducting both synchronic and diachronic analyses of solvers' discourse, gestures, drawings, and their interplay. The solvers who integrate motion into the interaction with the figural elements seem to reach detailed and complete products of GP actually because they seem to be able to conceive the figural elements as they handle objects.

  Moreover, dynamism plays a crucial role within the macro-process. A dynamic and a static approach can both lead to coherent products of GP. Nevertheless, the dynamic dimension seems to play a role in reaching products of GP that can lead the solver to a complete solution to the problem, which is a product of GP itself. Instead, a static approach can lead the solver to a coherent product of GP, but such product may not be the most complete.

  The most effective use of dynamism within the GP process occurs when it is coupled with good theoretical control. So, we advance the hypothesis that the dynamic dimension affects the interaction between the figural elements and the theoretical control. This is particularly evident when the solver undertakes GP processes only imagining the geometrical configuration. So, we advance the hypothesis that imagining continuous transformations of the figural elements can lead the solvers to express a product of GP without using any drawings.

- The products of GP can be *detailed* or *fuzzy*. The combination of incomplete drawings, *hedges* into the discourse, and uncertain or absent gestures reveals the fuzziness of a product of GP.

At the end of our analyses, we advance the hypothesis that a *detailed product of GP* corresponds to a geometrical object that for the solver has clear figural elements and well characterized theoretical elements. On the contrary, a *fuzzy product of GP* corresponds to a geometrical object that has figural and/or theoretical elements that are not so clear for the solver and in some sense fleeting. This dichotomy is also a marker of more or less effective GP processes. A process of GP is *effective* when it leads the solver to generate, and eventually communicate, a well-structured new geometrical object.

This dichotomy is reflected in a different interplay between solvers and drawings during the GP process. We find examples of solvers who interact with the drawing only in order to impose particular properties onto the figure. To do this they only describe or show through gestures the figural elements they were predicting; they do not draw anything but seem to control very carefully both theoretical and figural elements of the predicted configuration. The solvers do not need to get confirmation from the drawing and the process is mainly top-down; the drawings are used only for enriching an already communicated product of GP by making it explicit and better communicable to the interviewer.

On the contrary, when the products of GP appear to be fuzzy, the solvers heavily rely on the drawing to confirm, refine and talk about these products. In this case, the solvers do not seem to be confident about their predictions until the drawing confirms such predictions in a figural way; they seem to mainly use a bottom-up process to the extent that they could resort to physical tools or to a trial and errors approach; the drawings play a key role in guiding the process.

- Processes of GP are frequently accompanied by the production of several gestures that support the process itself and clarify the content of the other solvers' productions.

  Gestures are used for focusing on particular figural elements that were already drawn (*deictic gestures*) but they can also reveal a new product of GP. *Iconic-symbolic*, *iconic-physical* and *metaphoric* gestures are mostly used during the prediction process; they present a prominent dynamic dimension. Our analyses reveal two specific roles of gestures within the GP process: during the prediction process the gestures help the solver shape the products of GP; at the end of the process the gestures clarify the details of a product of GP. Moreover, gestures can announce the beginning of a new process of GP and can help the solver in constructing a more accurate product of GP.

Within the macro-process, the recurrence of gestures is particularly relevant. Since *catchments* show the cohesion within the discourse and reveal a recurrence in solver's thinking, they are useful for constructing the GPs' structure.

- One of our working hypotheses concerns the possible role of *anticipatory intuition* within the GP processes. Our analyses actually reveal that an anticipatory intuition can precede or accompany the communication of a product of GP; nevertheless, not every process of GP is accompanied by or coupled with anticipatory intuitions. More properly, anticipatory intuitions can support the process of GP, leading to products that share some features of the intuitions like immediacy, self-evidence, new insight, and sharp contrast with the previous attempts.

  Anticipatory intuitions seem to play a role both in the micro and in the macro process. In the first case, an anticipatory intuition intervenes between the figural elements and the theoretical control, but it also maintains a connection with the theoretical elements. Indeed, in our data, the evidences of anticipatory intuitions are accompanied by the communication of a new interesting position for a figural element of the geometrical configuration in order to maintain certain theoretical constraints. In the second case, an anticipatory intuition that announces an incoming process of GP can come after other GP processes, eventually with products that are in contrast with the new one.

  Moreover, anticipatory intuition is not connected with the coherence of the product of GP. Indeed, it could lead to both coherent and incoherent products of GP.

- We refer to *immediacy* as the quality of a reaction that is undertaken without a strong intervention of the interviewer and rapidly. In some cases, the process is so natural for the solver that it is trigger without any support from the interviewer. In other cases, the first question is sufficient to induce a prediction process. Immediacy can be a characteristic of the GP processes, but not all the processes are undertaken in an immediate way.

Moreover, it can happen that the process of GP is undertaken spontaneously, without questions from the interviewer. Immediacy seems to be mostly a quality of the process of GP undertaken by expert solvers.

Summarizing, during the resolution of the prediction open problems given, the solvers have accomplished different kinds of GP processes, even during the resolution of the same task. A micro process of GP can be accomplished:

- with dynamic or discrete interactions with the figural elements of the geometrical configuration;
- after or with the intervention of anticipatory intuitions;
- supported by both top-down and bottom-up processes, making use of gestures and drawings;
- in an immediate way.

During the resolution of a prediction open problem, the same solver can accomplish processes of GP which show with more or less continuity one or more features listed above. The most effective processes of GP seem to be those that follow a dynamic and top-down approach.

The model (Figure 60) allows us to describe the *experts' prototypical process of GP*. This description does not have an absolute value, but it is limited to the inferences that we can draw observing the expert solvers of our sample. When an expert solver is solving a prediction open problem, she can proceed as follows. Looking at the macro-process, we can draw the conclusions listed below.

- The first process of GP is undertaken quite rapidly: the first question is sufficient to trigger the process. In some cases, the first GP process is triggered even without the first question, by only reading the step-by-step construction.
- The macro-process proceeds naturally, and the solver undertakes several micro-processes of GP; each micro-process can be focused on different figural elements (see, for example, the interplay within the locus of M and the locus of P in Task 2).

Each micro-process is mainly a top-down process:

- in order to respect the given constraints, the solver imagines specific positions of a figural element within the configuration;
- she predicts the figural and theoretical consequence of such a choice, while the element is dynamically moved within the configuration;

- finally, she draws the figural elements of the new geometrical objects.

During the dynamic interaction with the configuration, expert solvers show strong theoretical control. At the end, the expert solver can crystallize the properties of the new geometrical element into a statement. However, sometimes the process is so immediate that the solver first communicates a crystallized statement that is then explained following the steps listed above.

10.1.3 In particular, what are the roles of the theoretical elements, of the figural elements, and of theoretical control?

The *theoretical elements* recalled by the solvers, according to their personal TEG, are strongly involved in GP processes. Because of the dual nature of geometrical objects, when the solvers talk about the geometrical configuration in focus it is quite natural for them to recall or introduce theoretical elements. This is evident mainly looking at one of the tools in the model: the funnels. Instead, what is notable is the quality of the theoretical elements. Indeed, they can be present in a different number, coherent and incoherent with respect to scholarly TEG, *new* or given in the step-by-step construction (*already known*).

The qualitative analysis of the elements reported in the transcription table, reveals that the theoretical elements play a role in selecting the constraints to be maintained. Indeed, each task explicitly gives a set of theoretical constraints that define the geometrical configuration; these constraints can induce additional constraints on the configuration that are logically derived from the given ones. However, there is an actual solver who should take into account the given and the derived constraints. Within the left side of the model, the solver can operate at two different stages: interpreting the given constraints and properly deriving further constraints.

By interpreting and deriving the proper theoretical elements, the solvers obtain a set of constraints that they try to maintain in order to solve the prediction open problem. Theoretical control allows the solver to properly manage all the theoretical elements. The funnels make explicit the theoretical elements considered by the solver during the prediction process.

We have found several instances of a misleading composition of the set of constraints to be maintained. In particular, we have observed the following cases which lead the solvers to *incoherent products* of GP:

- the given constraints are modified, and the solver obtains one or more new constraints;

- the given constraints are maintained, but the solver adds one or more constraints;
- one of the given constraints is completely removed or neglected.

According to our model, in the first case, the solver modified the constraints because of an imprecise interpretation of them. As a consequence, she may focus on incoherent theoretical elements that can be reported by researcher and shown in the funnel coupled with the figural elements that they address. Generally, the solvers do not seem to be aware of the inconsistency of the theoretical elements that they are focusing on and we can say that the theoretical control is not so efficient. Nevertheless, the solver can manipulate the figural elements according to such a set of incoherent theoretical elements, producing an incoherent product of GP. The lack of awareness on the incoherence of the product can be now explained considering that the product of GP is consistent for the solver according to *her TEG*.

In the second case, the solver properly interprets the given constraints, but deliberatively decides to introduce new constraints that are not derived from the given ones. This is evident looking at the funnel. When the solver tries to manipulate the figural elements maintaining the new set of theoretical elements, she can obtain inconsistent geometrical objects within the *scholarly TEG* (see for example a right and equilateral triangle) or the impossibility of manipulating the figural elements at all. The large number of theoretical constraints the solver wants to maintain is one of the causes of the fixedness of the figural elements, strictly connected with the impossibility of reaching a GP.

In the last case, the theoretical control is totally ineffective. The solver is not able to coherently interpret the given constraints because she ignores one or more theoretical elements. Nevertheless, she may be able to manipulate the figural elements, but without any theoretical control. In some sense the transformation is accomplished only within the figural domain, as if the figure *was* the drawing.

We have provided examples in Section 6.1. In all cases, the incoherence of the new geometrical object is evident to the researcher only: the solvers do not seem to be aware of such an inconsistency, or at least in their productions we do not find instances that suggest such an awareness; sometimes, they simply show some uncertainties.

Consequently, we can advance the hypothesis that if the solvers are theoretically poorly equipped and recall incoherent theoretical elements, they resort to using figural considerations totally disconnected with the theoretical constraints.

The theoretical elements introduced by the solver play a prominent role within the prediction process. In particular, the presence and the quality of the *new theoretical elements* are connected to the quality of the products of GP. More specifically, when there are new coherent theoretical elements, the products of GP are coherent. Depending on the number of new theoretical elements, the products of GP can be more or less detailed and well-described, and they provide new insight into the problem. Looking at the funnel, we have observed that a sequence of new coherent theoretical elements indicates the production of a more accurate and detailed product of GP (see Chapter 7). We can claim that the introduction of new coherent theoretical elements is a sort of catalyst for accomplishing a GP process which leads to a coherent product.

On the contrary:

- if the solver does not introduce new theoretical elements, the products are very simple, almost obvious, and they do not give any new information on the problem;
- when there are new theoretical elements and they are incoherent, the products of GP are strongly incoherent, very connected with figural elements, and seem to move the solver away from a coherent final GP.

Moreover, when the solvers communicate a product of GP passing through (eventually incoherent) theoretical elements introduced by themselves, they seem to be convinced of their findings even if the perceptual feedback appears inconsistent to the interviewer.

The tool "*funnel*" was designed a priori to explore the involvement of theoretical and figural elements within the GP process. In retrospect, funnels reveal the deep relationship between predictions and theoretical elements and seem to be powerful tools for predicting the quality (in terms of coherence and richness in mathematical details) of the products of GP.

Interestingly, the set of considered constraints, that a researcher can observe looking at the funnels, could have an influence also on the possible manipulation of the figural elements that solver can accomplish. We have found cases of solvers who have shown good theoretical control, but this was exercised on incoherent or a too large number of theoretical elements. The excessive number of theoretical constraints leads the solver to complete fixedness of the configuration which inhibits the process of GP.

The *figural elements* play the fundamental role of being referents of the solvers' manipulations, and they support the process of GP. Indeed, during the prediction process, solvers mimic through gestures, perform on the drawings, and talk about several figural transformations: for example, figures can be *mirrored*, points can be placed on the *opposite side*, segments can be *turned*. These all are transformations that can be performed within the figural domain but have a strong connection with some corresponding theoretical element. For example, "mirroring a figure" could mean making use of line symmetry. The final object may or may not be a coherent object within the TEG. What allows manipulations that lead to coherent objects is the theoretical control that coordinates between the figural and the theoretical elements.

So, a geometrical object can be manipulated but under the strict coordination of the theoretical control.

Figural elements can support the process because they actualize the imagined manipulation and the product of GP. Moreover, they can give a heuristic kind of help. For example, the case of the solver who finds on the drawing a particular position of a point (excerpt *Silvia_G10_T5_P1_(05:21 – 08:15)*, which suggests a coherent answer to the problem; such a finding triggers a new GP process.

Figural elements can play a role within the GP processes and influence the recalling of coherent theoretical elements. We have discussed the possible arising of the *prototype effect*. The stereotyped figural elements can impede the solver to properly consider the given constraints. For example, a prototypical orientation of the right triangle can contrast with a non-stereotyped given drawing, impeding the solver to recognize the proper hypotenuse. Moreover, when the given drawing does not propose, by design, a stereotyped image of a geometric figure the solver can restore to a prototypical orientation.

When the prototype effect is strongly dominant, it seems to influence the alternative arrangements that the solvers are willing to consider. See, for example, the difficulty to consider a triangle that is not isosceles (excerpts *Andrea_G9_T5_P1_(01:07 – 02:03)*, *Sergio_G10_T5_P1_(01:44 – 03:35)*)).

Our data suggest that theoretical control coupled with a dynamic approach can overcome the difficulties derived from the prototypical effect.

The model also shows the prominent role of *theoretical control*. When the solvers manifest a lack of theoretical control, they restore to one of the two domains: the figural or the theoretical. Manipulations that are not controlled by theoretical control and are accomplished only within the figural domain can lead to inconsistent geometrical objects. See, for example, the extreme example of the right and equilateral triangle proposed by several solvers (excerpts *Laura_G10_T4_P1_(00:00 – 01:16)*, *Giorgio_G13_T4_P1_(00:46 – 02:38)*). Moreover, restoring to figural considerations can lead to an overconfidence on the drawing or in general on the particular figural arrangement of the geometrical configuration, to drawing incoherent theoretical elements and therefore to incoherent products of GP.

When solvers are not able to manipulate the figural elements, they can restore to theoretical consideration, producing vague answers or the impossibility of producing a GP.

Lack of theoretical control also affects the awareness of eventually misleading constraints.

The role of coordination is confirmed by our findings on Task 6. Indeed, even if the solvers show good knowledge of the TEG, they can choose to apply to the configuration an incoherent theoretical element that is in sharp contrast with their ability to theoretically control the figural elements.

Because of the coordinating function of the theoretical control, we advance the hypothesis that when the solver is theoretically or figurally poorly equipped, theoretical control is not sufficient to overcome the difficulties.

So, the main common steps of a GP process are described below.

- After the step-by-step construction was accomplished, the theoretical control guides the interplay between the figural elements of the obtained figure and the given theoretical elements.
- The solver interprets the given theoretical elements, eventually considers other additional theoretical elements and selects the more suitable for producing a prediction. This step is accomplished by the interaction between the theoretical control and the theoretical elements.
- The theoretical and the figural elements in focus are made explicit through gestures and speech.

- Now the solver can transform the figural elements in order to maintain the selected theoretical elements through dynamic or discrete interactions. This step is accomplished by the interaction between the theoretical control and the figural elements.
- Finally, she produces a new geometrical object. This object can be drawn or only mentally controlled, according to the theoretical control that the solver can exercise.

The figural elements and the theoretical elements of the new geometrical object (the product of GP) can become the starting components of a new process of GP. After that a process of GP was accomplished, during a subsequent GP process, the solver will act on a new object that is the initial geometrical object enriched by the new insight taken from the first process.

As an example, now we can recall the excerpt reported in Chapter 3 and describe how our model can be used as a tool for analysis, unveiling the micro and the macro processes of GP which show some of the possible characteristics.

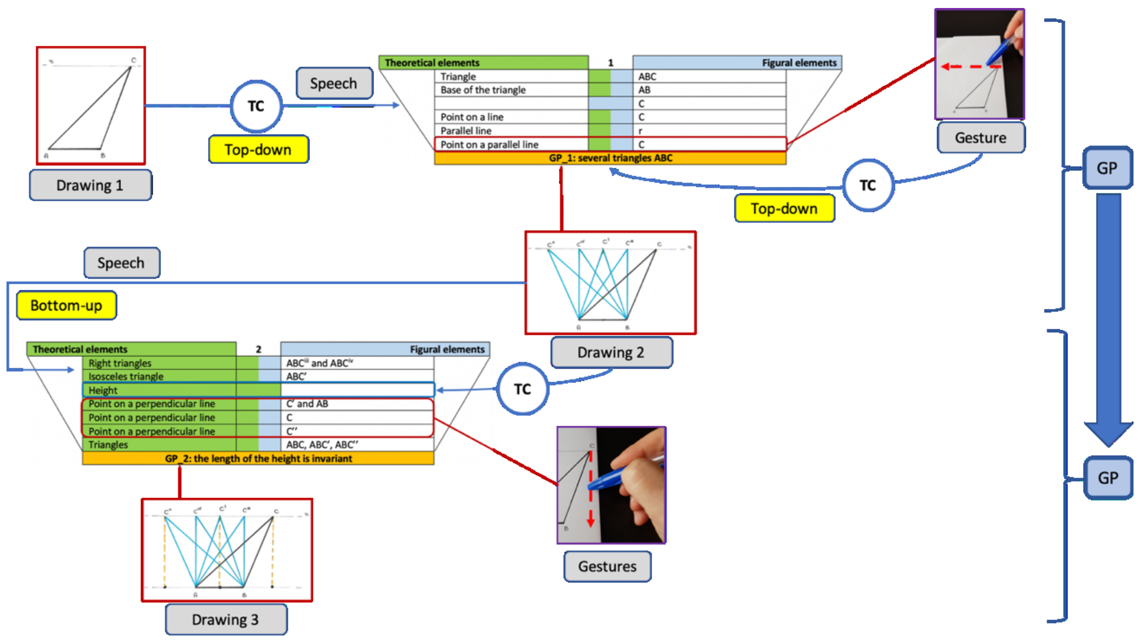


Figure 61 An example of the use of the model for analyzing the micro-processes and the macro-process

The visual diagram (Figure 61) shows two micro-processes that compose a macro-process (on the right); *TC* in the figure stands for *theoretical control*.

Following the thin arrows we can observe a first micro-process, starting with *Drawing 1* and ending with *Drawing 2*; a second micro-process starting with

Drawing 2 and ending with *Drawing 3*. On the right, there is the macro-process composed of the two micro-process connected with a big blue arrow.

After a step-by-step construction the task asks: *If C has to stay on a line r parallel to AB, what can you say about the configuration?*

The solver proceeds as follows.

First micro-process:

- She performs a drawing (*Drawing 1*) and explains though her utterance that there is a triangle with AB as the base, and a point C. These elements are made explicit in the first three entries of the funnel.
- She recalls the additional constraints given by the task and explicitly talks about C as a point on a line that is parallel.
- While she says that C can be moved back and forth on the line, she performs a dynamic gesture. This constitutes the last entries of the funnel.
- The product of GP refers to the possibility of obtaining several triangles while C is moving on the line. The dynamic interaction with drawing, supported by a top-down process of moving C on the line and the solver's theoretical control (TC) that allow her to conceive the consequences of this movement on the triangle, leads the solver to the product of GP.
- The solver performs a new drawing.

The process started with a triangle with a vertex on a parallel line; the outcome is several triangles or several instances of the same triangles with a variable vertex.

Second micro-process:

- The solver looks at *Drawing 2* and, supported by a bottom-up process, she realizes that there are also specific triangles. These are not products of GP, because are mere interpretation of the drawing. She makes explicit by speech and we reported in the funnel the kind of triangles she has observed.
- She says she can consider a "hypothetical height". This observation is accomplished by the solver's theoretical control which imposes the height of the triangles on the figure sketched out in *Drawing 2*.
- The height is conceived dynamically. Instead, she says that it "*is moved according to the point*". Three possible heights are mimicking through gesture.

- The solver recalls the various triangles of GP_1 and concludes that the length of the height *"is always the same"*. In this process, a non-stereotyped image of the height is fundamental for reaching the product of GP. Only a non-stereotyped image allows to consider the height as an invariant.
- She reproduces her findings in a drawing: *Drawing 3*.

The outcome of the process is composed of several triangles or several instances of the same triangles with a variable vertex and a height of the same length.

The macro-process (Figure 62) is composed of the two micro-processes that are connected because of the theoretical reference to "*several triangles*". The solver talks about several triangles when she communicates GP_1 and before GP_2. The macro-process can be sketched as follows (Figure 62).

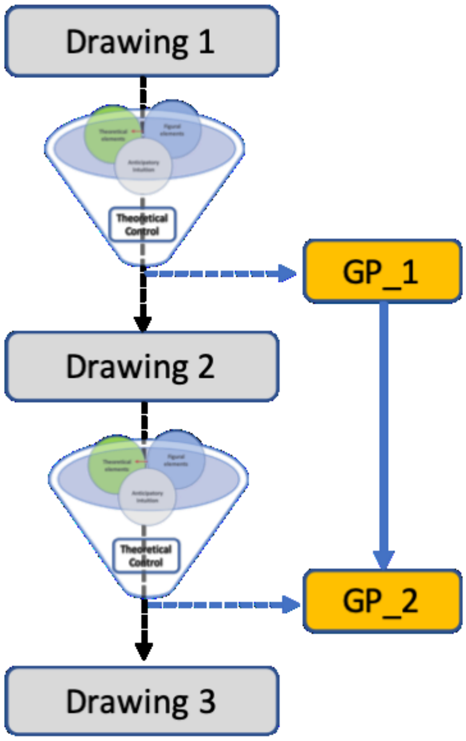


Figure 62 An example of macro-process of GP composed of two interacting micro-processes that led to GP1 and GP2

The macro-process is finished: the solvers does not undertake other GP processes because GP_2 is complete (it is in fact the most complete answer to the problem). Other possible considerations, for example about the invariance of the area, would not include predictions.

Summarizing, the components of our model and their interaction allow us to interpret the solver's thinking during the resolution of a *prediction open problem*. In particular, manipulation of figural elements and recalling of theoretical elements are different components that are constantly interacting through theoretical control. During a process of GP, the theoretical control allows the solvers to conceive step by step the given configuration as a geometrical object. This means that the solver is able to consider the figural elements taking into account their theoretical counterparts and not as mere pieces of a sketched or imagined object. In this way, theoretical control "fills the gap" between the geometrical figure and the drawing (eventually imagined).

10.1.4 A revised definition of *geometric prediction*

Geometric prediction is a complex and global process, where the dual nature of the geomatical objects plays a crucial role.

The main components of the model of GP are the *theoretical elements*, the *figural elements*, and the *theoretical control*.

At the end of our analyses and considering all the components that come into play, a micro *geometric prediction process* can be re-defined as follows.

Geometric prediction (**GP**) *is the process of generation of a new geometrical object through the manipulation of its figural elements that maintain invariant certain theoretical elements that belong to the solver's TEG.*

It is important to stress that the action of manipulation can only be imagined.

The new geometrical object is coherent or incoherent with respect to the scholarly TEG, according to:

- the coherence of solver's knowledge of the TEG with the formal TEG;
- the dynamic use of figural elements;
- the power of the solver's theoretical control.

As stressed in Section 10.1.1, a GP process does not produce either a pure theoretical or a pure figural object, but an object that is a composition of the two. In this perspective, the specificity of the problem of producing figural expectations within the geometrical domain becomes evident. The manipulations are not accomplished only within the figural domain, but they are accomplished under the monitoring of theoretical control which imposes on the figure the theoretical elements recalled by the solver.

At the end of this study, we have gained new insight into the prediction processes.

- We have discovered that there are different characteristics of the GP processes. Moreover, the same solver can accomplish processes of GP with different characteristics.
- We have constructed a model of the micro-process and outlined how the processes of GP interact within a macro-process.
- We have highlighted the features of the GP processes accomplished by expert solvers of our sample.
- The theoretical control, that at the beginning was only a construct that could interact with the GPs, is a crucial component of the process.
- The analyses have highlighted the crucial role of the solvers' knowledge of the TEG in producing coherent products of GP.
- The model gives researchers two operative tools for analyses: the funnels and the transcription coding. The tools operatively show how to recognize instances of GP and to explicitly report and see at a glance the figural and the theoretical elements using by the solvers during a GP process.

10.2 Contextualization within the literature

In this section we situate our findings within the field of mathematics education. In particular, we discuss how our results can be considered with respect to Fischbein's *Theory of Figural Concepts* and *intuitions*, Duval's framework of *cognitive apprehensions*, the debate about *mental imagery*. Moreover, we discuss how our model interacts with research on *visuo-spatial abilities*.

10.2.1 Our findings with respect to Fischbein's *Theory of Figural Concepts*

Fischbein's *Theory of Figural Concepts* (1993) provides an ontological point of view on the nature of geometrical objects. Moreover, it suggests the necessity to always take into account the conceptual and the figural components of a geometrical figure, when we want to study the processes involved in the resolution of a geometrical task.

More specifically, Fischbein (1993) talks about the "*manipulation of figural concepts*" as a means to get accesses to other cognitive processes, like induction:

> But, usually *in the process of mathematical invention* we try, we experiment, we resort to analogies and inductive processes by manipulating not crude images or pure,

> formal axiomatic constraints, but *figural concepts, images intrinsically controlled by concepts*. (p. 160, italics in the original)

Nevertheless, such a manipulation is not widely described. Starting from the theoretical assumptions given by the *Theory of Figural Concepts*, we modeled a process that allows us to shed light on the manipulation of figural concepts.

The particular design of *prediction open problems* allows us to observe solvers during a genuine mathematical experience of exploration and discovery. During the resolution of the given tasks, we can recognize several instances of manipulation of figural concepts.

What we actually observe are the transformations of figural components of the figural concepts accompanied by the description of the conceptual components. The transformation can be accomplished mentally or using physical supports, like drawing or gestures.

So, the *manipulation of figural concepts* during the resolution of *prediction open problems* is:

> *the (mental) transformation of the figural components of the figural concepts according to the conceptual components.*

Replacing the distinction assumed between a *figural concept* and a *geometrical figure* within the Fischbein's framework, the closer the solver's figural concept is to the actual geometrical figure, the more coherent the manipulation is with respect to the relations between the conceptual components. The outcome of the manipulation is a new figural concept, figurally and conceptually enriched by the new insight gained during the process of manipulation.

Another neglected aspect is the notion of *conceptual* or *theoretical control*. Although it is used many times (Fischbein, 1987, 1993), we did not find a clear definition of this term. We borrowed the definition recently given by Mariotti and Baccaglini-Frank (2018):

> [the act of] mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (ibid., p. 156).

This is coherent with the theoretical assumptions of the *Theory of Figural Concepts*. Supported by the findings of other research (Mariotti, 1995; Mariotti & Baccaglini-Frank, 2018); we have studied the possible role of theoretical control within the GP processes.

We can claim that theoretical control plays a crucial role in coordinating the figural and the conceptual components of a figural concept. The *geometrical reasoning* is interpreted in terms of harmony between the conceptual and figural components. From the cognitive point of view what guarantees this harmony is the theoretical control that the solver can exercise over the figural components. Our model reveals that even if the solver is able to properly consider the figural components and the conceptual components, it is the role of coordination played by theoretical control that produces an effective process of prediction or in other words effective geometrical reasoning.

10.2.2 Our findings with respect to Duval's *cognitive apprehensions*

Others research studies were grounded on theories having different backgrounds and different underlying assumptions about the status of the objects of Geometry, with respect to ours. For example, Duval's framework of *cognitive apprehensions* (1994, 1995) is grounded on a semiotic approach. We now try to situate our findings with respect to this framework and these studies.

The representation of a geometrical figure can trigger several cognitive apprehensions. More specifically, we refer to the role of *operative apprehension*. According to Duval (1994, 1995), through operative apprehensions the solvers can (mentally or physically) transform the given figure.

Mereologic, *optic* and *place way* describe transformations of the figure which maintain the ratio between the lengths of the *figural units* of the given figure. This becomes more evident looking at the examples proposed by Duval (1995). However, among the different ways through which a figure can be transformed, Duval does not mention the kind of continuous motion that a solver can imagine and perform. For example, the movement of a point along a locus that is being discovered cannot be explained using the transformations listed by Duval.

Our findings reveal that a solver may perform on the figure changes that are very similar to those that she can observe on a dynamic figure in a DGE when a point is dragged. In fact, this dynamic approach does not depend on any sort of *internalization* of dragging modalities (Mariotti & Baccaglini-Frank, 2011; Baccaglini-Frank & Antonini, 2016), because it is also evident in solvers' who make use of the DGE for the first time in their academic life.

For the sake of clarity, let us consider (Table 13) an example from (Duval, 1995).

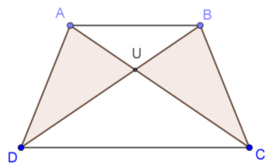
III	Statement of particular case	Statement of general case
	In the trapezium ABCD, U is the intersection of the two diagonals AC and BD. Compare the areas of the two shaded triangles.	In any trapezium, with unknown lengths, you draw shaded triangles do you get the same answer as in the previous question?
Responses	11%	0%

Table 13 Variation of task formulation and of performance in the resolution among 123 students aged 14 years (Duval, 1995, p. 144)

The task given in the "particular case" formulation can be solved making use of the operation of *reconfiguration* (Duval, 1995). In Figure 63 we show the solution of the particular case proposed by Duval following this approach.

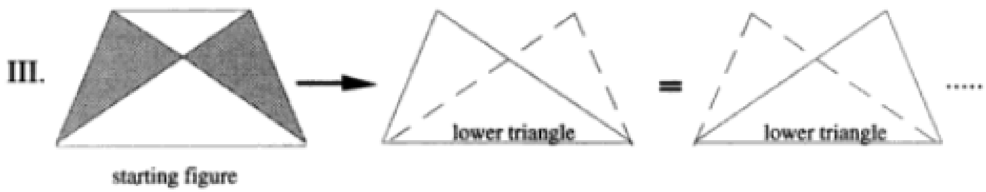


Figure 63 Figural heuristic processing of the task given in the particular case formulation (Duval, 1995, p. 150)

This resolution is very much connected with the recognition of two overlapping figures (*BDC* and *ADC*) and a common figure (*DUC*).

Moving to the general formulation, we can notice (Table 13) that no student from the sample was able to solve the problem. In this formulation, the task is very different: to solve the problem students have to recognize that the triangles *DBC* and *DAC* always have the same base and the same height, even if the kind of trapezium changes; for this reason, they have the same area. Finally, the two shaded triangles (*AUD* and *BUC*) have the same area, because they share the triangle *DUC*.

This resolution is difficult to reach making use of the transformation shown above. Instead, it is easy to find a solution if the solver imagines or draws two parallel lines, one through *DC* and another through *AB*, and imagines moving one of the vertices of the trapezium on such a line.

In Figure 64 we show a possible transformation of the figure, moving *B* on the line through *AB*.

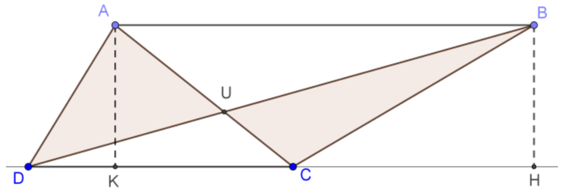


Figure 64 Possible representation of the figure after the movement of point B

We find this transformation difficult to depict using the transformations proposed by Duval (1995). We advance the hypothesis that this kind of transformation, that can be supported by a different task design, would increase the right answer given to the task.

The manipulations described by our model can be considered transformations of the figure suggested by *operational apprehensions*. However, our model adds another way through which a figure can be transformed. This new "way" is placed at the interplay between the *operative* and the *discursive* apprehensions, and it can be considered similarly to a *place and optic way transformation of a particular figural unit* that has an effect on the whole figure. More importantly, this transformation is allowed to be done according to the *discursive apprehensions*. Indeed, the speech acts (denomination, definition, primitive commands in a software) determine what the perceived figure represents (Duval, 1995). Therefore discursive apprehension convinces the solver about the consistency of the transformation within the reference theory.

This kind of transformation does not maintain the ratio between the length of the figural units, but it maintains the properties that the discursive apprehension allows the solver to recognize in the figure.

The product of GP can be seen as the outcome of a reconfiguration guided by operative and discursive apprehension. In this framework, the product of GP is a new semiotic representation of the figure. Moreover, according to Duval (1995) operations used in order to transform a figure could be accomplished only in the figure's register. Indeed, operative apprehension is close to perceptual apprehension. They share the same figural organization laws, but at a different level: in the second case it is an automatic and immediate process, in the first one it is a conscious process and could require a long time.

Our findings reveal that the operative apprehension is intertwined with the discursive apprehension. To be effective the transformations on the figural register must be coupled and directed by the discursive apprehensions.

The difficulties encountered by our solver because of the *prototypical effect* or a strong influence of the particular representation of the given figure can be explained by a too strong intervention of *perceptual apprehensions* which is dominant among the others.

Our findings also address the important role of *discursive apprehension* in interpreting the given theoretical constraints: mathematical knowledge supports the discursive apprehension which drives a coherent interpretation of the properties of a given figure.

10.2.3 Our findings with respect to research on *mental imagery* and *visuo-spatial abilities*

Concerning the research on mental images involved in doing mathematics (see, for example, Bishop, 1983; Presmeg, 1986, 1997; Owens, 1999, 2014), GP processes can be considered processes of elaboration and processing of mental images.

Our model explains how a solver engaged in the resolution of a prediction open problem can generate and transform mental images. The mental images in focus are strictly related to the reference theory (TEG). In this perspective, the outcome of a GP process is a mental image with attributes that belong to the solver's TEG.

Concerning the abilities theorized by Bishop (1983), we stress that one of the possible actions that the ability of VP allows the solver to make is transforming "one visual image into another" (ibid., p. 177). In this perspective a product of GP is the outcome of a process undertaken with the support of VP. Instead, the ability named IFI can support the initial process of interpretation and recalling of the theoretical elements.

Following the definition of visualization provided by Presmeg (2006), the processes of GP can be considered a particular *visualization process:*

> […] when a person creates a spatial arrangement (including a mathematical inscription) there is a visual image in the person's mind, guiding this creation. Thus visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (ibid, p. 206)

Indeed, processes of GP involve both mental and inscriptions (drawings) transformations and, in its new formulation, it is specific of the geometrical activities.

Moreover, following the definition provided by Bruce et al. (2017):

> the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects (ibid., p. 146)

the GP processes can be supported by the *spatial reasoning* when it is carried out in a geometrical context.

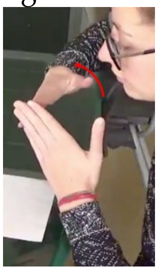
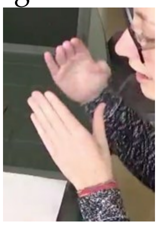
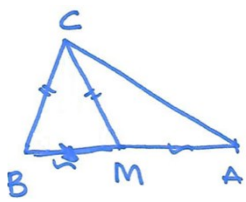
With respect to the research on *visuo-spatial abilities* (Cornoldi & Vecchi, 2004), our model takes into account the fundamental role of theoretical elements and theoretical control. These two components of the model represent the major difference with respect to those models grounded on perspectives (see, for example, Kosslyn, 1996) that consider the interaction between solvers and geometrical objects as pure scanning and manipulations of mental images, without the intervention of any reference mathematical theory (in our case TEG).

The outcome of a mental manipulation can interact with other elements deposited in long-term memory for interpreting the mental images. Our model stress that the role of the solver's geometrical knowledge and ability to impose on an image certain geometrical properties (i.e. theoretical control) do not only have an interpreting function but also a control function. We can claim that the manipulations of images that a solver can accomplish on a geometrical figure are driven by the solver's use of the TEG: the manipulations are *Theory-driven* by the theoretical control. The *theoretical control* is not only a system of control of the consistency of the outcomes but also a construct that acts *during* all the manipulation, guiding it.

The manipulations that a solver can accomplish on a geometrical figure during the GP processes convey a mutually determining phenomenon of acting and knowing: the GP process involves knowing that something occurs as a result of performing actions on a geometrical object.

Here is (Table 14) an example of an excerpt where we highlight the component of our model and the *visuo-spatial abilities*: *Marta_MS_T5_P1_(00:23 – 02:44)* already reported and commented in Section 6.3.2.

Who	*What is said*	*What is done*	*GP Model*	*Visuo-spatial abilities*
Int	Make a prediction: is it possible that CM is congruent to CB?			
Stud	Is it possible that CM is congruent to CB?			

Int	Mm mm.			
Stud	So...I have a triangle...M midpoint, perfect. CM...great, so it is a median. Ok.	She is looking at the step-by-step construction.	Recalling and introduction of theoretical elements	Visual reconstructive ability
Stud	CM can be congruent to CB...I think so, I mean that…ehm...	She is looking up and ahead.	Recalling and focusing on theoretical elements	
Stud	If CM is congruent to CB, I have a triangle BCM that is isosceles and, just a bit big, I have another triangle ABC that was my initial triangle, I mean. So…eh…	While she is talking about the triangle ABC, she rotates the right hand:  While she is talking about the initial triangle: 	Recalling and introduction of theoretical elements Manipulation of figural elements Theoretical control GP_1: BCM is an isosceles triangle	Imagery generation ability Imagery manipulation ability
Stud	Yes, I think that it is possible.			
Int	Ok. Make a drawing of what you imagined.			
[…]				
Stud	This is what I imagined.	 *Drawing 2a*	Recalling of theoretical elements (see the hash marks)	Imagery generation ability Planned visual scanning
[…]				
Int	Do you think that there are other positions for point			

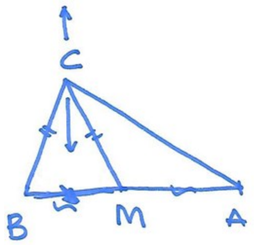
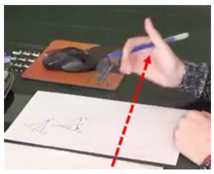
	C so that CM is congruent to CB?			
Stud	Ok, so any translation of C downwards or upwards, parallel to BA, **maintains this property**. So...	She draws two arrows:  *Drawing 2b* "downwards or upwards":  "parallel": 	Manipulation of figural elements Theoretical control GP_2: C on a line perpendicular to AB	Imagery manipulation ability Imagery generation ability

Table 14 An excerpt from Marta interview's during the resolution of Task 5. The fourth column reported the components of the GP model, the last column shows the visuo-spatial abilities possibly involved

The manipulations (mentally) accomplished by the solver can be supported by the visuo-spatial abilities reported in the last column. Nevertheless, the only use of those abilities does not explain why Marta decided to consider an isosceles triangle in her answer. More likely, her TEG supports the translation of the given information "*CM equal to CB*" into "*CMB is an isosceles triangle*"; then her theoretical control, possibly supported by the imagery manipulation ability, allows her to find the triangle ABC as a transformation of the triangle CMB.

For drawing the imagined figure, Marta can make use of the ability to generate and to visual scan images.

Again the manipulation which leads to the second product of GP can be supported by the imagery manipulation and generation ability but is strongly influenced by the theoretical control. This interpretation is supported by the solver's utterance: she explicitly says that the manipulation she is imagining is the most suitable for maintaining the given constraints. Such constraints are not figural but they belong to the TEG.

Summarizing, we can claim that during the processes of GP the solvers can accomplish manipulations that are supported by abilities like those described by in Cognitive Psychology, of which the visuo-spatial abilities are an example. However, the manipulations are not only supported by the solvers' TEG but they are also strongly controlled by the solvers' theoretical control, which interprets the outcome of and drives the manipulation.

Our findings also address the topic of *dynamism* in geometric reasoning. A priori, we have conjectured that the solvers can integrate motions into their interaction with drawings and in general into the manipulations of images. In retrospect, we can claim that, among the solvers of our sample, even if this is not the most popular approach to the task, a dynamic approach revealed to be the most effective in order to reach a coherent prediction. This is in line with Presmeg's findings on *dynamic imagery* (1997).

Recently, grounded on the classification of spatial skills as *static* versus *dynamic* skills and *intrinsic* versus *extrinsic* proposed by Uttal et al. (2013), McGarvey et al. (2012) focus on the dynamic dimension of students' geometrical reasoning discussing how student approach the task of drawing.

> […] there were two ways in which students approached the task of drawing: (1) drawings of multiple triangles, each one different from the other, and (2) drawings of single triangles undergoing transformation. (ibid., p. 144)

In McGarvey et al.'s study, the drawing task was given after an exploration of a dynamic triangle in a DGE, so students marked on the drawing how they perceived the dynamic shape.

> The first category of drawings focuses more on the multiple different shapes while the second focuses on the continuous transformation between shapes. […] The drawings in both categories show evidence of intrinsic-dynamic spatial reasoning as the children transform the triangle in time. (McGarvey et al., 2015, p. 115)

The authors conclude that:

> the drawings are less about the discrete example space of the dynamic triangle and more about its continuous changeability. (McGarvey et al., 2015, p. 115)

Looking at solvers like Marta (see the previous subsection), we observed that she seems to conceive the movement of C and the whole triangle as a continuously changing object. This solver's approach is independent of a dynamic exploration in a DGE because the task was accomplished first in a paper-and-pencil environment.

This finding stresses that motion can actually be embedded in the manipulation of a geometrical figure and in this case the manipulation is particularly effective in terms of exploring the possible solution of a given task.

Finally, our findings provide new insights into a kind of reasoning that is neither deductive nor inductive: *transformational reasoning* (Simon, 1996).

> Transformational reasoning is mental or physical enactment of an operation or set of operations of object or set of objects that allows one to envision the transformations these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider not a static state, but a dynamic process by which a new state or a continuum of states are generated. (Simon, 1996, p. 201)

Looking at the definition and at the example provided by Simon (1996), we can say that the kind of solvers' thinking underlying GP processes can be the *transformational reasoning.* So, our excerpts can be interpreted as additional examples of processes carried out by transformational reasoning.

Following the definition, in our case:

- the *objects* are the figural elements coupled with the theoretical elements;
- the *operation* is the manipulation of figural elements controlled by theoretical control;
- the *results* are the product of GP.

10.2.4 Our findings on Fischbein's *intuition*

Although our main focus was not on *intuitions*, this study also contributes to this topic. Since "intuition" is a controversial term, we explicitly followed Fischbein's notion of *intuition* (Fischbein, 1987), widely described in Chapter 3.

We decided to focus on *anticipatory intuition,* a kind of intuition specifically involved in problem-solving activities. This seems to be an open research topic: we did not find explicit examples of *anticipatory intuitions* within the literature (Fischbein, 1987; 1999a) or a description of the circumstances that lead to an anticipatory intuition.

We have found that *prediction open problems*, specifically designed for eliciting prediction processes, can also trigger processes that produce anticipatory intuitions. So, these kinds of intuition can arise during the resolution of an open problem. This provides a direction for the task-design of further research on the topic of intuition.

Moreover, we have found several anticipatory intuitions combined with the communication of a product of GP. We can recognize the arising of anticipatory intuition when the solver communicates a new piece of mathematical knowledge about the solution of the given task:

- suddenly;
- without an explicit or recognizable connection with the processes previously undertaken;
- after a long silence.

An anticipatory intuition may have one or more of these characteristics and it can support the process of GP, leading to products that share some features of the intuitions. Our data provide several examples of anticipatory intuitions: these episodes provide new insight into the Fischbein's taxonomy of the intuitions.

In some cases, anticipatory intuition provides a heuristic kind of help to the solvers (see, for example, excerpt like *Sergio_G10_T5_P1_(01:44 – 03:35)*). This is an interesting point for studying the role of anticipatory intuition within the resolution process of other kinds of problems.

Finally, we want to clarify that intuition and the product of GP are different outcomes. Supported by our data, we can claim that a product of GP is not an intuition, for the reasons listed below.

- Intuitions are characterized by self-evidence and vividness. If a product of GP is an intuition, we expect all of the products of GP to be detailed. Instead, we have both detailed and fuzzy products of GP.
- Intuitions are characterized by intrinsic certainty. Instead, we find several uncertainties within the solver utterances.
- Intuitions are immediate; instead, we also find a product of GP at the end of a long process.
- Intuitions are resistant to change; instead, often during the resolution process, the solvers change their idea about a product of GP.

In summary, the products of GP can share features of anticipatory intuitions, but not all of them are actually intuitions.

10.3 Educational implications

The need for *visual literacy* has become fundamental to the functioning of modern society, characterized by information-based communication where interfaces are increasingly less alphanumeric and more visuo-spatial (Mulligan, 2015). As Arcavi (2003) stresses

> we live in a world where information is transmitted mostly in visual wrappings, and technologies support and encourage communication which is essentially visual. (ibid., p. 215)

Because of mathematics educators' recent renewed interest in *visualization* and its related skills, our findings can be interesting from an educational point of view. In particular, it seems to be important that teachers develop the awareness that a student, who is solving a geometric task, can undertake GP processes even when this is not explicitly required.

Our data reveals that solvers engaged in a *prediction open problem* can spontaneously undertake GP processes, to the extent that the GP processes can be undertaken before the question, right after the step-by-step construction (see, for example, *Agnese_MS_T2_P1_(01:54 – 02:08)*, *Ilaria_G9_T4_P1*). As long as the question is not asked, the task is not yet a prediction open problem, but only the description of a given geometrical configuration that could be provided in another kind of problem. In other cases, the GP process was undertaken right after the first question, when a prediction had not yet been asked for explicitly.

This evidence supports the hypothesis that a GP process can be spontaneously undertaken by a solver when she approaches a geometrical task. So, from an educational point of view, it is important for the teacher to be aware of such a phenomenon, in particular because the products of GP processes undertaken can remain implicit and unconscious for the solver and unknown to the teacher. Indeed, the solvers of our sample communicate the product of their GP processes mainly because, by design, they are invited to do so explicitly; in a generic problem-solving context the products of GP can remain unexpressed. This claim is supported by the gestural instances of GP that are not accompanied by any discursive instances (see Section 6.4). Moreover, we have highlighted (see Section 8.1) that a product of GP is not neutral within the macro-process. Indeed, the products of GP can interact and influence each other within the macro-process. An incoherent answer to the problem can be supported by several products of GP that can remain unknown to the teacher if the solvers are not asked for making them explicit.

The teacher who knows the solver's products of GP has windows onto the solver's interpretation of the figure and of the task and, consequently, she can help students to overcome difficulties. So, we can draw the importance of discovering the solvers' products of GP.

More specifically, the process of making the products of GP explicit provides a window onto the solver's *TEG*. Our model stresses the fundamental role of the solvers' knowledge and use of the *TEG*. The funnels show the central role of the theoretical elements recalled or introduced by the solvers. These elements revealed to be fundamental for inferring the solver's interpretation of the given figure and in particular the theoretical constraints that the solver is imposing on the figure. We had the opportunity to capture solver's use of theoretical elements because of the particular design of the interview; nevertheless, the teacher can also ask the solver to make them explicit. A teacher, who knows what the "pieces of knowledge" are that the solvers are using for reaching a product of GP can coherently interpret the solvers' misleading interpretations and incoherent solutions. A coherent interpretation of the solvers' processes of GP leads the teacher to a proper intervention for supporting an effective resolution process. In Section 8.2, we highlight and discuss the elements that can inhibit an effective process of GP.

So, our model provides a theoretical lens through which the teachers can look at solvers' productions during the resolution of a geometrical task, for recognizing the possible obstacles and planning possible intervention to remove them.

In retrospect, data analysis also reveals particular observable markers that a GP process is occurring. In particular, during the processes of GP, we notice that:

- the solvers can silently perform gestures;
- the solvers use expressions like "I expect to", "I imagined", "what I was thinking is more intuitive", "intuitively", "hypothetical point/segment/line";
- the solvers' utterances contain the use of modal verbs and *hedges* in general;
- the solvers can explore a drawing previously performed, without adding other graphical elements.

A priori, we look at solvers' productions as windows onto the GP processes. In retrospect, the way through which the solvers make use of gestures, drawings, and utterances can reveal to an observer (eventually a teacher) that actually a GP process is running, even if the solver has not already communicated its products.

In terms of implications for supporting *visualization* and *spatial reasoning,* our study proposes a new kind of task: *prediction open problems*.

This kind of task provides an example of a geometrical problem that gives students a genuine research experience. This is in line with the Cuoco et al.'s (1996) perspective, concerning the *mathematical habits of mind* and the design of a mathematical curriculum which supports them.

> By "habits of mind," we mean ways of thinking that one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become, well, mental habits: not only *can* one draw upon them easily, one is *likely* to do so. (Goldenberg, 2014, p. 146)

The researchers describe the features of a mathematical curriculum that may "help high school students learn and adopt some of the ways that mathematicians think about problems" (Cuoco et al., p. 376). Among the several mathematical habits of mind, the researchers include *visualization* and more specifically the *visualization of the change*. The educational implication of such a perspective is stressed as follows:

> A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics *do* and what they say. (ibid., p. 376)

The *prediction open problems* proposed in our study have elicited GP processes strictly connected with the possibility of considering a situation that changes, potentially continuously (i.e. *visualizing the change*). Because of the explorative nature of the questions proposed in a prediction open problem, this kind of task seems to be particularly effective in promoting *visualization as a habit of mind*.

We advance the hypothesis that the implementation of *prediction open problems* in the classroom activities may support the development and refinement of abilities connected with visualization and spatial reasoning without losing the connection with the reference mathematical theory (in our case the TEG).

Even if the task-design in the case of *prediction open problems* has to be carefully constructed, it is not so difficult for teachers to find problems that can be reformulated as open problems. As shown in Section 3.3, in some cases it is sufficient to take the text of a geometrical problem asking for a proof and try to change the question, "opening" it. This can be a first approach to the task design of prediction open problems. Moreover, problems concerning locus constructions can potentially be fruitful in terms of eliciting GP processes and construct prediction open problems.

In terms of implications for the teaching and learning of geometry, searching for and recognizing invariants under transformations is not only fundamental in doing geometrical investigations, but it is recognized as an important achievement that should be developed by students attending Italian high schools, as reported in the government's document *Indicazioni Nazionali* (MIUR, 2010). According to our perspective, theoretical control allows the solver to maintain certain given properties during the manipulation of the figure. Moreover, a good theoretical control is strictly connected with reaching a coherent product of GP. So, we advance the hypothesis that educational activities that require the production of predictions can support a teaching and learning of geometry that addresses the issue of recognizing invariants.

Moreover, we have observed that when a solver works in a DGE, the products of the previously accomplished processes of GP influence the exploration of a dynamic figure corresponding to the figure reasoned upon in the paper-and-pencil environment. If the solver is quite convinced of her incoherent predictions, she seems to be blind to the feedbacks provided by the dynamic figure. Coupled with the immediacy and spontaneity of the GP processes, this finding is interesting from an educational point of view: it reveals that even when a solver starts solving a problem in a DGE, she may have figural expectations on the behavior of the dynamic figure that can influence the exploration. We again stress the importance of allowing the solver to make explicit their predictions about a given geometrical figure.

In Section 3.2.1, we touched on how students should develop all the elements involved in doing geometry and construct their figural concepts, as conjectured by Fischbein (1993). Indeed, concerning figural concepts, from the developmental point of view, initially the visual aspects are dominant, and gradually the role of formal constraints becomes more important, until the construction of the *figural concept* is reached (Mariotti, 2005). So, the *figural concept* is individually constructed during the learning experiences and can change. Moreover, personal *intuitions* change during the time (Fischbein, 1987).

> The educational problem is not to eliminate intuitive representations and interpretations. This, in our view, would be impossible and certainly not desirable. Rather, the educational problem is to develop the capacity of the student to analyse and keep under control his intuitive conceptions and to build new intuitions consistent with formal scientific requirements. (ibid., p. 206)

Therefore, we can claim that all the components of our model are modifiable passing time:

- the student's *TEG* changes because of the exposure to the mathematical culture and to the learning experiences;
- consequently, the student's figural components of figural concepts may also change, for example leaving some stereotyped images in favor of more versatile and transformable ones;
- finally, the theoretical control can also be developed, making use of appropriate educational activities.

Moreover, other abilities that could support the manipulation of figural elements (for example, *visuo-spatial abilities* or *VP*) are recognized to be not innate but learnable and malleable at any age (Newcombe, 2010; Bruce et al., 2017).

So we can advance the hypothesis that also the GP processes are trainable by designing effective educational activities, which specifically support and improve the dynamic dimension of the process. From this perspective, thanks to the natural integration of motion into the exploration of a figure in a DGE, we advance the additional hypothesis that DGE can offer powerful educational resources. In particular, exploring a dynamic figure can support and improve the solver's theoretical control and therefore the GP processes.

In addition, our findings concern the central role of theoretical control in coordinating the interplay between the figural and the conceptual components, and the importance to refer the theoretical control to the specific *TEG* developed at school. In particular, they suggest the importance of teaching students to carefully use intuitive reasoning, and the importance of promoting development of their consciousness regarding theoretical control as well as the possibilities of theoretical inconsistencies.

Finally, we want to stress that effective educational activities for improving GP processes must to take into account and support all the main components of the model: the manipulation of figural elements, the solver's *TEG* and, more importantly, the theoretical control.

10.4 Shortcomings and directions for further research

We would like to conclude this chapter by expressing some limitations of our study, introducing some general questions that arise from it and outlining possible directions for future research that could stem from it.

As for the analyses, because of the time constraints, we did not have the opportunity to adequately investigate the generation of predictions within the DGE. Since the interaction between a solver and a dynamic figure has specific features that are different with respect to the interaction between a solver and the drawing of a figure on a sheet of paper, when the process of GP is undertaken within a DGE we can observe other and potentially different characteristics of the process. This is a line for further research. Moreover, looking at the second part of the interview we observed that solvers make use of different dragging modalities and that in some cases these modalities evolve during the exploration, influenced by the previously accomplished processes of prediction. Analyzing the dragging modalities, their possible evolution during the dynamic exploration, and their interaction with prediction processes could be a fruitful direction for further research aimed at gaining a deeper insight into GP processes and their influence on the solvers' exploration of a dynamic figure.

As for the task design, we developed our tasks starting from a limited set of common mathematical notions and theoretical elements to be used for solving the proposed *prediction open problems* (see Section 5.3). This choice is supported by the idea of having a common mathematical background that all the solvers could recall more or less easily and that they have certainly developed during their academic life. For this reason, for example, we did not make use of tasks that involve loci like ellipses or parabolas. We proposed the same tasks both to high school students and more expert solvers. However, for the latter, not all the tasks were really challenging, to the extent that they solved the tasks easily only by recalling a well-known theorem or mathematical result. The most challenging task revealed to be Task 2; it could be interesting to explore the design of tasks that share a different set of mathematical notions and to study the solver's GP processes within these tasks.

Furthermore, since sometimes the questions that follow the very first one suggest a particular exploration and a specific focus (for instance, see the third question of Task 2) it might be interesting to see what happens if the question is changed, before focus is put on other configurations. For example, it might be interesting to observe also what happens if the request focuses on the general configuration, inducing the solver to focus on the final product of a possible transformation of the configuration and not only on a particular sub-configuration.

Moreover, in order to analyze in greater depth the role of theoretical elements recalled by the solvers during a process of GP, it could be interesting to interview solvers who belong to a more homogeneous sample. For example, we could analyze the features of GP processes undertaken by solvers who were previously exposed to a same set of educational activities designed by their teacher and aimed at introducing certain mathematical notions (for example, point symmetry and line symmetry). For the same reason, making use of suitable problems, it might be fruitful to conduct interviews with younger students that have experienced geometry activities only in an unformal way.

Concerning the third research question, a small remark is needed. The study aimed at describing the process of GP, also analyzing both the role of figural and theoretical elements. In retrospect, the main focus was on the latter. The reason is the starting point and the rationale of the study: the shortcomings of using a pure psychological perspective for interpreting the prediction processes when the reasoning is carried out in the context of the TEG. In particular, a pure psychological approach does not seem to take into account and to clarify the possible role of elements that belong to the domain of Euclidean Geometry. This is the reason why this study focused mainly on the fundamental role of the theoretical elements, and why the role of figural elements might seem somewhat neglected. For the sake of completeness, it is important to highlight that the figural elements also have an influence on the prediction processes. Moreover, the role of figural components in geometrical reasoning is well documented within the Mathematics Education literature (e.g., Mariotti, 1992, 1995; Fischbein, 1993; Duval, 1994, 1995). In this perspective, a deeper investigation on the role of gestures within the prediction processes could be a fruitful line of investigation. Indeed, as highlighted in Section 10.1.1, speech and gestures intervene during all the GP process. Nevertheless, similarly to drawings, gestures have a spatial component that makes them potentially more effective in dealing with the figural component; the dynamic nature of a gesture seems appropriate to provide and support a dynamic representation of a figural concept. As an example, the gesture of tracing a circle, usually accomplished by moving a pointing finger that mimes a point moving on a circular trajectory while another finger is fixed, provides support to the figural component of the figural concept of “circle”. Moreover, even if our findings reveal that to be effective the transformations on the figural register must be coupled and directed by the discursive apprehensions, also the figural register can be coupled and directed by an apprehension “supported by gestures”. Indeed, gesture and speech give a different contribution: since gesture is an iconic sign, better than speech it can afford the function of representing a manipulation of the figure. So, gestures can support perceptual apprehension in its dynamic dimension and can constitute the base of the prediction.

A discussion should be opened, stemming from our findings, about whether, as a mathematics education community, we are interested in fostering processes of prediction-generation as described by our model, and therefore whether specific educational activities *should* be taught as part of the mathematics curriculum. If we decide to support the development of effective students' GP processes, we should consider all the main components of our model as well as their interactions. Moreover, we need to take into account the solver's difficulties in producing coherent products of GP using properly both the theoretical constraints and the figural elements emerging from the given geometrical figure. Might it be possible, through particular teaching strategies, to avoid some of the potential cognitive difficulties that certain figural elements seem to induce? If not, what strategies might be developed to overcome such difficulties? Moreover, might it be possible, through particular teaching strategies, to support the development of the solver's theoretical control over the figures?

So, a possible line of research could address the issue of the trainability of effective GP processes; the hypothesis of the trainability of GP processes should be confirmed or disconfirmed by further investigation. Moreover, we had advanced the additional hypothesis that a DGE can offer powerful educational resources in order to foster a dynamic approach to the geometrical figures; in particular, exploring a dynamic figure could support and improve the solver's theoretical control and therefore the GP processes. This hypothesis offers an additional line of research.

As concerns the solvers' exploration of the tasks in a DGE after having faced the problem in a paper-and-pencil environment, we have unearthed an unexpected result. Indeed, we expected that a mismatch between the solver's expectations and the actual behavior of the dynamic figure will trigger the solver's surprise and consequently activate a new resolution process or new processes of prediction. Instead, we found that in some cases, if the products of GP are incoherent but quite convincing for the solvers, the dynamic exploration can be ineffective for changing the solvers' mind to the extent that they seem to be "blinded" to the contradictory feedbacks provided by the DGE. In this case, we did not find any observable instances of surprise and our hypothesis was partially disconfirmed. Moreover, this phenomenon seems not to be coherent with the *transformational-saliency* *hypothesis* (Battista, 2007). For these reasons, further investigations devoted to unveiling the reasons for this apparent blindness should be conducted. It might be interesting to investigate if acting in a DGE can support a solver in perceiving incoherence of which she was not previously aware.

An aspect that has not been considered in this study but that would be worth focusing on is the role of the interviewer (or possibly teacher if the interview activities are used in the classroom). We have decided to look at GP processes accomplished in the particular environment of the task-based interview. This choice was supported by our research aim of gaining insight into a cognitive process. However, a one-to-one task-based interview is not the usual context in which students face geometrical problems. So, two new issues arise. The first has to do with task design of educational activities that elicit GP processes and that involve a whole classroom of students instead of a single solver. Is it possible to design educational activities for the whole class that include *prediction open problems* or, in general, elicit GP processes? If so, how can we design these activities? How should the formulation of the tasks change? How should the teacher propose these activities? How should the teacher intervene (by design) to elicit the desired processes? How can GP processes be elicited during regular classroom activities in geometry?

The second issue addresses how the role of the teacher (as opposed to the unknown-to-the-students interviewer) might influence students' GP processes. What changes in students' processes when the teacher proposes a prediction open problem or educational activities that elicit GP processes? How can the teacher support the student's processes of prediction (potentially in different ways than through the supporting interview questions used by the researcher in this study)? What role is played by the *didactical contract*?

Furthermore, our data reveal that when a GP process has been completed, the solvers can crystallize the predicted property into a statement (see the excerpt *Marta_MS_T5_P1_(00:23 – 02:44)* in Section 6.3.2) that can be very close to a conditional statement. So, we can advance the hypothesis that the conjecture-generation process could be connected to and potentially could follow the prediction-generation process. The possible connection between the processes of prediction-generation and conjecture-generation can be further analyzed, as well as the situations in which a GP process does not lead to a conditional statement.

Moreover, looking at the macro-process of GP (Section 10.1) and how several processes of GP can be connected and lead to a more or less complete product of

GP, we can advance the additional hypothesis that the process of GP may be an initial phase of the problem-solving process that could be followed by more analytical investigations. Further research can address the issue of the possible role of GP processes within the general problem-solving process.

As highlighted in this chapter, the model allows us to describe the *experts' prototypical process of GP*. However our findings are limited to the inferences that we can draw observing the expert solvers of our sample. Moreover, in some cases, the processes of GP undertaken by expert solvers goes so fast and is so condensed that it can be difficult to grasp in detail all the steps and features of the process. Nevertheless, from an educational point of view and to gain further insight into the mathematical *habits of mind* that interact with the GP processes, it is potentially important to unveil the processes of prediction of expert solvers. For this reason, a possible line of further research is to investigate the approach to more mathematically advanced prediction open problems of mathematicians who are active in research in different mathematical fields.

We conclude with an important issue in the field of math education: the problem of designing tasks that are in line with the educational goals behind a specific activity, that is, the problem of generating "good problems" aimed at achieving certain educational goals. This issue touches research in curriculum design and further research could explore if *prediction open problems* should have a place in school mathematics, and if so which, at what grades, and to help achieve which educational goals.

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Glossary

Table of abbreviation

GP	Process of GP
GP_N	Product of GP number "N"
TEG	Theory of Euclidean Geometry
TC	Theoretical Control
DGE	Dynamic Geometry Environment
P	in place of "*the point P*" we reported only the letter P
MB	in place of "*the segment MB*" we reported only the letters MB
C(C,CA)	circle centered at C and with radius CA
Name of the solver_GNumber	when the interview was video recorded, the solver was attending the "Number" class
Name of the solver_MS	when the interview was video recorded, the solver has already reached a bachelor's degree (the Italian "*Laurea Triennale*") and was attending the postgraduate classes in Mathematics (the Italian "*Laurea Magistrale*")
Name of the solver_MD	when the interview was video recorded, the solver has already reached a master's degree in mathematics (the Italian "*Laurea Magistrale*")
Name of the solver_PhD	when the interview was video recorded, the solver was a PhD student in Mathematics.

What is meant by…

Expert solver	A solver who was exposed for a long time to the mathematical knowledge and, by virtue of this, is supposed to be expert.
Theoretical elements	Elements that belong to the formal Theory of Euclidean Geometry. They include: all the properties that a solver gives to the configuration or that she gives to part of it; theorems and mathematical results.
Figural elements	Elements that belong to the figural domain in a specific moment (seen on the drawings or on the screen), related to the configuration in front of the solver.
Theoretical control	We use the expression *theoretical control* intending "[the act of] mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (Mariotti & Baccaglini-Frank, 2018, p. 156).
Anticipatory Intuition	According to Fischbein's definition (1987), we label *anticipatory intuition* moments in which a solver produces a sentence or a gesture suddenly without an explicit link to the previous process of solution and which led to new insight into the problem. At such a moment, we infer that an intuition has occurred, and we can observe an evidence of it.
Window gestures	These are gestures that communicates a product of GP. Gestures may or may not be coupled with discursive elements.
Instance of surprise	Gestural or discursive expressions which reveal a surprise.

Appendix A

Here are the two version of the original consent forms used before the interviews.

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Io sottoscritto/a ____

autorizzo

la dott.ssa Elisa Miragliotta, dottoranda presso l'Università di Modena-Reggio Emilia in convenzione con le Università di Ferrara e Parma, sotto la direzione scientifica della prof. Anna Baccaglini-Frank (Università di Pisa) e la supervisione della prof. Alessandra Fiocca (Università degli Studi di Ferrara), all'uso di foto e video delle attività didattiche svolte tra dicembre 2017 e maggio 2018, nell'ambito del progetto di ricerca riguardante i processi coinvolti nella risoluzione di compiti geometrici, per realizzare materiale documentativo mediante, anche, riduzioni o adattamenti.

Il materiale documentativo sarà utilizzato senza limite di spazio e di tempo per scopi:

- informativi – formativi;
- didattici;
- tecnico – scientifici.

Non è richiesto, in proposito, alcun tipo di compenso.

DATA ______________________________

FIRMA ____

DOCUMENTAZIONE VIDEO FOTOGRAFICA

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Io sottoscritto/a ____

genitore o rappresentante legale dello studente ______________________________

autorizzo

la dott.ssa Elisa Miragliotta, dottoranda presso l'Università di Modena-Reggio Emilia in convenzione con le Università di Ferrara e Parma, sotto la direzione scientifica della prof. Anna Baccaglini-Frank (Università di Pisa) e la supervisione della prof. Alessandra Fiocca (Università degli Studi di Ferrara), all'uso di foto e video delle attività didattiche svolte tra dicembre 2017 e maggio 2018, nell'ambito del progetto di ricerca riguardante i processi coinvolti nella risoluzione di compiti geometrici, per realizzare materiale documentativo mediante, anche, riduzioni o adattamenti.

Il materiale documentativo sarà utilizzato senza limite di spazio e di tempo per scopi:

- informativi – formativi;
- didattici;
- tecnico – scientifici.

Non è richiesto, in proposito, alcun tipo di compenso.

DATA	FIRMA
____________________	______________________________

Appendix B

The original Italian version of all the excerpts that have been used in the thesis is available here:

https://drive.google.com/open?id=11tiLfw8b33R2vUGnsXccUabsEZzrM7cE

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