Evolution fractional differential problems with impulses and nonlocal conditions

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Dedicated to Professor Patrizia Pucci on the occasion of her 65th birthday

Abstract

We obtain existence results for mild solutions of a fractional differential inclusion subjected to impulses and nonlocal initial conditions. By means of a technique based on the weak topology in connection with the Glicksberg-Ky Fan Fixed Point Theorem we are able to avoid any hypothesis of compactness on the semigroup and on the nonlinear term and at the same time we do not need to assume hypotheses of monotonicity or Lipschitz regularity neither on the nonlinear term, nor on the impulse functions, nor on the nonlocal condition. An application to a fractional diffusion process complete the discussion of the studied problem.

Keywords: Caputo fractional derivative; Impulse functions; Nonlocal initial conditions; Evolution inclusions in abstract spaces.

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1 Introduction

Fractional calculus allows to consider integration and differentiation of any order, not necessarily integer. In recent years more and more attention has been given to this area of research. The reason is not merely theoretical: one of the most important advantage of fractional order models in comparison with integer order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials. In fact integer order derivatives are local operators, while the fractional order derivative of a function in a point depends on the past values of such function. This features motivated the successful use of fractional calculus in nonlocal diffusion processes.

There are some different definitions of fractional derivatives: Riemann-Liouville, Hadamard and Caputo are examples of fractional derivatives. For a survey on the subject see e.g. [18, 23, 26]. In particular, the Caputo fractional derivative is especially suitable for physical applications. Unlike the Riemann-Liouville fractional derivative, the Caputo derivative of a constant is zero and it allows a physical interpretation of the initial conditions as well as of boundary conditions.
By replacing the time derivative with a fractional derivative of order α, with $0 < \alpha < 2$, in the classical partial differential equations describing diffusion or wave propagation, we obtain processes which interpolate (if $1 < \alpha < 2$) or extrapolate (if $0 < \alpha < 1$) the classical phenomena. The former are referred to as intermediate processes, the latter as ultraslow processes. We focus our study on the ultraslow processes and we merge these problems in the unitary framework of abstract fractional semilinear differential equations or inclusions, i.e.

$$\begin{cases} {}^C D^\alpha x(t) \in Ax(t) + F(t, x(t)), \quad \text{for a.a. } t \in [0, b] \\ 0 < \alpha < 1 \end{cases} \tag{1.1}$$

where x is a function with values in a reflexive Banach space E, ${}^C D^\alpha$ means the Caputo fractional derivative, $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}, F : [0, b] \times E \multimap E$ is a given map or multivalued map (multimap for short).
In order to better describe natural phenomena it is useful to consider non necessarily continuous propagation of the studied process, allowing that the model is subjected to short-term perturbations in time, the so-called impulses. For instance in the periodic treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics impulses may correspond to abrupt changes of prices. For these reasons, we consider the fractional evolution inclusion (1.1) in the presence of impulse effects, i.e.:

$$\begin{cases} {}^C D^\alpha x(t) \in Ax(t) + F(t, x(t)), \quad \text{for a.a. } t \in [0, b], t \neq t_1, \dots, t_N \\ x(t_k^+) = x(t_k) + I_k(x(t_k)),\ k = 1, \dots, N \\ 0 < \alpha < 1 \end{cases} \tag{1.2}$$

where $I_k : E \to E$, $k = 1, \cdots, N$ are given maps and $x(t^+) = \lim_{s \to t^+} x(s)$ with $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = b$.
Real life problems are often associated with periodic, anti-periodic, mean value or multipoint boundary conditions on the solutions. A unitary framework including all these conditions is the one of nonlocal boundary problems, that is to associate with the differential equation or inclusion an initial condition depending on the behaviour of the whole solution. More precisely, we consider problem (1.2) associated with the nonlocal boundary condition

$$x(0) \in M(x), \tag{1.3}$$

with $M : \mathcal{C}([0, b]; E) \multimap E$ a multivalued not necessarily linear operator (multioperator for short), where $\mathcal{C}([0, b]; E)$ is the space of piecewise continuous functions.
The boundary condition considered is fairly general and includes the initial value problem as well as several nonlocal conditions. For instance, the following particular cases are covered by our general approach:

(i) $M(x) = \dfrac{1}{b} \displaystyle\int_0^b p(t)x(t)\, dt$ with $p \in L^1([0, b]; \mathbb{R})$.

(ii) $M(x) = \sum_{i=1}^{n} \alpha_i x(s_i) + x_0$, with $x_0 \in E$, $\alpha_i \neq 0$, $s_i \in [0,b]$, $i = 1, \ldots, n$.

(iii) $M(x) \equiv B$, with B a prescribed set.

There are mainly two approaches to define the solution of Caputo fractional differential equations with impulses. One keeps the lower limit 0 of the Caputo derivative for all $t \geq 0$, considering the same fractional equation on each subinterval (t_k, t_{k+1}), but with different initial conditions (see, for example, [14, 32, 33, 34]). The other one is based on the fact that the Caputo fractional derivative depends significantly on the initial point, leading to a change of the equation on each interval (t_k, t_{k+1}). Thus with this approach the lower limit of the Caputo fractional derivative is given by the impulse time t_k (see, for example, [1, 2, 8, 7, 12, 24, 28, 29]).

More precisely, the approach in [32] is based on the possibility to split a nonhomogeneous linear fractional problem with fixed impulses into a nonhomogeneous nonimpulsive problem and a homogeneous impulsive one. The solution of the original problem is then written as the sum of the solutions of the splitted ones. This method doesn't work in our case, because in presence of a nonlinear term and of nonconstant impulse functions, after each jump a new dynamic starts.

Thus, following the approach in [28], firstly we give a definition of mild solution defined step by step, based on the definition of mild solution for the Cauchy initial problem with starting point $a > 0$ associated to a fractional differential inclusion of type (1.1). Then we give a new concept of mild solution through four operators recursively defined, for details see Section 4. In [28], see Remark 3.2, the authors declare that the mild solutions for impulsive Caputo differential equations can be expressed only by using piecewise functions, however with this approach we are able to give a unique formula for the solution. This definition takes into account the fact that the families of operators $\{S_\alpha(t)\}_{t\in[0,b]}$ and $\{T_\alpha(t)\}_{t\in[0,b]}$ defined respectively in (2.1) and (2.2) do not satisfy the semigroup properties, that the solutions of an impulsive equation are no longer continuous and that the Caputo derivative strongly depends on the initial time. In our opinion it is particularly suitable to prove existence results in the presence of nonlocal conditions.

By means of a technique based on the weak topology and developed in [3], applying the Glicksberg-Ky Fan fixed point Theorem, we are able to prove the existence of at least one solution of problem (1.2)-(1.3). With this approach, we avoid the compactness of the semigroup generated by the linear part and we do not need to assume any hypothesis of monotonicity, Lipschizianity, or compactness neither on the nonlinear term F, nor on the impulse functions, nor on the nonlocal condition. We apply a similar approach in the framework of fractional differential inclusion in [4, 5] and in comparison with the literature on the subject, this is the main novelty of the paper. For instance, in [2, 8, 12, 29] the existence, uniqueness and controllability ([29]) of the solution of a problem similar to (1.2) via fixed point theorems is proved under Lipschitz regularity assumptions on the nonlinear part, the nonlocal condition and the impulse functions; applying the monotone iterative technique in the presence of upper and lower solutions, in [24] the existence of extremal solutions is obtained under monotonicity and compactness like assumptions on the nonlinear term and on the nonlocal condition and under monotonicity assumptions on the impulse functions; in [1] the compactness of the α-resolvent family generated by the linear part is assumed; in [7] and in [28] the Lipschitz regularity of the nonlinear term, the nonlocal condition and the impulse functions, or alternatively the compactness of the $\alpha-$resolvent family generated by the linear part, of the nonlinear term, of the nonlocal condition and of the impulse function are taken as main hypotheses.

We complete our study with an application to a ultraslow process of this kind

$$D_t^\alpha u = \Delta u + \left[f_1 \left(t, x, \int_\Omega k_1(x,\xi)u(t,\xi)\, d\xi \right), f_2 \left(t, x, \int_\Omega k_2(x,\xi)u(t,\xi)\, d\xi \right) \right] \psi(t,x), \quad (1.4)$$

with $t \in [0,b], t \neq t_k,\ k = 1, \dots, N, x \in \Omega$, where Ω is a nonempty domain in $\mathbb{R}^n$, with impulses

$$u(t_k^+, x) = u(t_k, x) + c_k, \quad k = 1, ..., N,\ x \in \Omega$$

and subjected to the boundary conditions

$$\begin{array}{l} u(t,x) = 0, \quad t \in [0,b], x \in \partial\Omega \\ u(0,x) = \sum_{i=1}^J \alpha_i u(s_i, x), \quad x \in \Omega, s_1 \leq 0 < \cdots < s_J \leq b. \end{array}$$

The problem (1.4) is a perturbation by means of a nonlocal forcing term of the diffusion of particles verifying a generalized Fick's second law. In particular, the multivalued nonlinearity represents the external influence on the process which is known up to some degree of uncertainty and the integral term describes the property that the state of the problem at a given point may include states in a suitable neighborhood. After the pioneering work of Nigmatullin in [25], who explicitly introduced in physics the fractional diffusion equation to describe diffusion in media with fractal geometry, Mainardi in [22] has provided a physical interpretation of it in the framework of dynamic viscoelasticity, pointing out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media which exhibit a power-law creep. Kochubei in [19] uses the semigroup theory in Banach spaces to study the existence and the properties of the solutions of the fractional diffusion equation with variable coefficients. More recently, in collaboration with Eidelman he constructed and investigated the fundamental solution of the Cauchy problem associated to the fractional diffusion equation, see [10]. Later on, several perturbation of the linear fractional diffusion equation have been studied in literature, for instance in [24, 29, 32, 34] the authors study specific ultraslow diffusion types of porous medium with impulses and nonlocal initial conditions arising in heat conduction similar to (1.4).

2 Basic results and notation

Let $(E, \|\cdot\|)$ be a reflexive Banach space and E_w denote the space E endowed with the weak topology. We denote by B the closed unit ball in E and for a set $A \subset E$, the symbol $\overline{A}^w$ means the weak closure of A. In the whole paper we denote by $\|\cdot\|_p$ and $\|\cdot\|_0$ the $L^p([0,b];\mathbb{R})$-norm, $1 < \frac{1}{\alpha} < p < \infty$, and the $C[0,b];E)$-norm respectively; we consider the norm of a set $\Omega \subset E$ defined as

$$\|\Omega\| := \sup\{\|x\| : x \in \Omega\}$$

and by ν we denote the Lebesgue measure on $[0,b]$.

Let us briefly recall that a multimap $\Phi \colon X \multimap Y$ of topological spaces X and Y is a relation that assigns to every point $x \in X$ a nonempty set $\Phi(x) \subset Y$. A multimap Φ of Banach spaces is called weakly sequentially closed, provided the conditions $x_n \rightharpoonup x_0$, $y_n \rightharpoonup y_0$, and $y_n \in \Phi(x_n)$, imply $y_0 \in \Phi(x_0)$. It is clear that this condition is equivalent to the hypothesis that Φ has a weakly sequentially closed graph. A multimap $\Phi\colon X \multimap Y$ is said to be upper semicontinuous (u.s.c. for short), if the set $\Phi^{-1}(V) := \{x \in X : \Phi(x) \subset V\}$ is open for every open subset $V \subseteq Y$.

Let $\mathcal{C}([0,b];E)$ be the space of all piecewise continuous functions $x : [0,b] \to E$ with discontinuity points at $t = t_k$, $k = 1,\dots,N$ such that all values $x(t_k^+) = \lim_{s\to t_k^+} x(s)$ and $x(t_k^-) = \lim_{s\to t_k^-} x(s)$ are finite and $x(t_k) = x(t_k^-)$ for all such points. The space $\mathcal{C}([0,b];E)$ is a normed space endowed with the $\|\cdot\|_0-$norm.

Let $BV([0,b];E)$ be the space of functions of bounded variation. We recall (see [6, Theorem 4.3]) that a sequence $\{x_n\} \subset BV([0,b];E)$ weakly converges to an element $x \in BV([0,b];E)$ if and only if

1. $\|x_n(t)\| \le N$, for each $n \in \mathbb{N}$ and for each $t \in [0,b]$, for some constant $N > 0$;
2. $x_n(t) \rightharpoonup x(t)$ for every $t \in [0,b]$.

Thus, the above characterization of weakly convergent sequences holds also for the space $\mathcal{C}([0,b];E)$. Moreover, we recall that a map $\phi\colon X \to Y$ is weakly sequentially continuous if the weak convergence of a sequence $\{x_n\} \subset X$ to an element $x \in X$ implies the weak convergence in Y of the sequence $\{\phi(x_n)\}$ to $\phi(x)$.

The proof of the main result is based on the following Glicksberg-Ky Fan Theorem ([13], [15]).

Theorem 2.1. *Let X be a Hausdorff locally convex topological vector space, K a compact convex subset of X and $G : K \multimap K$ a upper semicontinuous multimap with closed, convex values. Then G has a fixed point $x_* \in K$ i.e. $x_* \in G(x_*)$.*

For a map $f : [c,d] \to E$, the definition of the Riemann-Liouville fractional derivative with $0 < \alpha < 1$ is the following

$$[D^\alpha f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_c^t \frac{f(s)}{(t-s)^\alpha}\, ds,$$

with Γ the Euler function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx;$$

while the corresponding fractional integral is defined as

$$\frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} x(s)\, ds.$$

The Caputo fractional derivative is defined through the Riemann-Liouville fractional derivative as

$$[^C D^\alpha f](t) = D^\alpha[f(\cdot) - f(c)](t).$$

In the whole paper we assume that $A : D(A) \subset E \to E$ is a linear, not necessarily bounded operator generating a bounded C_0-semigroup $U : \mathbb{R}_+ \to \mathcal{L}(E)$, i.e. a family of bounded linear operators $U(t) : E \to E$, for $t \in \mathbb{R}_+$ such that

(a) $U(0) = I$;

(b) $U(t + r) = U(t)U(r) = U(r)U(t)$ for every $t, r \in \mathbb{R}_+$;

(c) the function $t \in \mathbb{R}_+ \to U(t)x \in E$ is continuous for every $x \in E$;

(d) $D := \sup_{t\in[0,\infty]} \|U(t)\| < +\infty$.

Define the families of operators $\{S_\alpha(t)\}_{t\in[0,b]}$ and $\{T_\alpha(t)\}_{t\in[0,b]}$ in E by the formulas

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(s)U(t^\alpha s)x\,ds, \tag{2.1}$$

where ϕ_α is the probability density function

$$\phi_\alpha(s) = \frac{1}{\alpha} s^{-\frac{\alpha+1}{\alpha}} \psi_\alpha(s^{-1/\alpha}),$$

$$\psi_\alpha(s) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} s^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha),$$

and

$$T_\alpha(t)x = \alpha \int_0^\infty s\phi_\alpha(s)U\left(t^\alpha s\right) x\,ds. \tag{2.2}$$

Remark 2.1. *(See, e.g., [34])* $\int_0^\infty s\phi_\alpha(s)\,ds = \dfrac{1}{\Gamma(\alpha+1)}$ *and* $\int_0^\infty \phi_\alpha(s)\,ds = 1$.

By Propositions 2.1 and 2.2. in [35] the next regularity result holds.

Lemma 2.1. *The operator functions S_α and T_α possess the following properties:*

a) for every $t \in [0, b]$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators. More precisely,

$$\|S_\alpha(t)\| \le D \text{ and } \|T_\alpha(t)\| \le \frac{D\alpha}{\Gamma(1+\alpha)} \quad \text{for every } t \in [0, b];$$

b) S_α and T_α are strongly continuous, i.e., for each $x \in E$, the functions $S_\alpha(\cdot)x : [0, b] \to E$ and $T_\alpha(\cdot)x : [0, b] \to E$ are continuous.

3 Main assumptions

We will study problem (1.2) - (1.3) under the following assumptions.

(A) A is the generator of a bounded C_0-semigroup $\{U(t)\}_{t\ge0}$.

Concerning the multivalued nonlinearity $F : [0, b] \times E \multimap E$ we will suppose that it has closed bounded and convex values and, moreover, the following conditions hold true:

$(F1)$ the multifunction $F(\cdot, c) : [0, b] \multimap E$ has a strongly measurable selection for every $c \in E$, i.e., there exists a measurable function $f : [0, b] \to E$ such that $f(t) \in F(t, c)$ for a.e. $t \in [0, b]$;

$(F2)$ the multimap $F(t,\cdot) : E \multimap E$ is weakly sequentially closed for a.e. $t \in [0,b]$;

We assume that operators M and I_k satisfy the following conditions.

(M) $M : \mathcal{C}([0,b];E) \multimap E$ is a weakly sequentially closed multioperator, with convex, closed and bounded values, mapping bounded sets into bounded sets and such that

$$\limsup_{\|u\|_0 \to \infty} \frac{\|M(u)\|}{\|u\|_0} = l \ \text{ with } l < \frac{1}{D^{N+1}+1}. \tag{3.1}$$

(I_k) the functions $I_k : E \to E$, $k = 1, \dots, N$ satisfy the following assumptions:

(i) are weakly sequentially continuous;

(ii) map bounded sets into bounded sets;

(iii) $\limsup\limits_{\|c\| \to \infty} \dfrac{\|I_k(c)\|}{\|c\|} = 0$, $k = 1, \dots, N$.

For all our preliminary results we always assume the following condition of local integral boundedness on the multivalued map F.

(F3) for every $r > 0$ there exists a function $\mu_r \in L^p([0,b];\mathbb{R}_+)$ with $p > \frac{1}{\alpha}$, such that for each $c \in E$, $\|c\| \le r$:

$$\|F(t,c)\| \le \mu_r(t) \ \text{ for a.a. } t \in [0,b].$$

For our main result (see Theorem 5.1), instead of condition (F3), we need the stronger assumption below:

$(F3')$ for every $n \in \mathbb{N}$ there exists a function $\varphi_n \in L^p([0,b];\mathbb{R})$, with $p > \frac{1}{\alpha}$ such that, for a.a. $t \in [0,b]$,

$$\sup_{\|x\| \le n} \|F(t,x)\| \le \varphi_n(t)$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \left\{ \int_0^b |\varphi_n(s)|^p \, ds \right\}^{\frac{1}{p}} = 0. \tag{3.2}$$

Note that, under hypotheses (F1-3), given $q \in \mathcal{C}([0,b];E)$, the superposition multioperator $\mathcal{P}_F(q) : \mathcal{C}([0,b];E) \multimap L^p([0,b];E)$, with

$$\mathcal{P}_F(q) = \{ f \in L^p([0,b];E) \ : \ f(t) \in F(t,q(t)) \text{ a.a. } t \in [0,b] \},$$

is well defined (see [4, Propositon 3.1]).

Remark 3.1. It is usual in literature to assume that the limit in (3.1) is equal to zero. This is the case when the multimap M is globally bounded, for example when M is as in (iii). On the contrary, we are able to consider operator M satisfying a linear growth condition, as in (i) and (ii). In these cases condition (3.1) respectively reads as $\|p\|_1 < \frac{b}{D^{N+1}+1}$ and $\sum_{i=1}^n |\alpha_i| < \frac{1}{D^{N+1}+1}$.

In the next Section, in order to state our main results, we construct the definition of mild solution we will consider.

4 New concept of solution

It is well known that a function $x \in C([0,b];E)$ is a mild solution of the Cauchy problem

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + f(t), \quad \text{for a.e. } t \in [0,b], \\ x(0) = x_0 \\ 0 < \alpha < 1, \end{cases} \tag{4.1}$$

with $A : D(A) \subset E \to E$ a generator of a bounded C_0-semigroup $\{U(t)\}_{t\geq 0}$ and $f \in L^p([0,b];E)$, $p > \frac{1}{\alpha}$, if it satisfies the integral formula

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds, \tag{4.2}$$

see e.g. [35, Definition 2.2]. Notice that, since $p > \frac{1}{\alpha}$ the function $r \to r^{\alpha-1}$ belongs to $L^{p'}([0,b];E)$, with p' the conjugate exponent of p, and by Hölder inequality the integral in (4.2) is well defined. Following the same reasoning as in [35, Lemma 2.3], it is possible to justify the notion of the mild solution of problem (4.1) with starting point $a > 0$ by the next arguments. Let $x \in C([a,b];E)$ be a strong solution of (4.1) with starting point $a > 0$, i.e. $x : [a,b] \to E$ satisfies the inclusion in (4.1) for a.e. $t \in [a,b]$ and $x(a) = x_0$. Then $x : [a,b] \to E$ is a solution of the following integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{1-\alpha} (Ax(s) + f(s))\, ds, \ t \in [a,b]. \tag{4.3}$$

We extend the function $x \in C([a,b];E)$ from the interval $[a,b]$ to the interval $[0,b]$ as follows:

$$y(t) = \begin{cases} 0 & t \in [0,a), \\ x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{1-\alpha} (Ax(s) + f(s))\, ds & t \in [a,b]. \end{cases}$$

Hence defining the Heaviside map $H : \mathbb{R} \to \mathbb{R}$ as

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

we have that $y(t) = H(t-a)x(t)$, i.e.

$$y(t) = H(t-a)x_0 + H(t-a)\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{1-\alpha} Ax(s)\, ds + H(t-a)\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{1-\alpha} f(s)\, ds.$$

Thus, defining $\widetilde{f} : [0,b] \to E$ as

$$\widetilde{f}(t) = \begin{cases} 0 & 0 \leq t < a, \\ f(t) & a \leq t \leq b, \end{cases}$$

we get

$$y(t) = H(t-a)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} Ay(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} \widetilde{f}(s)\, ds. \tag{4.4}$$

Notice that the last two terms on the right hand side of equation (4.4) are the convolution products of the function $\delta : [0,b] \to \mathbb{R}$ defined as $\delta(t) = \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}$ with Ay and $\widetilde{f}$ respectively. Let $\lambda > 0$ and denote by $\mathcal{L}$ the Laplace transform. It is well known that

$$\mathcal{L}(\delta)(\lambda) = \frac{1}{\lambda^{\alpha}},$$
$$\mathcal{L}(H(\cdot - a))(\lambda) = \frac{e^{-\lambda a}}{\lambda}.$$

Thus, denoting with v the Laplace transform of y, i.e.

$$v(\lambda) = \int_0^{\infty} e^{-\lambda s} y(s)\, ds,$$

and applying the Laplace transform to (4.4), recalling that the Laplace transform of the convolution product of two functions is the product of the corresponding Laplace transforms, we obtain

$$v(\lambda) = \frac{e^{-\lambda a}}{\lambda} x_0 + \frac{1}{\lambda^{\alpha}} A v(\lambda) + \frac{1}{\lambda^{\alpha}} \widetilde{\omega}(\lambda),$$

where $\widetilde{\omega}(\lambda) = \mathcal{L}(\widetilde{f})$. Hence

$$\left(I - \frac{1}{\lambda^{\alpha}} A\right) v(\lambda) = \frac{e^{-\lambda a}}{\lambda} x_0 + \frac{1}{\lambda^{\alpha}} \widetilde{\omega}(\lambda), \quad \lambda > 0,$$

where I is the identity operator defined on E. Thus,

$$\lambda^{-\alpha} \left(\lambda^{\alpha} I - A\right) v(\lambda) = \frac{e^{-\lambda a}}{\lambda} x_0 + \frac{1}{\lambda^{\alpha}} \widetilde{\omega}(\lambda), \quad \lambda > 0.$$

Applying the operator $\left(\lambda^{\alpha} I - A\right)^{-1} \lambda^{\alpha}$ to both sides of the previous equality, we have

$$v(\lambda) = \left(\lambda^{\alpha} I - A\right)^{-1} \lambda^{\alpha - 1} e^{-\lambda a} x_0 + \left(\lambda^{\alpha} I - A\right)^{-1} \widetilde{\omega}(\lambda).$$

It is possible to prove that for any $x \in E$, $(\lambda^{\alpha} I - A)^{-1} x$, $\lambda > 0$ is the Laplace transform of the map $t \to t^{\alpha-1} T_{\alpha}(t) x$ and that $(\lambda^{\alpha} I - A)^{-1} \lambda^{\alpha-1} x$, $\lambda > 0$ is the Laplace transform of the map $t \to S_{\alpha}(t) x$. Therefore, inverting the Laplace transform and exploiting the definition of the Laplace transform of a convolution product of two maps, we obtain

$$y(t) = H(t-a) S_{\alpha}(t-a) x_0 + \int_0^t (t-\theta)^{\alpha-1} T_{\alpha}(t-\theta) \widetilde{f}(\theta)\, d\theta,\ t \in [0,b].$$

Thus, by the definition of the map $\widetilde{f}$, we have

$$x(t) = S_{\alpha}(t-a) x_0 + \int_a^t (t-\theta)^{\alpha-1} T_{\alpha}(t-\theta) f(\theta)\, d\theta,\ t \in [a,b]. \tag{4.5}$$

According to (4.2) and (4.5), the solution of (4.1) when $f \equiv 0$ is $x(t) = S_{\alpha}(t) x_0$, while the solution of the problem

$$\begin{cases} {}^{C}D^{\alpha} x(t) = A x(t), & \text{for a.a. } t \in [a,b], \\ x(a) = S_{\alpha}(a) x_0 \end{cases} \tag{4.6}$$

is $x(t) = S_\alpha(t-a)x(a) = S_\alpha(t-a)S_\alpha(a)x_0$. Since the family $\{S_\alpha(t)\}_{t\geq 0}$ does not satisfy the semigroup property, it follows that

$$S_\alpha(t-a)S_\alpha(a)x_0 \neq S_\alpha(t)x_0.$$

Therefore the solution of (4.1) with $f \equiv 0$ is not a solution of (4.6).
Thus, we have to define the solution of the impulsive problem step by step. Namely, consider the following Cauchy problem with one impulse

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + f(t), & \text{for a.a. } t \in [0, t_2], \\ x(t_1^+) = x(t_1^-) + I_1(x(t_1)) \\ x(0) = x_0 \\ 0 < \alpha < 1, \end{cases} \tag{4.7}$$

where $0 < t_1 < t_2$ and, as before, $f \in L^p([0, t_2]; E)$, $p > \frac{1}{\alpha}$. Its mild solution is

$$x(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\, ds & 0 \leq t \leq t_1 \\ S_\alpha(t-t_1)x(t_1^+) + \int_{t_1}^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\, ds & t_1 < t \leq t_2. \end{cases}$$

Indeed, by the above reasonings x satisfies the integral equation (4.2) in the interval $[0, t_1]$ and the integral equation (4.5) with starting point $t = t_1$ and initial value $x(t_1) = x(t_1^+)$ in the interval $]t_1, t_2]$. Hence, from $x(t_1^+) = x(t_1^-) + I_1(x(t_1))$ we get

$$x(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\, ds & 0 \leq t \leq t_1 \\ S_\alpha(t-t_1)[S_\alpha(t_1)x_0 + \int_0^{t_1} (t_1-s)^{\alpha-1}T_\alpha(t_1-s)f(s)ds + I_1(x(t_1))]+ & \\ + \int_{t_1}^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\, ds & t_1 < t \leq t_2. \end{cases}$$

Reasoning by induction we get that the mild solution of the Cauchy problem with $N \geq 1$ impulses

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + f(t), & \text{for a.a. } t \in [0, b], t \neq t_1, \ldots, t_N \\ x(t_k^+) = x(t_k) + I_k(x(t_k)),\ k = 1, \ldots, N \\ x(0) = x_0 \\ 0 < \alpha < 1 \end{cases} \tag{4.8}$$

can be defined through four operators defined recursively. More precisely, we define the maps $\psi_k : (t_{k-1}, t_k] \to \mathcal{L}(E)$, $k = 1, \ldots, N+1$, as

$$\begin{cases} \psi_1(t) = S_\alpha(t), \\ \psi_k(t) = S_\alpha(t-t_k)S_\alpha(t_k - t_{k-1}) \ldots S_\alpha(t_1), & k = 2, \ldots, N+1 \end{cases}$$

$\phi_k : C([0, b]; E) \to C((t_k, b]; E)$, $k = 1, \ldots, N$, as

$$\begin{cases} \phi_1(x)(t) = S_\alpha(t-t_1)I_1(x(t_1)), \\ \phi_k(x)(t) = S_\alpha(t-t_k)[\phi_{k-1}(x)(t_k) + I_k(x(t_k))], & k = 2, \ldots, N \end{cases}$$

and r_k: $L^p([0, b]; E) \to C((t_k, b]; E)$, $k = 1, \ldots, N$, as

$$\begin{cases} r_1(f)(t) = S_\alpha(t-t_1) \displaystyle\int_0^{t_1} (t_1-s)^{\alpha-1}T_\alpha(t_1-s)f(s)\, ds \\ r_k(f)(t) = S_\alpha(t-t_k) \left[r_{k-1}(f)(t_k) + \displaystyle\int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}T_\alpha(t_k-s)f(s)\, ds \right] & k = 2, \ldots, N. \end{cases}$$

Let the operators $\mathcal{A} : [0,b] \to \mathcal{L}(E)$, $\mathcal{B} : \mathcal{C}([0,b];E) \to \mathcal{C}([0,b];E)$, $\mathcal{D}\colon L^p([0,b];E) \to \mathcal{C}([0,b];E)$ and $\mathcal{F}\colon L^p([0,b];E) \to \mathcal{C}([0,b];E)$ be defined respectively as

$$
\begin{aligned}
\mathcal{A}(t) &= \chi_{\{0\}}(t)I + \sum_{k=1}^{N+1} \chi_{(t_{k-1},t_k]}(t)\psi_k(t)\\
\mathcal{B}(x)(t) &= \sum_{k=1}^{N} \chi_{(t_k,t_{k+1}]}(t)\phi_k(x)(t)\\
\mathcal{D}(f)(t) &= \sum_{k=1}^{N} \chi_{(t_k,t_{k+1}]}(t)r_k(f)(t)\\
\mathcal{F}(f)(t) &= \sum_{k=1}^{N+1} \chi_{(t_{k-1},t_k]}(t)\int_{t_{k-1}}^{t} (t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds,
\end{aligned}
$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is the characteristic function and $I : E \to E$ is the identity map in E. Now, we can define the mild solution $x\colon [0,b] \to E$ of (4.8) as

$$x(t) = \mathcal{A}(t)x_0 + \mathcal{B}(x)(t) + \mathcal{D}(f)(t) + \mathcal{F}(f)(t),\ t \in [0,b].$$

Hence, we open the way towards the following concept of the mild solution to (1.1)-(1.3).

Definition 4.1. A function $x \in \mathcal{C}([0,b];E)$ is a *mild solution* to problem (1.1)-(1.3) if and only if there exist $\omega \in M(x)$ and a map $f \in L^p([0,b];E), p > \frac{1}{\alpha}$, with $f(t) \in F(t,x(t))$ for a.e. $t \in [0,b]$ such that

$$x(t) = \mathcal{A}(t)\omega + \mathcal{B}(x)(t) + \mathcal{D}(f)(t) + \mathcal{F}(f)(t),\ \ t \in [0,b].$$

The operators $\mathcal{B}, \mathcal{D}$ and $\mathcal{F}$ satisfy the following regularity conditions.

Lemma 4.1. *The operator* $\mathcal{B}$ *is weakly sequentially continuous.*

Proof. Let $\{q_n\} \subset \mathcal{C}([0,b];E)$ be such that $q_n \rightharpoonup q$. From the weak convergence it follows that there exists a constant $r > 0$ such that $\|q_n\|_0 < r$ for every $n \in \mathbb{N}$ and $q_n(t) \rightharpoonup q(t)$ for every $t \in [0,b]$. Then by the weak sequential continuity of the functions I_k, we have that $I_k(q_n(t_k)) \rightharpoonup I_k(q(t_k))$ for any $k = 1,\dots,N$. Thus, according to Lemma 2.1, $\phi_k(q_n)(t) \rightharpoonup \phi(q)(t)$ for every $t > t_k$ and $\mathcal{B}(q_n)(t) \rightharpoonup \mathcal{B}(q)(t)$ for every $t \in [0,b]$. Moreover, (I_k) (ii) yields the existence of $R > 0$ such that

$$\|I_k(q_n(t_k))\| \le R$$

for every $k = 1,..,N$ and $n \in \mathbb{N}$. Notice that

$$
\begin{aligned}
\|\phi_3(q_n)(t)\| \le D(\|\phi_2(q_n)(t_3)\| + R) \ &\le D[D(\|\phi_1(q_n)(t_2)\| + R) + R]\\
&\le D[D(DR + R) + R] \le DR(D+1)^2.
\end{aligned}
$$

Hence, reasoning by induction, we get that $\|\phi_k(q_n)(t)\| \le DR(D+1)^{k-1}$ for every $t > t_k, k = 1,...,N$ and $n \in \mathbb{N}$. Thus

$$\|\mathcal{B}(q_n)(t)\| \le DR(D+1)^{N-1} \tag{4.9}$$

for every $t \in [0,b]$ and $n \in \mathbb{N}$, implying the weak convergence of $\mathcal{B}(q_n)$ to $\mathcal{B}(q)$ in $\mathcal{C}([0,b];E)$. □

Lemma 4.2. *The operators $\mathcal{D}$ and $\mathcal{F}$ are linear and bounded.*

Proof. The linearity follows from the linearity of the integral operator and from Lemma 2.1. We now prove the boundedness. For every $\tau_1, \tau_2 \in [0,b]$ we have

$$\left(\int_{\tau_1}^{\tau_2} \left((\tau_2 - s)^{\alpha-1}\right)^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \leq \left[\frac{p-1}{\alpha p - 1}\right]^{\frac{p-1}{p}} b^{\alpha - \frac{1}{p}}. \tag{4.10}$$

Thus, from Hölder inequality and Lemma 2.1, we get for any $f \in L^p([0,b];E)$ and $\tau_1, \tau_2 \in [0,b]$

$$\left\|\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} T_\alpha(\tau_2 - s) f(s)\, ds\right\| \leq \frac{D\alpha}{\Gamma(1+\alpha)} \left[\frac{p-1}{\alpha p - 1}\right]^{\frac{p-1}{p}} b^{\alpha - \frac{1}{p}} \|f\|_p.$$

Denoting by

$$H := \frac{D\alpha}{\Gamma(1+\alpha)} \left[\frac{p-1}{\alpha p - 1}\right]^{\frac{p-1}{p}} b^{\alpha - \frac{1}{p}}, \tag{4.11}$$

it follows that $\|r_k(f)(t)\| \leq DH(D+1)^{k-1}\|f\|_p$ for every $t > t_k$ and $k = 1, ..., N$, hence for every $t \in [0,b]$ we obtain

$$\|\mathcal{D}(f)(t)\| \leq DH(D+1)^{N-1}\|f\|_p \tag{4.12}$$

and

$$\|\mathcal{F}(f)(t)\| \leq H\|f\|_p \tag{4.13}$$

for every $f \in L^p([0,b];E)$ and the thesis is proved. □

5 Existence results

Consider the multioperator $\Psi : \mathcal{C}([0,b];E) \multimap \mathcal{C}([0,b];E)$ defined as

$$\Psi(q) = \{x \in \mathcal{C}([0,b];E) : x = \mathcal{A}(\cdot)\omega + \mathcal{B}(q) + \mathcal{D}(f) + \mathcal{F}(f) : \omega \in M(q), f \in \mathcal{P}_f(q)\}.$$

Then the fixed points of Ψ are mild solutions of problem (1.1). Now we can state the main result of the paper.

Theorem 5.1. *Under assumptions* $(A), (F1), (F2), (F3'), (M)$ *and* (I_k) *the problem* (1.2)-(1.3) *has at least one mild solution.*

To prove the Theorem 5.1 we show that the multioperator Ψ satisfies all the hypotheses of the Fixed Point Theorem 2.1. For the reader's convenience we split the proof of the regularity conditions satisfied by the multioperator Ψ in three propositions.

Proposition 5.1. *The multioperator Ψ is weakly sequentially closed.*

Proof. Let $\{q_n\} \subset \mathcal{C}([0,b];E)$ and $\{x_n\} \subset \mathcal{C}([0,b];E)$ satisfying $x_n \in \Psi(q_n)$ for all $n, q_n \rightharpoonup q$ and $x_n \rightharpoonup x$ in $\mathcal{C}([0,b];E)$. We will prove that $x \in \Psi(q)$.
The fact that $x_n \in \Psi(q_n)$ means that there exist a sequence $\{f_n\}$, with $f_n \in \mathcal{P}_F(q_n)$, and a sequence $\{\omega_n\}$, with $\omega_n \in M(q_n)$, such that

$$x_n = \mathcal{A}(\cdot)\omega_n + \mathcal{B}(q_n) + \mathcal{D}(f_n) + \mathcal{F}(f_n).$$

From Lemma 4.1 we get that $\mathcal{B}q_n \rightharpoonup \mathcal{B}q$ in $\mathcal{C}([0,b];E)$.
Moreover, by the weak convergence of the sequence $\{q_n\}$ in $\mathcal{C}([0,b];E)$, it follows that there exists a constant $r > 0$ such that $\|q_n\|_0 < r$ for every $n \in \mathbb{N}$ and $q_n(t) \rightharpoonup q(t)$ for every $t \in [0,b]$. Therefore, according to (F3), there exists $\mu_r \in L^p([0,b];\mathbb{R}_+)$ with $p > \frac{1}{\alpha}$, such that $\|f_n(t)\| \leq \mu_r(t)$ for a.a. t and every $n \in \mathbb{N}$, i.e. $\{f_n\}$ is integrably bounded. By the reflexivity of the space $L^p([0,b];E)$, we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_n \rightharpoonup g$ in $L^p([0,b];E)$. Lemma 4.2 implies that $\mathcal{D}(f_n) \rightharpoonup \mathcal{D}(g)$ and $\mathcal{F}(f_n) \rightharpoonup \mathcal{F}(g)$ in $\mathcal{C}([0,b];E)$.
The operator M maps bounded sets into bounded sets and it is weakly sequentially closed, hence, up to subsequence, $w_n \rightharpoonup w$ in E, with $w \in M(x)$ and there exists $L > 0$ such that

$$\|\omega_n\| \leq L$$

for every $n \in \mathbb{N}$.
Notice finally that, again from Lemma 2.1, $\psi_k(t)\omega_n \rightharpoonup \psi_k(t)\omega$ and $\|\psi_k(t)\| \leq D^k$ for every $t \in (t_{k-1}, t_k]$ and $k = 1, ..., N+1$. Thus $\mathcal{A}(t)\omega_n \rightharpoonup \mathcal{A}(t)\omega$ and, since $D \geq 1$,

$$\|\mathcal{A}(t)\omega_n\| \leq (D^{N+1} + 1)L \tag{5.1}$$

for every $t \in [0,b]$, yielding that $\mathcal{A}(\cdot)\omega_n \rightharpoonup \mathcal{A}(\cdot)\omega$ in $C([0,b];E)$.
So, we have

$$x_n \rightharpoonup \mathcal{A}(\cdot)\omega + \mathcal{B}q + \mathcal{D}g + \mathcal{F}g =: \overline{x},$$

and thus, by the uniqueness of the weak limit, we obtain that $\overline{x} = x$.
To conclude, we have only to prove that $g(t) \in F(t, q(t))$ for a.a. $t \in [0,b]$.
To this aim, by Mazur's convexity theorem (see e.g. [11]) we have a sequence

$$\tilde{f}_n = \sum_{i=0}^{h_n} \lambda_{ni} f_{n+i}, \qquad \lambda_{ni} \geq 0, \ \sum_{i=0}^{h_n} \lambda_{ni} = 1$$

satisfying $\tilde{f}_n \to g$ in $L^1([0,b];E)$, thus, up to subsequence, there is $\Omega_0 \subset [0,b]$ with Lebesgue measure zero such that $\tilde{f}_n(t) \to g(t)$ for all $t \in [0,b] \setminus \Omega_0$ (see [27, Chapter IV, Theorem 38]). With no loss of generality we can also assume that $F(t,\cdot)\colon E \multimap E$ is weakly sequentially closed and $\sup\limits_{\|x\| \leq r} \|F(t,x)\| \leq \mu_r(t)$ for every $t \notin \Omega_0$.
We now prove, by contradiction that $g(t_0) \in F(t_0, q(t_0))$ for every $t_0 \notin \Omega_0$. By the reflexivity of the space E and (F3) the restriction $F_{rB}(t_0,\cdot)$ of the multimap $F(t_0,\cdot)$ on the set rB is weakly compact. Hence, we have that $F_{rB}(t_0,\cdot)$ is a weakly closed multimap and by [16, Theorem 1.1.5] it is weakly u.s.c. Since $\|q(t_0)\| \leq \liminf_{n\to\infty} \|q_n(t_0)\| \leq r$ and since $F_{rB}(t_0, q(t_0))$ is closed and convex, from the Hahn–Banach theorem it follows that there is a weakly open convex set $V \supset F_{rB}(t_0, q(t_0))$ satisfying $g(t_0) \notin \overline{V}^w$. Being $F_{rB}(t_0,\cdot)$ weakly u.s.c., we can also find a weak neighborhood V_1 of $q(t_0)$ such that $F_{rB}(t_0, y) \subset V$ for all $y \in V_1$ with $\|y\| \leq r$. Notice that $\|q_n(t_0)\| \leq r$ for all n. The convergence $q_n(t_0) \rightharpoonup q(t_0)$ as $n \to \infty$ then implies the existence of $n_0 \in \mathbb{N}$ such that $q_n(t_0) \in V_1$ for all $n > n_0$. Therefore $f_n(t_0) \in F_{rB}(t_0, q_n(t_0)) \subset V$ for all $n > n_0$. The convexity of V implies that $\tilde{f}_n(t_0) \in V$ for all $n > n_0$ and, by the convergence, we arrive to the contradictory conclusion that $g(t_0) \in \overline{V}^w$.
We obtain that $g(t) \in F(t, q(t))$ for a.a. $t \in [0,b]$. □

Proposition 5.2. *The multioperator Ψ is weakly compact.*

Proof. By the Eberlein Smulian theorem (see [17, Theorem 1, p. 219]) it is sufficient to prove that Ψ is weakly sequentially compact.
Let $\{q_n\} \subset \mathcal{C}([0,b];E)$ be a bounded sequence and $\{x_n\} \subset \mathcal{C}([0,b];E)$ satisfy $x_n \in \Psi(q_n)$ for all n. By the definition of the multioperator Ψ, there exist a sequence $\{f_n\}$, with $f_n \in \mathcal{P}_F(q_n)$, and a sequence $\{\omega_n\}$, with $\omega_n \in M(q_n)$, such that

$$x_n = \mathcal{A}(\cdot)\omega_n + \mathcal{B}(q_n) + \mathcal{D}(f_n) + \mathcal{F}(f_n).$$

Reasoning as in Proposition 5.1, we have that there exists a subsequence, denoted as the sequence $\{f_n\}$, and a function g such that $f_n \rightharpoonup g$ in $L^p([0,b];E)$. Moreover, since M maps bounded sets into bounded sets and $\{q_n\}$ is bounded, we obtain that, up to subsequence, $\omega_n \rightharpoonup \omega \in E$ as $n \to \infty$. Therefore

$$x_n \rightharpoonup \overline{x} := \mathcal{A}(\cdot)\omega + \mathcal{B}q + \mathcal{D}g + \mathcal{F}g,$$

in $\mathcal{C}([0,b];E)$, i.e. Ψ is weakly sequentially compact, and the assertion is proved. □

Proposition 5.3. *The multioperator Ψ has convex and weakly compact values.*

Proof. Fix $q \in \mathcal{C}([0,b];E)$. Since F and M are convex valued, the set $\Psi(q)$ is convex from the linearity of $S_\alpha(t)$ for every $t \in [0,b]$ and of the operators $\mathcal{D}$ and $\mathcal{F}$. The weak compactness of $\Psi(q)$ follows from Propositions 5.1 and 5.2. □

We are able now to prove the Theorem 5.1.

Proof. Fix $n \in \mathbb{N}$, consider Q_n the closed ball of radius n of $\mathcal{C}([0,b];E)$ centered at the origin. We show that there exists $\overline{n} \in \mathbb{N}$ such that the operator Ψ maps the ball $Q_{\overline{n}}$ into itself.
According to (3.2), there exists a subsequence, still denoted as the sequence $\{\varphi_n\}$, such that

$$\lim_{n\to\infty} \frac{1}{n} \left\{ \int_a^b |\varphi_n(s)|^p \, ds \right\}^{\frac{1}{p}} = 0. \tag{5.2}$$

Assume to the contrary, that there exist two sequences $\{q_n\}$ and $\{x_n\}$ such that $q_n \in Q_n$, $x_n \in \Psi(q_n)$ and $x_n \notin Q_n$ for all $n \in \mathbb{N}$. By the definition of Ψ, there exist a sequence $\{f_n\}$, with $f_n \in \mathcal{P}_F(q_n)$, and a sequence $\{\omega_n\}$, with $\omega_n \in M(q_n)$, such that

$$x_n = \mathcal{A}(\cdot)\omega_n + \mathcal{B}q_n + \mathcal{D}f_n + \mathcal{F}f_n.$$

From the assumption $x_n \notin Q_n$, according to (5.1), (4.9), (4.12) and (4.13) and recalling that $\|\omega_n\| \le \|M(q_n)\|$, we must have, for any n,

$$n < \|x_n\|_0 \le (D^{N+1}+1)\|M(q_n)\| + D(D+1)^{N-1} \max_{k=1,\dots N} \|I_k(q_n(t_k))\| + [D(D+1)^{N-1}+1]H\|f_n\|_p,$$

where H is defined in (4.11). Moreover $q_n \in Q_n$ implies, by (F3′), that $\|f_n(t)\| \le \varphi_n(t)$ for a.a. $t \in [0,b]$, hence $\|f_n\|_p \le \|\varphi_n\|_p$. Consequently

$$n < (D^{N+1}+1)\|M(q_n)\| + D(D+1)^{N-1} \max_{k=1,\dots N} \|I_k(q_n(t_k))\| + [D(D+1)^{N-1}+1]H\|\varphi_n\|_p.$$

Therefore

$$1 < (D^{N+1}+1)\frac{\|M(q_n)\|}{n}+D(D+1)^{N-1}\frac{\max_{k=1,\dots,N}\|I_k(q_n(t_k))\|}{n}+[D(D+1)^{N-1}+1]H\frac{\|\varphi_n\|_p}{n}.$$

Notice that if $\{q_n\}$ is bounded, then

$$\lim_{n\to\infty}\frac{\|M(q_n)\|}{n}=0,$$

because M maps bounded sets into bounded sets.
If $\limsup_{n\to\infty}\|q_n\|_0=+\infty$, by (3.1) we have

$$\limsup_{n\to\infty}\frac{\|M(q_n)\|}{n}\le\limsup_{n\to\infty}\frac{\|M(q_n)\|}{\|q_n\|_0}\le\lim_{\|u\|_0\to\infty}\frac{\|M(u)\|}{\|u\|_0}=l<\frac{1}{D^{N+1}+1}.$$

So, in both cases

$$\limsup_{n\to\infty}\frac{\|M(q_n)\|}{n}<\frac{1}{D^{N+1}+1}.$$

Moreover, fix $k\in\{1,...,N\}$. If $\|q_n(t_k)\|$ is bounded, then, since I_k maps bounded sets into bounded sets for any $k=1,\dots,N$, it follows

$$\lim_{n\to\infty}\frac{\|I_k(q_n(t_k))\|}{n}=0.$$

If $\limsup_{n\to\infty}\|q_n(t_k)\|=+\infty$, by (I_k) (iii) we have

$$\lim_{n\to\infty}\frac{\|I_k(q_n(t_k))\|}{n}\le\lim_{n\to\infty}\frac{\|I_k(q_n(t_k))\|}{\|q_n(t_k)\|}\le\lim_{\|c\|\to\infty}\frac{\|I_k(c)\|}{\|c\|}=0.$$

In conclusion

$$\lim_{n\to\infty}\frac{\max_{k=1,\dots N}\|I_k(q_n(t_k))\|}{n}=0.$$

Hence, by (5.2),

$$\begin{aligned}1\le\ &\limsup_{n\to\infty}\Bigg[(D^{N+1}+1)\frac{\|M(q_n)\|}{n}+D(D+1)^{N-1}\frac{\max_{k=1,\dots,N}\|I_k(q_n(t_k))\|}{n}+\\&[D(D+1)^{N-1}+1]H\tfrac{\|\varphi_n\|_p}{n}\Bigg]<1,\end{aligned}$$

giving the contradiction.
Now, fix $\overline{n}\in\mathbb{N}$ such that $\Psi(Q_{\overline{n}})\subseteq Q_{\overline{n}}$. By Proposition 5.2 the set $V_{\overline{n}}=\overline{\Psi(Q_{\overline{n}})}^{w}$ is a weakly compact set. Let $W_{\overline{n}}=\overline{\mathrm{co}}(V_{\overline{n}})$, where $\overline{\mathrm{co}}(V_{\overline{n}})$ denotes the closed convex hull of $V_{\overline{n}}$. By the Krein-Smulian theorem (see [9, p. 434]) $W_{\overline{n}}$ is weakly compact. Moreover from the fact that $\Psi(Q_{\overline{n}})\subset Q_{\overline{n}}$ and since $Q_{\overline{n}}$ is a convex closed set we get $W_{\overline{n}}\subset Q_{\overline{n}}$ and hence

$$\Psi(W_{\overline{n}})=\Psi(\overline{\mathrm{co}}(\Psi(Q_{\overline{n}})))\subseteq\Psi(Q_{\overline{n}})\subseteq\overline{\Psi(Q_{\overline{n}})}^{w}=V_{\overline{n}}\subset W_{\overline{n}}.$$

Therefore from Proposition 5.1 and we obtain that the restriction of the multimap Ψ on W_n is weakly closed and, hence, it is weakly u.s.c (see [16, Theorem 1.1.5]). The conclusion then follows from Theorem 2.1. □

The next theorem shows that, when we investigate the existence of a solution for the Cauchy problem, an existence result can be proved under weaker growth conditions on F and on I_k.

Theorem 5.2. *Assume* $(A), (F1)$, *and* $(F2)$. *Moreover suppose that*

$(F3'')$ *there exists* $\varphi \in L^p([0,b];\mathbb{R})$, *with* $p > \frac{1}{\alpha}$, *such that, for every* $x \in E$ *and a.a.* $t \in [0,b]$,

$$\|F(t,x)\| \leq \varphi(t)(1 + \|x\|).$$

(I'_k) *the functions* $I_k : E \to E$, $k = 1, \ldots, N$ *are weakly sequentially continuous and there exists a constant* $m > 0$ *such that for every* $c \in E$ *and* $k = 1, ..., N$

$$\|I_k(c)\| \leq m\|c\|.$$

Then the problem

$$\begin{cases} {}^C D^\alpha x(t) \in Ax(t) + F(t, x(t)), \quad \text{for a.a. } t \in [0,b], t \neq t_1, \ldots, t_N \\ x(t_k^+) = x(t_k) + I_k(x(t_k)),\ k = 1, \ldots, N \\ x(0) = x_0 \end{cases}$$

has at least one mild solution.

Proof. According to (4.10) and the Hölder inequality, for every $\tau_1, \tau_2 \in E$ and $\varphi \in L^p([0,b],E)$, we have

$$\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}\varphi(s)ds \leq \left[\frac{p-1}{\alpha p - 1}\right]^{\frac{p-1}{p}} b^{\alpha - \frac{1}{p}} \|\varphi\|_p.$$

Denote

$$C := (D^{N+1} + 1)\|x_0\| + [D(D+1)^{N-1} + 1]H\|\varphi\|_p,$$

where H is defined in (4.11). Consider the function $h : [0,b] \times [0,b] \times [0,\infty) \to \mathbb{R}$ defined as

$$h(t,s,\ell) = \begin{cases} \frac{e^{\ell(s-t)}\varphi(s)}{(t-s)^{1-\alpha}} & t > s \\ 0 & t \leq s. \end{cases}$$

By Hölder inequality we have that

$$\int_0^t \frac{e^{\ell(s-t)}}{(t-s)^{1-\alpha}}\varphi(s)ds \leq \|\varphi\|_p \left[\int_0^t \left(\frac{e^{\ell(s-t)}}{(t-s)^{1-\alpha}}\right)^{\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}}.$$

Moreover, notice that

$$\left[\int_0^t \left(\frac{e^{\ell(s-t)}}{(t-s)^{1-\alpha}}\right)^{\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}} \leq \left[\int_0^b \left(\frac{e^{-\ell r}}{r^{1-\alpha}}\right)^{\frac{p}{p-1}} dr\right]^{\frac{p-1}{p}},$$

where by the Lebesgue Dominate Convergence theorem the last integral tends to zero as ℓ goes to infinity. Thus, it is possibile to find two positive constants L and R such that

$$\left[D(D+1)^{N-1}m + \frac{D\alpha}{\Gamma(1+\alpha)}\left(D(D+1)^{N-1} + 1\right)\right] \max_{t\in[0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}}\varphi(s)ds := \overline{\beta} < 1$$

and

$$R \geq C(1-\overline{\beta})^{-1}.$$

Define

$$Q = \{q \in \mathcal{C}([0,b],E) : \|q(t)\| \leq Re^{Lt} \text{ for all } t \in [0,b]\}.$$

It is clear that Q is bounded, convex and closed. For $q \in Q$ consider

$$\Psi(q)(t) = \mathcal{A}(t)x_0 + \mathcal{B}(q)(t) + \mathcal{D}(f)(t) + \mathcal{F}(f)(t), \quad t \in [0,b],$$

with $f \in L^p([0,b],E), f(s) \in F(s,q(s))$ for a.a. $s \in [0,b]$.
Notice that, from Lemma 2.1 and (I_k'), reasoning as to get (4.9), we obtain, for any $q \in Q, k = 1,...,N,$

$$\phi_k(q)(t) \leq D(D+1)^{k-1} mRe^{Lt_k} \leq D(D+1)^{N-1} mRe^{Lt}$$

for every $t > t_k$, thus

$$\mathcal{B}(q)(t) \leq D(D+1)^{N-1} mRe^{Lt}$$

for every $t \in E$.
Moreover, from Lemma 2.1 and (F3$''$) , we get for any $q \in Q, f \in \mathcal{P}_F(q)$ and $\tau_1, \tau_2 \in [0,b]$

$$\begin{aligned}
\left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} T_\alpha(\tau_2 - s) f(s)\, ds \right\| \leq\ & \frac{D\alpha}{\Gamma(1+\alpha)} \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} \|f(s)\| ds \\
\leq\ & \frac{D\alpha}{\Gamma(1+\alpha)} \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} \varphi(s)(1 + \|q(s)\|) ds \\
\leq\ & H\|\varphi\|_p + \tfrac{D\alpha}{\Gamma(1+\alpha)} e^{L\tau_2} R \int_0^{\tau_2} \frac{e^{L(s-\tau_2)}}{(\tau_2 - s)^{1-\alpha}} \varphi(s) ds \\
\leq\ & H\|\varphi\|_p + \frac{D\alpha}{\Gamma(1+\alpha)} e^{L\tau_2} R \max_{t \in [0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}} \varphi(s) ds.
\end{aligned}$$

Then it easily follows that

$$\begin{aligned}
\|r_k(f)(t)\| \leq\ & D(D+1)^{k-1} \left[H\|\varphi\|_p + \frac{D\alpha}{\Gamma(1+\alpha)} e^{Lt_k} R \max_{t \in [0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}} \varphi(s) ds \right] \\
\leq\ & D(D+1)^{N-1} \left[H\|\varphi\|_p + \frac{D\alpha}{\Gamma(1+\alpha)} e^{Lt} R \max_{t \in [0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}} \varphi(s) ds \right]
\end{aligned}$$

for every $t > t_k$ and $k = 1,...,N$, hence

$$\|\mathcal{D}(f)\| \leq D(D+1)^{N-1} \left[H\|\varphi\|_p + \frac{D\alpha}{\Gamma(1+\alpha)} e^{Lt} R \max_{t \in [0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}} \varphi(s) ds \right]$$

and

$$\|\mathcal{F}(f)(t)\| \leq H\|\varphi\|_p + \frac{D\alpha}{\Gamma(1+\alpha)} e^{Lt} R \max_{t \in [0,b]} \int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}} \varphi(s) ds.$$

Hence

$$\begin{aligned}\|\Psi(q)(t)\| \leq\ & (D^{N+1}+1)\|x_0\| + D(D+1)^{N-1}mRe^{Lt} + [D(D+1)^{N-1}+1]H\|\varphi\|_p + \\ & \frac{D\alpha}{\Gamma(1+\alpha)}[D(D+1)^{N-1}+1]e^{Lt}R\max_{t\in[0,b]}\int_0^t \frac{e^{L(s-t)}}{(t-s)^{1-\alpha}}\varphi(s)ds \\ =\ & C + Re^{Lt}\overline{\beta} \\ \leq\ & R(1-\overline{\beta}) + Re^{Lt}\overline{\beta} \\ \leq\ & Re^{Lt}(1-\overline{\beta}) + Re^{Lt}\overline{\beta} = Re^{Lt},\end{aligned}$$

yielding $\Psi(Q) \subseteq Q$. Now, since the proofs of Propositions 5.1, 5.2 and 5.3 rely on the integral boundedness condition $(F3)$, a weaker assumption than $(F3'')$, we can reason as at the end of the proof of Theorem 5.1, obtaining the claimed result. □

6 A fractional nonlocal diffusion process

To demonstrate the effectiveness of the theoretical results proved in the previous Section, we consider the following nonlocal diffusion process:

$$D_t^\alpha u = \Delta u + \left[f_1\left(t,x,\int_\Omega k_1(x,\xi)u(t,\xi)\,d\xi\right), f_2\left(t,x,\int_\Omega k_2(x,\xi)u(t,\xi)\,d\xi\right)\right]\psi(t,x), \quad (6.1)$$

with $t \in [0,b], t \neq t_k,\ k = 1,\dots,N, x \in \Omega$, where Ω is a nonempty domain in $\mathbb{R}^n$, with impulses

$$u(t_k^+,x) = u(t_k,x) + c_k, \quad k = 1,...,N,\ x \in \Omega \quad (6.2)$$

and subjected to the boundary conditions

$$\begin{array}{l} u(t,x) = 0, \quad t \in [0,b], x \in \partial\Omega \\ u(0,x) = \sum_{i=1}^J \alpha_i u(s_i,x), \quad x \in \Omega, s_1 \leq 0 < \dots < s_J \leq b. \end{array} \quad (6.3)$$

We assume the following hypotheses:

(k) $k_i(x,\cdot) \in L^2(\Omega,\mathbb{R})$, with $\|k_i(x,\cdot)\|_2 \leq 1$, $i = 1,2$, for almost all $x \in \Omega$

(ψ) $\psi\colon [0,b]\times\Omega \to \mathbb{R}$ is measurable with $\psi(t,\cdot) \in L^2(\Omega;\mathbb{R})$ for a.a. $t \in [0,b]$ and there exists $V > 0$ such that $\|\psi(t,\cdot)\|_2 \leq V$ for almost all $t \in [0,b]$;

(f1) functions $f_1, f_2\colon [0,b]\times\Omega\times\mathbb{R} \to \mathbb{R}$ are such that for each $y \in L^2(\Omega,\mathbb{R})$ there is a measurable function $z\colon [0,b]\times\Omega \to \mathbb{R}$ such that for almost all $t \in [0,b]$ the inequalities

$$f_1\left(t,x,\int_\Omega k_1(x,\xi)y(\xi)\,d\xi\right) \leq z(t,x) \leq f_2\left(t,x,\int_\Omega k_2(x,\xi)y(\xi)\,d\xi\right)$$

hold for almost all $x \in \Omega$;

(f2) for almost all $t \in [0,b]$ and for almost all $x \in \Omega$ the function $f_1(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and the function $f_2(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous;

(f3) there exists $\varphi \in L^p([0,b],[0,\infty))$, with $p > \frac{1}{\alpha}$ and a non decreasing function $\mu : [0,\infty) \to [0,\infty)$ such that for every $t \in [0,b]$, almost all $x \in \Omega$ and each $r > 0$

$$|f_i(t,x,r)| \leq \varphi(t)\mu(|r|) \qquad i = 1,2,$$

with

$$\liminf_{r\to\infty} \frac{\mu(r)}{r} = 0; \tag{6.4}$$

(α) $\alpha_i \in \mathbb{R} \setminus \{0\}$ satisfy $\sum_{i=1}^n |\alpha_i| < \frac{1}{2}$.

We consider problem (6.1)-(6.2)-(6.3) in the state space $E = L^2(\Omega;\mathbb{R})$. In order to apply Theorem 5.1 we note that the linear operator $A : D(A) \subset E \to E$ defined by

$$\begin{cases} D(A) = \{y \in W_0^{1,2}(\Omega;\mathbb{R}),\ \Delta y \in L^2(\Omega;\mathbb{R})\}, \\ Ay = \Delta y, \quad y \in D(A), \end{cases}$$

generates a strongly continuous semigroup of contractions in E (see e.g. [31] Theorem 4.1.2). For $y \in E$ we define $F(t,y)$ as the set of all functions $g \in E$ satisfying $g(x) = h(x)\psi(t,x)$, with $h : \Omega \to \mathbb{R}$ a measurable function such that

$$h(x) \in \left[f_1\Big(t,x,\int_\Omega k_1(x,\xi)y(\xi)\,d\xi\Big), f_2\Big(t,x,\int_\Omega k_2(x,\xi)y(\xi)\,d\xi\Big)\right] \tag{6.5}$$

for almost all $x \in \Omega$. According to (k) and the Hölder inequality we have for a.a. $x \in \Omega$ and every $y \in E$, that

$$\left|\int_\Omega k_i(x,\xi)y(\xi)d\xi\right| \leq \int_\Omega |k_i(x,\xi)||y(\xi)|d\xi \leq \|k(x,\cdot)\|_2\|y\|_2 \leq \|y\|_2, \quad i = 1,2.$$

Thus, for $y \in E$ denoting with $P_y^i : \Omega \to \mathbb{R}$ the map

$$P_y^i(x) = \int_\Omega k_i(x,\xi)y(\xi)d\xi, \qquad i = 1,2,$$

(f3) implies, for every $t \in [0,b]$, for a.a. $x \in \Omega$ and every $y \in E$, that

$$|f_i\left(t,x,P_y^i(x)\right)| \leq \varphi(t)\mu\left(\left|P_y^i(x)\right|\right) \leq \varphi(t)\,\mu(\|y\|_2), \quad i = 1,2.$$

Hence, for every $g \in F(t,y)$, we get

$$\|g\|_2 \leq \|\psi(t,\cdot)\|_2\,\varphi(t)\,\mu(\|y\|_2),$$

obtaining that $F : [0,b] \times E \multimap E$ is well defined. With this definition of F we can rewrite (6.1)-(6.2)-(6.3) in the abstract form (1.2)-(1.3) with $I_k : E \to E$, $I_k(y) \equiv c_k$ and $M : \mathcal{C}([0,b];E) \to E$ defined as $M(w)(x) = \sum_{i=1}^n \alpha_i w(s_i)(x)$ for a.a. $x \in \Omega$. We verify now that all hypotheses of Theorem 5.1 are satisfied.
First of all notice that by (6.4) and (ψ) it follows

$$\liminf_{n\to\infty} \frac{1}{n}\int_0^b \|\psi(t,\cdot)\|_2\varphi(t)\mu(n)dt \leq \liminf_{n\to\infty} V\frac{\mu(n)}{n}\|\varphi\|_1 = 0,$$

proving that the growth condition $(F3')$ is satisfied with $\varphi_n(t) = V\varphi(t)\mu(n)$, $n \in \mathbb{N}$.
For every $y \in E$ let $z : [0,b] \times \Omega \to \mathbb{R}$ be the measurable function satisfying $(f1)$. Now $v : [0,b] \times \Omega \to \mathbb{R}$ defined as $v(t,x) = z(t,x)\psi(t,x)$ is a measurable map. Since $v(t,\cdot) \in E$ for almost all $t \in E$, from [30, Theorem 4.4.2], we obtain that if $v\colon [0,b] \times \Omega \to \mathbb{R}$ is measurable with $v(t,\cdot) \in E$ for almost all $t \in E$ then $t \mapsto v(t,\cdot)$ is measurable as a function from $[0,b]$ into E. Thus condition $(F1)$ follows trivially.
Now we verify (F2). We fix a value t in $[0,b]$ for which (f2) is satisfied and consider two sequences $\{g_n\}, \{y_n\} \subset E$ such that $g_n \rightharpoonup g$, $y_n \rightharpoonup y$ in E and $g_n \in F(t,y_n)$ for all $n \in \mathbb{N}$, i.e. $g_n(x) = h_n(x)\psi(t,x)$ for a.a. $x \in \Omega$, with h_n satisfying (6.5). Notice that the weak convergence of y_n and (k) imply $P^i_{y_n}(x) \to P^i_y(x)$ for all $x \in \Omega$, $i = 1,2$. Applying Mazur's convexity lemma, for each n there exist $k_n \in \mathbb{N}$ and positive numbers δ_{ni}, $i = 0,1,...,k_n$ such that $\sum_{i=0}^{k_n} \delta_{ni} = 1$ and the sequence $\{\widetilde{g}_n\}$,

$$\widetilde{g}_n(x) := \sum_{i=0}^{k_n} \delta_{ni} g_{n+i}(x), \quad x \in \Omega$$

converges to g with respect to the norm of $L^2(\Omega;\mathbb{R})$. Passing if necessary to a subsequence we can assume that $\{\widetilde{g}_n\}$ converges to g almost everywhere in Ω. Since

$$\sum_{i=0}^{k_n} \delta_{ni} f_1(t,x,P^1_{y_{n+i}}(x))\psi(t,x) \le \widetilde{g}_n(x) \le \sum_{i=0}^{k_n} \delta_{ni} f_2(t,x,P^2_{y_{n+i}}(t))\psi(t,x), \quad \text{for a.a. } x \in \Omega,$$

passing to the limit as $n \to \infty$ and according to (f2), we obtain that

$$f_1(t,x,P^1_y(x))\psi(t,x) \le g(x) \le f_2(t,x,P^2_y(x))\psi(t,x)$$

a.e. in Ω.
Thus, considering the map $h : \Omega \to \mathbb{R}$ defined as

$$h(x) = \begin{cases} \frac{g(x)}{\psi(t,x)} & x \in \Omega \text{ such that } \psi(t,x) \neq 0 \\ z(t,x) & x \in \Omega \text{ such that } \psi(t,x) = 0 \end{cases}$$

where $z : [0,b] \times \Omega \to \mathbb{R}$ is the measurable function from $(f1)$, we get that $g(x) = h(x)\psi(t,x)$ and so $g \in F(t,y)$, proving that F is weakly sequentially closed.
Finally, it is easy to see that the multimap F has bounded, closed and convex values.
Moreover, the constant functions $I_k, k = 1,...,N$ are trivially sequentially continuous with respect to the weak topology, map bounded sets into bounded ones and satisfy (3.2). Finally, if $\sum_{i=1}^{J} |\alpha_i| < \frac{1}{2}$ condition (M) is satisfied. Indeed M is a linear and bounded single valued operator, hence it is a weakly sequentially closed multioperator. Furthermore for $w \in \mathcal{C}([0,b];E)$ we have

$$\frac{\| \sum_{i=1}^{J} \alpha_i w(s_i)\|_2}{\|w\|_0} \le \frac{\sum_{i=1}^{J} |\alpha_i| \|w(s_i)\|_2}{\|w\|_0} \le \frac{\|w\|_0 \sum_{i=1}^{J} |\alpha_i|}{\|w\|_0} = \sum_{i=1}^{J} |\alpha_i|.$$

Hence

$$\limsup_{\|w\|_0 \to \infty} \frac{\| \sum_{i=1}^{J} \alpha_i w(s_i)\|_2}{\|w\|_0} \le \sum_{i=1}^{J} |\alpha_i| < \frac{1}{2}.$$

Thus all the assumptions of Theorem 5.1 are satisfied and the existence of a solution $u \in \mathcal{C}([0,b];L^2(\Omega;\mathbb{R}))$ of (6.1)-(6.2)-(6.3) is proved.

Remark 6.1. We point out the fact that in order to obtain the existence of at least one solution of (6.1) we do not need any measurability assumption on the map $f_i(\cdot,\cdot,r)$, $r \in \mathbb{R}$, but only the much weaker hypothesis (f1).

Remark 6.2. The problem (6.1) is quite general, it includes the case of unbounded domains Ω and as a nonlinear term an interval between two different values, that can represent the range of a nonlocal forcing term acting on the process and that is the most natural example of a multivalued map. This is due to the fact that with our approach we can handle the case of a non compact semigroup generated by the linear part and at the same time non compact valued multimaps.

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