ERROR BOUNDS FOR SOLUTIONS OF SYSTEMS OF QUASILINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN BICHARACTERISTIC FORM

RITA CEPPITELLI (Perugia) - MASSIMO VILLARINI (Perugia) (*) (**)

The purpose of this paper is to obtain a theoretical estimate of the effect of a nonlinear perturbation on the solution of a boundary value problem for quasilinear hyperbolic systems in bicharacteristic canonic form with the same boundary data which L. Cesari introduced in [9], [10]. The estimate is given by the difference between the solutions of the linear and the perturbed systems.

1. Introduction.

In this paper we consider a boundary value problem for quasilinear hyperbolic systems in bicharacteristic canonic form with the same boundary data which L. Cesari introduced in [9], [10]. For this problem Cesari in [10] proved a theorem of existence, uniqueness and continuous dependence on the data.

The formulation of systems studied in the present paper is equivalent to one of Cesari's paper [10], but here we split the linear from the nonlinear terms. Precisely, having introduced two parameters $\mu \geq 0$,

---

(*) Entrato in Redazione il 4 ottobre 1988.

(**) This research was carried out within the Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni, Consiglio Nazionale delle Ricerche.

$\eta \geq 0$, we consider the quasilinear system

$$\sum_{j=1}^{m}(\mu B_{ij}(x,y)+\eta\Gamma_{ij}(x,y,z))\cdot$$

(c.1)
$$\cdot\left[\frac{\partial z_j}{\partial x}+\sum_{k=1}^{r}(\mu\theta_{ik}(x,y)+\eta\lambda_{ik}(x,y,z))\frac{\partial z_j}{\partial y_k}\right]=$$

$$=\mu v_i(x,y)+\eta w_i(x,y,z), \qquad (x,y)\in[0,a]\times E^r,\ i=1,2,\dots,m.$$

Here η is the nonlinearity parameter. We consider this system either with Cauchy data

$$z_i(0,y)=\phi_i(y), \qquad y\in E^r,\ i=1,\dots,m,$$

or with Cesari [9], [10] boundary data

$$\sum_{j=1}^{m}b_{ij}(y)z_j(a_i,y)=\psi_i(y) \qquad y\in E^r,\ i=1,\dots,m,$$

where $\phi_i : E^r \to E$, $b_{ij} : E^r \to E$, $\psi_i : E^r \to E$, $i,j=1,\dots,m$, are given functions, the a_i, $0\leq a_i\leq a$, $i=1,\dots,m$, are given numbers and the $m\times m$ matrix (b_{ij}) has dominant diagonal.

Adapting Cesari's proof of theorems I and II in [10] to system (c.1) we reprove in Sections 2 and 4 his theorems of existence, uniqueness and continuous dependence on the data. Our version of these theorems holds in a suitable slab $[0,a]\times E^r$, $0<a\leq a_0$, where the constant "a" satisfies a system of algebraical inequalities depending on the parameters μ and η. In detail, in Section 2 we prove a theorem of existence, uniqueness and continuous dependence on the initial data for the Cauchy problem relative to system (c.1). Then we use, according to Cesari's method [10], this result in Section 4 to obtain the mentioned theorem for the boundary value problem.

The aim of this paper is to give a theoretical estimate of the effect of the nonlinear perturbation on the solution. That is, we estimate the difference between the solutions of the nonlinear problem and these of the purely linear one. Indeed, in Section 3, by virtue of Cesari's demonstrative technique, we obtain an evaluation of $||z^{(\eta)}-z^{(0)}||$, where $z^{(\eta)}$ and $z^{(0)}$ are the solutions of the Cauchy problem for (c.1), studied in Section 2, in the nonlinear case ($\eta>0$) and the linear case ($\eta=0$), respectively.

In Section 5 we obtain an evaluation of $||z^{(\eta)} - z^{(0)}||$, where here $z^{(\eta)}$ and $z^{(0)}$ are the solutions of the boundary problem for (c.1) in the case $\eta > 0$, $\eta = 0$, respectively.

The results obtained in this note are a continuation of the one proved in [7]. Indeed in [7], by using Cesari's method introduced in [9], we obtain analogous estimates of $||z^{(\eta)} - z^{(0)}||$ for quasilinear hyperbolic systems in diagonal form.

The results in [8], [9], [10] have been applied in [11] to the analysis of the phenomenon of duplication of frequency of laser radiation through a nonlinear medium (problem of Graffi-Cesari). From a numerical point of view, P. Bassanini [1,2,3] continued the work of Cesari. Numerical results were obtained in [12], [5], [6], [14]. A survey of a number of these results is presented in [4]. In all these numerical papers the solutions of the nonlinear problems have been confronted with the solutions of the linear one.

We wish to thank Professor L. Cesari who suggested this research and supported it with stimulating conversations, and Professors C. Vinti and P. Brandi for their helpful advice.

2. The Cauchy Problem.

Here, x is a scalar, $y = (y_1, \ldots, y_r)$ is a vector in E^r and $z = (z_1, \ldots, z_m)$ is a vector in E^m. Let $|y| = \max_k |y_k|$, $|z| = \max_i |z_i|$ be the norms in E^r and E^m, respectively.

We consider the following Cauchy problem (C)

$$\sum_{j=1}^{m}(\mu B_{ij}(x,y) + \eta\Gamma_{ij}(x,y,z))\cdot$$

$$\text{(c.1)} \qquad \cdot\left[\frac{\partial z_j}{\partial x} + \sum_{k=1}^{r}(\mu\theta_{ik}(x,y) + \eta\lambda_{ik}(x,y,z))\frac{\partial z_j}{\partial y_k}\right] =$$

$$= \mu v_i(x,y) + \eta w_i(x,y,z) \text{ a.e. in } D_a = [0,a] \times E^r,\ i = 1, \ldots, m$$

$$\text{(c.2)} \qquad z_i(0,y) = \phi_i(y) \text{ for every } y \in E^r,\ i = 1, \ldots, m.$$

We remark that (c.1) is a system in bicharacteristic canonic form. If $\mu = 0$, $\eta = 1$ the problem (C) has been studied in [10], while if $\eta = 0$ it is a linear system.

In this section we state a theorem of existence, uniqueness and continuous dependence on initial data for the solution of (C). This theorem is essentially a particular case of an analogous one in [10], with few slightly modifications, but we will however briefly sketch the proof. Indeed, adapting Cesari's proof in [10] to the particular form of problem (C), we obtain a sharper estimate of the size of the slab domain of the solution. Moreover, and above all, a sketch of the proof gives the algorithm of the solution that we will use to obtain the requested estimates of approximation error with respect to the parameter of non-linearity η.

THEOREM 1. (An existence theorem for the Cauchy problem (C)). *Let Ω denote a given positive constant and the interval $[-\Omega,\Omega]^m \subset E^m$, too. Let μ,η be nonnegative constants. Let $B_{ij}(x,y)$, $\Gamma_{ij}(x,y,z)$, $i,j=1,\dots,m$, be continuous functions defined in $D_{a_0}=[0,a_0]\times E^r$, $D_{a_0}\times\Omega$ respectively, $a_0>0$, with det $(B_{ij})\geq\tau>0$, $det(\mu B_{ij}+\eta\Gamma_{ij})\geq\tau>0$ in their respective domain for some constant τ. Let us assume that there exist constant $H>0$, $C\geq 0$, and a function $\mathring{\mathrm{m}}(x)\geq 0$, $0\leq x\leq a_0$, $\mathring{\mathrm{m}}\in L_1[0,a_0]$, such that, for every (x,y), $(\bar{x},y)$, $(x,\bar{y})\in D_{a_0}$, (x,y,z), $(x,\bar{y},\bar{z})$, $(\bar{x},y,z)\in D_{a_0}\times\Omega$ and $i,j=1,\dots,m$, we have*

$$|B_{ij}(x,y)|\leq H,\quad |\Gamma_{ij}(x,y,z)|\leq H; \tag{2.1}$$

$$\begin{aligned}&|B_{ij}(x,y)-B_{ij}(x,\bar{y})|\leq C|y-\bar{y}|,\\ &|\Gamma_{ij}(x,y,z)-\Gamma_{ij}(x,\bar{y},\bar{z})|\leq C[|y-\bar{y}|+|z-\bar{z}|];\end{aligned} \tag{2.2}$$

$$\begin{aligned}&|B_{ij}(x,y)-B_{ij}(\bar{x},y)|\leq \Big|\int_x^{\bar{x}}\mathring{\mathrm{m}}(t)dt\Big|,\\ &|\Gamma_{ij}(x,y,z)-\Gamma_{ij}(\bar{x},y,z)|\leq \Big|\int_x^{\bar{x}}\mathring{\mathrm{m}}(t)dt\Big|.\end{aligned} \tag{2.3}$$

Let $\theta_{ik}(x,y)$, $v_i(x,y)$, $i=1,\dots,m$, $k=1,\dots,r$, be given functions defined in D_{a_0}, which are measurable in x for every y and continuous in y for every x. Let $\lambda_{ik}(x,y,z)$, $w_i(x,y,z)$, $i=1,\dots,m$, $k=1,\dots,r$, be functions defined in $D_{a_0}\times\Omega$, all measurable in x for every (y,z) and continuous in (y,z) for every x.

Let us assume that there are nonnegative summable functions $m(x)$, $1(x)$, $n(x)$, $l_1(x)$, $0 \leq x \leq a_0$, *such that for every* (x,y), $(x,\bar{y}) \in D_{a_0}$, (x,y,z), $(x,\bar{y},\bar{z}) \in D_{a_0} \times \Omega$ *and* $i = 1,\dots,m$, $k = 1,\dots,r$ *we have*

$$(2.4) \qquad \begin{aligned} &|\theta_{ik}(x,y)| \leq m(x), \quad |\lambda_{ik}(x,y,z)| \leq m(x),\\ &|v_i(x,y)| \leq n(x), \quad |w_i(x,y,z)| \leq n(x); \end{aligned}$$

$$(2.5) \qquad \begin{aligned} &|\theta_{ik}(x,y) - \theta_{ik}(x,\bar{y})| \leq 1(x)|y-\bar{y}|,\\ &|\lambda_{ik}(x,y,z) - \lambda_{ik}(x,\bar{y},\bar{z})| \leq 1(x)[|y-\bar{y}| + |z-\bar{z}|] \end{aligned}$$

$$(2.6) \qquad \begin{aligned} &|v_i(x,y) - v_i(x,\bar{y})| \leq l_1(x)|y-\bar{y}|,\\ &|w_i(x,y,z) - w_i(x,\bar{y},\bar{z})| \leq l_1(x)[|y-\bar{y}| + |z-\bar{z}|]. \end{aligned}$$

Let $\phi(y) = (\phi_1(y),\dots,\phi_m(y))$, $y \in E^r$ *be a given vectorial function and let* ω, Λ *be nonnegative costants with* $0 \leq \omega \leq \Omega$, *such that, for every* y, $\bar{y} \in E^r$ *and* $i = 1,\dots,m$ *we have*

$$(2.7) \qquad |\phi_i(y)| \leq \omega, \quad |\phi_i(y) - \phi_i(\bar{y})| \leq \Lambda|y-\bar{y}|.$$

Then, there exist a sufficiently small a, $0 < a \leq a_0$, *a costant* $Q > 0$, *a summable function* $\chi(x) \geq 0$, $0 \leq x \leq a$, *and a vectorial function* $z(x,y) = (z_1,\dots,z_m)$, *continuous in* $D_a = [0,a] \times E^r$, *such that for every* (x,y), $(x,\bar{y})$, $(\bar{x},y) \in D_a$ *and* $i = 1,\dots,m$ *we have*

$$(2.8) \qquad |z_i(x,y)| \leq \Omega, \quad |z_i(x,y) - z_i(x,\bar{y})| \leq Q|y-\bar{y}|,$$

$$(2.9) \qquad |z_i(x,y) - z_i(\bar{x},y)| \leq \left|\int_x^{\bar{x}} \chi(t)dt\right|,$$

satisfying (c.1) a.e. in D_a *and (c.2) everywhere in* E^r. *Furthermore this solution is unique and depends continuously on* $\phi = (\phi_1,\dots,\phi_m)$.

Proof. For every $(x,y,z) \in D_{a_0} \times \Omega$ and every $i,j = 1,\dots,m$, $k = 1,\dots,r$ we put

$$(2.10) \qquad \begin{aligned} A_{ij}(x,y,z) &= \mu B_{ij}(x,y) + \eta\Gamma_{ij}(x,y,z),\\ \rho_{ik}(x,y,z) &= \mu\theta_{ik}(x,y) + \eta\lambda_{ik}(x,y,z),\\ f_i(x,y,z) &= \mu v_i(x,y) + \eta w_i(x,y,z). \end{aligned}$$

The functions $A_{ij}(x,y,z)$, $i,j=1,\dots,m$, are continuos in $D_{a_0}\times\Omega$ and the functions $\rho_{ik}(x,y,z)$, $f_i(x,y,z)$, $i=1,\dots,m$, $k=1,\dots,r$ are all measurable in x for every (y,z) and continuous in (y,z) for every x. Moreover by (2.1) - (2.6) the following properties hold in $D_{a_0}\times\Omega$:

$$|A_{ij}(x,y,z)|\leq(\mu+\eta)H, \tag{2.11}$$

$$|A_{ij}(x,y,z)-A_{ij}(x,\bar{y},\bar{z})|\leq(\mu+\eta)C|y-\bar{y}|+\eta C|z-\bar{z}|, \tag{2.12}$$

$$|A_{ij}(x,y,z)-A_{ij}(\bar{x},y,z)|\leq(\mu+\eta)\left|\int_x^{\bar{x}}\mathring{m}(t)dt\right|, \tag{2.13}$$

$$\begin{aligned}&|\rho_{ik}(x,y,z)|\leq(\mu+\eta)m(x),\ |f_i(x,y,z)|\leq(\mu+\eta)n(x),\\&|\rho_{ik}(x,y,z)-\rho_{ik}(x,\bar{y},\bar{z})|\leq(\mu+\eta)1(x)|y-\bar{y}|+\eta 1(x)|z-\bar{z}|,\end{aligned} \tag{2.14}$$

$$|f_i(x,y,z)-f_i(x,\bar{y},\bar{z})|\leq(\mu+\eta)1_1(x)|y-\bar{y}|+\eta 1_1(x)|z-\bar{z}|. \tag{2.15}$$

Let us remark that by hypothesis on (B_{ij}), (Γ_{ij}), μ, η, it result that $\det(A_{ij})\geq\tau>0$.

So, the functions $A_{ij}(x,y,z)$, $\rho_{ik}(x,y,z)$, $f_i(x,y,z)$, $i,j=1,\dots,m$, $k=1,\dots,r$, satisfy the same hypotheses of Cesari's theorem I in [10], with suitable constants and summable functions. Let us point out that in (2.12) and (2.15) we have distinguished the Lipschitz property of the functions with respect to the variables y and z.

From now on, we will follow the outline of Cesari's proof of theorem I in [10].

(a) *Choice of constants* p,a *and function* χ. As $\det(A_{ij})\geq\tau>0$, $i,j=1,\dots,m$, we can introduce the transpose of the inverse of the matrix (A_{ij}) that we denote by $(\alpha_{ij})=(\alpha_{ij}(x,y,z))$. So $(\alpha_{ij})=((A_{ij})^{-1})^t$. The relations (2.11)-(2.13) yeld analogous relations for the elements α_{ij} (x,y,z). Thus, there are constants $H'>0$, $C'\geq 0$ and a function $\mathring{m}'(x)\geq 0$, $0\leq x\leq a_0$, $\mathring{m}'\in L_1[0,a_0]$, such that

$$|\alpha_{ij}(x,y,z)|\leq H', \tag{2.16}$$

$$(2.17)\qquad |\alpha_{ij}(x,y,z)-\alpha_{ij}(x,\bar{y},\bar{z})| \le C'[|y-\bar{y}|+|z-\bar{z}|],$$

$$(2.18)\qquad |\alpha_{ij}(x,y,z)-\alpha_{ij}(\bar{x},y,z)| \le \left|\int_x^{\bar{x}} \mathring{\mathrm{m}}'(t)dt\right|.$$

For every a, $0 < a \le a_0$, we define the following constants:

$$M_a = \int_0^a m(x)dx, \quad N_a = \int_0^a n(x)dx, \quad \mathring{\mathrm{M}}_a = \int_0^a \mathring{\mathrm{m}}(x)dx,$$

$$L_a = \int_0^a \mathrm{l}(x)dx, \quad L_{1a} = \int_0^a \mathrm{l}_1(x)dx.$$

Let us choose constants p, Q, k, R_0, R_1, R_2, R_3 with

$$(2.19)\qquad 0<p<1, \quad Q>\Lambda(1+m^2HH'(2+p)(\mu+\eta)), \quad 0<k<1,$$

$$(2.20)\qquad R_0 > m(\mu+\eta)H', \quad R_1, R_2 > 0, \quad R_3 > m^2H'H\Lambda(1-k)^{-1}(\mu+\eta)^2.$$

Let us take

$$\chi(x) = R_0 n(x) + R_1\mathring{\mathrm{m}}(x) + R_2\mathring{\mathrm{m}}'(x) + R_3 m(x), \ 0 \le x \le a_0,$$

and for every a, $0 < a \le a_0$, we define

$$\Xi_a = \int_0^a \chi(x)dx.$$

We shall have to impose some limitations on the size of a. Though this could well be done at this stage, we prefer to introduce the restrictions on a as need comes in the course of the argument.

(b) *The classes* $\mathcal{K}_0$ *and* $\mathcal{K}_1$. Let $I_a = [0,a]$ and $D_a = I_a \times E^r$, $\Delta_a = I_a \times I_a \times E^r$. Let $\mathcal{K}_0$ be the class of all systems

$$g = [g_{ik}(\xi;x,y), \ i=1,\dots,m, \ k=1,\dots,r]$$

of continuous functions g_{ik} in Δ_a satisfying

$$(2.21)\qquad g_{ik}(x;x,y) = y_k \quad \text{for all } (x,y) \in D_a,$$

(2.22) $$|g_{ik}(\xi;x,y)-g_{ik}(\bar{\xi};x,y)|\leq|\int_{\xi}^{\bar{\xi}}m(t)dt|(\mu+\eta),$$

(2.23) $$|g_{ik}(\xi;x,y)-g_{ik}(\xi;x,\bar{y})-y_k+\bar{y}_k|\leq p|y-\bar{y}|$$
$$\text{for all } (\xi;x,y),(\bar{\xi};x,y),(\xi;x,\bar{y})\in\Delta_a.$$

For every $i=1,\dots,m$ we put $\tilde{g}_i(\xi;x,y)=(g_{ik}(\xi;x,y),\ k=1,\dots,r)$. Moreover let $\mathcal{K}_1$ be the class of all vectorial continuous functions $z=[z_i(x,y),i=1,\dots,m]$, satisfying the following conditions

(2.24) $$|z_i(x,y)|\leq\Omega,$$

(2.25) $$|z_i(x,y)-z_i(x,\bar{y})|\leq Q|y-\bar{y}|,$$

(2.26) $$|z_i(x,y)-z_i(\bar{x},y)|\leq|\int_x^{\bar{x}}\chi(t)dt|$$

for all (x,y), $(x,\bar{y})$, $(\bar{x},y)\in D_a$.

(c) *The characteristic system of* (C).

Let us consider the characteristic integral system (Δ) of problem (C)

(Δ.1) $$g_{ik}(\xi;x,y)=y_k-\int_{\xi}^{x}\rho_{ik}(t,\tilde{g}_i(t;x,y),z(t,\tilde{g}_i(t;x,y)))dt \text{ a.e. in } \Delta_a$$

(Δ.2) $$u_i(x,y)=\phi_i(y)+\sum_{s=1}^{m}\alpha_{si}(x,y,z(x,y))\cdot$$
$$\cdot\Big\{\sum_{h=1}^{m}A_{sh}(0,\tilde{g}_s(0;x,y),z(0,\tilde{g}_s(0;x,y)))\phi_h(\tilde{g}_s(0;x,y))+$$
$$-\sum_{h=1}^{m}A_{sh}(x,\tilde{g}_s(x;x,y),z(x,\tilde{g}_s(x;x,y)))\phi_h(\tilde{g}_s(x;x,y))+$$
$$+\int_0^x[\sum_{h=1}^{m}(dA_{sh}(\xi,\tilde{g}_s(\xi;x,y),z(\xi,\tilde{g}_s(\xi;x,y)))/d\xi)\cdot$$
$$\cdot u_h(\xi,\tilde{g}_s(\xi;x,y))+$$
$$+f_s(\xi,\tilde{g}_s(\xi;x,y),z(\xi,\tilde{g}_s(\xi;x,y)))]d\xi\Big\} \text{ a.e. in } D_a$$

$$(\Delta.3) \qquad z_i(x,y) = u_i(x,y) \quad \text{everywhere in } D_a$$

with initial data (Δ')

$$(\Delta'.1) \qquad g_{ik}(x;x,y) = y_k, \ (x,y) \in D_a,$$

$$(\Delta'.2) \qquad u_i(0,y) = \phi_i(y), \ y \in E^r,$$

$$(\Delta'.3) \qquad z_i(0,y) = \phi_i(y), \ y \in E^r, \ i = 1, \dots, m, \ k = 1, \dots, r.$$

If (g,u,z), $g \in \mathcal{K}_0$, $u, z \in \mathcal{K}_1$, is a solution of (Δ) with initial data (Δ'), then z is a solution of Cauchy problem (C) (see [13]). To prove the existence of a unique solution of problem $(\Delta) - (\Delta')$, we introduce three transformations which we will prove to be contractions with respect to uniform topology.

(d) *The tranformation* T_z. Let $z \in K_1$ be fixed. We define in $\mathcal{K}_0$ the transformation T_z by $G = T_z g$, $g \in \mathcal{K}_0$ where

$$G_{ik}(\xi;x,y) = y_k - \int_{\xi}^{x} \rho_{ik}(t, \tilde{g}_i(t;x,y), z(t, \tilde{g}_i(t;x,y)))dt.$$

At first, we prove that T_z has range in $\mathcal{K}_0$. The continuity and the properties (2.21), (2.22) are straightforward consequences of the definition of T_z. Moreover by (2.15), (2.23), (2.25)

$$|G_{ik}(\xi;x,y) - G_{ik}(\xi;x,\bar{y}) - y_k - \bar{y}_k| \le ((\mu+\eta) + \eta Q)L_a(1+p)|y-\bar{y}|$$

and so, for a sufficiently small a, $0 < a \le a_0$, such that

$$(2.27) \qquad (\mu + \eta(1+Q))L_a(1+p) \le p$$

the property (2.23) holds.

As shown in [10] in the proof of theorem I, step (c), to prove that the transformation T_z has a unique fixed point we show that $||G - G'|| \le k||g - g'||$, where $0 < k < 1$, $G = T_z g$, $G' = T_z g'$, g, $g' \in \mathcal{K}_0$.

From (2.15), (2.25)

$$|G_{ik}(\xi;x,y) - G'_{ik}(\xi;x,y)| \le (\mu + \eta(1+Q))L_a||g - g'||$$

for every $(\xi; x, y) \in \Delta_a$. So, if a, $0 < a \leq a_0$ is sufficiently small such that

$$(\mu + \eta(1 + Q))L_a \leq k < 1 \tag{2.28}$$

the transformation T_z has a unique fixed point $g = g[z]$.

Each component g_{ik} $(\xi; x, y)$ of $g[z]$ is absolutely continuous in x for every (ξ, x).

Indeed, for every $(\xi; x, y)$, $(\xi; \bar{x}, y) \in \Delta_a$ we have

$$\begin{aligned} |g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)| &\leq (\mu + \eta)|\int_x^{\bar{x}} m(t)dt| + \\ &+ |\int_\xi^x [(\mu + \eta)1(t)|\tilde{g}_i(t; x, y) - \tilde{g}_i(t; \bar{x}, y)| + \\ &+ \eta 1(t)Q|\tilde{g}_i(t; x, y) - \tilde{g}_i(t; \bar{x}, y)|]dt| \end{aligned} \tag{2.29}$$

where we have used (2.14), (2.15) and (2.25).

If, for every fixed $x, \bar{x}, y$ and i, we define

$$\delta = \max_k \max_{0 \leq \xi \leq a} |g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, a)|$$

from (2.29) it follows that

$$\delta \leq (\mu + \eta) \left| \int_x^{\bar{x}} m(t)dt \right| + ((\mu + \eta) + \eta Q)L_a \delta$$

or

$$|g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)| \leq (\mu + \eta)\lambda \left| \int_x^{\bar{x}} m(t)dt \right| \tag{2.30}$$

where we have put

$$\lambda = [1 - (\mu + \eta(1 + Q))L_a]^{-1}. \tag{2.31}$$

Finally, we prove that $g[z]$ depends continuously on z in $\mathcal{K}_1$. Let $z, z' \in \mathcal{K}_1$ and let $g = g[z]$, $g' = g[z']$. By (2.15), (2.25) we have

$$|g_{ik}(\xi; x, y) - g'_{ik}(\xi; x, y)| \leq (\mu + \eta)L_a||g - g'|| + \eta L_a(Q||g - g'|| + ||z - z'||)$$

and so

$$\|g - g'\| \leq \eta L_a \lambda \|z - z'\|. \tag{2.32}$$

(e) *The transformation* $T^*_{z\phi}$; Let $\mathcal{I}$ be the class of all functions $\phi(y) = (\phi_1, \dots, \phi_m)$, $y \in E^r$, which satisfy (2.7). Moreover, let $\mathcal{K}_{1\phi}$ be the class of all systems $z = (z_1, \dots, z_m)$, such that $z \in \mathcal{K}_1$ and

$$z_i(0, y) = \phi_i(y), \; i = 1, \dots, m, \; y \in E^r. \tag{2.33}$$

For every fixed $z \in \mathcal{K}_1$, let $g = g[z]$ be the corresponding fixed point of transformation T_z. Now, fixed $z \in \mathcal{K}_1$ and $\phi \in \mathcal{I}$, we consider the linear transformation $U = T^*_{z\phi} u$ defined in $\mathcal{K}_{1\phi}$ by

$$U_i(x, y) = \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \cdot$$

$$\cdot \left\{ \sum_{h=1}^{m} A_{sh}(0, \tilde{g}_s(0; x, y), z(0, \tilde{g}_s(0; x, y)))\phi_h(\tilde{g}_s(0; x, y)) + \right.$$

$$+ \int_0^x \left[\sum_{h=1}^{m} (dA_{sh}(t, \tilde{g}_s(t; x, y), z(t, \tilde{g}_s(t; x, y)))/dt) u_h(t, \tilde{g}_s(t; x, y)) + \right.$$

$$\left. \left. + f_s(t, \tilde{g}_s(t; x, y), z(t, \tilde{g}_s(t; x, y))) \right] dt \right\}, (x, y) \in D_a, i = 1, \dots, m.$$

First we prove that $T^*_{z\phi}$ has range in $\mathcal{K}_{1\phi}$. From the definition of $T^*_{z\phi}$ it is straightforward that the function $U = T^*_{z\phi} u$ satisfies the property (2.33). To prove the properties (2.24)-(2.26) is useful to consider the following equivalent definition of $T^*_{z\phi}$ (see [10]):

$$U_i(x, y) = \phi_i(y) + \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \, (\Delta_{s1} + \Delta_{s2} + \Delta_{s3}) \tag{2.34}$$

where

$$\Delta_{s1}(x, y) = \int_0^x f_s(t, \tilde{g}_s(t; x, y), z(t, \tilde{g}_s(t; x, y)))dt,$$

$$\Delta_{s2}(x,y) = \sum_{h=1}^{m} A_{sh}(0, \tilde{g}_s(0;x,y), \tag{2.35}$$
$$z(0, \tilde{g}_s(0;x,y))) \cdot [\phi_h(\tilde{g}_s(0;x,y)) - \phi_h(\tilde{g}_s(x;x,y))],$$

$$\Delta_{s3}(x,y) = \int_0^x \sum_{h=1}^{m} (dA_{sh}(t, \tilde{g}_s(t;x,y), z(t, \tilde{g}_s(t;x,y)))/dt) \cdot$$
$$\cdot [u_h(t, \tilde{g}_s(t;x,y)) - \phi_h(\tilde{g}_s(x;x,y))] dt.$$

By the usual chain rule and properties (2.12), (2.13), (2.22), (2.25), (2.26) we obtain

$$|dA_{sh}/dt| \leq (\mu+\eta)\mathring{m}(t) + (\mu+\eta)rC(\mu+\eta+m\eta Q)m(t) + m\eta C\chi(t). \tag{2.36}$$

Again by using the chain rule and the properties (2.22), (2.25), (2.26) we have

$$|du_h/dt| \leq \chi(t) + (\mu+\eta)rQm(t). \tag{2.37}$$

Moreover, put

$$\Delta' = u_h(\bar{t}, \tilde{g}_s(\bar{t};x,y)) - \phi_h(\tilde{g}_s(x;x,y)) =$$
$$= [u_h(\bar{t}, \tilde{g}_s(\bar{t};x,y)) - u_h(0, \tilde{g}_s(\bar{t};x,y))] +$$
$$+ [u_h(0, \tilde{g}_s(\bar{t};x,y)) - u_h(0, \tilde{g}_s(x;x,y))]$$

by the properties (2.22), (2.25), (2.26) we have

$$|\Delta'| \leq \Xi_a + (\mu+\eta)QM_a. \tag{2.38}$$

So, by (2.7), (2.11), (2.14), (2.36) and (2.38)

$$|\Delta_{s1}| \leq (\mu+\eta)N_a, \quad |\Delta_{s2}| \leq m(\mu+\eta)^2 H\Lambda M_a$$
$$|\Delta_{s3}| \leq m[(\mu+\eta)(\mathring{M}_a + rC(\mu+\eta+m\eta Q)M_a) + \tag{2.39}$$
$$+ m\eta C\Xi_a](\Xi_a + (\mu+\eta)QM_a)$$

and hence

$$|U_i(x,y)| \leq \omega + mH'\{(\mu+\eta)N_a + m(\mu+\eta)^2 H\Lambda M_a +$$
$$+ m[(\mu+\eta)(\mathring{M}_a + rC(\mu+\eta+m\eta Q)M_a) +$$
$$+ m\eta C\Xi_a](\Xi_a + (\mu+\eta)QM_a)\}.$$

If we assume a sufficiently small in order that

(2.40)
$$mH'\{(\mu+\eta)N_a + m(\mu+\eta)^2 H\Lambda M_a + \\ + m[(\mu+\eta)\cdot(\mathring{M}_a + rC(\mu+\eta+m\eta Q)M_a) + m\eta C\Xi_a](\Xi_a + \\ + (\mu+\eta)QM_a)\} \leq \Omega - \omega$$

then the function $U(x,y)$ satisfies (2.24).

By using (2.34) we can write

$$U_i(x,y) - U_i(x,\bar{y}) = \delta_0 + \delta_0' + \delta_1 + \delta_2 + \delta_3$$

Where

$$\delta_0 = \phi_i(y) - \phi_i(\bar{y}),$$

$$\delta_0' = \sum_{s=1}^{m}[\alpha_{si}(x,y,z(x,y)) - \alpha_{si}(x,\bar{y},z(x,\bar{y}))](\Delta_{s1}(x,y) + \\ + \Delta_{s2}(x,y) + \Delta_{s3}(x,y)),$$

$$\delta_1 = \sum_{s=1}^{m}\alpha_{si}(x,\bar{y},z(x,\bar{y}))[\Delta_{s1}(x,y) - \Delta_{s1}(x,\bar{y})].$$

$$\delta_2 = \sum_{s=1}^{m}\alpha_{si}(x,\bar{y},z(x,\bar{y}))[\Delta_{s2}(x,y) - \Delta_{s2}(x,\bar{y})],$$

$$\delta_3 = \sum_{s=1}^{m}\alpha_{si}(x,\bar{y},z(x,\bar{y}))[\Delta_{s3}(x,y) - \Delta_{s3}(x,\bar{y})].$$

By (2.7)

$$|\delta_0| \leq \Lambda|y-\bar{y}|.$$

By the properties (2.17) and (2.39)

$$|\delta_0'| \le mC'(1+Q)[(\mu+\eta)N_a + m(\mu+\eta)^2 H\Lambda M_a + m[(\mu+\eta)(\mathring{M}_a + rC(\mu+\eta+m\eta Q)M_a) + m\eta C\Xi_a] \cdot (\Xi_a + (\mu+\eta)QM_a)]|y-\bar{y}|.$$

By using (2.15), (2.16), (2.23) and (2.25)

$$|\delta_1| \le mH'(1+p)(\mu+\eta+\eta Q)L_{1a}|y-\bar{y}|.$$

Moreover, by (2.7), (2.11), (2.12), (2.16), (2.22), (2.23) and (2.25)

$$|\delta_2| \le m^2 H'(\mu+\eta)[H\Lambda(2+p) + C\Lambda(\mu+\eta+\eta Q)(1+p)M_a]|y-\bar{y}|.$$

Finally, by (2.7), (2.12), (2.16), (2.23), (2.25) (2.26), (2.35), (2.36) and (2.37) and integration by parts we have

$$|\delta_3| \le m^2 H'\{(\mu+\eta+\eta Q)C\Xi_a + (\mu+\eta+\eta Q)C(1+p)(\mu+\eta)QM_a + (\mu+\eta+\eta Q)C(1+p)(\Xi_a + rQM_a) + ((\mu+\eta)\mathring{M}_a + (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a + m\eta C\Xi_a)\cdot(Q(1+p)+\Lambda)\}|y-\bar{y}|.$$

Combining the previous estimates we have

$$|U_i(x,y) - U_i(x,\bar{y})| \le [\gamma_1 N_a + \gamma_2 \mathring{M}_a + \gamma_3 L_{1a} + \gamma_4 M_a + \gamma_5 \Xi_a + \Lambda(1+(\mu+\eta)m^2 H'H(2+p))]|y-\bar{y}|,$$

where

$$\gamma_1 = (\mu+\eta)mC'(1+Q),$$

$$\gamma_2 = (\mu+\eta)m^2[C'(1+Q)(\Xi_a + (\mu+\eta)QM_a) + H'(Q(1+p)+\Lambda)],$$

$$\gamma_3 = mH'(1+p)(\mu+\eta+\eta Q),$$

$$\begin{aligned}\gamma_4 = (\mu+\eta)m^2[&C'\Lambda H(1+Q) + rCC'(1+Q)(\mu+\eta+m\eta Q)\cdot \\ &\cdot(\Xi_a + (\mu+\eta)QM_a) + H'C\Lambda(\mu+\eta+\eta Q)(1+p) + \\ &+ (r+1)H'CQ(\mu+\eta+\eta Q)(1+p) + rH'C(\mu+\eta+m\eta Q)\cdot \\ &\cdot(Q(1+p)+\Lambda)],\end{aligned} \tag{2.41}$$

$$\gamma_5 = m^2[m\eta CC'(1+Q)(\Xi_a + (\mu+\eta)QM_a) + H'C(\mu+\eta+\eta Q)(2+p) + mH'\eta C(Q(1+p)+\Lambda)].$$

Hence, if a is sufficiently small in order that

$$(2.42)\quad \gamma_1 N_a+\gamma_2\mathring{\mathrm{M}}_a+\gamma_3 L_{1a}+\gamma_4 M_a+\gamma_5\Xi_a \le Q-\Lambda(1+(\mu+\eta)m^2H'H(2+p))$$

then the function $U(x,y)$ satisfies the property (2.25).

In order to prove that $U(x,y)$ satisfies (2.26) we put

$$U_i(x,y)-U_i(\bar{x},y)=\sigma_0+\sigma_1+\sigma_2+\sigma_3$$

where

$$(2.43)\quad \begin{aligned}\sigma_0 &= \sum_{s=1}^{m}[\alpha_{si}(x,y,z(x,y))-\alpha_{si}(\bar{x},y,z(\bar{x},y))]\cdot\\ &\quad\cdot(\Delta_{s1}(x,y)+\Delta_{s2}(x,y)+\Delta_{s3}(x,y))\\ \sigma_j &= \sum_{s=1}^{m}\alpha_{si}(\bar{x},y,z(\bar{x},y))(\Delta_{sj}(x,y)-\Delta_{sj}(\bar{x},y)),\, j=1,2,3.\end{aligned}$$

By (2.17), (2.18), (2.26) and (2.39) we have

$$\begin{aligned}|\sigma_0| &\le m[(\mu+\eta)N_a+mH\Lambda(\mu+\eta)^2M_a+m((\mu+\eta)\cdot\\ &\quad\cdot(\mathring{\mathrm{M}}_a+rC(\mu+\eta+m\eta Q)M_a)+m\eta C\Xi_a)\cdot\\ &\quad\cdot(\Xi_a+(\mu+\eta)QM_a)]\cdot\left(\left|\int_x^{\bar{x}}\mathring{\mathrm{m}}'(t)dt\right|+C'\left|\int_x^{\bar{x}}\chi(t)dt\right|\right).\end{aligned}$$

By the relations (2.14), (2.15), (2.16) and (2.30)

$$|\sigma_1|\le mH'(\mu+\eta)\left[\lambda(\mu+\eta+\eta Q)L_{1a}\left|\int_x^{\bar{x}}m(t)dt\right|+\left|\int_x^{\bar{x}}n(t)dt\right|\right].$$

By (2.7), (2.8), (2.11), (2.12), (2.16), (2.22), (2.30) and (2.35)

$$\begin{aligned}|\sigma_2| &\le m^2H'(\mu+\eta)\Bigg[H\Lambda(\mu+\eta)\lambda\left|\int_x^{\bar{x}}m(t)dt\right|+\\ &\quad+C(\mu+\eta+\eta Q)\Lambda\lambda M_a\left|\int_x^{\bar{x}}m(t)dt\right|\Bigg].\end{aligned}$$

Finally, by (2.12), (2.22), (2.25), (2.30), (2.36) and (2.37) and integration by parts

$$|\sigma_3| \leq m^2 H' \Big\{ (\Xi_a +$$

$$+ QM_a) \left| \int_x^{\bar{x}} [(\mu+\eta)\mathring{\mathrm{m}}(t) + (\mu+\eta) r C(\mu +\right.$$

$$\left. + \eta + m\eta Q) m(t) + m\eta C \chi(t)] dt \right| +$$

$$+ 2C(\mu+\eta+\eta Q)(\mu+\eta)\lambda(\Xi_a + (\mu+\eta) r Q M_a) \left| \int_x^{\bar{x}} m(t) dt \right| +$$

$$+ Q\lambda(\mu+\eta)[(\mu+\eta)\mathring{\mathrm{M}}_a +$$

$$+ (\mu+\eta) r C(\mu+\eta+m\eta Q) M_a + m\eta C \Xi_a] \left| \int_x^{\bar{x}} m(t) dt \right| \Big\}.$$

Combining the previous estimates we deduce

$$|U_i(x,y) - U_i(\bar{x},y)| \leq mH'(\mu+\eta) \left| \int_x^{\bar{x}} n(t) dt \right| +$$

$$+ m^2 H' H (\mu+\eta)^2 \Lambda\lambda \left| \int_x^{\bar{x}} m(t) dt \right| + \gamma_1' \left| \int_x^{\bar{x}} \mathring{\mathrm{m}}(t) dt \right| +$$

$$+ \gamma_2' \left| \int_x^{\bar{x}} \mathring{\mathrm{m}}'(t) dt \right| + \gamma_3' \left| \int_x^{\bar{x}} m(t) dt \right| + \gamma_0' \left| \int_x^{\bar{x}} \chi(t) dt \right|,$$

where

$$\begin{aligned}
\gamma_1' &= m^2H'(\mu+\eta)(\Xi_a+QM_a),\\
\gamma_2' &= m[(\mu+\eta)N_a+mH\Lambda(\mu+\eta)^2M_a+m((\mu+\eta)\cdot\\
&\quad\cdot(\mathring{\mathrm{M}}_a+rC(\mu+\eta+m\eta Q)M_a)+m\eta C\Xi_a)\cdot\\
&\quad\cdot(\Xi_a+(\mu+\eta)QM_a)],\\
\gamma_3' &= mH'(\mu+\eta)\lambda(\mu+\eta+\eta Q)L_{1a}+\\
&\quad+m^2H'(\mu+\eta)C(\mu+\eta+\eta Q)\Lambda\lambda M_a+\\
&\quad+rm^2H'(\mu+\eta)C\cdot(\mu+\eta+m\eta Q)(\Xi_a+QM_a)+\\
&\quad+2m^2H'C(\mu+\eta+\eta Q)(\mu+\eta)\lambda(\Xi_a+(\mu+\eta)rQM_a)+\\
&\quad+m^2H'Q\lambda(\mu+\eta)[(\mu+\eta)\mathring{\mathrm{M}}_a+\\
&\quad+(\mu+\eta)rC(\mu+\eta+m\eta Q)M_a+m\eta C\Xi_a],\\
\gamma_0' &= mC'[(\mu+\eta)N_a+mH\Lambda(\mu+\eta)^2M_a+\\
&\quad+m((\mu+\eta)(\mathring{\mathrm{M}}_a+rC(\mu+\eta+m\eta Q)M_a)+\\
&\quad+m\eta C\Xi_a)\cdot(\Xi_a+(\mu+\eta)QM_a)]+m^3H'\eta C(\Xi_a+QM_a).
\end{aligned}\tag{2.44}$$

From the relation (2.20) we derive

$$1-R_0^{-1}mH'(\mu+\eta)>0,\quad 1-R_3^{-1}m^2H'H\Lambda\lambda(\mu+\eta)^2>0$$

so we can take a sufficiently small such that

$$\begin{aligned}
&\gamma_0'<1-R_0^{-1}mH'(\mu+\eta),\ \gamma_0'<1-R_3^{-1}m^2H'H\Lambda\lambda(\mu+\eta)^2,\\
&\gamma_1'\le(1-\gamma_0')R_1,\ \gamma_2'\le(1-\gamma_0')R_2,\\
&\gamma_3'\le(1-\gamma_0')R_3-m^2H'H(\mu+\eta)^2\Lambda\lambda.
\end{aligned}\tag{2.45}$$

Then by (2.44), (2.45) and the definition of the function $\chi(t)$ we obtain

$$\begin{aligned}
|U_i(x,y)-U_i(\bar{x},y)| &\le \left|\int_x^{\bar{x}}[R_0n(t)+R_1\mathring{\mathrm{m}}(t)+R_2\mathring{\mathrm{m}}'(t)+R_3m(t)]dt\right| =\\
&=\left|\int_x^{\bar{x}}\chi(t)dt\right|.
\end{aligned}$$

So, we have proved that $T^*_{z\phi}$ maps $\mathcal{K}_{1\phi}$ into itself.

Let us prove that $T^*_{z\phi}$ is a contraction. Let $z \in \mathcal{K}_{1\phi}$ and $g = g[z]$ be the relative fixed point of T_z. Let $U = T^*_{z\phi}u$, $U' = T^*_{z\phi}u'$, where u, $u' \in \mathcal{K}_{1\phi}$.

By (2.36) we have

$$|U_i(x,y) - U'_i(x,y)| \leq \gamma ||u - u'||, i = 1, \dots, m, \text{ where}$$

$$\gamma = m^2 H'[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta) rC(\mu+\eta+m\eta Q)M_a + m\eta C\Xi_a]. \tag{2.46}$$

Then

$$||U - U'|| \leq \gamma ||u - u'||$$

and so by taking a sufficiently small in order that

$$\gamma \leq k < 1 \tag{2.47}$$

the transformation $T^*_{z\phi}$ is a contraction in $\mathcal{K}_{1\phi}$. Therefore $T^*_{z\phi}$ has a unique fixed point $u = u[z,\phi]$, which is the solution of the integral equation (Δ.2) with initial data (Δ'.2).

Let us prove that this solution $u = u[z,\phi]$ is a continuous function of z and ϕ.

Let $z, z' \in \mathcal{K}_1$ and ϕ, $\phi' \in \mathcal{I}$. Let $g = g[z]$, $g' = g[z']$ be the corresponding elements of $\mathcal{K}_0$, $g = T_z g$, $g' = T_{z'}g'$ and let $u = u[z,\phi]$, $u' = u[z',\phi']$ be the corresponding elements of $\mathcal{K}_{1\phi}$, $u = T^*_{z\phi}u$, $u' = T^*_{z'\phi'}u'$. With these notations we have $u(x,y) - u'(x,y) = \phi(y) - \phi'(y) + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, where

$$\begin{aligned} \varepsilon_0 &= \sum_{s=1}^{m} [\alpha_{si}(x,y,z(x,y)) - \alpha_{si}(x,y,z'(x,y))][\Delta_{s1}(x,y) + \\ &\quad + \Delta_{s2}(x,y) + \Delta_{s3}(x,y)] \\ \varepsilon_j &= \sum_{s=1}^{m} \alpha_{si}(x,y,z'(x,y))(\Delta_{sj}(x,y) - \Delta'_{sj}(x,y)), \ j = 1,2,3. \end{aligned} \tag{2.48}$$

By (2.17) and (2.39)

$$\begin{aligned} |\varepsilon_0| \leq mC'\{(\mu+\eta)N_a + m(\mu+\eta)^2 H\Lambda M_a + m[(\mu+\eta)(\mathring{\mathrm{M}}_a + \\ + rC(\mu+\eta+m\eta Q)M_a) + m\eta C\Xi_a] \cdot (\Xi_a + (\mu+\eta)QM_a)\}||z - z'||. \end{aligned}$$

By (2.15), (2.16), (2.25) and (2.32)

$$|\varepsilon_1| \leq mH'\eta L_{1a}[1 + (\mu + \eta + \eta Q)\lambda L_a]||z - z'||$$

and by (2.7), (2.11), (2.12), (2.16), (2.22) and (2.32)

$$|\varepsilon_2| \leq 2m^2(\mu + \eta)H'H||\phi - \phi'|| + m^2 H'\Lambda[2(\mu + \eta)H\lambda L_a\eta + \\ + \eta C(1 + (\mu + \eta + \eta Q)\lambda L_a)M_a]||z - z'||.$$

Moreover by (2.12), (2.22), (2.25), (2.26), (2.36), (2.37), (2.39) and integration by parts

$$|\varepsilon_3| \leq m^2 H'\{\eta C\Xi_a + 2C(\eta + (\mu + \eta + \eta Q)\eta\lambda L_a)(\Xi_a + (\mu + \eta)rQM_a) + \\ + [(\mu + \eta)\mathring{\mathrm{M}}_a + (\mu + \eta)rC(\mu + \eta + m\eta Q)M_a + m\eta C\Xi_a]Q\eta\lambda L_a\}||z - z'|| + \\ + m^2 H'[(\mu + \eta)\mathring{\mathrm{M}}_a + (\mu + \eta)rC(\mu + \eta + m\eta Q)M_a + \\ + m\eta C\Xi_a](||u - u'|| + ||\phi - \phi'||).$$

Combining the previous estimates we have

$$|u_i(x, y) - u_i'(x, y)| \leq \gamma||u - u'|| + (1 + \gamma + 2m^2(\mu + \eta)HH')||\phi - \phi'|| + \bar{\gamma}||z - z'||$$

where γ is defined in (2.46) and

$$\bar{\gamma} = mC'\{(\mu + \eta)N_a + m(\mu + \eta)^2 H\Lambda M_a + \\ + m[(\mu + \eta)(\mathring{\mathrm{M}}_a + rC(\mu + \eta + m\eta Q)M_a) + \\ + m\eta C\Xi_a](\Xi_a + (\mu + \eta)QM_a)\} + m\eta H'L_{1a}[1 + \\ + (\mu + \eta + \eta Q)\lambda L_a] + m^2 H'\Lambda[2\eta(\mu + \eta)H\lambda L_a + \eta C(1 + \\ + (\mu + \eta + \eta Q)\lambda L_a)M_a] + \\ + m^2 H'\{\eta C\Xi_a + 2\eta C(1 + (\mu + \eta + \eta Q)\lambda L_a)(\Xi_a + \\ + (\mu + \eta)rQM_a) + [(\mu + \eta)\mathring{\mathrm{M}}_a + (\mu + \eta)rC(\mu + \eta + m\eta Q)M_a + \\ + m\eta C\Xi_a]Q\eta\lambda L_a\}. \tag{2.49}$$

Then by (2.47), if we assume $a > 0$ sufficiently small such that

$$(1 - \gamma)^{-1}\bar{\gamma} \leq k < 1 \tag{2.50}$$

the estimates above yield

$$\begin{aligned}(2.51)\qquad &\|u[z,\phi]-u[z',\phi']\| \le \\ &\le (1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH')\|\phi-\phi'\| + k\|z-z'\|\end{aligned}$$

i.e. the requested continuity of $u[z,\phi]$ with respect to z and ϕ.

(f) *The transformation* $\mathcal{T}_\phi$. Since $\mathcal{K}_{1\phi} \subset \mathcal{K}_1$, fixed $\phi \in \mathcal{I}$ we can define the transformation $\mathcal{T}_\phi : \mathcal{K}_{1\phi} \longrightarrow \mathcal{K}_{1\phi}$ by $\mathcal{T}_\phi z = u[z,\phi]$, i.e. $\mathcal{T}_\phi z$ is the fixed point of operator $T^*_{z\phi}$. The transformation $\mathcal{T}_\phi$ so defined is a contraction in the uniform topology of $\mathcal{K}_{1\phi}$. Indeed, put $u = \mathcal{T}_\phi z$, $u' = \mathcal{T}_\phi z'$, z, $z' \in \mathcal{K}_{1\phi}$, $u = u[z,\phi]$, $u' = u\ [z',\phi]$, by (2.51) we have

$$\|u-u'\| = \|u[z,\phi] - u'[z',\phi]\| \le k\|z-z'\|,\ 0<k<1.$$

Thus, for every $\phi \in \mathcal{I}$, the functions $z = z[\phi] \in \mathcal{K}_{1\phi}$, $u = u[z,\phi] \in \mathcal{K}_{1\phi}$, $g = g[z] \in \mathcal{K}_0$ represent the unique solution of the integral system (Δ) with inital data (Δ'). So, $z = z[\phi]$ is the unique solution of the Cauchy problem (C).

Finally, let us prove that $z[\phi]$ depends continuously on $\phi \in \mathcal{I}$. Let $\phi, \phi' \in \mathcal{I}$ and $z = z[\phi]$, $z' = z[\phi']$.

By (2.51) we derive

$$\begin{aligned}\|z-z'\| = \|z[\phi]-z[\phi']\| = \|u[z,\phi]-u[z',\phi']\| \le \\ \le k\|z-z'\| + (1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH')\|\phi-\phi'\|\end{aligned}$$

where $0<k<1$ and so

$$(2.52)\qquad \|z-z'\| \le (1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH')\|\phi-\phi'\|.$$

Let us remark that the restrictions we had to impose on the size of a, $0<a\le a_0$, are relations (2.27), (2.28), (2.40), (2.42), (2.45), (2.47) and (2.50). If $\mu = 0$ and $\eta = 1$ the above relations are exactly the same of theorem I in [10].

3. An estimate of approximation error for the Cauchy problem.

In this section we will refer to Cauchy problem (C) as problem (C,η). In this way we point out the dependence of the hyperbolic differential system (c.1) on parameter η, $\eta \geq 0$. Here the parameter $\mu, \mu > 0$ is considered as a fixed number. The solution of problem (C,η) will be denoted by $z^{(\eta)}$. Thus, for $\eta = 0$, $z^{(0)}$ is the solution of the Cauchy problem relative to the linear system

$$(3.1) \qquad \sum_{j=1}^{m} \mu B_{ij}(x,y)\left[\frac{\partial z_j}{\partial x} + \sum_{k=1}^{r} \mu\theta_{ik}(x,y)\frac{\partial z_j}{\partial y_k}\right] = \mu v_i(x,y)$$

with inital data (c.2).

Applying the successive approximation method provided by theorem I in [10], we obtain an estimate of $||z^{(\eta)} - z^{(0)}||$ with respect to the parameter of nonlinearity η. This estimate is well-defined for every value of $\eta > 0$ because both the solution $z^{(\eta)}$ and $z^{(0)}$ are defined in the slab $[0, a(\eta)] \times E^r$, where $a(\eta)$ is the size of a, $0 < a \leq a_0$ in the Cauchy problem (C,η). Indeed from inequalities (2.27), (2.28), (2.40), (2.42), (2.45), (2.47) and (2.50) it follows that $a = a(\eta)$ is a nonincreasing function of η, so that $a(\eta) \leq a(0)$ for every $\eta > 0$. Let us point out that in the present Section we will suppose that the constant H' introduced in (2.16) of Section 2, satisfies the inequality $||(B_{ij})^{-1}|| \leq \mu H'$, too.

Let $z \in \mathcal{K}_1$ be fixed. Here we name $T_z^{(\eta)}$ the transformation defined in step (d) of theorem 1: so we point out its dependence on η, $\eta \geq 0$. Moreover let $g^{(\eta)}$, $n \geq 0$ be the fixed point of the transformation $T_z^{(\eta)}$. For every $\eta \geq 0$ and for arbitrarily fixed $g \in \mathcal{K}_0$, $z \in \mathcal{K}_1$ let us inductively define $g^{(\eta,0)} = g$, $g^{(\eta,n+1)} = T_z^{(\eta)}\ g^{(\eta,n)}$. Thus we have $g^{(\eta)} = \lim_{n\to+\infty} g^{(\eta,n)}$ without respect to the choice of function $g = g^{(\eta,0)} \in \mathcal{K}_0$.

At first, fixed arbitrarily $\eta > 0$, we estimate $||g^{(\eta)} - g^{(0)}||$. As we have remarked above, it is not restrictive to choose $g^{(0,0)} = g^{(\eta,0)} = g \in \mathcal{K}_0$.

For every $(\xi; x, y) \in \Delta_a$, $i = 1, \ldots, m$, $k = 1, \ldots, r$, we have

$$|g_{ik}^{(\eta,1)}(\xi; x,y) - g_{ik}^{(0,1)}(\xi; x,y)| \leq$$
$$\leq \eta \left| \int_{\xi}^{x} \lambda_{ik}(t, \tilde{g}_i(t;x,y), z(t, \tilde{g}_i(t;x,y)))dt \right| \leq \eta M_a$$

and so

$$(3.2) \qquad ||g^{(\eta,1)} - g^{(0,1)}|| \leq \eta M_a.$$

For $q > 1$, $(\xi; x, y) \in \Delta_a$, $i = 1, \ldots, m$, $k = 1, \ldots, r$, we have

$$|g_{ik}^{(\eta,q)}(\xi; x, y) - g_{ik}^{(0,q)}(\xi; x, y)| \leq \mu \left| \int_{\xi}^{x} [\theta_{ik}(t, \tilde{g}_i^{(\eta,q-1)}(t; x, y)) - \right.$$
$$\left. - \theta_{ik}(t, \tilde{g}_i^{(0,q-1)}(t; x, y)] dt \right| +$$
$$+ \eta \left| \int_{\xi}^{x} \lambda_{ik}(t, \tilde{g}_i(t, \tilde{g}_i^{(\eta,q-1)}(t; x, y), z(t, \tilde{g}_i^{(\eta,q-1)}(t; x, y))) dt \right| \leq$$
$$\leq \mu L_a ||g^{(\eta,q-1)} - g^{(0,q-1)}|| + \eta M_a$$

which leads to

$$||g^{(\eta,q)} - g^{(0,q)}|| \leq \mu L_a ||g^{(\eta,q-1)} - g^{(0,q-1)}|| + \eta M_a. \tag{3.3}$$

By recursion, from (3.2) and (3.3) we have

$$||g^{(\eta,q)} - g^{(0,q)}|| \leq \eta M_a \sum_{h=0}^{q-1} (\mu L_a)^h, \text{ for every } q \in \mathbb{N}^+.$$

As by (2.28), $\mu L_a < 1$, we obtain

$$||g^{(\eta)} - g^{(0)}|| \leq \eta M_a (1 - \mu L_a)^{-1} \tag{3.4}$$

which is the requested estimate relative to transformation T_z. We remark that in the linear case $\eta = 0$ the transformation $T_z^{(0)}$ does not depend on the choice of the function $z \in \mathcal{K}_1$. Therefore, the estimate (3.4) does not depend on the choice of $z \in \mathcal{K}_1$, too.

Let us estimate now $||u^{(\eta)} - u^{(0)}||$, where $u^{(\eta)}$, $\eta \geq 0$ is the fixed point in $\mathcal{K}_{1\phi}$ of transformation $T_{z\phi}^{*(\eta)}$, $z \in \mathcal{K}_1$, $\phi \in \mathcal{I}$. We have emphasized the dependence of the transformation $T_{z\phi}^*$, defined in step (e) of theorem 1, on parameter $\eta, \eta \geq 0$.

From the definition of $T_{z\phi}^*$ and (2.10) we have

$$U_i^{(\eta)}(x, y) = (T_{z\phi}^{*(\eta)} u(x, y))_i = \sum_{s=1}^{m} ((\mu B + \eta \Gamma)^{-1})_{si}^t (x, y, z(x, y)) \cdot \tag{3.5}$$
$$\cdot (S_\eta u)(x, y), \quad (x, y) \in D_a, i = 1, \ldots, m$$

where B, Γ denote the matrices $B = (B_{ij})$, $\Gamma = (\Gamma_{ij})$ and

$$
\begin{aligned}
(S_\eta u)(x,y) = & \sum_{h=1}^{m} (\mu B + \eta\Gamma)_{sh}(0, \tilde{g}_s^{(\eta)}(0; x, y), \\
& z(0, \tilde{g}_s^{(\eta)}(0; x, y)))\phi_h(\tilde{g}_s^{(\eta)}(0; x, y)) + \\
& + \int_0^x \left[\sum_{h=1}^{m} (d(\mu B + \eta\Gamma)_{sh}(t, \tilde{g}_s^{(\eta)}(t; x, y)), \right. \\
& z(t, \tilde{g}_s^{(\eta)}(t; x, y)))/dt) \cdot \\
& \cdot u_h(t, \tilde{g}_s^{(\eta)}(t; x, y)) + \mu v_s(t, \tilde{g}_s^{(\eta)}(t; x, y)) + \\
& \left. + \eta w_s(t, \tilde{g}_s^{(\eta)}(t; x, y)), z(t, \tilde{g}_s^{(\eta)}(t; x, y))) \right] dt.
\end{aligned} \tag{3.6}
$$

For every $\eta \geq 0$ and for every fixed $z \in \mathcal{K}_1$, $\phi \in \mathcal{I}$ let us define by induction $u^{(\eta,0)} = u \in \mathcal{K}_{1\phi}$, $u^{(\eta,n)} = T_{z\phi}^{*(\eta)} u^{(\eta,n-1)}$, $n \in \mathbb{N}^+$. Therefore, $u^{(\eta)} = \lim_{n\to+\infty} u^{(\eta,n)}$, $n \geq 0$, without respect to the choice of u in $\mathcal{K}_{1\phi}$.

By (3.5) and (3.6), for every $(x,y) \in D_a$, $i = 1, \ldots, m$, we have

$$
\begin{aligned}
|u_i^{(\eta,1)}(x,y) - u_i^{(0,1)}(x,y)| \leq & \sum_{s=1}^{m} [|((\mu B + \eta\Gamma)^{-1})_{si}^t(x, y, z(x,y)) - \\
& - ((\mu B)^{-1})_{si}^t(x,y)| \cdot |(S_\eta u)(x,y)| + \\
& + |((\mu B)^{-1})_{si}^t(x,y)| \cdot |(S_\eta u)(x,y) - (S_0 u)(x,y)|].
\end{aligned} \tag{3.7}
$$

From now on we suppose that

$$
||B^{-1}|| \leq \mu H' \tag{3.8}
$$

where H' is the same constant of (2.16). Then

$$
\begin{aligned}
||((\mu B + \eta\Gamma)^{-1})^t - ((\mu B)^{-1})^t|| = & ||(\mu B + \eta\Gamma)^{-1} - (\mu B)^{-1}|| \leq \\
\leq & ||((\mu B + \eta\Gamma)^{-1} - (\mu B)^{-1})(\mu B + \eta\Gamma)|| \, ||\alpha|| \leq \\
= & ||(\mu B)^{-1}\eta\Gamma|| \, ||\alpha|| \leq \eta H'^2 H.
\end{aligned} \tag{3.9}
$$

Here α is the matrix $\alpha = (\alpha_{ij})$. By using (2.2), (2.3), (2.22), (2.25), (2.26) and the chain rule, for every $(t,x,y) \in \Delta_a$ we have

$$
|dB_{sh}(t, \tilde{g}_s(t; x, y))/dt| \leq \mathring{m}(t) + rC(\mu + \eta)m(t), \tag{3.10}
$$

$$(3.11)\qquad \begin{aligned}&|d\Gamma_{sh}(t,\tilde{g}_s(t;x,y),z(t,\tilde{g}_s(t;x,y)))/dt| \le \mathring{\mathrm{m}}(t)+\\&+rC(\mu+\eta)m(t)(1+mQ)+mC\chi(t).\end{aligned}$$

So by (2.7), (2.11), (2.14), (2.24), (3.10) and (3.11) for every $(x,y)\in D_a$ we have

$$(3.12)\qquad \begin{aligned}|(S_\eta u)(x,y)| \le m\{&(\mu+\eta)H\omega+\mu(\mathring{\mathrm{M}}_a+rC(\mu+\eta)M_a)\Omega+\\&+\eta(\mathring{\mathrm{M}}_a+rC(\mu+\eta)M_a(1+mQ)+mC\Xi_a)\Omega\}+\\&+(\mu+\eta)N_a.\end{aligned}$$

By (3.8)

$$(3.13)\qquad ||((\mu B)^{-1})^t|| = ||(\mu B)^{-1}|| \le H'.$$

From relations (2.1), (2.2), (2.4), (2.6), (2.7), (2.25), (2.37), (3.10), (3,11) and integration by parts, for every $(x,y)\in D_a$ we have

$$(3.14)\qquad \begin{aligned}&|(S_\eta u)(x,y)-(S_0u)(x,y)| \le\\
&\le \sum_{h=1}^{m}\{|(\mu B+\eta\Gamma)_{sh}(0,\tilde{g}_s^{(\eta)}(0;x,y),z(0,\tilde{g}_s^{(\eta)}(0;x,y)))-\\
&-(\mu B)_{sh}(0,\tilde{g}_s^{(0)}(0;x,y))|\cdot|\phi_h(g_s^{(\eta)}(0;x,y))|+\\
&+|(\mu B)_{sh}(0,\tilde{g}_s^{(0)}(0;x,y))|\cdot|\phi_h(\tilde{g}_s^{(\eta)}(0;x,y))-\phi_h(\tilde{g}_s^{(0)}(0;x,y))|+\\
&+\mu\int_0^x |d(B_{sh}(t,\tilde{g}_s^{(\eta)}(t;x,y))-B_{sh}(t,\tilde{g}_s^{(0)}(t;x,y)))/dt|\cdot\\
&\cdot|u_h(t,\tilde{g}_s^{(\eta)}(t;x,y))|dt+\mu\int_0^x |dB_{sh}(t,\tilde{g}_s^{(0)}(t;x,y))/dt|\cdot\\
&\cdot|u_h(t,\tilde{g}_s^{(\eta)}(t;x,y))-u_h(t,\tilde{g}_s^{(0)}(t;x,y))|dt+\\
&+\eta\int_0^x |d\Gamma_{sh}(t,\tilde{g}_s^{(\eta)}(t;x,y),z(t,\tilde{g}_s(t;x,y)))/dt|\cdot\\
&\cdot|u_h(t,\tilde{g}_s^{(\eta)}(t;x,y))|dt\}+\mu\int_0^x |v_s(t,\tilde{g}_s^{(\eta)}(t;x,y))-v_s(t,\tilde{g}_s^{(0)}(t;x,y)|dt+\\
&+\eta\int_0^x |w_s(t,\tilde{g}_s^{(\eta)}(t;x,y),z(t,\tilde{g}_s^{(\eta)}(t;x,y)))|dt \le\end{aligned}$$

$$\leq \sum_{h=1}^{m} \{(\mu C||g^{(\eta)} - g^{(0)}|| + \eta H)\omega + \mu H\Lambda||g^{(\eta)} - g^{(0)}||+$$

$$+ \mu|B_{sh}(0, \tilde{g}_s^{(\eta)}(0; x, y) - B_{sh}(0, \tilde{g}_s^{(0)}(0; x, y))|\ |u_h(0, g_s^{(\eta)}(0; x, y))|+$$

$$+ \mu \int_0^x |B_{sh}(t, \tilde{g}_s^{(\eta)}(t; x, y)) - B_{sh}(t, \tilde{g}_s^{(0)}(t; x, y))|\ |du_h(t, \tilde{g}_s^{(\eta)}(t; x, y))/dt|dt+$$

$$+ \mu(\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a)Q||g^{(\eta)} - g^{(0)}|| + \eta(\mathring{\mathrm{M}}_a+$$

$$+ rC(\mu + \eta)M_a(1 + mQ) + mC\Xi_a)\Omega\} + \mu L_{1a}||g^{(\eta)} - g^{(0)}||+$$

$$+ \eta N_a \leq m\{(\mu C||g^{(\eta)} - g^{(0)}|| + \eta H)\omega + \mu H\Lambda||g^{(\eta)} - g^{(0)}||+$$

$$+ \mu C||g^{(\eta)} - g^{(0)}||\Omega + \mu C||g^{(\eta)} - g^{(0)}||(\Xi_a + (\mu + \eta)rQM_a)+$$

$$+ \mu(\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a)Q||g^{(\eta)} - g^{(0)}|| + \eta\Omega(\mathring{\mathrm{M}}_a+$$

$$+ rC(\mu + \eta)M_a(1 + mQ) + mC\Xi_a)\} + \mu L_{1a}||g^{(\eta)} - g^{(0)}|| + \eta N_a.$$

By (3.7), (3.9), (3.12), (3.13) and (3.14), for every $(x, y) \in D_a$, $i = 1, \ldots, m$, we deduce

$$|u_i^{(\eta,1)}(x, y) - u_i^{(0,1)}(x, y)| \leq \bar{\varepsilon}_1\ ||g^{(\eta)} - g^{(0)}|| + \bar{\varepsilon}_2$$

where

$$\bar{\varepsilon}_1 = m^2\mu H'[C\omega + H\Lambda + \c{C}\Omega+$$
$$+ C(\Xi_a + (\mu + \eta)rQM_a) + (\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a)Q] + m\mu H' L_{1a},$$
$$\bar{\varepsilon}_2 = m^2\eta H H'^2[(\mu + \eta)H\omega + \mu(\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a)\Omega+$$

$$+ \eta(\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a(1 + mQ) + mC\Xi_a)\Omega]+ \tag{3.15}$$
$$+ m\eta H H'^2(\mu + \eta)N_a+$$
$$+ m^2 H'[\eta H\omega + \eta\Omega(\mathring{\mathrm{M}}_a + rC(\mu + \eta)M_a(1 + mQ)+$$
$$+ mC\Xi_a)] + m\eta H' N_a$$

and so

$$||u^{(\eta,1)} - u^{(\eta,0)}|| \leq \bar{\varepsilon}_1||g^{(\eta)} - g^{(0)}|| + \bar{\varepsilon}_2. \tag{3.16}$$

Now let q be a positive integer. From (3.5) and (3.6), for every $(x,y) \in D_a$ and $i = 1, \dots, m$ we have

$$|u_i^{(\eta,q)}(x,y) - u_i^{(0,q)}(x,y)| \leq \sum_{s=1}^{m} [|((\mu B + \eta\Gamma)^{-1})_{si}^{t}(x,y,z(x,y)) -$$
$$- ((\mu B)^{-1})_{si}^{t}(x,y)| \cdot |(S_\eta u^{(\eta,q-1)})(x,y)| +$$
$$+ |((\mu B)^{-1})_{si}^{t}(x,y)| \, |(S_\eta u^{(\eta,q-1)})(x,y) - (S_0 u^{(0,q-1)})(x,y)|].$$

Only the last term has to be estimated.
In analogous way as we have obtained (3.14), we have

$$|(S_\eta u^{(\eta,q-1)})(x,y) - (S_0 u^{(0,q-1)})(x,y)| \leq$$
$$\leq \sum_{h=1}^{m} \{(\mu C ||g^{(\eta)} - g^{(0)}|| + \eta H)\omega + \mu H \Lambda ||g^{(\eta)} - g^{(0)}|| +$$
$$+ \mu \int_0^x |d(B_{sh}(t, \tilde{g}_s^{(\eta)}(t,x,y)) - B_{sh}(t, \tilde{g}_s^{(0)}(t;x,y))/dt| \cdot$$
$$\cdot |u_h^{(\eta,q-1)}(t, \tilde{g}_s^{(\eta)}(t;x,y))| dt + \mu \int_0^x |dB_{sh}(t, \tilde{g}_s^{(0)}(t;x,y))/dt| \cdot$$
$$\cdot [|u_h^{(\eta,q-1)}(t, \tilde{g}_s^{(\eta)}(t;x,y)) - u_h^{(\eta,q-1)}(t, \tilde{g}_s^{(0)}(t;x,y))| +$$
$$+ |u_h^{(\eta,q-1)}(t, \tilde{g}_s^{(0)}(t;x,y)) - u_h^{(0,q-1)}(t, \tilde{g}_s^{(0)}(t;x,y))|] dt +$$
$$+ \eta \int_0^x |d\Gamma_{sh}(t, \tilde{g}_s^{(\eta)}(t;x,y), z(t, \tilde{g}_s^{(\eta)}(t;x,y)))/dt| \cdot$$
$$\cdot |u_h^{(\eta,q-1)}(t, \tilde{g}_s^{(\eta)}(t;x,y))/dt\} + \mu L_{1a} ||g^{(\eta)} - g^{(0)}|| + \eta N_a \leq$$
$$\leq m\{(\mu C ||g^{(\eta)} - g^{(0)}|| + \eta H)\omega + \mu H \Lambda ||g^{(\eta)} - g^{(0)}|| +$$
$$+ \mu C ||g^{(\eta)} - g^{(0)}|| \Omega + \mu C ||g^{(\eta)} - g^{(0)}|| (\Xi_a + (\mu + \eta) r Q M_a) +$$
$$+ \mu(\mathring{M}_a + rC(\mu + \eta) M_a)[Q ||g^{(\eta)} - g^{(0)}|| + ||u^{(\eta,q-1)} - u^{(0,q-1)}||] +$$
$$+ \eta \Omega (\mathring{M}_a + rC(\mu + \eta) M_a (1 + mQ) +$$
$$+ mC\Xi_a)\} + \mu L_{1a} ||g^{(\eta)} - g^{(0)}|| + \eta N_a.$$

So, by (3.9), (3.12), (3.13) and (3.17), for every $(x,y) \in D_a$ and

$i = 1, \dots, m$ we have

$$|u_i^{(\eta,q)}(x,y) - u_i^{(0,q)}(x,y)| \leq m^2\eta H H'^2[(\mu+\eta)H\omega + \mu(\mathring{\mathrm{M}}_a + rC(\mu+ \\ +\eta)M_a)\Omega + \eta(\mathring{\mathrm{M}}_a + rC(\mu+\eta)M_a(1+mQ) + mC\Xi_a)\Omega]+ \\ + m\eta H H'^2(\mu+\eta)N_a+ \\ + m^2H'\eta[H\omega + \Omega(\mathring{\mathrm{M}}_a + rC(\mu+\eta)M_a(1+mQ) + mC\Xi_a)]+ \\ + mH'\eta N_a + \{m^2H'\mu[C\omega + H\Lambda + C\Omega + C(\Xi_a + (\mu+\eta)rQM_a)+ \\ + (\mathring{\mathrm{M}}_a + rC(\mu+\eta)M_a)Q] + mH'\mu L_{1a}\}||g^{(\eta)} - g^{(0)}||+ \\ + m^2H'\mu(\mathring{\mathrm{M}}_a + rC(\mu+\eta)M_a)||u^{(\eta;q-1)} - u^{(0,q-1)}||.$$

From (3.4) and (3.15) we deduce

$$||u^{(\eta,q)} - u^{(0,q)}|| \leq \bar{\varepsilon}_2 + \bar{\varepsilon}_1\eta M_a(1-\mu L_a)^{-1} + \bar{\varepsilon}_3||u^{(\eta,q-1)} - u^{(0,q-1)}|| \tag{3.18}$$

where

$$\bar{\varepsilon}_3 = m^2H'\mu(\mathring{\mathrm{M}}_a + rC(\mu+\eta)M_a). \tag{3.19}$$

By recursion from (3.16) and (3.18) we have

$$||u^{(\eta,q)} - u^{(0,q)}|| \leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1\eta M_a(1-\mu L_a)^{-1}]\sum_{h=0}^{q-1}\bar{\varepsilon}_3^h.$$

As by (2.47) $\bar{\varepsilon}_3 < 1$ we obtain

$$||u^{(\eta)} - u^{(0)}|| \leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1\eta M_a(1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1}. \tag{3.20}$$

Now we are able to give an estimate of $||z^{(\eta)} - z^{(0)}||$.

Let $\phi \in \mathcal{I}$ and let $\mathcal{T}_\phi^{(\eta)} : \mathcal{K}_{1\phi} \to \mathcal{K}_{1\phi}$ be the transformation defined in step (f) of theorem 1, in which we have emphasized, as usual, the dependence on the parameter η, $\eta \geq 0$. For every $n \geq 0$ and for every fixed $\phi \in \mathcal{I}$ let us inductively define $z^{(\eta,0)} = z \in \mathcal{K}_{1\phi}$, $z^{(\eta,n)} = \mathcal{T}_\phi^{(\eta)} z^{(\eta,n-1)}$, $n \in \mathbb{N}^+$.

So, denoted by $z^{(\eta)}$ the fixed point of $\mathcal{T}_\phi^{(\eta)}$ in $\mathcal{K}_{1\phi}$, we have $z^{(\eta)} = \lim_{n\to+\infty} z^{(\eta,n)}$, $\eta \geq 0$, without respect to the choice of z in $\mathcal{K}_{1\phi}$. On the other hand, from the definition of $\mathcal{T}_\phi^{(\eta)}$ we deduce

$$z^{(\eta,n)} = u^{(\eta)}[z^{(\eta,n-1)}, \phi], \; n \in N, \; \eta \geq 0 \tag{3.21}$$

where $u^{(\eta)}[z,\phi]$ is the fixed point of the transformation $T^{*(\eta)}_{z\phi}$.

It is immediate to observe that $\mathcal{T}^{(0)}_{\phi} : \mathcal{K}_{1\phi} \to \mathcal{K}_{1\phi}$ is a constant transformation such that $\mathcal{T}^{(0)}_{\phi} z = z^{(0)}$, for every choice of $z \in \mathcal{K}_{1\phi}$. So for $\eta = 0$ the following equalities hold

$$z^{(0)} = z^{(0,n)} = u^{(0)}[z,\phi] \tag{3.22}$$

for every $n \in \mathbb{N}^+$ and without respect to the choice of z in $\mathcal{K}_{1\phi}$. Moreover, the estimate (3.20), which we can write more precisely as

$$||u^{(\eta)}[z,\phi] - u^{(0)}[z,\phi]|| \leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} \tag{3.20$'$}$$

does not depend on $z \in \mathcal{K}_{1\phi}$. So, by (3.20)$'$, (3.21) and (3.22) for every $n \in \mathbb{N}^+$ we have

$$\begin{aligned} ||z^{(\eta,n)} - z^{(0,n)}|| = ||u^{(\eta)}[z^{(\eta,n-1)},\phi] - \\ - u^{(0)}[z,\phi]|| \leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} \end{aligned}$$

or

$$||z^{(\eta)} - z^{(0)}|| \leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} \tag{3.23}$$

which is the requested estimate.

If the exact solution $z^{(0)}$ fo linear Cauchy problem is known, (3.23) represents a bound for the perturbation due to the nonlinearity.

4. The boundary value problem.

In this section we consider the following hyperbolic system in bicharacteristic canonic form:

$$\begin{aligned} &\sum_{j=1}^{m} (\mu B_{ij}(x,y) + \eta \Gamma_{ij}(x,y,z)) \cdot \\ &\cdot \left[\frac{\partial z_j}{\partial x} + \sum_{k=1}^{r} (\mu \theta_{ik}(x,y) + \eta \lambda_{ik}(x,y,z)) \frac{\partial z_j}{\partial y_k} \right] = \\ &= \mu v_i(x,y) + \eta w_i(x,y,z), \end{aligned} \tag{c.1}$$

a.e. in $D_a = [0,a] \times E^r$, $i = 1, \dots, m$

with boundary conditions

$$(b.2) \qquad \sum_{j=1}^{m} b_{ij}(y) z_j(a_i, y) = \Psi_i(y), \ y \in E^r, \ i = 1, \dots, m$$

where $b_{ij} : E^r \to E$, $\Psi_i : E^r \to E$, $i, j = 1, \dots, m$ are given functions with det $(b_{ij}) \neq 0$ and a_i, $i = 1, \dots, m$, are given numbers, $0 \leq a_1 \leq \dots \leq a_n \leq a$. We name this boundary value problem, problem (B).

We will briefly sketch the proof of a theorem of existence, uniqueness and continuous dependence on the boundary data for problem (B). This theorem is essentially the same of theorem II in [10], but its proof provides the algorithm of the solution of problem that will be used in the next section to obtain the requested estimate of approximation error.

We assume that the $m \times m$ matrices (b_{ij}) and $(A_{ij}) = (\mu B_{ij} + \eta \Gamma_{ij})$ (see definition 2.10 in Section 2) have «dominant» diagonal terms. By possibly multiplying each equation (c.1) and (b.2) by suitable nonzero factors, we shall assume that

$$(4.1) \qquad \begin{aligned} A_{ij}(x, y, z) &= \delta_{ij} + \tilde{A}_{ij}(x, y, z), \ (x, y, z) \in \Delta_a, \\ b_{ij}(y) &= \delta_{ij} + \tilde{b}_{ij}(y), \ y \in E^r, \ i, j = 1, \dots, m, \end{aligned}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$.

As in Section 2 we denote by (α_{ij}) the transpose of the inverse of the matrix (A_{ij}), and we take

$$(4.2) \qquad \alpha_{ij}(x, y, z) = \delta_{ij} + \tilde{\alpha}_{ij}(x, y, z), \ (x, y, z) \in \Delta_a, \ i, j = 1, \dots, m.$$

Now let

$$(4.3) \qquad \sigma_0 = \max_i \sup \sum_{h=1}^{m} |\tilde{b}_{ih}(y)|,$$

where sup is taken for all $y \in E^r$, and

$$(4.4) \qquad \begin{aligned} \sigma_1 &= \max_i \sup \sum_{h=1}^{m} |\tilde{A}_{ih}(x, y, z)|, \\ \sigma_2 &= \max_i \sup \sum_{h=1}^{m} |\tilde{\alpha}_{hi}(x, y, z)|, \\ \sigma_3 &= \max_i \sup \sum_{s=1}^{m} \sum_{h=1}^{m} |\tilde{\alpha}_{si}(x, y, z)| \, |\tilde{A}_{sh}(x, y, z)|, \end{aligned}$$

where sup is taken for all $(x,y,z) \in \Delta_a$ and let

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3. \tag{4.5}$$

THEOREM 2. (An existence theorem for the boundary value problem) *Let* $\Omega > 0$, $B_{ij}(x,y)$, $\Gamma_{ij}(x,y,z)$, $\theta_{ik}(x,y)$, $\lambda_{ik}(x,y,z)$, $v_i(x,y)$, $w_i(x,y,z)$, $i,j = 1,\dots,m$, $k = 1,\dots,r$, *satisfy the same hypotheses of theorem 1.*

Let $\Psi_i(y)$, $b_{ij}(y)$, $y \in E^r$, $i,j = 1,\dots,m$ *be given continuous functions, and let us assume that there are constants* $\omega_0, \Lambda_0, \tau_0$, $0 < \omega_0 < \Omega$, $\Lambda_0 \geq 0$, $\tau_0 \geq 0$ *such that for all* y, $\bar{y} \in E^r$ *and* $i = 1,\dots,m$, *we have*

$$\begin{gathered} |\Psi_i(y)| \leq \omega_0, \ |\Psi_i(y) - \Psi_i(\bar{y})| \leq \Lambda_0 |y - \bar{y}|, \\ \sum_{j=1}^{m} |b_{ij}(y) - b_{ij}(\bar{y})| \leq \tau_0 |y - \bar{y}|. \end{gathered} \tag{4.6}$$

With the notations (4.1), (4.3), (4.4) let us assume that

$$\sigma + \sigma_0 + \sigma\sigma_0 < 1. \tag{4.7}$$

Then, for a, ω_0, τ_0, C, C' *(see (2.12) and (2.17)) sufficiently small,* $0 < a \leq a_0$, $\omega_0, \tau_0, C, C' > 0$ *and for every system of numbers* a_i, $0 \leq a_i \leq a, i = 1,\dots,m$ *there are a constant* $Q > 0$, *a function* $\chi(x) \geq 0$, $0 \leq x \leq a$, $\chi \in L_1[0,a]$, *and a vector function* $z(x,y) = (z_1,\dots,z_m)$, $(x,y) \in D_a$, *continuous in* D_a, *satisfying (c.1) a.e. in* D_a *and (b.2) everywhere in* E^r. *Moreover* $z(x,y)$ *is unique, depends continuously on boundary data* $\Psi(y)$ *and for all* (x,y), $(\bar{x},y)$ $(x,\bar{y}) \in D_a$, $i = 1,\dots,m$ *satisfies the following properties*

$$\begin{gathered} |z_i(x,y)| \leq \Omega, \ |z_i(x,y) - z_i(x,\bar{y})| \leq Q|y - \bar{y}|, \\ |z_i(x,y) - z_i(\bar{x},y)| \leq \left| \int_x^{\bar{x}} \chi(t)dt \right|. \end{gathered} \tag{4.8}$$

Proof. Let us recall the definitions (2.10):

$$A_{ij}(x,y,z) = \mu B_{ij}(x,y) + \eta\Gamma_{ij}(x,y,z),$$

$$\rho_{ik}(x,y,z) = \mu\theta_{ik}(x,y) + \eta\lambda_{ik}(x,y,z),$$

$$f_i(x,y,z) = \mu v_i(x,y) + \eta w_i(x,y,z),$$

$$(x,y,z) \in \Delta_a, \ i,j = 1,\dots,m, \ k = 1,\dots,r.$$

The functions $A_{ij}(x,y,z)$, $\rho_{ik}(x,y,z)$, $f_i(x,y,z)$ satisfy the same hypotheses of theorem II of Cesari's paper [10] with suitable constants and summable functions. So we can report the Cesari's proof of theorem II in [10].

Let $\mathcal{I}$ be the class of functions defined in step (e) of theorem 1. Let us choose ω_0, $0 < \omega_0 < \omega < \Omega$, and $\Lambda_0, \Lambda > \Lambda_0$ such that

$$(\sigma + \sigma_0 + \sigma\sigma_0)\omega < \omega - \omega_0, \quad \Lambda_0 + (\sigma + \sigma_0 + \sigma\sigma_0)\Lambda < \Lambda. \tag{4.9}$$

In section 2, for every fixed $\phi \in \mathcal{I}$, we have determined a unique element $z = z[\phi]$ and a corresponding element $g = g[z]$, $z \in \mathcal{K}_{1\phi} \subset \mathcal{K}_1$, $g \in \mathcal{K}_0$, satisfying the characteristic integral system (Δ). In particular, we recall that $g = g[z]$ is the unique Carathéodory solution of the Cauchy problem $(\Delta.1) - (\Delta'.1)$ and so the function $\tilde{g}_i = [g_{ik}, k = 1, \ldots, r]$ allows us to define the $1-1$ transformation of D_a onto itself (see Cesari [10], pag. 321) given by

$$y = \tilde{g}_i(x; 0, \nu), \; \nu = \tilde{g}_i(0; x, y). \tag{4.10}$$

We consider now the transformation T^{**}, or $\Phi = T^{**}\phi$, $\phi \in \mathcal{I}$, defined by

$$\begin{aligned} &\Phi_i(\nu) = [\Phi_i(\tilde{g}_i(0; a_i, y))]_{y=\tilde{g}_i(a_i;0,\nu)}, \nu \in E^r, i = 1, \ldots, m, \\ &\Phi_i(\tilde{g}_i(0; a_i, y)) = \Psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y) z_j(a_i, y) - \zeta_i(a_i, y), \\ &y \in E^r, \; i = 1, \ldots, m, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \zeta_i(x,y) = &\sum_{h=1}^{m} \tilde{A}_{ih}(0, \tilde{g}_i(0; x, y), z(0, \tilde{g}_i(0; x, y)))\phi_h(\tilde{g}_i(0; x, y)) + \\ &+ \sum_{h=1}^{m} \tilde{\alpha}_{hi}(x, y, z(x, y))\phi_h(\tilde{g}_h(0; x, y)) + \\ &+ \sum_{s=1}^{m}\sum_{h=1}^{m} \tilde{\alpha}_{si}(x, y, z(x, y))\tilde{A}_{sh}(x, y, z(x, y))\phi_h(\tilde{g}_s(0; x, y)) + \\ &+ \sum_{s=1}^{m}\sum_{h=1}^{m} \tilde{\alpha}_{si}(x, y, z(x, y))[\tilde{A}_{sh}(0, \tilde{g}_s(0; x, y), z(0, \tilde{g}_s(0; x, y))) - \end{aligned} \tag{4.12}$$

$$- \tilde{A}_{sh}(x, y, z(x, y))]\phi_h(\tilde{g}_s(0; x, y))+$$
$$+ \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \int_0^x \Big[\sum_{h=1}^{m} (dA_{sh}(t, \tilde{g}_s(t; x, y),$$
$$z(t, \tilde{g}_s(t, x, y)))/dt)\cdot$$
$$\cdot z_h(t, \tilde{g}_s(t; x, y)) + f_s(t, \tilde{g}_s(t; x, y), z(t, \tilde{g}_s(t; x, y)))\Big] dt =$$
$$= \varepsilon_1' + \varepsilon_2' + \varepsilon_3' + \varepsilon_4' + \varepsilon_5'.$$

We will prove that T^{**} is a contraction in the uniform topology on $\mathcal{I}$. At first we show that its range is contained in $\mathcal{I}$.

By (2.7) (2.12), (2.13), (2.25), (2.26) and (4.4) we have

$$|\varepsilon_4'| \leq m\sigma_2\omega[(\mu + \eta)\mathring{\mathrm{M}}_a + \eta C\chi_a + (\mu + \eta)C(\mu + \eta + \eta Q)M_a]$$

and by (2.14), (2.16) and (2.36) we have

$$|\varepsilon_5'| \leq M^2 H'[(\mu + \eta)\mathring{\mathrm{M}}_a + (\mu + \eta)rC(\mu + \eta + m\eta Q)M_a+$$
$$+ m\eta C\Xi_a]||z|| + mH'(\mu + \eta)N_a.$$

By these partial estimates and (2.7),

$$|\zeta_i(x, y)| \leq \sigma_1\omega + \sigma_2\omega + \sigma_3\omega + m\sigma_2\omega\cdot$$
$$\cdot [(\mu + \eta)\mathring{\mathrm{M}}_a + \eta C\Xi_a + (\mu + \eta)C(\mu + \eta + \eta Q)M_a]+$$
$$+ m^2 H'[(\mu + \eta)\mathring{\mathrm{M}}_a + (\mu + \eta)rC(\mu + \eta + m\eta Q)M_a + m\eta C\Xi_a]||z||+$$
$$+ mH'(\mu + \eta)N_a.$$

By using (2.46) and (4.5) we have

(4.13) $$|\zeta_i(x, y)| \leq \sigma\omega + \gamma||z|| + R_a$$

where

(4.14) $$R_a = m\sigma_2\omega[(\mu + \eta)\mathring{\mathrm{M}}_a + \eta C\Xi_a+$$
$$+ (\mu + \eta)C(\mu + \eta + \eta Q)M_a] + mH'(\mu + \eta)N_a.$$

As shown in [10] we can write

$$z_i(x,y) = \phi_i(\tilde{g}_i(0;x,y)) + \zeta_i(x,y) \tag{4.15}$$

and so by (2.7), (4.13)

$$||z|| \leq \omega + \sigma\omega + \gamma||z|| + R_a$$

or

$$||z|| \leq (1-\gamma)^{-1}((1+\sigma)\omega + R_a) \tag{4.16}$$

and finally

$$\begin{aligned} |\zeta_i(x,y)| &\leq \sigma\omega + \gamma(1-\gamma)^{-1}((1+\sigma)\omega + R_a) + R_a = \\ &= (1-\gamma)^{-1}(\sigma\omega + \gamma\omega + R_a). \end{aligned} \tag{4.17}$$

By (4.6), (4.12) and (4.17)

$$\begin{aligned} |\Phi_i(\nu)| = |\Phi_i(\tilde{g}_i(0;x,y))| &\leq |\Psi_i(y)| + \sum_{j=1}^{m} |\tilde{b}_{ij}(y)|\,|z_j(a_i,y)| + |\zeta_i(a_i,y)| \leq \\ &\leq \omega_0 + \sigma_0(1-\gamma)^{-1}((1+\sigma)\omega + R_a) + (1-\gamma)^{-1}(\sigma\omega + \gamma\omega + R_a) = \\ &= \omega_0 + (1-\gamma)^{-1}[(\sigma + \sigma_0 + \sigma\sigma_0)\omega + R_a(1+\sigma_0) + \gamma\omega]. \end{aligned}$$

If we determine a sufficiently small such that

$$(1-\gamma)^{-1}[(1+\sigma_0)R_a + \gamma\omega + (\sigma + \sigma_0 + \sigma\sigma_0)\omega] \leq \omega - \omega_0 \tag{4.18}$$

we can deduce

$$|\Phi_i(\nu)| \leq \omega.$$

Now we prove that Φ satisfies a Lipschitz condition with costant Λ. For any two $y,\ \bar{y} \in E^r$ and $i = 1,\ldots,m$, by using (4.12) and the same manipulations applied by Cesari in step (c) of theorem II (see [10], pag. 348) we can write

$$\xi_i(x,y) - \zeta_i(x,\bar{y}) = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$$

where $\delta_i, i, = 1,\ldots,5$ have the same meaning as in Cesari's proof.

Let us denote by $\tilde{H}$, $\tilde{H}'$ constants such that

$$|\tilde{A}_{ij}(x,y,z)| \leq \tilde{H}, \ |\tilde{\alpha}_{ij}(x,y,z)| \leq \tilde{H}' \tag{4.19}$$

for all $(x,y,z) \in \Delta_a$ and $i,j = 1,\ldots,m$. So we can take $H, H' \leq 1+\sigma, \tilde{H}, \tilde{H}' \leq \sigma$.

By force of (2.7), (2.12), (2.13), (2.15), (2.17), (2.18), (2.22), (2.23), (2.25), (2.26), (2.36), (4.4) and (4.16) we have

$$|\delta_1| \leq \{\sigma + m\sigma_2[(\mu+\eta)\mathring{\mathrm{M}}_a + \eta C \Xi_a +$$

$$+ (\mu+\eta)C(\mu+\eta+\eta Q)M_a]\}\Lambda(1+p)|y-\bar{y}|;$$

$$|\delta_2| \leq [m(\mu+\eta+\eta Q)(1+p)\cdot$$

$$\cdot (1+m\tilde{H}')C\omega + m(1+Q)(1+m\tilde{H})C'\omega]|y-\bar{y}|;$$

$$|\delta_3| \leq mC'(1+Q)[m((\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a +$$

$$+ m\eta C\Xi_a)(1-\gamma)^{-1}((1+\sigma)\omega + R_a) + (\mu+\eta)N_a]|y-\bar{y}|;$$

$$|\delta_4| \leq mH'L_{1a}(\mu+\eta+\eta Q)(1+p)|y-\bar{y}|.$$

By integration by parts and the use of (2.12), (2.23), (2.25), (2.36), (2.37) and (4.16) we also have

$$|\delta_5| \leq m^2 H'[C(\mu+\eta+\eta Q)(1-\gamma)^{-1}((1+\sigma)\omega + R_a) +$$

$$+ C(\mu+\eta+\eta Q)(1+p)(1-\gamma)^{-1}((1+\sigma)\omega + R_a) +$$

$$+ C(\mu+\eta+\eta Q)(1+p)(\Xi_a + (\mu+\eta)rQM_a) +$$

$$+ ((\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a +$$

$$+ m\eta C\Xi_a)Q(1+p)]|y-\bar{y}|.$$

Combining the estimate above we obtain

$$|\zeta_i(x,y) - \zeta_i(x,\bar{y})| \leq (K' + S_a)|y-\bar{y}| \tag{4.20}$$

where

$$(4.21)\quad \begin{aligned} K' =& \sigma\Lambda(1+p) + m(\mu+\eta+\eta Q)(1+p)(1+m\tilde{H}')C\omega + m(1+Q)(1+ \\ &+ m\tilde{H})C'\omega + m^2H'(\mu+\eta+\eta Q)(1+\sigma)(1-\gamma)^{-1}(2+p)C\omega, \\ S_a =& m\sigma_2\Lambda(1+p)[(\mu+\eta)\mathring{\mathrm{M}}_a + \eta C\Xi_a + (\mu+\eta)C(\mu+\eta+\eta Q)M_a]+ \\ &+ m^2C'(1+Q)(1-\gamma)^{-1}[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+ \\ &+ m\eta Q)M_a + m\eta C\Xi_a]((1+\sigma)\omega + R_a) + mC'(1+Q)(\mu+\eta)N_a+ \\ &+ mH'(\mu+\eta+\eta Q)(1+p)L_{1a} + m^2H'C(\mu+\eta+\eta Q)(1-\gamma)^{-1}\cdot \\ &\cdot (2+p)R_a + m^2H'C(\mu+\eta+\eta Q)\cdot \\ &\cdot (1+p)(\Xi_a + (\mu+\eta)rQM_a) + m^2H'Q(1+p)[(\mu+\eta)\mathring{\mathrm{M}}_a+ \\ &+ (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a + m\eta C\Xi_a]. \end{aligned}$$

By force of (2.7), (2.23), (4.15) and (4.20) we have also

$$(4.22)\quad \begin{aligned} &|z_i(x,y) - z_i(x,\bar{y})| \le |\phi_i(\tilde{g}_i(0;x,y)) - \phi_i(\tilde{g}_i(0;x,\bar{y}))|+ \\ &+ |\zeta_i(x,y) - \zeta_i(x,\bar{y})| \le [\Lambda(1+p) + (K'+S_a)]|y-\bar{y}|. \end{aligned}$$

So by (4.3), (4.6), (4.11), (4.16) and (4.22) we deduce

$$\begin{aligned} |\Phi_i(\tilde{g}_i(0;a_i,y)) - \Phi_i(\tilde{g}_i(0;a_i,\bar{y}))| \le& |\Psi_i(y) - \Psi_i(\bar{y})|+ \\ &+ \sum_{j=1}^{m} |\tilde{b}_{ij}(y) - \tilde{b}_{ij}(\bar{y})|\ |z_j(a_i,y)|+ \\ &+ \sum_{j=1}^{m} |\tilde{b}_{ij}(\bar{y})|\ |z_j(a_i,y) - z_j(a_i,\bar{y})|+ \\ &+ |\zeta_i(a_i,y) - \zeta_i(a_i,\bar{y})| \le [\Lambda_0 + \tau_0(1-\gamma)^{-1}((1-\sigma)\omega + R_a)+ \\ &+ \sigma_0(\Lambda(1+p) + K' + S_a) + K' + S_a]|y-\bar{y}| = \\ &= [\Lambda_0 + (1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda + S_0'\tau_0\omega + S_1'\omega C + S_2'C'\omega+ \\ &+ (1+\sigma_0)S_a + \tau_0(1-\gamma)^{-1}R_a]|y-\bar{y}| \end{aligned}$$

where

$$S_0' = (1+\sigma)(1-\gamma)^{-1},$$

$$S_1' = m(1+\sigma_0)(\mu+\eta+\eta Q)[(1+p)(1+mH')+ \\ + mH'(1+\sigma)(1-\gamma)^{-1}(2+p)],$$

$$S_2' = m(1+\sigma_0)(1+Q)(1+m\tilde{H}).$$

It we choose a sufficiently small in order that

$$\Lambda_0 + (1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda + S_0'\tau_0\omega + S_1'\omega C + S_2'\omega C' + \\ + (1+\sigma_0)S_a + \tau_0(1-\gamma)^{-1}R_a \leq (1+p)^{-1}\Lambda \tag{4.23}$$

we obtain

$$|\Phi_i(\tilde{g}_i(0;a_i,y) - \Phi_i(\tilde{g}_i(0;a_i,\bar{y}))| \leq (1+p)^{-1}\Lambda|y-\bar{y}| \tag{4.24}$$

and so, by (4.11), we have for all $\nu, \bar{\nu} \in E^r$ and $i = 1, \dots, m$

$$|\Phi_i(\nu) - \Phi_i(\bar{\nu})| \leq (1+p)^{-1}\Lambda(1+p)|\nu-\bar{\nu}| = \Lambda|\nu-\bar{\nu}|.$$

Thus we have proved that the transformation T^{**} maps $\mathcal{I}$ into $\mathcal{I}$.

Now let us prove that the transformation $T^{**}: \mathcal{I} \to \mathcal{I}$ is a contraction. Let ϕ, $\phi' \in \mathcal{I}$, and $z, z' \in \mathcal{K}_{1\phi}$, $g = g[z]$, $g' = g[z']$, $\Phi = T^{**}\phi$, $\Phi' = T^{**}\phi'$, $\zeta_i(x,y)$, $\zeta_i'(x,y)$ be the corresponding elements. Then, by using the same manipulations of step (d), pag. 352, of Cesari's theorem II in [10], we have

$$\zeta_i(x,y) - \zeta_i'(x,y) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5$$

with the same meaning of the symbols. We now estimate these terms ε_i, $i = 1, \dots, 5$ one by one. By (2.12), (2.13), (2.15)-(2.17), (2.22), (2.36), (4.4), (4.5), (4.16) and (4.19) we obtain

$$|\varepsilon_1| \leq \{\sigma + m\sigma_2[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)C(\mu+\eta+\eta Q)M_a + \eta C\Xi_a]\} \cdot \\ \cdot (\|\phi-\phi'\| + \Lambda\|g-g'\|);$$

$$|\varepsilon_2| \leq m(\mu+\eta+\eta Q)(1+m\tilde{H}')C\omega\|g-g'\| + m[(1+m\tilde{H}')\eta C\omega + \\ + (1+m\tilde{H})C'\omega]\|z-z'\|;$$

$$|\varepsilon_3| \leq mC'\{m(1-\gamma)^{-1}[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+ \\ + m\eta Q)M_a + m\eta C\Xi_a] \cdot ((1+\sigma)\omega + R_a) + (\mu+\eta)N_a\}\|z-z'\|;$$

$$|\varepsilon_4| \leq mH'L_{1a}[(\mu+\eta+\eta Q)\|g-g'\| + \eta\|z-z'\|].$$

By integration by parts we have also

$$|\varepsilon_5| \leq m^2 H'\eta C||z-z'||(1-\gamma)^{-1}((1+\sigma)\omega+R_a)+$$
$$+m^2H'C[(\mu+\eta+\eta Q)||g-g'||+\eta||z-z'||](1-\gamma)^{-1}((1+\sigma)\omega+R_a)+$$
$$+m^2H'C[(\mu+\eta+\eta Q)||g-g'||+\eta||z-z'||](\Xi_a+(\mu+\eta)rQM_a)+$$
$$+m^2H'[(\mu+\eta)\mathring{\mathrm{M}}_a+(\mu+\eta)rC(\mu+\eta+m\eta Q)M_a+$$
$$+m\eta C\Xi_a](Q||g-g'||+||z-z'||).$$

By using the previous estimates and (2.32) we obtain

$$|\zeta_i(x,y)-\zeta_i'(x,y)| \leq |\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|+|\varepsilon_4|+|\varepsilon_5| \leq$$
$$\leq \{\sigma+m\sigma_2[(\mu+\eta)\mathring{\mathrm{M}}_a+(\mu+\eta)C(\mu+\eta+\eta Q)M_a+\eta C\Xi_a]\}\cdot$$
$$\cdot[||\phi-\phi'||+\eta\Lambda\lambda L_a||z-z'||]+\{m(\mu+\eta+\eta Q)(1+$$
$$+m\tilde{H}')C\omega\lambda\eta L_a+m[(1+m\tilde{H}')\eta C\omega+(1+m\tilde{H})C'\omega]\}||z-z'||+$$
$$+mC'\{m(1-\gamma)^{-1}[(\mu+\eta)\mathring{\mathrm{M}}_a+(\mu+\eta)rC(\mu+\eta+$$
$$+m\eta Q)M_a+m\eta C\Xi_a]\cdot((1+\sigma)\omega+R_a)+(\mu+\eta)N_a\}||z-z'||+$$
$$+m\eta H'L_{1a}[(\mu+\eta+\eta Q)L_a\lambda+1]||z-z'||+$$
$$+m^2H'\eta C(1-\gamma)^{-1}((1+\sigma)\omega+R_a)||z-z'||+$$
$$+m^2\eta H'C(1-\gamma)^{-1}((1+\sigma)\omega+R_a)[(\mu+\eta+\eta Q)\lambda L_a+1]||z-z'||+$$
$$+m^2H'C\eta(\Xi_a+(\mu+\eta)rQM_a)[(\mu+\eta+\eta Q)\lambda L_a+1]||z-z'||+$$
$$+m^2H'[(\mu+\eta)\mathring{\mathrm{M}}_a+(\mu+\eta)rC(\mu+\eta+m\eta Q)M_a+$$
$$+m\eta C\Xi_a](Q\eta\lambda L_a+1)||z-z'||.$$

So by (2.52) we can write

$$|\zeta_i(x,y)-\zeta_i'(x,y)| \leq (L+T_a)||\phi-\phi'|| \tag{4.25}$$

where

$$L=\sigma+[m(1+m\tilde{H}')\eta C\omega+m(1+m\tilde{H})C'\omega+2m^2\eta H'(1+\gamma)^{-1}\cdot$$
$$\cdot(1+\sigma)C\omega](1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH'), \tag{4.26}$$

and

$$
\begin{aligned}
T_a = {} & m\sigma_2[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)C(\mu+\eta+\eta Q)M_a + \eta C\Xi_a] + \\
& + \{[\sigma + m\sigma_2((\mu+\eta)\mathring{\mathrm{M}}_a + \\
& + (\mu+\eta)C(\mu+\eta+\eta Q)M_a + \eta C\Xi_a)]\eta\Lambda\lambda L_a + \\
& + m(\mu+\eta+\eta Q)(1+m\tilde{H}')C\omega\eta\lambda L_a + mC'(\mu+\eta)N_a + \\
& + m^2C'(1-\gamma)^{-1}[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a + \\
& + m\eta C\Xi_a]\cdot((1+\sigma)\omega + R_a) + m\eta H'[1+(\mu+\eta+\eta Q)\lambda L_a]L_{1a} + \\
& + m^2H'\eta C(1-\gamma)^{-1}R_a + m^2H'\eta C(1-\gamma)^{-1}(1+\sigma)\omega(\mu+\eta+ \\
& + \eta Q)\lambda L_a + m^2H'\eta C(1-\gamma)^{-1}[1+(\mu+\eta+\eta Q)\lambda L_a)R_a + \\
& + m^2H'\eta C[1+(\mu+\eta+\eta Q)\lambda L_a](\Xi_a + (\mu+\eta)rQM_a) + \\
& + m^2H'[(\mu+\eta)\mathring{\mathrm{M}}_a + (\mu+\eta)rC(\mu+\eta+m\eta Q)M_a + m\eta C\Xi_a]\cdot \\
& \cdot(1+Q\eta\lambda L_a)\}(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH').
\end{aligned} \tag{4.27}
$$

Finally, by (2.7), (2.32), (2.52), (4.3), (4.11), (4.15) and (4.25) we have

$$
\begin{aligned}
& |\Phi_i(\tilde{g}_i(0;a_i,y)) - \Phi'(\tilde{g}'_i(0;a_i,y))| \leq \\
& \leq \sum_{j=1}^{m} |\tilde{b}_{ij}(y)|[|\phi_j(\tilde{g}_j(0;a_i,y)) - \phi'_j(\tilde{g}'_j(0;a_i,y))| + \\
& + |\zeta_j(a_i,y) - \zeta'_j(a_i,y)|] + |\zeta_i(a_i,y) - \zeta'_i(a_i,y)| \leq \\
& \leq \sigma_0[||\phi-\phi'|| + \Lambda||g-g'|| + (L+T_a)||\phi-\phi'||] + \\
& + (L+T_a)||\phi-\phi'|| \leq \sigma_0[||\phi-\phi'|| + \eta\Lambda\lambda L_a||z-z'|| + \\
& + (L+T_a)||\phi-\phi'||] + (L+T_a)||\phi-\phi'|| \leq \\
& \leq [\sigma_0 + (1+\sigma_0)L + \sigma_0\eta\Lambda\lambda L_a(1-k)^{-1}(1-\gamma)^{-1}\cdot \\
& \cdot(1+\gamma+2m^2(\mu+\eta)HH') + (1+\sigma_0)T_a]||\phi-\phi'||.
\end{aligned} \tag{4.28}
$$

Also

$$|\Phi_i(\tilde{g}_i(0;a_i,y)) - \Phi_i'(\tilde{g}_i(0;a_i,y)| \leq$$
$$\leq |\Phi_i(\tilde{g}_i(0;a_i,y)) - \Phi_i'(\tilde{g}_i'(0;a_i,y)| + \eta\Lambda\lambda L_a||z - z'||.$$

So, by using (4.28) and (2.52) we obtain

$$\begin{aligned} ||\Phi - \Phi'|| \leq [\sigma_0 + (1+\sigma_0)L+ \\ + (1+\sigma_0)\eta\Lambda\lambda L_a(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH')+ \\ + (1+\sigma_0)T_a]||\phi - \phi'||. \end{aligned} \tag{4.29}$$

Now by force of (4.26), (4.27) and by manipulations in the previous inequality we have

$$\begin{aligned} ||\Phi - \Phi'|| \leq [(\sigma + \sigma_0 + \sigma\sigma_0) + T_1'C\omega + T_2'C'\omega+ \\ + (1+\sigma_0)\eta\Lambda\lambda L_a(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH')+ \\ + (1+\sigma_0)T_a]||\phi - \phi'|| \end{aligned}$$

where

$$\begin{aligned} T_1' = (1+\sigma_0)[m\eta(1+m\tilde{H}') + 2m^2\eta H'(1-\gamma)^{-1}(1+\sigma)]\cdot \\ \cdot(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2(\mu+\eta)HH'), \\ T_2' = (1+\sigma_0)m(1+m\tilde{H})(1-k)^{-1}(1-\gamma)^{-1}\cdot \\ \cdot(1+\gamma+2m^2(\mu+\eta)HH'). \end{aligned} \tag{4.30}$$

If we choose a sufficiently small in order that

$$\begin{aligned} (\sigma + \sigma_0 + \sigma\sigma_0) + T_1'C\omega + T_2'C'\omega + (1+\sigma_0)(1-k)^{-1}(1-\gamma)^{-1}\cdot \\ \cdot(1+\gamma+2m^2(\mu+\eta)HH')\eta\Lambda\lambda L_a + (1+\sigma_0)T_a \leq k' \end{aligned} \tag{4.31}$$

where k' denotes any number such that $0 < k' < 1$ we have

$$||\Phi - \Phi'|| \leq k'||\phi - \phi'||. \tag{4.32}$$

Thus, $T^{**}\mathcal{I} \to \mathcal{I}$ is a contraction. By Banach's fixed point theorem there is a unique element $\phi \in \mathcal{I}$ with $\phi = T^{**}\phi$. For this element by (4.11) and (4.15) we have

$$\phi_i(\tilde{g}_i(0; a_i, y)) = \Psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij} z_j(a_i, y) - \zeta_i(a_i, y)$$

or

$$z_i(a_i y) = \Psi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij} z_j(a_i, y)$$

and by using (4.1) also

$$\sum_{j=1}^{m} b_{ij}(y) z_j(a_i, y) = \Psi_i(y), \; y \in E^r, \; i = 1, \ldots, m.$$

Thus $z = z[\phi]$ is the unique solution of the boundary value problem (B).

As in theorem II of Cesari's paper [10], we prove that the fixed point $\phi \in \mathcal{I}$ of transformation T^{**} is also a continuous function of Ψ. Let Ψ, Ψ' two functions satisfying (4.6) and z, z' the corresponding elements.

By (4.11) and (4.32) we have

$$||\phi - \phi'|| \leq ||\Psi - \Psi'|| + k'||\phi - \varphi'||$$

or

$$||\phi - \phi'|| < (1 - k')^{-1}||\Psi - \Psi'||$$

and by (2.52)

$$\begin{aligned} ||z - z'|| \leq (1 - k)^{-1}(1 - \gamma)^{-1}(1 + \gamma + \\ + 2m^2(\mu + \eta)HH')(1 - k')^{-1}||\Psi - \Psi'||. \end{aligned} \tag{4.33}$$

By (4.33) we have proved the continuous dependence of the solution of the boundary value problem on the boundary data.

We remark that the new restrictions imposed on the size of a, $0 < a \leq a_0$, derive from (4.18), (4.23) and (4.31).

5. An estimate of approximation error for the boundary value problem.

In this section we will denote the boundary value problem (B) by (B, η): in this way we point out explicitely the dependence of the system (c.1) on parameter $\eta, \eta \geq 0$. Analogously, we refer to the Cauchy problem (C) as problem (C, η). The parameter $\mu, \mu \geq 0$, will be considered fixed. We name $z^{(\eta)}, \eta \geq 0$, the solution of the problem (B, η). When $\eta = 0$, $z^{(0)}$ is the solution of the boundary value problem for the linear system (3.1) with data (b.2). As in Section 3, we suppose that $||(B_{ij})^{-1}|| \leq \mu H'$, where H' is the same constant introduced in (2.16). Moreover, to apply theorem 2 of Section 4 to the linear system (3.1) we suppose that the matrix $(A_{ij}) = (\mu B_{ij})$ has «dominant» diagonal terms, i.e. it satisfies (4.7) with the notations (4.1)-(4.5).

Let $\varphi^{(\eta)}$ be the fixed point of the transformation $T_\eta^{**} : \mathcal{I} \to \mathcal{I}$. In these notations we have emphasized the dependence of the transformation T^{**} on η. At first, we estimate $||\phi^{(\eta)} - \phi^{(0)}||$. For every $\eta \geq 0$, let us inductively define $\phi^{(\eta,0)} = \phi \in \mathcal{I}$, $\phi^{(\eta,n)} = T_\eta^{**}\phi^{(\eta,n-1)}$ and so $\phi^{(\eta)} = \lim_{n\to+\infty} \phi^{(\eta,n)}$, without respect to the choice of function $\phi = \phi^{(\eta,0)} \in \mathcal{I}$. Therefore it is not restrictive to choose $\phi^{(\eta,0)} = \phi^{(0,0)} = \phi$. Let $z^{(\eta)}[\phi]$, $\eta \geq 0$, be the solution of the Cauchy problem (C, η) with initial data $\phi \in \mathcal{I}$, and let $g^{(\eta)}[\phi] = g[z^{(\eta)}[\phi]]$ be the corresponding function in $\mathcal{K}_0$.

We remark again that the estimate (3.4) is independent on the choice of function $z \in \mathcal{K}_1$, so the estimate (3.4) is independent on the choice of function $\phi \in \mathcal{I}$, too. Indeed, from the definition of the transformation T_z it easily follows that $g^{(0)}[z^{(0)}[\phi]]$ does not depend on ϕ: from now on we will simply write $g^{(0)}$ instead of $g^{(0)}[\phi]$.

For every $y \in E^r$, $i = 1, \dots, m$ we have

$$
\begin{aligned}
&|\phi_i^{(\eta,1)}(\tilde{g}_i^{(\eta)}[\phi](0; a_i, y)) - \phi_i^{(0,1)}(\tilde{g}_i^{(\eta)}[\phi](0; a_i, y))| \leq \\
&\leq |\phi_i^{(\eta,1)}(\tilde{g}_i^{(\eta)}[\phi](0; a_i, y)) - \phi_i^{(0,1)}(\tilde{g}_i^{(0)}(0; a_i, y))| + \\
&+ |\phi_i^{(0,1)}(\tilde{g}_i^{(0)}(0; a_i, y)) - \phi_i^{(0,1)}(\tilde{g}_i^{(\eta)}[\phi](0; a_i, y))|.
\end{aligned}
\tag{5.1}
$$

By definition (4.11) and (4.15) we obtain

$$\begin{aligned}
&|\phi_i^{(\eta,1)}(\tilde{g}_i^{(\eta)}[\phi](0;a_i,y)) - \phi_i^{(0,1)}(\tilde{g}_i^{(0)}(0;a_i,y))| \leq \\
(5.2)\qquad &\leq \sum_{j=1}^{m} |\tilde{b}_{ij}(y)|\ |z_j^{(\eta)}[\phi](a_i,y) - z_j^{(0)}[\phi](a_i,y)| + \\
&+ |\zeta_i^{(\eta)}[\phi](a_i,y) - \zeta_i^{(0)}[\phi](a_i,y)|,
\end{aligned}$$

where $\zeta_i^{(\eta)}[\phi] = \zeta_i^{(\eta)}[\phi, z^{(\eta)}[\phi]]$, $\eta \geq 0$, $i = 1, \dots, m$, and also, by (2.7),

$$\begin{aligned}
&|\zeta_i^{(\eta)}[\phi](a_i,y) - \zeta_i^{(0)}[\phi](a_i,y)| \leq |z_i^{(\eta)}[\phi](a_i,y) - z_i^{(0)}[\phi](a_i,y)| + \\
(5.3)\qquad &+ |\phi_i(\tilde{g}_i^{(\eta)}[\phi](0;a_i,y)) - \phi_i(\tilde{g}_i^{(0)}(0;a_i,y))| \leq \\
&\leq ||z^{(\eta)}[\phi] - z^{(0)}[\phi]|| + \Lambda ||g^{(\eta)}[\phi] - g^{(0)}||.
\end{aligned}$$

Therefore for every $i = 1, \dots, m$ and $y \in E^r$ we deduce

$$\begin{aligned}
(5.4)\qquad &|\phi_i^{(\eta,1)}(\tilde{g}_i^{(\eta)}[\phi](0;a_i,y)) - \phi_i^{(0,1)}(\tilde{g}_i^{(0)}(0;a_i,y))| \leq \\
&\leq (1+\sigma_0) ||z^{(\eta)}[\phi] - z^{(0)}[\phi]|| + \Lambda ||g^{(\eta)}[\phi] - g^{(0)}||
\end{aligned}$$

and so, by (3.4), (3.23) we can conclude

$$\begin{aligned}
(5.5)\qquad &||\phi^{(\eta,1)} - \phi^{(0,1)}|| \leq (1+\sigma_0)[\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} + \\
&+ 2\Lambda \eta M_a (1-\mu L_a)^{-1}.
\end{aligned}$$

Let q be a positive integer, $q > 1$. We have

$$\begin{aligned}
&|\phi_i^{(\eta,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0;a_i,y)) - \phi_i^{(0,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0;a_i,y))| \leq \\
(5.6)\qquad &\leq |\phi_i^{(\eta,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0;a_i,y)) - \phi_i^{(0,q)}(\tilde{g}_i^{(0)}(0;a_i,y))| + \\
&+ |\phi_i^{(0,q)}(\tilde{g}_i^{(0)}(0;a_i,y)) - \phi_i^{(0,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0;a_i,y))|.
\end{aligned}$$

At first we estimate the term

$$
\begin{aligned}
&|\phi_i^{(\eta,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0;a_i,y)) - \phi_i^{(0,q)}(\tilde{g}_i^{(0)}(0;a_i,y))| \leq \\
&\leq \sum_{j=1}^{m} |\tilde{b}_{ij}(y)|\ |z_j^{(\eta)}[\phi^{(\eta,q-1)}](a_i,y) - z_j^{(0)}[\phi^{(0,q-1)}](a_i,y)| + \\
&+ |\zeta_i^{(\eta)}[\phi^{(\eta,q-1)}](a_i,y) - \zeta_i^{(0)}[\phi^{(0,q-1)}](a_i,y)| \leq \\
&\leq \sigma_0[|z_j^{(\eta)}[\phi^{(\eta,q-1)}](a_i,y) - z_j^{(0)}[\phi^{(\eta,q-1)}](a_i,y)| + \\
&+ |z_j^{(0)}[\phi^{(\eta,q-1)}](a_i,y) - z_j^{(0)}[\phi^{(0,q-1)}](a_i,y)|] + \\
&+ |\zeta_i^{(\eta)}[\phi^{(\eta,q-1)}](a_i,y) - \zeta_j^{(0)}[\phi^{(\eta,q-1)}](a_i,y)| + \\
&+ |\zeta_i^{(0)}[\phi^{(\eta,q-1)}](a_i,y) - \zeta_i^{(0)}[\phi^{(0,q-1)}](a_i,y)| \leq \\
&\leq \sigma_0||z^{(\eta)} - z^{(0)}|| + \sigma_0[|\phi_j^{(\eta,q-1)}(\tilde{g}_j^{(0)}(a_i;x,y)) - \phi_j^{(0,q-1)}(\tilde{g}_j^{(0)}(a_i;x,y))| + \\
&+ |\zeta_j^{(0)}[\phi^{(\eta,q-1)}](a_i,y) - \zeta_i^{(0)}[\phi^{(0,q-1)}](a_i,y)|] + \\
&+ |z_i^{(\eta)}[\phi^{(\eta,q-1)}](a_i,y) - z_i^{(0)}[\phi^{(\eta,q-1)}](a_i y)| + \\
&+ |\phi^{(\eta,q-1)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](a_i;x,y)) - \phi^{(\eta,q-1)}(\tilde{g}_i^{(0)}(a_i;x,y))| + \\
&+ |\zeta_i^{(0)}[\phi^{(\eta,q-1)}](a_i,y) - \zeta_i^{(0)}[\phi^{(0,q-1)}](a_i,y)| \leq \\
&\leq \sigma_0||z^{(\eta)} - z^{(0)}|| + \sigma_0(1 + L^{(0)} + T_a^{(0)})||\phi^{(\eta,q-1)} - \phi^{(0,q-1)}|| + \\
&+ ||z^{(\eta)} - z^{(0)}|| + \Lambda||g^{(\eta)} - g^{(0)}|| + (L^{(0)} + T_a^{(0)})||\phi^{(\eta,q-1)} - \phi^{(0,q-1)}||.
\end{aligned}
$$

Here, we have used the continuous dependence with respect to the data of the function ζ (see (4.25) in Section 4). We remark that we have applied (4.25) in the case $\eta = 0$. So (4.25) signifies

$$(4.25)' \qquad ||\zeta^{(0)}[\phi] - \zeta^{(0)}[\phi']|| \leq (L^{(0)} + T_a^{(0)})||\phi - \phi'||$$

with

$$(2.46)' \qquad \gamma^{(0)} = m^2 H'(\mu\mathring{M}_a + \mu^2 rCM_a),$$

$$(4.14)' \qquad R_a^{(0)} = m\sigma_2\omega(\mu\mathring{M}_a + \mu^2 CM_a) + m\mu H' N_a,$$

$$(4.26)' \quad L^{(0)} = \sigma + m(1 + m\tilde{H})C'\omega(1-k)^{-1}(1-\gamma^{(0)})^{-1}(1+\gamma^{(0)}+2m^2\mu H H'),$$

and

$$(4.27)' \quad \begin{aligned} T_a^{(0)} &= m\sigma_2(\mu M_a + \mu^2 C M_a) + [m\mu C' N_a + \\ &+ m^2 C'(1-\gamma^{(0)})^{-1}(\mu \mathring{M}_a + \mu^2 r C M_a)((1+\sigma)\omega + R_a^{(0)}) + \\ &+ m^2 H'(\mu \mathring{M}_a + \mu^2 m C M_a)](1-k)^{-1}(1+\gamma^{(0)})^{-1} \cdot \\ &\cdot (1+\gamma^{(0)} + 2m^2 \mu H H'). \end{aligned}$$

So, by using (3.4) and (3.23), for every $y \in E^\tau$ and $1 = 1, \dots, m$ we have

$$(5.7) \quad \begin{aligned} &||\phi_i^{(\eta,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0; a_i, y) - \phi_i^{(0,q)}(\tilde{g}_i^{(\eta)}[\phi^{(\eta,q-1)}](0; a_i, y))|| \leq \\ &\leq (1+\sigma_0)[\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} + \\ &+ \Lambda \eta M_a (1-\mu L_a)^{-1} + [\sigma_0 + (1+\sigma_0)(L^{(0)} + \\ &+ T_a^{(0)})]||\phi^{(\eta,q-1)} - \phi^{(0,q-1)}||. \end{aligned}$$

Then, (5.6) and (5.7) provide

$$(5.8) \quad \begin{aligned} ||\phi^{(\eta,q)} - \phi^{(0,q)}|| &\leq (1+\sigma_0)[\bar{\varepsilon}_2 + \\ &+ \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} + 2\Lambda\eta M_a(1-\mu L_a)^{-1} + \\ &+ [\sigma_0 + (1+\sigma_0)(L^{(0)} + T_a^{(0)})]||\phi^{(\eta,q-1)} - \phi^{(0,q-1)}||. \end{aligned}$$

By recursion, from (5.5) and (5.8) it follows that

$$\begin{aligned} ||\phi^{(\eta,q)} - \phi^{(0,q)}|| \leq \{&(1+\sigma_0)[\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} + \\ &+ 2\Lambda\eta M_a(1-\mu L_a)^{-1}\} \cdot \\ &\cdot \sum_{h=0}^{q-1} [\sigma_0 + (1+\sigma_0)(L^{(0)} + T_a^{(0)})]^h. \end{aligned}$$

As $[\sigma_0 + (1+\sigma_0)(L^{(0)} + T_a^{(0)})] < 1$ (see (4.31) of Section 4) we conclude that

$$(5.9) \quad \begin{aligned} ||\phi^{(\eta)} - \phi^{(0)}|| \leq \{&(1+\sigma_0)[\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1-\mu L_a)^{-1}](1-\bar{\varepsilon}_3)^{-1} + \\ &+ 2\Lambda\eta M_a(1-\mu L_a)^{-1}\} \frac{1}{1 - [\sigma_0 + (1+\sigma_0)(L^{(0)} + T_a^{(0)})]}. \end{aligned}$$

Finally, we can evaluate an estimate of $||z^{(\eta)} - z^{(0)}||$. By using (3.23) and the continuous dependence on the data of the function z (see (2.52) of Section 2) applied in the linear case $\eta = 0$, we have

$$
\begin{aligned}
&||z^{(\eta)} - z^{(0)}|| = ||z^{(\eta)}[\phi^{(\eta)}] - z^{(0)}[\phi^{(0)}]|| \leq \\
&\leq ||z^{(\eta)}[\phi^{(\eta)}] - z^{(0)}[\phi^{(\eta)}]|| + ||z^{(0)}[\phi^{(\eta)}] - z^{(0)}[\phi^{(0)}]|| \leq \\
&\leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1 - \mu L_a)^{-1}](1 - \bar{\varepsilon}_3)^{-1} + \\
&+ (1-k)^{-1}(1-\gamma^{(0)})^{-1}(1 + \gamma^{(0)} + 2m^2 \mu H H')||\phi^{(\eta)} - \phi^{(0)}|| \leq \\
&\leq [\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1 - \mu L_a)^{-1}](1 - \bar{\varepsilon}_3) + \\
&+ (1-k)^{-1}(1-\gamma^{(0)})^{-1}(1 - \gamma^{(0)} + 2m^2 \mu H H') \cdot \\
&\cdot \{(1+\sigma_0)[\bar{\varepsilon}_2 + \bar{\varepsilon}_1 \eta M_a (1 - \mu L_a)^{-1}](1 - \bar{\varepsilon}_3)^{-1} + \\
&+ 2\Lambda \eta M_a (1 - \mu L_a)^{-1}\} \frac{1}{1 - [\sigma_0 + (1+\sigma_0)(L^{(0)} + T_a^{(0)}]}
\end{aligned} \tag{5.10}
$$

which is the approximation estimate we were looking for.

Let us remark that the estimate (5.10) vanishes as η approaches to zero.

REFERENCES

[1] Bassanini P., *A nonlinear hyperbolic problem arising from a question of nonlinear optics,* J. Appl. Math. Phys. (ZAMP), part. I, **27** (1976), 409-422.

[2] Bassanini P., *Su una recente dimostrazione circa il problema di Cauchy per sistemi quasi lineari iperbolici,* Boll. UMI, (5), **13**-B (1976), 322-335.

[3] Bassanini P, *On a recent proof concerning a boundary value problem for quasi linear hyperbolic systems in the Schauder canonic form,* Boll. UMI, (5), **14**-A (1977), 325-332.

[4] Bassanini P., Cesari L., *La duplicazione di frequenza nella radiazione laser,* Rend. Accad. Naz. Lincei (8), **69** (1980), 166-173.

[5] Bassanini P., Salvatori M.C., *Problemi ai limiti per sistemi iperbolici quasi lineari e generazione di armoniche ottiche,* Riv. Mat. Univ. Parma (4), **5** (1979), 55-76.

[6] Bassanini P., Salvatori M.C., *A theorem of existence and uniqueness in nonlinear optics,* Boll. UMI, **16**-B, (1979), 597-611.

[7] Ceppitelli R., Villarini M., *Error bounds for solutions of systems of quasi linear hyperbolic partial differential equations in diagonal form,* Atti Sem. Mat. Fis. Univ. Modena (to appear).

[8] Cesari L., *Un problema ai limiti per sistemi di equazioni iperboliche quasi lineari alle derivate parziali,* Rend. Accad. Naz. Lincei (8), **56**, (1974), 1-4, (8), **57**, (1974), 303-307.

[9] Cesari L., *A boundary value problem for quasi linear hyperbolic systems,* Rivista Mat. Univ. Parma (3), **3** (1974), 107-131.

[10] Cesari L., *A boundary value problem for quasi linear hyperbolic systems in the bicharacteristic canonic form,* Annali Scuola Norm. Sup. Pisa (4), **1** (1974), 311-358.

[11] Cesari L., *Sistemi iperbolici ed oscillazioni nonlineari,* Rend. Sem. Mat. Fis. Milano **44** (1974), 139-154.

[12] Cesari L., Hou S.H., Turner J., *The duplication of frequency of laser radiation through nonlinar media,* Report Univ. of Michigan, Dept. of Math. (1978).

[13] Cinquini Cibrario M., Cinquini S., *Equazioni alle Derivate Parziali di Tipo Iperbolico,* Ed. Cremonese Roma (1964).

[14] Filiaggi E, *Metodi numerici per sistemi iperbolici quasi lineari e applicazioni al problema del laser,* Ph. D. Thesis, Univ. of Perugia (1979).

[15] Graffi D, *Problemi non lineari nella teoria del campo elettromagnetico,* Mem. Accad. Sci. Modena (6) **11**, (1967), 172-196.

[16] Graffi D., *Nonlinear Partial Differential Equations in Physical Problems,* Pitman (1980).

Dipartimento di Matematica
Università di Perugia
Via Vanvitelli, 1 - 06100 Perugia (Italy)