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A profusion of 1/2 BPS Wilson loops in $\mathcal{N} = 4$ Chern-Simons-matter theories

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Abstract: We initiate the study of 1/2 BPS Wilson loops in $\mathcal{N} = 4$ Chern-Simons-matter theories in three dimensions. We consider a circular or linear quiver with Chern-Simons levels k, $-k$ and 0, and focus on loops preserving one of the two SU(2) subgroups of the R-symmetry. In the cases with no vanishing Chern-Simons levels, we find a pair of Wilson loops for each pair of adjacent nodes on the quiver connected by a hypermultiplet (nodes connected by twisted hypermultiplets have Wilson loops preserving another set of supercharges). We expect this classical pairwise degeneracy to be lifted by quantum corrections. In the case with nodes with vanishing Chern-Simons terms connected by twisted hypermultiplets, we find that the usual 1/4 BPS Wilson loops are automatically enlarged to 1/2 BPS, as happens also in 3-dimensional Yang-Mills theory. When the nodes with vanishing Chern-Simons levels are connected by untwisted hypermultiplets, we do not find any Wilson loops coupling to those nodes which are classically invariant. Rather, we find several loops whose supersymmetry variation, while non zero, vanishes in any correlation function, so is weakly zero. We expect only one linear combination of those Wilson loops to remain BPS when quantum corrections are included. We analyze the M-theory duals of those Wilson loops and comment on their degeneracy. We also show that these Wilson loops are cohomologically equivalent to certain 1/4 BPS Wilson loops whose expectation value can be evaluated by the appropriate localized matrix model.

Keywords: Supersymmetry and Duality, Wilson, 't Hooft and Polyakov loops, Chern-Simons Theories, M-Theory

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1 Introduction

It is by now a widely substantiated fact that BPS Wilson loops provide a powerful probe of supersymmetric gauge theories. While Wilson loops in general enable us to study gauge theories and their phases, the BPS ones are particularly interesting as they also allow to perform exact calculations, via localization [1]. The prototypical example of such operators is the 1/2 BPS Wilson loop of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions, which is defined in terms of a gauge connection augmented by a scalar coupling [2–4]. In the case of the circular geometry, this is evaluated by the Gaussian matrix model [5, 6]. The same is true for a much larger class of theories with $\mathcal{N} = 2$ supersymmetry, where again a scalar coupling allows to turn the loop for any gauge group in a quiver gauge theory (or a theory of class $\mathcal{S}$) to be BPS and they provide rich probes of the theory [1, 7–9].

In three dimensions the story is quite similar, if we consider Chern-Simons (CS) theories with $\mathcal{N} = 2$ supersymmetry. The 1/2 BPS Wilson loops there again involve a coupling to a scalar from the vector multiplet [10] and can be evaluated by localization to a matrix model [11]. The same is true for Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry [12]. Things are however more complicated when considering Chern-Simons theories with more extended supersymmetry. In the ABJ(M) theory [13, 14], which has $\mathcal{N} = 6$ supersymmetries, the loops with only scalar couplings turn out to be only 1/6 BPS [15–17], while the 1/2 BPS ones require a much more complicated structure, involving couplings to both gauge groups and the inclusion of fermionic terms [18]. Technically, the proof of supersymmetry invariance for those 1/2 BPS loops becomes more complicated, since the connection itself is not invariant under the supersymmetry variations, but gives a total derivative which needs to be integrated along the Wilson loops.

In this paper, we initiate the study of 1/2 BPS Wilson loops in theories with Chern-Simons couplings and $\mathcal{N} = 4$ supersymmetry. The case of CS theories with $\mathcal{N} = 3$ supersymmetry was already studied in [19], where it was found that the only supersymmetric loops are 1/3 BPS. The structure of $\mathcal{N} = 4$ theories is much more restrictive and we may hope that there could be 1/2 BPS loops. Indeed we shall present an embarrassingly large number of Wilson loops that seem to be 1/2 BPS and will discuss how this degeneracy may be lifted.

Before outlining the results, we proceed by specifying the theories analyzed in this paper.

1.1 The theories

The first $\mathcal{N} = 4$ superconformal Chern-Simons-matter theories in three dimensions were constructed in [20]. These were generalized in [21] by the inclusion of "twisted" hypermultiplets and further generalized in [22], whose notations we mainly use, see appendix A for

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details. The theories we consider are circular or linear quivers with $U(N_I)$ gauge group nodes. Adjacent nodes can be connected by bifundamental fields which are either hypermultiplets or twisted hypermultiplets. We assume that there are p hypermultiplets and q twisted hypermultiplets. A circular quiver has then $p+q$ nodes, while a linear quiver has $p+q+1$ nodes. The supersymmetry enhancement of these extended theories was studied in [23] and, in particular, it was shown that ABJ(M) lies in this class of theories.

The vector multiplets associated to the nodes with vanishing CS levels may be integrated out to obtain a non-linear theory [24]. We do not adopt this approach and work with the UV description.

We label our nodes by the index I, so the vector field is $A_{(I)\mu}$. The gaugino and auxiliary scalar (in $\mathcal{N}=2$ language) are $\lambda_{(I)}$ and $\varphi_{(I)}$, though we will integrate them out for nodes with non-vanishing CS levels. The hypermultiplets in the bifundamental representation of nodes I and $I+1$ have a scalar $q^{\bar{A}}_{(I)}$ and fermion $\psi_{(I)\underline{B}}$. They are doublets of the $\mathrm{SU}(2)_A$ and $\mathrm{SU}(2)_B$ subgroups of the $\mathrm{SO}(4)=\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ R-symmetry group and are indicated by underlined and overlined indices, respectively. The same multiplet includes also $\bar{q}_{(I)\bar{A}}$ and $\bar{\psi}^{\underline{B}}_{(I)}$. The field content of the twisted hypermultiplets is obtained by exchanging underlined and overlined indices.

The CS level of the I^{th} node, k_I, is fixed by the condition

$$k_I=\frac{k}{2}(s_I-s_{I+1})\,,\qquad s_I=\pm 1\,, \tag{1.1}$$

where $s_I=1$ for a hypermultiplet at the I^{th} link and $s_I=-1$ for a twisted hypermultiplet.[1] With these conditions, it is apparent that $k_I\in\{k,-k,0\}$.

1.2 The Wilson loops

The theories we study have an $\mathrm{SO}(4)=\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ R-symmetry. We shall look for Wilson loops which preserve the $\mathrm{SU}(2)_A$ subgroup. There will be other Wilson loops which preserve the $\mathrm{SU}(2)_B$ subgroup, of course. Those can be studied by replacing the theory with another one where all hypermultiplets are exchanged with twisted hypermultiplets and vice-versa.

To be specific, we choose to preserve the supersymmetries generated by the four parameters (see the supersymmetry transformations in appendix A)

$$\xi^{+}_{\bar{A}\underline{1}}\,,\quad \xi^{-}_{\bar{A}\underline{2}}\,,\qquad \bar{A}=\bar{1},\bar{2}\,. \tag{1.2}$$

The Wilson loops preserving those supercharges are straight lines in Euclidean space and will also preserve four superconformal generators. In section 3 we write down circular loops, which preserve eight linear combinations of the Poincaré and conformal supersymmetries, but will also preserve $\mathrm{SU}(2)_A$.

An important point about the supersymmetries in (1.2) is that there is a pairwise symmetry under exchange of chirality $\pm$ and $\mathrm{SU}(2)_B$ index $\underline{1}$, $\underline{2}$. The construction of the Wilson loops below mirrors that in ABJM theory and includes fermions from the

[1] We adopt the convention that in quiver diagrams I increases from left to right.

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hypermultiplet, which carry the same type of indices. Hence the choice of fermionic coupling in the Wilson loops breaks this pairwise symmetry and consequently we find a pairwise degeneracy in all our constructions. We refer to those below as the "ψ_1-loops" and "ψ_2-loops", reflecting the $\mathrm{SU}(2)_B$ label of the fermionic coupling.

There are several different cases of loops, the details of which are presented in the next section, but most possess this degeneracy. We do not see a trace of this degeneracy in M-theory and we expect it to be lifted by quantum corrections. We discuss this in some more detail in the discussion section.

The paper is organized as follows. In the next section we present the construction of the 1/2 BPS Wilson loops. We start with the case of segments of the quiver with alternating $k, -k$ CS levels. We then consider linear quivers, by removing a hypermultiplet or twisted hypermultiplet from a circular quiver. Finally, we consider the cases of segments of the quiver with vanishing CS levels. In section 3 we study the case of the circular Wilson loop. In section 4 we discuss how to calculate those Wilson loops using localization. The main point is that all those Wilson loops for quivers with alternating CS levels are classically cohomologically equivalent to 1/4 BPS loops, which can be evaluated in the matrix models of these theories. We expect, though, that this analysis receives quantum corrections and only a certain linear combination of those Wilson loops will be in fact quantum mechanically equivalent to the 1/4 BPS loops. For segments of the quiver comprising successive hypermultiplets, we have been able to show only for one of our proposed BPS loops that it is cohomologically equivalent to a 1/4 BPS loop. The situation with the other possible BPS Wilson loops supported on this part of the quiver the situation is not clear. In section 5 we discuss the M2-brane duals of these Wilson loops and comment on their degeneracies. We finally conclude with a discussion of the many remaining questions left open. The notations and some technicalities are relegated to appendices.

During the course of our work, a manuscript addressing the same question has appeared [25]. This prompted us to present the rich class of observables we have found, leaving their further study to the future. The Wilson loop found in [25] is one of those presented below for the particular theories which are orbifolds of ABJM. To be specific, it is the loop coupling to ψ_1 and in a representation of *all* of the nodes of the quiver.

2 Infinite straight line

Wilson loops are traditionally defined as the holonomy of the gauge connection. It was however found in [18] that the 1/2 BPS Wilson loops in ABJ(M) theory must be expressed in terms of the holonomy of a *superconnection* $\mathcal{L}$, with modified connections for the two $U(N_1)$ and $U(N_2)$ vector multiplets on the diagonal blocks and bifundamental Fermi fields in the off-diagonal blocks. The Wilson loop is then

$$\mathcal{W} = \mathrm{tr}_{\mathcal{R}}\, P \exp\left(-i \int d\tau \mathcal{L}(\tau)\right), \tag{2.1}$$

where $\mathcal{R}$ is a representation of the supergroup $U(N_1|N_2)$.

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The supersymmetry variation of the superconnection does not vanish, rather it is a total differential. In [26] this was expressed in terms of a supercovariant derivative of a supermatrix valued in the superalgebra, i.e.,

$$\delta\mathcal{L} = \mathfrak{D}_\tau G \equiv \partial_\tau G - i\{\mathcal{L}, G\}\,. \tag{2.2}$$

This is enough to guarantee that the Wilson loop is invariant under the corresponding supersymmetries. One must furthermore ensure that the trace is such that the boundary terms from integrating this term cancel each other. Our construction below of 1/2 BPS loops in $\mathcal{N}=4$ theories will be based on similar principles.

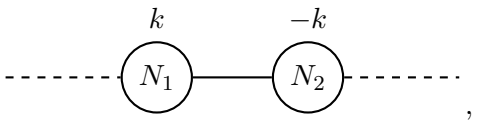
In this section, we start by considering an infinite spacelike line in the x^1-direction in $\mathbb{R}^3$ parametrized by

$$x^1 = \tau\,, \qquad x^2 = x^3 = 0\,. \tag{2.3}$$

2.1 Alternating CS levels

The theories we consider have a number of vector multiplets coupled by p hypermultiplets and q twisted hypermultiplets forming a circular or linear quiver. We study different possible sections of the quiver and find the Wilson loops supported on the relevant nodes.

We start by considering a segment of the quiver of the form



where a solid link corresponds to a hypermultiplet and a dashed link to a twisted hypermultiplet. Each node represents a $U(N_I)$ vector multiplet with Chern-Simons level k_I indicated above.

We begin by considering the variation of the gauge connection of the first node. Before proceeding, we define following [22], the useful bilinears

$$\begin{aligned}
&\nu_{(I)} = q^{\bar{A}}_{(I)}\bar{q}_{(I)\bar{A}}, && \tilde{\nu}_{(I)} = \bar{q}_{(I)\bar{A}}q^{\bar{A}}_{(I)}\,,\\
&(\mu_{(I)})^{\bar{A}}{}_{\bar{B}} = q^{\bar{A}}_{(I)}\bar{q}_{(I)\bar{B}} - \frac{1}{2}\delta^{\bar{A}}{}_{\bar{B}}(\nu_{(I)})^{\bar{C}}{}_{\bar{C}}, && (\tilde{\mu}_{(I)})^{\bar{A}}{}_{\bar{B}} = \bar{q}_{(I)\bar{B}}q^{\bar{A}}_{(I)} - \frac{1}{2}\delta^{\bar{A}}{}_{\bar{B}}(\tilde{\nu}_{(I)})^{\bar{C}}{}_{\bar{C}}\,,\\
&j^{\bar{A}\underline{B}a}_{(I)} = \sqrt{2}q^{\bar{A}}_{(I)}\bar{\psi}^{\underline{B}a}_{(I)} - \sqrt{2}\epsilon^{\bar{A}\bar{C}}\epsilon^{\underline{BD}}\psi^a_{(I)\underline{D}}\bar{q}_{(I)\bar{C}}\,, && \tilde{j}^{\bar{A}\underline{B}a}_{(I)} = \sqrt{2}\bar{\psi}^{\underline{B}a}_{(I)}q^{\bar{A}}_{(I)} - \sqrt{2}\epsilon^{\bar{A}\bar{C}}\epsilon^{\underline{BD}}\bar{q}_{(I)\bar{C}}\psi^a_{(I)\underline{D}}\,.
\end{aligned} \tag{2.4}$$

The currents j and $\tilde{j}$ are descendents of the moment maps μ and $\tilde{\mu}$.

The variation with supersymmetry parameters (1.2) is (see appendix A)

$$\delta A_{(1)1} = \frac{1}{k}\xi^+_{\bar{A}\underline{1}}\left(j^{\bar{A}\underline{1}}_{(1)+} - \tilde{j}^{\underline{1}\bar{A}}_{(0)+}\right) - \frac{1}{k}\xi^-_{\bar{A}\underline{2}}\left(j^{\bar{A}\underline{2}}_{(1)-} - \tilde{j}^{\underline{2}\bar{A}}_{(0)-}\right). \tag{2.5}$$

In CS-matter theories, it is natural to allow for a bilinear of the scalars in the connection [10, 15–18]. The variation of the moment map (2.4) associated to the twisted hypermultiplet on the left is

$$\delta(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} = \frac{i}{2}\xi^+_{\bar{A}\underline{1}}\tilde{j}^{\bar{A}\underline{1}}_{(0)+} - \frac{i}{2}\xi^-_{\bar{A}\underline{2}}\tilde{j}^{\bar{A}\underline{1}}_{(0)-}\,. \tag{2.6}$$

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We can eliminate all the terms in the variation of $A_{(1)}$ (2.5) that depend on the twisted hypermultiplets by taking the linear combination

$$\delta\left[A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}\right]=\frac{1}{k}\xi^{+}_{\bar{A}\underline{1}}j^{\bar{A}\underline{1}}_{(1)+}-\frac{1}{k}\xi^{-}_{\bar{A}\underline{2}}j^{\bar{A}\underline{2}}_{(1)-}\,. \tag{2.7}$$

Since $\tilde{\mu}$ is traceless, this is the same as adding $\frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{2}}{}_{\underline{2}}$.

We can also include terms from the scalars in the untwisted hypermultiplets. A term proportional to their moment map will lead to a connection invariant under half of the supercharges in (1.2), that is a 1/4 BPS Wilson loop. But our choice of of supersymmetries (1.2) distinguish between the twisted and untwisted fields. So for the untwisted fields consider the variation of ν

$$\delta\nu_{(1)}=i\sqrt{2}\left(\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^{-}_{\bar{B}\underline{2}}\psi_{(1)\underline{1}-}\bar{q}_{\bar{A}}+\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^{+}_{\bar{B}\underline{1}}\psi_{(1)\underline{2}+}\bar{q}_{\bar{A}}+\xi^{+}_{\bar{A}\underline{1}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{1}}_{(1)+}+\xi^{-}_{\bar{A}\underline{2}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{2}}_{(1)-}\right). \tag{2.8}$$

This does not package nicely in terms of the currents, so let us also expand (2.7) in terms of the component fields

$$\begin{aligned}&\delta\left[A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}\right]\\&\quad=\frac{\sqrt{2}}{k}\left(\xi^{+}_{\bar{A}\underline{1}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{1}}_{(1)+}-\xi^{-}_{\bar{A}\underline{2}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{2}}_{(1)-}-\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^{+}_{\bar{B}\underline{1}}\psi_{(1)\underline{2}+}\bar{q}_{(1)\bar{A}}+\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^{-}_{\bar{B}\underline{2}}\psi_{(1)\underline{1}-}\bar{q}_{(1)\bar{A}}\right).\end{aligned} \tag{2.9}$$

We see that we have all the same terms in those two expressions, but with different signs. We can therefore add or subtract $\nu_{(1)}$ from the gauge connection and reduce the variation to just two terms

$$\delta\left[A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}-\frac{i}{k}\nu_{(1)}\right]=\frac{2\sqrt{2}}{k}\left[\xi^{+}_{\bar{A}\underline{1}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{1}}_{(1)+}+\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^{-}_{\bar{B}\underline{2}}\psi_{(1)\underline{1}-}\bar{q}_{(1)\bar{A}}\right], \tag{2.10a}$$

$$\delta\left[A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}+\frac{i}{k}\nu_{(1)}\right]=-\frac{2\sqrt{2}}{k}\left[\xi^{-}_{\bar{A}\underline{2}}q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{2}}_{(1)-}+\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^{+}_{\bar{B}\underline{1}}\psi_{(1)\underline{2}+}\bar{q}_{(1)\bar{A}}\right]. \tag{2.10b}$$

In each case we find a non-vanishing variation, which we will have to cancel by considering a superconnection, with fermionic couplings either $\bar{\psi}^{\underline{1}}_{(1)+}$ and $\psi_{(1)\underline{1}-}$ for the first sign choice, or $\bar{\psi}^{\underline{2}}_{(1)-}$ and $\psi_{(1)\underline{2}+}$ in the second case. We refer in the following to the two respective loops as "ψ_1-loops" and "ψ_2-loops".

2.1.1 The ψ_1-loop

Thus, following [18, 26], we try introducing a super-connection whose top left block is given by (2.10a). On dimensional and Graßmann odd/even grounds, it can be seen that the (1,2) and (2,1) component of the superconnection will be of the form, $\bar{c}^{A}_{\underline{a}}\psi^{\underline{a}}_{(1)\underline{A}}$ and $c^{\underline{a}}_{\underline{A}}\bar{\psi}^{\underline{A}}_{(1)\underline{a}}$, where the spinor couplings $\bar{c}^{A}_{\underline{a}}$ and $c^{\underline{a}}_{\underline{A}}$ are Graßmann even. The supersymmetry conditions (2.2) tell us that we must write the variation of this entry with respect to the supersymmetries as a covariant derivative with respect to some specified modified bosonic connections. By assumption, we firstly consider taking the modified bosonic connection of the first node is

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given by (2.10a), and we fix the form of the second connection such that the conditions (2.2) are satisfied.

As such, we must choose the fermionic couplings such that the covariant derivative in the variation of the fermions is projected along the x^1 direction. This requires $c^-_{\underline{1}} = c^+_{\underline{2}} = \bar{c}^{\underline{1}}_- = \bar{c}^{\underline{2}}_+ = 0$. Motivated by the lack of an appearance of ψ_2 and $\bar{\psi}^2$ in (2.10a) in the context of the supersymmetry conditions (2.2), let us further assume that $c^-_{\underline{2}} = \bar{c}^{\underline{2}}_- = 0$. We thus write the superconnection as

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \bar{c}\psi^+_{(1)\underline{1}} \\ c\bar{\psi}^{\underline{1}}_{(1)+} & \star \end{pmatrix}, \tag{2.11}$$

where c, $\bar{c}$ and '$\star$' remain to be fixed.

As the variation of the bosonic connection contains no derivatives, the supermatrix G of (2.2) is of the form

$$G^{\psi_1} = \begin{pmatrix} 0 & g \\ \bar{g} & \star \end{pmatrix}. \tag{2.12}$$

Towards this, consider

$$\begin{aligned} \delta\psi^+_{(1)\underline{1}} = {} & \sqrt{2}(\not{D})^+{}_+ q^{\bar{A}}_{(1)}\, \xi^+_{\bar{A}\underline{1}} - \frac{\sqrt{2}}{k}\xi^+_{\bar{A}\underline{1}}\left(\nu_{(1)}q^{\bar{A}}_{(1)} - q^{\bar{A}}_{(1)}\tilde{\nu}_{(1)}\right) \\ & - \frac{2\sqrt{2}}{k}\xi^+_{\bar{A}\underline{1}}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} q^{\bar{A}}_{(1)} + \frac{2\sqrt{2}}{k}\xi^+_{\bar{A}\underline{1}} q^{\bar{A}}_{(1)}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}}. \end{aligned} \tag{2.13}$$

Thus, we may write

$$\bar{c}\,\delta\psi^+_{(1)\underline{1}} = \mathcal{D}_\tau g, \tag{2.14}$$

with

$$\begin{aligned} & g \equiv \sqrt{2}(\bar{c}^{\underline{1}})_+ q^{\bar{A}}_{(1)}(\xi_{\bar{A}\underline{1}})^+, \\ & \mathcal{D}_\tau g \equiv D_\tau g + i\left(\frac{i}{k}\nu_{(1)} + \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}\right) g - ig\left(\frac{i}{k}\tilde{\nu}_{(1)} + \frac{2i}{k}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}}\right), \\ & D_\tau g \equiv \partial_\tau g - iA_{(1)1} g + igA_{(2)1}, \end{aligned} \tag{2.15}$$

in agreement with the modified connection for the loop in (2.10a). We are now furthermore find that the bottom right block has be be $A_{(2)} - \frac{2i}{k}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)}$.

Similarly, we may use the variation of the bottom left block

$$\begin{aligned} \delta\bar{\psi}^{\underline{1}}_{(1)+} = {} & \sqrt{2}(\not{D})_{+-}\bar{q}^{\bar{A}}_{(1)}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^-_{\bar{B}\underline{2}} + \frac{\sqrt{2}}{k}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}+}\left[\tilde{\nu}_{(1)}\bar{q}_{(1)\bar{A}} - \bar{q}_{(1)\bar{A}}\nu_{(1)}\right] \\ & - \frac{2\sqrt{2}}{k}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}+}\bar{q}_{(1)\bar{A}}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} + \frac{2\sqrt{2}}{k}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}+}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}}\bar{q}_{(1)\bar{A}}, \end{aligned} \tag{2.16}$$

to write

$$c\,\delta\bar{\psi}^{\underline{1}}_{(1)+} = \mathcal{D}_\tau\bar{g}, \tag{2.17}$$

where

$$\bar{g} \equiv -\sqrt{2}\,c\,\bar{q}^{\bar{A}}_{(1)}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}+}, \tag{2.18}$$

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and the modified bosonic connections agree with (2.15). It now remains to check the variations of the bosonic connections satisfy the conditions coming from (2.2), i.e.,

$$\delta\left[A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}}-\frac{i}{k}\nu_{(1)}\right]=-i\bar{c}\,(\psi_{(1)\underline{1}})^{+}\bar{g}+igc\,(\bar{\psi}^{\underline{1}}_{(1)})_{+}, \tag{2.19a}$$

$$\delta\left[A_{(2)1}-\frac{2i}{k}(\mu_{(2)})^{1}{}_{\underline{1}}-\frac{i}{k}\tilde{\nu}_{(1)}\right]=-ic\,(\bar{\psi}^{\underline{1}}_{(1)})_{+}g+i\bar{g}\bar{c}\,(\psi_{(1)\underline{1}})^{+}. \tag{2.19b}$$

A simple calculation shows that these conditions hold, provided $c\bar{c}=-\frac{2i}{k}$. The value of c itself is immaterial, as it always will appear in the combination $c\bar{c}$ in the trace of the superconnection. We choose therefore $c=\bar{c}=\frac{1-i}{\sqrt{k}}$. In summary, we find that the loop with super-connection

$$\mathcal{L}^{\psi_1}=\begin{pmatrix}A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}}-\frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\,\psi^{+}_{(1)\underline{1}}\\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{1}}_{(1)+} & A_{(2)1}-\frac{2i}{k}(\mu_{(2)})^{1}{}_{\underline{1}}-\frac{i}{k}\tilde{\nu}_{(1)}\end{pmatrix} \tag{2.20}$$

preserves the supersymmetries (1.2).

2.1.2 The ψ_2-loop

We now consider the second modified bosonic connection (2.10b) $A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}}+\frac{i}{k}\nu_{(1)}$ and proceed similarly to the previous section.

We find that this works with

$$\bar{g}\equiv\sqrt{2}\,c\,\bar{q}_{(1)\bar{A}}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi_{\bar{B}\underline{1}-}, \tag{2.21}$$

and the super-connection of the ψ_2-loop is given by

$$\mathcal{L}^{\psi_2}=\begin{pmatrix}A_{(1)1}-\frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}}+\frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\,\psi^{-}_{(1)\underline{2}}\\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{2}}_{(1)-} & A_{(2)1}-\frac{2i}{k}(\mu_{(2)})^{1}{}_{\underline{1}}+\frac{i}{k}\tilde{\nu}_{(1)}\end{pmatrix}. \tag{2.22}$$

Indeed using

$$\begin{aligned}\delta\psi^{-}_{(1)\underline{2}}=&\sqrt{2}(\not{D})^{-}{}_{-}q^{\bar{A}}_{(1)}\,\xi^{-}_{\bar{A}\underline{2}}-\frac{\sqrt{2}}{k}\xi^{-}_{\bar{A}\underline{2}}\left(\nu_{(1)}q^{\bar{A}}_{(1)}-q^{\bar{A}}_{(1)}\tilde{\nu}_{(1)}\right)\\&-\frac{2\sqrt{2}}{k}\xi^{-}_{\bar{A}\underline{2}}(\tilde{\mu}_{(0)})^{\underline{2}}{}_{\underline{2}}q^{\bar{A}}_{(1)}+\frac{2\sqrt{2}}{k}\xi^{-}_{\bar{A}\underline{2}}q^{\bar{A}}_{(1)}(\mu_{(2)})^{\underline{2}}{}_{\underline{2}},\end{aligned} \tag{2.23}$$

and the variation of the bosonic connections we find that the supermatrix appearing in the covariant derivative (2.2) is

$$G^{\psi_2}=\begin{pmatrix}0 & -\frac{\sqrt{2}(1-i)}{\sqrt{k}}\,q^{\bar{A}}_{(1)}\,\xi^{-}_{\bar{A}\underline{2}}\\ \frac{\sqrt{2}(1-i)}{\sqrt{k}}\,\bar{q}_{(1)\bar{A}}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi_{\bar{B}\underline{1}-} & 0\end{pmatrix}. \tag{2.24}$$

We find for each hypermultiplet in a quiver connecting nodes with CS levels k to $-k$ a pair of superconnections. Out of each of them we can construct a Wilson loop in an arbitrary representation of the supergroup $U(N_1|N_2)$. We in fact expect to find only one Wilson loop for each representation of such a pair. We comment about this in the discussion section.

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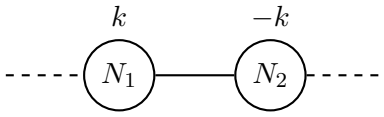
2.2 Linear quivers

Before proceeding to the case of quivers with nodes with vanishing CS levels, let us comment on linear quivers. We can construct any linear quiver by starting from a circular quiver and removing a hypermultiplet or a twisted hypermultiplet (or a pair and the intermediate vector multiplet). The relevant loops are gotten from those constructed above by erasing the coupling to the removed fields.

We start from the segment of the quiver considered above



The loops we constructed couple to scalars and fermions of the hypermultiplet between the nodes and to the scalars of the two adjacent twisted hypermultiplets.

Consider removing the twisted hypermultiplet to the right of the second node. Erasing it from the ψ_1-loop (2.20) gives

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\,\psi^{+}_{(1)\underline{1}} \\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{1}}_{(1)+} & A_{(2)1} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \tag{2.25}$$

and it is easy to check that this is indeed is 1/2 BPS, with the appropriate supersymmetry transformations of the linear quiver in appendix A. Similarly, the ψ_2-loop (2.22) becomes

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}} + \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\,\psi^{-}_{(1)\underline{2}} \\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{2}}_{(1)-} & A_{(2)1} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}. \tag{2.26}$$

Likewise we can remove the twisted hypermultiplet from the left which will remove the coupling to the $\tilde{\mu}_{(0)}$ moment map from the above.

If we instead remove the hypermultiplet connecting the N_1 and N_2 nodes we expect to lose the structure of the superconnection. Indeed, in that case we find two independent 1/2 BPS Wilson loops, each with a gauge connection and a coupling to the twisted moment maps

$$\mathcal{L}_{(1)} = A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{1}{}_{\underline{1}}\,, \qquad \mathcal{L}_{(2)} = A_{(2)1} - \frac{2i}{k}(\mu_{(2)})^{1}{}_{\underline{1}}\,. \tag{2.27}$$

Note that both ψ_1 and ψ_2-loops give the same loops via this process, so we lose that two-fold degeneracy. Also, these loops are essentially the same as the usual 1/4 BPS loops one gets from a modified connection with the scalar φ of the vector multiplet. In this formulation this scalar has been integrated out giving rise to the couplings to μ. Those loops are 1/2 BPS in theories with $\mathcal{N} = 2$ supersymmetry and in this particular case get enhanced to preserve 4 supercharges, rather than 2.

A similar story holds when considering linear quivers that end on nodes with vanishing CS levels. The case with vanishing levels inside a quiver are studied in the following two sections. Again, one can remove a hypermultiplet or twisted hypermultiplet to open up the quiver and the Wilson loops with those fields removed would remain 1/2 BPS. We do not list all the examples or repeat the algebra for all of them.

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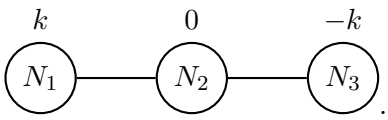
2.3 Vanishing CS levels: Repeated untwisted hypermultiplets

We have thus far considered Chern-Simons-matter quivers with alternating $\pm k$ Chern-Simons levels. Here we study the Wilson loops in the case with extra nodes of vanishing CS levels [20, 22] that preserve $\mathcal{N} = 4$ supersymmetry.

Let us consider the quiver with a node with vanishing CS coupling to a *pair* of untwisted hypermultiplets



We have removed for convenience the twisted hypermultiplets from the left and the right, but they can be incorporated by adding the moment maps to the gauge connection.

We start by mimicking the construction of superconnections in the alternating levels case and find several different possible connections involving pairs of nodes. Then in section 2.3.4 we find a new structure of a superconnection which couples to all the different fields of the *three* nodes. We expect this structure to be more fundamental than the ones involving only pairs of nodes.

2.3.1 The ψ_1-loop for the first link

We begin by constructing the ψ_1-loop for the first link. The analysis proceeds as in section 2.1.1: We take the same connection for the first node, since it couples to the same fields as previously (only that now, since we don't have twisted hypermultiplets, we don't need to include a $\tilde{\mu}_{(0)}$ contribution). We then continue to study the fermions and the connection on the second node.

Requiring that the variation of the bosonic connection in the first node is a commutator with $\psi_{(1)}$ gives[2]

$$\mathcal{L}^{\psi_1}_{(1)} = \begin{pmatrix} A_{(1)1} - \frac{i}{k}\nu_{(1)} & \bar{c}\,\psi^{+}_{(1)\underline{1}} \\ c\,\bar{\psi}^{\underline{1}}_{(1)+} & \star \end{pmatrix}, \tag{2.28}$$

where it remains to fix '$\star$' and the fermionic couplings c and $\bar{c}$.

We fix the form of the bosonic connection of the second node by studying the variation of the fermionic terms

$$\begin{aligned} \delta\psi^{+}_{(1)\underline{1}} = \sqrt{2}D_\tau q^{\bar{A}}_{(1)}\xi^{+}_{\bar{A}\underline{1}} - \frac{\sqrt{2}}{k}\xi^{+}_{\bar{A}\underline{1}}\left[\nu_{(1)}q^{\bar{A}}_{(1)} - q^{\bar{A}}_{(1)}\tilde{\nu}_{(1)}\right] \\ - \frac{2\sqrt{2}}{k}\xi^{+}_{\bar{A}\underline{1}}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}q^{\bar{A}}_{(1)} - \sqrt{2}\xi^{+}_{\bar{A}\underline{1}}q^{\bar{A}}_{(1)}(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}}. \end{aligned} \tag{2.29}$$

If we require that the variation of the fermion is a total derivative this fixes the second connection to $A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)}$.

Furthermore, it fixes

$$g_{(1)} = \sqrt{2}\bar{c}q^{\bar{A}}_{(1)}\xi^{+}_{\bar{A}\underline{1}}. \tag{2.30}$$

[2]The '(1)' subscript for $\mathcal{L}$ indicates that we are attempting to construct a $U(N_1|N_2)$ valued connection.

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It is also in agreement with the bosonic connection of the first node given above. The variation of the conjugate fermion suggests the same bosonic connections as well as

$$\bar{g}_{(1)} = -\sqrt{2}c\,\bar{q}_{(1)\bar{A}}\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}+}, \tag{2.31}$$

cf. the calculation in section 2.1.1. This suggests that the form of the supermatrix $G_{(1)}$ is identical to that of the alternating level case.

We see that if we take

$$\mathcal{L}^{\psi_1}_{(1)} = \begin{pmatrix} A_{(1)1} - \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\,\psi^{+}_{(1)\underline{1}} \\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{1}}_{(1)+} & A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \tag{2.32}$$

then we verified that the variation of all but the bottom right block are total covariant superderivatives. The variation of that block is

$$\begin{aligned} &\delta\left[A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)}\right] \\ &= \frac{2\sqrt{2}}{k}\left[\xi^{+}_{\bar{A}\underline{1}}\bar{\psi}^{\underline{1}}_{(1)+}q^{\bar{A}}_{(1)} + \xi^{\bar{A}\underline{1}-}\bar{q}_{(1)\bar{A}}\psi_{(1)\underline{1}-}\right] + \frac{1}{2k}\left(\xi^{+}_{\bar{A}\underline{1}}\left[j^{\bar{A}\underline{1}}_{(2)+} - \tilde{j}^{\bar{A}\underline{1}}_{(1)+}\right] - \xi^{-}_{\bar{A}\underline{2}}\left[j^{\bar{A}\underline{2}}_{(2)-} - \tilde{j}^{\bar{A}\underline{2}}_{(1)-}\right]\right). \end{aligned} \tag{2.33}$$

The first two terms are exactly as for the variation of the bosonic connection of the second node in section 2.1.1. Thus we can write

$$\delta\mathcal{L}^{\psi_1}_{(1)} \equiv \mathfrak{D}^{\psi_1}_{(1)\tau}G^{\psi_1}_{(1)} + \Delta J\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta J = \frac{1}{2k}\left(\xi^{+}_{\bar{A}\underline{1}}\left[j^{\bar{A}\underline{1}}_{(2)+} - \tilde{j}^{\bar{A}\underline{1}}_{(1)+}\right] - \xi^{-}_{\bar{A}\underline{2}}\left[j^{\bar{A}\underline{2}}_{(2)-} - \tilde{j}^{\bar{A}\underline{2}}_{(1)-}\right]\right), \tag{2.34}$$

where $\mathfrak{D}^{\psi_1}_{(1)\tau}$ is a supercovariant derivative with respect the superconnection $\mathcal{L}^{\psi_1}_{(1)}$.

We shall come back to discuss the ΔJ term in section 2.3.5. For now we examine more possibilities for superconnections.

2.3.2 The ψ_2-loop for the first link

We can also take the superconnection

$$\mathcal{L}^{\psi_2}_{(1)} = \begin{pmatrix} A_{(1)1} + \frac{i}{k}\nu_{(1)} & \bar{c}\,\psi^{-}_{(1)\underline{2}} \\ c\,\bar{\psi}^{\underline{2}}_{(1)-} & A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \tag{2.35}$$

and find

$$\delta\mathcal{L}^{\psi_2}_{(1)} \equiv \mathfrak{D}^{\psi_2}_{(1)\tau}G^{\psi_2}_{(1)} + \Delta J\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.36}$$

where $G^{\psi_2}_{(1)}$ is the supermatrix for the alternating level case.

2.3.3 Superconnections in the second link

We can also consider superconnections involving the vector fields $A_{(2)}$ and $A_{(3)}$ and the hypermultiplet connecting them. By reflection it is clear that the resulting loops would have superconnections

$$\mathcal{L}^{\psi_1}_{(1)} = \begin{pmatrix} A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\nu_{(2)} & \frac{1-i}{\sqrt{k}}\,\psi^{+}_{(2)\underline{1}} \\ \frac{1-i}{\sqrt{k}}\,\bar{\psi}^{\underline{1}}_{(2)+} & A_{(3)1} - \frac{i}{k}\tilde{\nu}_{(2)} \end{pmatrix}, \tag{2.37}$$

satisfying

$$\delta\mathcal{L}^{\psi_1}_{(2)} = \mathfrak{D}^{\psi_1}_{(2)\tau} G^{\psi_1}_{(2)} - \Delta J \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.38}$$

where $G^{\psi_1}_{(2)}$ is the supermatrix found in the alternating level case.

Likewise

$$\mathcal{L}^{\psi_2}_{(2)} = \begin{pmatrix} A_{2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{k}\nu_{(2)} & \frac{1-i}{\sqrt{k}}\psi^{-}_{(2)\underline{2}} \\ \frac{1-i}{\sqrt{k}}\bar{\psi}^{\underline{2}}_{(2)-} & A_{(3)1} + \frac{i}{k}\tilde{\nu}_{(2)} \end{pmatrix}, \tag{2.39}$$

satisfies

$$\delta\mathcal{L}^{\psi_1}_{(2)} = \mathfrak{D}^{\psi_2}_{(2)\tau} G^{\psi_2}_{(2)} - \Delta J \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.40}$$

where $G^{\psi_2}_{(2)}$ is the supermatrix found in the alternating level case.

2.3.4 A 4×4 construction

Thus far we have found four superconnections involving pairs of vector fields and the relevant hypermultiplets. We can construct from them Wilson loops in arbitrary representations of $U(N_1|N_2)$ and $U(N_2|N_3)$. We can also combine pairs of them together, to form block diagonal superconnections with a 4×4 block structure, with the central node represented twice. Those will allow to write Wilson loops in arbitrary representations of $U(N_1 + N_3|2N_2)$.

In fact, this 4×4 ansatz allows for a more general superconnection, which is not block diagonal. In addition to bosonic entries transforming in the adjoints of the individual nodes, and fermions in the bifundamentals of adjacent nodes, we have two bosonic entries transforming in bifundamentals of $U(N_1) \times U(N_3)$. On dimensional and representation grounds, they have to take the form $\bar{d}_{\bar{A}\bar{B}} q^{\bar{A}}_{(1)} q^{\bar{B}}_{(2)}$ and $d^{\bar{A}\bar{B}} \bar{q}_{(2)\bar{A}} q_{(1)\bar{B}}$, respectively. We got the construction below to work with the antisymmetric couplings $\bar{d}_{\bar{A}\bar{B}} = \bar{d}\epsilon_{\bar{A}\bar{B}}$ and $d^{\bar{A}\bar{B}} = d\epsilon^{\bar{A}\bar{B}}$.

Starting from a general ansatz we have been able to show that the structure of the superconnection has to be either the block diagonal ones made of pairs of superconnections discussed above or the superconnection

$$\mathcal{L} = \begin{pmatrix} A_{(1)1} & \bar{c}^{\underline{1}}_{(1)} \psi^{+}_{(1)\underline{1}} & \bar{c}^{\underline{2}}_{(1)} \psi^{-}_{(1)\underline{2}} & \bar{d}\epsilon_{\bar{A}\bar{B}} q^{\bar{A}}_{(1)} q^{\bar{B}}_{(2)} \\ c_{(1)\underline{1}} \bar{\psi}^{\underline{1}}_{(1)+} & \mathcal{A}_{(2)} & 0 & \bar{c}^{\underline{2}}_{(2)} \psi^{-}_{(2)\underline{2}} \\ c_{(1)\underline{2}} \bar{\psi}^{\underline{2}}_{(1)-} & 0 & \mathcal{A}'_{(2)} & \bar{c}^{\underline{1}}_{(2)} \psi^{+}_{(2)\underline{1}} \\ d\epsilon^{\bar{A}\bar{B}} \bar{q}_{(2)\bar{A}} q_{(1)\bar{B}} & c_{(2)\underline{2}} \bar{\psi}^{\underline{2}}_{(2)-} & c_{(2)\underline{1}} \bar{\psi}^{\underline{1}}_{(2)+} & A_{(3)1} \end{pmatrix}, \tag{2.41}$$

with $\mathcal{A}_{(2)} \equiv A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{2k}\left(\nu_{(2)} - \tilde{\nu}_{(1)}\right)$ and $\mathcal{A}'_{(2)} \equiv A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{2k}\left(\nu_{(2)} - \tilde{\nu}_{(1)}\right)$. The connection in the top left and bottom right corners involve only the gauge fields (and one could include the appropriate moment maps, if coupling to extra twisted hypermultiplets). It is rather interesting that in the case of alternating levels we had to augment all gauge connections by a coupling to ν with a coefficient $\pm i/k$. Indeed that is also what we found in the loops associated to the first and second link. But here we have the third option of not including this coupling for the external nodes.[3]

[3] We were not able to find BPS Wilson loops with any other values for the coupling to ν.

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The supermatrix G is given by

$$G = \begin{pmatrix} 0 & g_{(1)\underline{1}} & g_{(1)\underline{2}} & 0 \\ \bar{g}^{\underline{1}}_{(1)} & 0 & 0 & g_{(2)\underline{2}} \\ \bar{g}^{\underline{2}}_{(1)} & 0 & 0 & g_{(2)\underline{1}} \\ 0 & \bar{g}^{\underline{2}}_{(2)} & \bar{g}^{\underline{1}}_{(2)} & 0 \end{pmatrix}, \tag{2.42}$$

where

$$\begin{aligned} g_{(1)\underline{1}} &= \sqrt{2}\bar{c}^{\underline{1}}_{(1)} q^{\bar{A}}_{(1)} \xi^{+}_{\bar{A}\underline{1}}\,, & g_{(1)\underline{2}} &= -\sqrt{2}\bar{c}^{\underline{2}}_{(1)} q^{\bar{A}}_{(1)} \xi^{-}_{\bar{A}\underline{2}}\,, \\ g_{(2)\underline{1}} &= \sqrt{2}\bar{c}^{\underline{1}}_{(2)} q^{\bar{A}}_{(2)} \xi^{+}_{\bar{A}\underline{1}}\,, & g_{(2)\underline{2}} &= -\sqrt{2}\bar{c}^{\underline{2}}_{(2)} q^{\bar{A}}_{(2)} \xi^{-}_{\bar{A}\underline{2}}\,, \\ \bar{g}^{\underline{1}}_{(1)} &= -\sqrt{2} c_{(1)\underline{1}} \bar{q}_{(1)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{12}} \xi_{\bar{B}\underline{2}+}\,, & \bar{g}^{\underline{2}}_{(1)} &= \sqrt{2} c_{(1)\underline{2}} \bar{q}_{(1)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{21}} \xi_{\bar{B}\underline{1}-}\,, \\ \bar{g}^{\underline{1}}_{(2)} &= -\sqrt{2} c_{(2)\underline{1}} \bar{q}_{(2)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{12}} \xi_{\bar{B}\underline{2}+}\,, & \bar{g}^{\underline{2}}_{(2)} &= \sqrt{2} c_{(2)\underline{2}} \bar{q}_{(2)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{21}} \xi_{\bar{B}\underline{1}-}\,. \end{aligned} \tag{2.43}$$

We firstly consider the variations of the bosonic components, listing below the corresponding conditions which must be imposed for supersymmetry.

$$\begin{aligned} &\text{11 entry:} && c^{\underline{1}}_{(1)} \bar{c}_{(1)\underline{1}} = c^{\underline{2}}_{(1)} \bar{c}_{(1)\underline{2}} = -\frac{i}{k}\,, \\ &\text{44 entry:} && c^{\underline{1}}_{(2)} \bar{c}_{(2)\underline{1}} = c^{\underline{2}}_{(2)} \bar{c}_{(2)\underline{2}} = -\frac{i}{k}\,, \\ &\text{14 entry:} && -\bar{c}^{\underline{1}}_{(1)} \bar{c}^{\underline{2}}_{(2)} = \bar{c}^{\underline{2}}_{(1)} \bar{c}^{\underline{1}}_{(2)} = \bar{d}\,, \\ &\text{41 entry:} && -c_{(1)\underline{1}} c_{(2)\underline{2}} = c_{(1)\underline{2}} c_{(2)\underline{1}} = d\,. \end{aligned} \tag{2.44}$$

The supersymmetry conditions corresponding to the 22 and 33 entries are then automatically satisfied.

As for the central 23 and 32 entries in (2.41) the difference between the supersymmetry variation and the super-covariant derivative of G gives

$$\begin{aligned} (\delta\mathcal{L} - \mathfrak{D}_\tau G)_{23} &= -i \left(c_{(1)\underline{1}} \bar{c}^{\underline{2}}_{(1)} \tilde{j}^{\bar{A}\underline{1}}_{(1)+} + c_{(2)\underline{1}} \bar{c}^{\underline{2}}_{(2)} j^{\bar{A}\underline{1}}_{(2)+} \right) \xi^{-}_{\bar{A}\underline{2}}, \\ (\delta\mathcal{L} - \mathfrak{D}_\tau G)_{32} &= i \left(c_{(1)\underline{2}} \bar{c}^{\underline{1}}_{(1)} \tilde{j}^{\bar{A}\underline{2}}_{(1)-} + c_{(2)\underline{2}} \bar{c}^{\underline{1}}_{(2)} j^{\bar{A}\underline{2}}_{(2)-} \right) \xi^{+}_{\bar{A}\underline{1}}, \end{aligned} \tag{2.45}$$

where $\mathfrak{D}_\tau$ is the super-covariant derivative with respect to the super-connection $\mathcal{L}$. Imposing the bilinear constraints (2.44), we see that $(\delta\mathcal{L} - \mathfrak{D}_\tau G)_{23}$ and $(\delta\mathcal{L} - \mathfrak{D}_\tau G)_{32}$ are proportional to $j^{\bar{A}\underline{1}}_{(2)+} - \tilde{j}^{\bar{A}\underline{1}}_{(1)+}$ and $j^{\bar{A}\underline{2}}_{(2)-} - \tilde{j}^{\bar{A}\underline{2}}_{(1)-}$, respectively. This is similar to the 2×2 cases discussed above.

With the constraints (2.44), the supersymmetry conditions for the fermionic entries of (2.41) are satisfied, up to a remainder term, which is proportional to $(\mu_{(2)})^{\bar{A}}{}_{\bar{B}} - (\tilde{\mu}_{(1)})^{\bar{A}}{}_{\bar{B}}$. We present the example of the 12 entry. Consider

$$(\mathfrak{D}_\tau G)_{12} = \partial_\tau g_{(1)\underline{1}} - i A_{(1)1} g_{(1)\underline{1}} + i g_{(1)\underline{1}} \mathcal{A}_{(2)} - i\bar{d} \epsilon_{\bar{A}\bar{B}} q^{\bar{A}}_{(1)} q^{\bar{B}}_{(2)} \bar{g}^{\underline{2}}_{(2)}. \tag{2.46}$$

With some manipulation it may be seen that

$$\begin{aligned} \epsilon_{\bar{A}\bar{B}} q^{\bar{A}}_{(1)} q^{\bar{B}}_{(2)} \bar{g}^{\underline{2}}_{(2)} = \sqrt{2} c_{(2)\underline{2}} \Big(& \nu_{(1)} q^{\bar{A}}_{(1)} - \frac{1}{2} q^{\bar{A}}_{(1)} \left(\tilde{\nu}_{(1)} + \nu_{(2)} \right) \xi^{+}_{\bar{A}\underline{1}} \\ & - \epsilon_{\bar{A}\bar{B}} q^{\bar{A}}_{(1)} \left((\mu_{(2)})^{\bar{B}}{}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}{}_{\bar{C}} \right) \epsilon^{\bar{C}\bar{D}} \xi_{\bar{D}\underline{1}-} \Big). \end{aligned} \tag{2.47}$$

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This identity, coupled with (2.44) gives us that

$$(\delta\mathcal{L} - \mathfrak{D}_\tau G)_{12} = -i\bar{d}\epsilon_{\bar{A}\bar{B}}q^{\bar{A}}_{(1)}\left((\mu_{(2)})^{\bar{B}}{}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}{}_{\bar{C}}\right)\epsilon^{\bar{C}\bar{D}}\xi_{\bar{D}\underline{1}-}. \tag{2.48}$$

Indeed all fermionic entries have such a remainder term, proportional to $(\mu_{(2)})^{\bar{B}}{}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}{}_{\bar{C}}$. We discuss these remainder terms in the next section.

We have also tried to construct Wilson loops with 3×3 and 6×6 block structure, but couldn't find any useful ones. We expect then that the Wilson loops will be given by this connection, or that made of a pair of the previous 2×2 connections, or a linear combination thereof. In any case the Wilson loops will be classified by representations of $U(N_1 + N_3|2N_2)$.

The story for longer segments of the quiver with vanishing CS levels should be similar. The even entries will be along the diagonal or in the bifundamental of $U(N_I) \times U(N_{I+2})$ and made of a bilinear of the scalars. In the 4-node case the resulting connection would be a $U(N_1 + 2N_3|2N_2 + N_4)$ matrix, and the generalization to longer quivers is obvious.

2.3.5 On ΔJ and $\Delta\mu$

In the alternating level case we found a pair of superconnections whose variation is a total derivative. In the case with a single $k = 0$ node we found five possible superconnections but in all cases we found an extra term in the variation proportional to ΔJ (2.34). One can add more hypermultiplets and more nodes with vanishing CS levels and the same construction leads to superconnections whose variation includes terms proportional to ΔJ terms on the different nodes with $k = 0$. For the construction with the 4×4 connection we also found the remainder $\Delta\mu$ term (2.48)

In the case of alternating levels we found a superconnection whose variation doesn't vanish, but like in ABJM theory, it is a total derivative. So the variation of the Wilson loop constructed out of the superconnection does vanish. In this case, the variation of the Wilson loop built out of any of the four superconnections will not vanish, but will rather give the insertion of the integral of ΔJ into the Wilson loop.

Still, we expect there to be a BPS Wilson loop associated with this segment of the quiver. Indeed, examining the action [22] for the vector multiplets with $k = 0$ one notices that the gaugino λ appears only linearly and multiplying ΔJ. Thus the variation of the Wilson loop is an insertion of $\frac{\delta}{\delta\lambda}$ into the Wilson loop and the path integral over λ of any observable will be a total Graßmann derivative and will therefore vanish. We conclude that the variation of the Wilson loop is zero in the weak sense — all correlation functions with it vanish.

As for the $\Delta\mu$ piece (2.48). Again the action of the vector multiplet includes this combination multiplied by an auxiliary field. Integrating it out identifies $(\mu_{(2)})^{\bar{B}}{}_{\bar{C}} = (\tilde{\mu}_{(1)})^{\bar{B}}{}_{\bar{C}}$, so this term does not obstruct the supersymmetry analysis, and the 4×4 loops are thus also supersymmetric in the weak sense.

2.4 Vanishing CS levels: Repeated twisted hypermultiplets

One last simple case to consider is a segment of the quiver of the form

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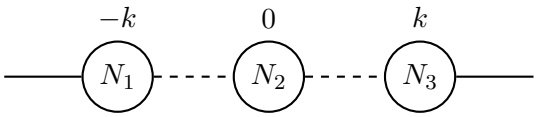


The nodes on the left and on the right will couple to extra nodes through the hypermultiplets (solid lines) and form superconnections with them (or if we remove the hypermultiplets and consider a linear quiver, will have 1/2 BPS connections like in (2.27)). In either case we can construct a 1/2 BPS Wilson loop coupling to the central node alone

$$\mathcal{L}_{(2)} = A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\left((\mu_{(2)})^{\underline{1}}{}_{\underline{1}} + (\tilde{\mu}_{(1)})^{\underline{1}}{}_{\underline{1}}\right). \tag{2.49}$$

3 Circular Wilson loops

For every straight BPS Wilson loop we expect to also find a circular one which will have a finite expectation value calculable using the localization matrix model. We study those here. The circle is given by

$$x^1 = \cos\tau\,, \qquad x^2 = \sin\tau\,, \qquad x^3 = 0\,. \tag{3.1}$$

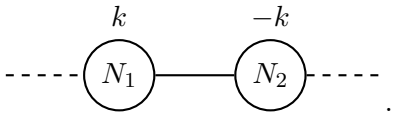
Whereas the straight line preserved half of the Poincaré supersymmetries and half of the superconformal ones, we expect the circular loop to preserve eight linear combinations of the two. Concretely, we look for Wilson loops that preserve supersymmetries where the superconformal variation parameters η are related to the superpoincaré ones ξ by

$$\eta^a_{\bar{A}\underline{1}} = i(\sigma^2)^a{}_b\,\xi^b_{\bar{A}\underline{1}}\,, \qquad \eta^a_{\bar{A}\underline{2}} = -i(\sigma^2)^a{}_b\,\xi^b_{\bar{A}\underline{2}}\,. \tag{3.2}$$

The superconformal transformations of the fields are given in the usual way [27] by replacing $(\xi_{\bar{A}\underline{B}})^a \to x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{A}\underline{B}})^b$, except for the variations of the fermions which pick up an extra term (see appendix A).

3.1 Alternating levels

Let us consider the same segment as for the straight line case in section 2.1



As in the straight line case, contribution to the variation of the gauge field from the adjoining twisted hypermultiplet may be cancelled by considering the combination

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$A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}}$. Futhermore, we may consider the two combinations

$$\begin{aligned}
&\delta\left[\dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\nu_{(1)}\right] \\
&= \frac{2\sqrt{2}}{k}\left\{\left[(1-\sin\tau)\xi^+_{\bar{A}\underline{1}} + \cos\tau\,\xi^-_{\bar{A}\underline{1}}\right] q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{1}}_{(1)+} + \left[\cos\tau\,\xi^+_{\bar{A}\underline{1}} + (1+\sin\tau)\xi^-_{\bar{A}\underline{1}}\right] q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{1}}_{(1)-}\right. \\
&\quad + \left[(1+\sin\tau)\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^+_{\bar{B}\underline{2}} - \cos\tau\,\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^-_{\bar{B}\underline{2}}\right]\psi_{(1)\underline{1}+}\bar{q}^{\bar{A}}_{(1)} \\
&\quad \left. + \left[-\cos\tau\,\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi^+_{\bar{B}\underline{2}} + (1-\sin\tau)\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}}\right]\psi_{(1)\underline{1}-}\bar{q}^{\bar{A}}_{(1)}\right\},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\delta\left[\dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{k}\nu_{(1)}\right] \\
&= -\frac{2\sqrt{2}}{k}\left\{\left[(1+\sin\tau)\xi^+_{\bar{A}\underline{2}} - \cos\tau\,\xi^-_{\bar{A}\underline{2}}\right] q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{2}}_{(1)+} + \left[-\cos\tau\,\xi^+_{\bar{A}\underline{2}} + (1-\sin\tau)\xi^-_{\bar{A}\underline{2}}\right] q^{\bar{A}}_{(1)}\bar{\psi}^{\underline{2}}_{(1)-}\right. \\
&\quad + \left[(1-\sin\tau)\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^+_{\bar{B}\underline{1}} + \cos\tau\,\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^-_{\bar{B}\underline{1}}\right]\psi_{(1)\underline{2}+}\bar{q}^{\bar{A}}_{(1)} \\
&\quad \left. + \left[\cos\tau\,\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^+_{\bar{B}\underline{1}} + (1+\sin\tau)\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{21}}\xi^-_{\bar{B}\underline{1}}\right]\psi_{(1)\underline{2}-}\bar{q}^{\bar{A}}_{(1)}\right\}.
\end{aligned} \tag{3.4}$$

From the straight line case, we expect that these two choices correspond to the ψ_1 and ψ_2-loops respectively. We assume the form of the superconnections and verify that they are indeed supersymmetric.

3.1.1 The ψ_1-loop

We take the superconnection to be of the form

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} \dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \bar{c}_a\psi^a_{(1)\underline{1}} \\ c^a\bar{\psi}^{\underline{1}a}_{(1)} & \dot{x}^\mu A_{(2)\mu} - \frac{2i}{k}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \tag{3.5}$$

and the supermatrix G to be of the form

$$G = \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix}. \tag{3.6}$$

As in [18], we consider the projector

$$\mathcal{P}^+ \equiv \delta^a{}_b + \dot{x}^\mu(\gamma_\mu)^a{}_b = \begin{pmatrix} 1-\sin\tau & \cos\tau \\ \cos\tau & 1+\sin\tau \end{pmatrix}, \tag{3.7}$$

and demand that the fermionic couplings be eigenstates of this projector. In particular, we choose

$$c^a = (\cos\tau,\, 1+\sin\tau)^a\, c(\tau)\,, \qquad \bar{c}_a = \begin{pmatrix} 1-\sin\tau \\ \cos\tau \end{pmatrix}_a \bar{c}(\tau), \tag{3.8}$$

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where $c(\tau)$ and $\bar{c}(\tau)$ are functions of τ. The variation is given by

$$\begin{aligned}\delta\left[\bar{c}_a\psi^a_{(1)\underline{1}}\right] =& 2\sqrt{2}D_\tau q^{\bar{A}}\bar{c}_a\xi^a_{\bar{A}\underline{1}} - \frac{2\sqrt{2}}{k}\left[(\nu_{(1)})^{\bar{B}}{}_{\bar{B}}q^{\bar{A}}_{(1)} - q^{\bar{A}}_{(1)}(\tilde{\nu}_{(1)})^{\bar{B}}{}_{\bar{B}}\right]\bar{c}_a\xi^a_{\bar{A}\underline{1}} \\ &+ i\sqrt{2}\bar{c}_a(\sigma^2)^a{}_b\xi^b_{\bar{A}\underline{1}}q^{\bar{A}}_{(1)}.\end{aligned} \tag{3.9}$$

We wish to write this as a covariant derivative. We may impose

$$\partial_\tau\bar{c}_a = \frac{i}{2}\bar{c}_b(\sigma^2)^b{}_a, \tag{3.10}$$

which tells us that

$$\bar{c}(\tau) = \frac{\bar{C}}{\cos\frac{\tau}{2} - \sin\frac{\tau}{2}}, \tag{3.11}$$

where $\bar{C}$ is a constant. With this choice of $\bar{c}(\tau)$, we have

$$\delta\left[\bar{c}_a(\psi_{(1)\underline{1}})^a\right] = \mathcal{D}_\tau\left(2\sqrt{2}q^{\bar{A}}\bar{c}_a\xi^a_{\bar{A}\underline{1}}\right), \tag{3.12}$$

where $\mathcal{D}_\tau q^{\bar{A}}_{(1)} \equiv D_\tau q^{\bar{A}} + \frac{1}{k}\nu_{(1)}q^{\bar{A}} - \frac{1}{k}q^{\bar{A}}\tilde{\nu}_{(1)}$. The supersymmetry conditions demand

$$g = 2\sqrt{2}q^{\bar{A}}_{(1)}\bar{c}_a\xi^a_{\bar{A}\underline{1}}. \tag{3.13}$$

Similarly, we find

$$\delta\left[c^a(\bar{\psi}^{\underline{1}}_{(1)})_a\right] = \mathcal{D}_\tau\left(-2\sqrt{2}\bar{q}_{(1)\bar{A}}c^a\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}a}\right), \tag{3.14}$$

where we have imposed that

$$\partial_\tau c^a = -\frac{i}{2}c^b(\sigma^2)^a{}_b, \tag{3.15}$$

which is solved by

$$c(\tau) = \frac{C}{\cos\frac{\tau}{2} + \sin\frac{\tau}{2}}, \tag{3.16}$$

where C is a constant. We thus have

$$\bar{g} = -2\sqrt{2}\bar{q}_{(1)\bar{A}}c^a\epsilon^{\bar{A}\bar{B}}\epsilon^{\underline{12}}\xi_{\bar{B}\underline{2}a}. \tag{3.17}$$

A simple calculation, as in the straight line case in section 2.3.1 gives that $C\bar{C} = -\frac{i}{k}$.

The story for the segment of the quiver with vanishing CS level is very similar, where for example the circular analog of the straight line presented in section 2.3.1 is the same as (3.5) with the lower right corner replaced by $\dot{x}^\mu A_{(2)\mu} + i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)}$, which is essentially the same as in (2.32). The variation of the super-connection is then

$$\delta\mathcal{L}^{\psi_1} \equiv \mathfrak{D}^{\psi_1}_\tau G + \Delta J\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{3.18}$$

with G as in (3.6) and ΔJ is as defined in (2.34). The other loops for this case may be similarly obtained in analogy with the straight line case.

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3.1.2 The ψ_2-loop

To construct a loop coupling to $\psi_{\underline{2}}$, we take the fermionic couplings to be eigenstates of the projector $\mathcal{P}^- = \delta^a{}_b - \dot{x}^\mu(\gamma_\mu)^a{}_b$, given explicitly by

$$\mathcal{P}^- = \begin{pmatrix} 1+\sin\tau & -\cos\tau \\ -\cos\tau & 1-\sin\tau \end{pmatrix}. \tag{3.19}$$

We take the fermionic couplings to be

$$c^a = c(\tau)\left(-\cos\tau,\, 1-\sin\tau\right)^a, \qquad \bar{c}_a = \bar{c}(\tau)\begin{pmatrix} 1+\sin\tau \\ -\cos\tau \end{pmatrix}_a, \tag{3.20}$$

with the superconnection given by

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} \dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{k}\nu_{(1)} & \bar{c}_a\psi^-_{(1)\underline{2}} \\ c^a\bar{\psi}^{\underline{2}}_{(1)-} & \dot{x}^\mu A_{(2)\mu} - \frac{2i}{k}(\mu_{(2)})^{\underline{1}}{}_{\underline{1}} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}. \tag{3.21}$$

Proceeding as in the previous section, studying the variations of the fermions gives us

$$\begin{aligned} g &= -2\sqrt{2}q^{\bar{A}}\bar{c}_a(\xi_{\bar{A}\underline{2}})^a\,, & \bar{c}(\tau) &= \frac{\bar{C}}{\cos\frac{\tau}{2}+\sin\frac{\tau}{2}}\,, \\ \bar{g} &= 2\sqrt{2}\,\bar{q}_{\bar{A}}c^a(\xi^{\bar{A}\underline{2}})_a\,, & c(\tau) &= \frac{C}{\cos\frac{\tau}{2}-\sin\frac{\tau}{2}}\,, \end{aligned} \tag{3.22}$$

as well as compatibility with the conjectured bosonic connection of the two nodes. An examination of the variation of the bosonic connections gives $C\bar{C} = \frac{i}{k}$.

3.1.3 The boundary conditions

We have shown that the supersymmetric variation of $\mathcal{L}$ may be written as a super-covariant derivative. In order for the circle, which is compact, to be invariant under supersymmetry we must ensure that there are no boundary terms when integrating this total derivative term. We follow the analysis in [26].

We define $\mathfrak{W}$ to be the untraced Wilson loop and analyze its transformation by the total derivative viewed as a super gauge transformation. Consider the matrix G (for either the ψ_1 and ψ_2-loops) found above, then we find

$$G(2\pi) = -G(0)\,. \tag{3.23}$$

This can be written as

$$G(2\pi) = \mathcal{T}G(0)\mathcal{T}^{-1}\,, \qquad \text{with} \qquad \mathcal{T} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.24}$$

This implies that the finite gauge transformation $U = \exp(iG)$ also satisfies

$$U(2\pi) = \mathcal{T}U(0)\mathcal{T}^{-1}\,. \tag{3.25}$$

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Now we consider the transformation of $\mathfrak{WT}$. It is invariant under the supersymmetry/supergauge transformations.

$$\mathrm{sTr}(\mathfrak{WT}) \to \mathrm{sTr}(U^{-1}(0)\mathfrak{W}U(2\pi)\mathcal{T}) = \mathrm{sTr}(U^{-1}(0)\mathfrak{WT}U(0)) = \mathrm{sTr}(\mathfrak{WT})\,. \tag{3.26}$$

We thus should consider the supertrace of $\mathfrak{WT}$ rather than of $\mathfrak{W}$. Equvalently, examining that form of $\mathcal{T}$, we see that the trace of $\mathfrak{W}$ (rather than the supertrace) is a supersymmetric operator. This is similar to the result for ABJM in [18].

4 Localization and the matrix model

We would now like to show that we can calculate the expectation value of the 1/2 BPS Wilson loops using localization. The idea, as in [18], is to show that the 1/2 BPS loops are cohomologically equivalent, under the localization supercharge, to certain combinations of 1/4 BPS loops for which the localization calculation has already been done. This reduces the calculation of the Wilson loop to that of an observable in a matrix model, as in [28–31]. We shall leave the solution of the matrix model to future work.

4.1 The straight line

We now demonstrate, following closely [18], that both the ψ_1-loops and ψ_2-loops are in the same cohomology class — that of a certain 1/4 BPS straight loop. We should note that this is a classical calculation and we expect it to be modified by quantum corrections and in fact only a single linear combination of these loops will be exactly equivalent to the 1/4 BPS loop. We discuss the lifting of the degeneracy in section 6.

The appropriate 1/4 BPS operator has a purely bosonic connection from two adjacent nodes

$$\mathcal{L}^{1/4} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}\left((\mu_{(0)})^{\underline{1}}{}_{\underline{1}} + (\mu_{(1)})^{\bar{1}}{}_{\bar{1}}\right) & 0 \\ 0 & A_{(2)1} - \frac{2i}{k}\left((\tilde{\mu}_{(1)})^{\bar{1}}{}_{\bar{1}} + (\mu_{(2)})^{\underline{1}}{}_{\underline{1}}\right) \end{pmatrix}. \tag{4.1}$$

We note that this shares its four supersymmetries with our 1/2 BPS loops. As our localizing supercharge, we consider the linear combination

$$Q \equiv Q_{-}^{\bar{1}2} + Q_{+}^{\bar{2}1}. \tag{4.2}$$

4.1.1 The ψ_1-loop

The difference between the ψ_1-loop (2.20) found in section 2.1.1 and the 1/4 BPS loop is given by

$$\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = \mathrm{tr}\,\mathcal{P}\left[e^{-i\int d\tau\,\mathcal{L}^{\psi_1}} - e^{-i\int d\tau\,\mathcal{L}^{1/4}}\right]. \tag{4.3}$$

We wish to demonstrate that this is a Q-exact quantity by finding a V such that $\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = QV$. Following [18], we take

$$V^{\psi_1} = -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\,\mathcal{L}^{1/4}}\Lambda^{\psi_1}(\tau)e^{-i\int_{\tau}^{\infty} d\tau_2\,\mathcal{L}^{\psi_1}}\right], \tag{4.4}$$

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where Λ^{ψ_1} is to be determined by the requirement $Q\Lambda^{\psi_1} = \mathcal{L}_F^{\psi_1}$ and $\mathcal{L}_F^{\psi_1}$ is the connection with fermionic entries equal to that of $\mathcal{L}^{\psi_1}$ and zeros everywhere else. We find

$$\Lambda^{\psi_1} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c}\, q^{\bar{2}}_{(1)} \\ -c\, \bar{q}_{(1)\bar{2}} & 0 \end{pmatrix}, \qquad (\Lambda^{\psi_1})^2 = \frac{1}{2}\tilde{\mathcal{L}}_B^{\psi_1} \equiv \frac{1}{2}\left(\mathcal{L}_B^{\psi_1} - \mathcal{L}_B^{1/4}\right), \tag{4.5}$$

where the subscript 'B' indicates the bosonic part of and $\tilde{\mathcal{L}}^{\psi_1} \equiv \mathcal{L}^{\psi_1} - \mathcal{L}^{1/4}$.

Acting with Q on V^{ψ_1} gives

$$QV^{\psi_1} = -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\, \mathcal{L}^{1/4}} \left(\mathcal{L}_F^{\psi_1}(\tau) + \Lambda^{\psi_1} Q\right) e^{-i\int_{\tau}^{\infty} d\tau_2\, \mathcal{L}^{\psi_1}}\right]. \tag{4.6}$$

We pick up a boundary term from acting with the supercharges on the exponent. Noting that

$$Q\psi^{+}_{(1)\underline{1}} = \sqrt{2}\mathcal{D}_\tau q^{\bar{2}}_{(1)}\,, \qquad Q\bar{\psi}^{\underline{1}}_{(1)+} = -\sqrt{2}\mathcal{D}_\tau \bar{q}_{(1)\bar{2}}\,, \tag{4.7}$$

we find

$$\begin{aligned} QV^{\psi_1} &= -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\, \mathcal{L}^{1/4}} \left(\mathcal{L}_F^{\psi_1}(\tau) + \tilde{\mathcal{L}}_B^{\psi_1}\right) e^{-i\int_{\tau}^{\infty} d\tau_2\, \mathcal{L}^{\psi_1}}\right], \\ &= -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\, \mathcal{L}^{1/4}} \tilde{\mathcal{L}}^{\psi_1}(\tau) e^{-i\int_{\tau}^{\infty} d\tau_2\, \mathcal{L}^{\psi_1}}\right]. \end{aligned} \tag{4.8}$$

As in [18], upon Taylor expansion, we see that this is equal to the expansion of

$$\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = \mathrm{tr}\,\mathcal{P}\left[e^{-i\int_{-\infty}^{\infty}\mathcal{L}^{1/4}} \sum_{p=1}^{\infty}(-i)^p \int d\tau_1 \cdots d\tau_p\, \tilde{\mathcal{L}}^{\psi_1}(\tau_1)\cdots\tilde{\mathcal{L}}^{\psi_1}(\tau_p)\right], \tag{4.9}$$

demonstrating the cohomological equivalence of the ψ_1-loop and 1/4 BPS loop.

4.1.2 The ψ_2-loop

The ψ_2-loop (2.22) of section 2.1.2 is also cohomologically equivalent to the 1/4 BPS loop. The result follows almost identically to that of the ψ_1-loop, with

$$V^{\psi_2} = -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\, \mathcal{L}^{1/4}} \Lambda^{\psi_2}(\tau) e^{-i\int_{\tau}^{\infty} d\tau_2\, \mathcal{L}^{\psi_1}}\right], \tag{4.10}$$

where

$$\Lambda^{\psi_2} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c}\, q^{\bar{1}}_{(1)} \\ c\, \bar{q}_{(1)\bar{1}} & 0 \end{pmatrix}, \qquad (\Lambda^{\psi_1})^2 = \frac{1}{2}\tilde{\mathcal{L}}_B^{\psi_2} \equiv \frac{1}{2}\left(\mathcal{L}_B^{\psi_2} - \mathcal{L}_B^{1/4}\right). \tag{4.11}$$

Noting also that

$$Q\psi^{-}_{(1)\underline{2}} = -\sqrt{2}\mathcal{D}_\tau q^{\bar{1}}_{(1)}\,, \qquad Q\bar{\psi}^{\underline{2}}_{(1)-} = -\sqrt{2}\mathcal{D}_\tau \bar{q}_{(1)\bar{1}}\,, \tag{4.12}$$

it is straightforward to see that both loops are cohomologically equivalent to each other.

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4.1.3 Vanishing CS levels

In the case of the quivers with vanishing CS levels discussed in section 2.3 We have not demonstrated the cohomological equivalence to 1/4 BPS loops for the loops based on 2×2 blocks. Below we do it for the 4×4 superconnection (2.41). For the 1/2 BPS loop in section 2.4 it is in fact identical to the 1/4 BPS loop.

We found the $U(N_1+N_3|2N_2)$ valued superconnection (2.41) in section 2.3.4 We now apply the arguments of the previous section to this loop. It is useful to rearrange the superconnection to make manifest its $U(N_1+N_3|2N_2)$ structure:

$$\mathcal{L}=\begin{pmatrix} A_{(1)1} & \bar{d}\epsilon_{\bar{A}\bar{B}}q^{\bar{A}}_{(1)}q^{\bar{B}}_{(2)} & \bar{c}^{\underline{1}}_{(1)}\psi^{+}_{(1)\underline{1}} & \bar{c}^{\underline{2}}_{(1)}\psi^{-}_{(1)\underline{2}} \\ d\epsilon^{\bar{A}\bar{B}}\bar{q}_{(2)\bar{A}}q_{(1)\bar{B}} & A_{(3)1} & c_{(2)\underline{2}}\bar{\psi}^{\underline{2}}_{(2)-} & c_{(2)\underline{1}}\bar{\psi}^{\underline{1}}_{(2)+} \\ c_{(1)\underline{1}}\bar{\psi}^{\underline{1}}_{(1)+} & \bar{c}^{\underline{2}}_{(2)}\psi^{-}_{(2)\underline{2}} & \mathcal{A}_{(2)} & 0 \\ c_{(1)\underline{2}}\bar{\psi}^{\underline{2}}_{(1)-} & \bar{c}^{\underline{1}}_{(2)}\psi^{+}_{(2)\underline{1}} & 0 & \mathcal{A}'_{(2)} \end{pmatrix}. \tag{4.13}$$

As in (4.4) we define

$$V^{4\times 4}=-i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty}d\tau\, e^{-i\int_{-\infty}^{\tau}d\tau_1\,\mathcal{L}^{1/4}}\Lambda^{4\times 4}(\tau)e^{-i\int_{\tau}^{\infty}d\tau_2\,\mathcal{L}^{\psi_1}}\right], \tag{4.14}$$

where $\mathcal{L}^{1/4}$ is the combination of 1/4 BPS connections

$$\begin{aligned}\mathcal{L}^{1/4}=\mathrm{diag}\Big\{&A_{(1)1}-\frac{2i}{k}(\mu_{(1)})^{\bar{1}}{}_{\bar{1}},A_{(3)1}-\frac{2i}{k}(\tilde{\mu}_{(2)})^{\bar{1}}{}_{\bar{1}}, \\ &A_{(2)1}+i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}}-\frac{i}{k}\left((\tilde{\mu}_{(1)})^{\bar{1}}{}_{\bar{1}}+(\mu_{(2)})^{\bar{1}}{}_{\bar{1}}\right),\ A_{(2)1}+i(\varphi_{(2)})^{\underline{1}}{}_{\underline{1}}-\frac{i}{k}\left((\tilde{\mu}_{(1)})^{\bar{1}}{}_{\bar{1}}+(\mu_{(2)})^{\bar{1}}{}_{\bar{1}}\right)\Big\}.\end{aligned} \tag{4.15}$$

We determine $\Lambda^{4\times 4}$ by the requirement that $Q\Lambda^{4\times 4}=\mathcal{L}_F$, with $\mathcal{L}_F$ the fermionic part of $\mathcal{L}$.

Given the same supercharge and many of the same fermionic entries as in the examples above, we may read build $\Lambda^{4\times 4}$ from (4.5), (4.11)

$$\Lambda^{4\times 4}=-\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\bar{c}^{\underline{1}}_{(1)}q^{\bar{2}}_{(1)} & \bar{c}^{\underline{2}}_{(1)}q^{\bar{2}}_{(1)} \\ 0 & 0 & c_{(2)\underline{2}}\bar{q}_{(2)\bar{1}} & c_{(2)\underline{1}}\bar{q}_{(2)\bar{2}} \\ c_{(1)\underline{1}}\bar{q}_{(1)\bar{2}} & \bar{c}^{\underline{2}}_{(1)}q^{\bar{1}}_{(2)} & 0 & 0 \\ c_{(1)\underline{2}}\bar{q}_{(1)\bar{1}} & -\bar{c}^{\underline{1}}_{(1)}q^{\bar{2}}_{(2)} & 0 & 0 \end{pmatrix}. \tag{4.16}$$

A short computation (making use of the constraints (2.44)) shows that

$$(\Lambda^{4\times 4})^2=\frac{1}{2}\tilde{\mathcal{L}}_B-\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{(1)\underline{1}}\bar{c}^{\underline{2}}_{(1)}\left((\mu_{(2)})^{\bar{1}}{}_{\bar{2}}-(\tilde{\mu}_{(1)})^{\bar{1}}{}_{\bar{2}}\right) \\ 0 & 0 & c_{(1)\underline{2}}\bar{c}^{\underline{1}}_{(1)}\left((\mu_{(2)})^{\bar{2}}{}_{\bar{1}}-(\tilde{\mu}_{(1)})^{\bar{2}}{}_{\bar{1}}\right) & 0 \end{pmatrix}. \tag{4.17}$$

As before, $\tilde{\mathcal{L}}_B$ is the bosonic part of the difference $\mathcal{L}-\mathcal{L}^{1/4}$. We note that both of the non-zero entries are proportional to $(\mu_{(2)})^{\bar{A}}{}_{\bar{B}}-(\tilde{\mu}_{(1)})^{\bar{A}}{}_{\bar{B}}$, for fixed $\bar{A}$, $\bar{B}$. For notational convenience, we define $(\Lambda^{4\times 4})^2\equiv\frac{1}{2}\tilde{\mathcal{L}}_B+\Delta\mu$.

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Continuing as before, using the action of the supercharges on the fermions given in (4.7) and (4.12), we see that

$$Q\mathcal{L}_F = -2i\mathfrak{D}_\tau \Lambda, \tag{4.18}$$

where $\mathfrak{D}_\tau$ is the supercovariant derivative with respect to $\mathcal{L}$.

Acting with the supercharge on V thus gives

$$\begin{aligned} QV &= -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\,\mathcal{L}^{1/4}}\left(\mathcal{L}_F(\tau)+\Lambda Q\right) e^{-i\int_{\tau}^{\infty} d\tau_2\,\mathcal{L}}\right] \\ &= -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\,\mathcal{L}^{1/4}}\left(\mathcal{L}_F(\tau)+\tilde{\mathcal{L}}_B+\Delta\mu\right) e^{-i\int_{\tau}^{\infty} d\tau_2\,\mathcal{L}^{\psi_1}}\right] \\ &= -i\,\mathrm{tr}\,\mathcal{P}\left[\int_{-\infty}^{\infty} d\tau\, e^{-i\int_{-\infty}^{\tau} d\tau_1\,\mathcal{L}^{1/4}}\left(\tilde{\mathcal{L}}(\tau)+\Delta\mu\right) e^{-i\int_{\tau}^{\infty} d\tau_2\,\mathcal{L}}\right]. \end{aligned} \tag{4.19}$$

As in the previous cases, the first term in brackets matches the expansion of $\mathcal{W}-\mathcal{W}^{1/4}$ exactly, where $\mathcal{W}$ is the loop defined with superconnection $\mathcal{L}$. The second term is however new.

In section 2.3.5, we argued that the variation of the loop (2.41) vanishes inside correlation functions. Indeed there too we found terms proportional to $\mu_{(2)}-\tilde{\mu}_{(1)}$, which vanish once the auxiliary field is integrated out. This is enough to conclude that the expectation value of $\mathcal{W}$ and $\mathcal{W}^{1/4}$ are equal.

4.2 The circle

We now consider the circle. This will be cohomologically equivalent to the circular analogs of the 1/4 BPS loops above, which have a finite expectation value, which can be calculated by localization [11]. We take Q to be the same supercharge used for localization

$$Q = (Q_-^{\bar{1}2} - S_+^{\bar{1}2}) + (Q_-^{\bar{2}1} + S_+^{\bar{2}1}). \tag{4.20}$$

4.2.1 The ψ_1-loop

We begin by considering the difference $\mathcal{W}^{\psi_1}-\mathcal{W}^{1/4}$ between the ψ_1-loop (3.5) found in section 3.1.1 and the 1/4 BPS circle, similar to (4.1). We again look for V^{ψ_1} as in (4.4), this time perturbatively in a power series. It will be useful to consider the insertion

$$\Lambda^{\psi_1} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & \bar{c}(\tau)q_{(1)}^{\bar{2}} \\ -c(\tau)\bar{q}_{(1)\bar{2}} & 0 \end{pmatrix}, \tag{4.21}$$

with $c(\tau)$ and $\bar{c}(\tau)$ as in (3.16), (3.11)). It satisfies

$$Q\Lambda^{\psi_2} = \mathcal{L}_F^{\psi_2}\,, \qquad 4\cos\tau(\Lambda^{\psi_1})^2 = \tilde{\mathcal{L}}_B^{\psi_1}\,. \tag{4.22}$$

Furthermore, the action of the supercharges on the fermionic components of the ψ_1-loop is given by

$$Q\mathcal{L}_F^{\psi_1} = 2\sqrt{2}\mathcal{D}_\tau\left[\cos\tau\begin{pmatrix} 0 & \bar{c}(\tau)q_{(1)}^{\bar{2}} \\ -c(\tau)\bar{q}_{(1)\bar{2}} & 0 \end{pmatrix}\right] = -4i\mathcal{D}_\tau\left[\cos\tau\,\Lambda^{\psi_1}\right]. \tag{4.23}$$

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Order-by-order, we define

$$
\begin{aligned}
V_1^{\psi_1} &= -i\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\,\Lambda^{\psi_1}\right\},\\
V_2^{\psi_1} &= -\frac{1}{2}\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\int_{\tau_2}^{2\pi}d\tau_3\,\left(\Lambda^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)-\mathcal{L}_F^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)\right)\right\}\\
V_3^{\psi_1} &= \mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\left(-\int_0^{2\pi}d\tau_2\int_{\tau_2}^{2\pi}d\tau_3\left(\tilde{\mathcal{L}}_B^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)+\Lambda^{\psi_1}(\tau_2)\tilde{\mathcal{L}}_B^{\psi_1}(\tau_3)\right)\right.\right.\\
&\quad +i\int_{\tau_2<\tau_3<\tau_4}d\tau_2\,d\tau_3\,d\tau_4\,\left(\Lambda^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)\mathcal{L}_F^{\psi_1}(\tau_4)+\mathcal{L}_F^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)\mathcal{L}_F^{\psi_1}(\tau_4)\right.\\
&\quad \left.\left.\left.+\mathcal{L}_F^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)\Lambda^{\psi_1}(\tau_4)\right)\right)\right\}.
\end{aligned}
\tag{4.24}
$$

The action of the supercharges on these operators is given by

$$
\begin{aligned}
QV_1^{\psi_1} &= -i\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\,\mathcal{L}_F^{\psi_1}\right\},\\
QV_2^{\psi_1} &= -\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\left(\int_{\tau_2}^{2\pi}d\tau_3\,\mathcal{L}_F^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)+i\tilde{\mathcal{L}}_B^{\psi_1}(\tau_2)\right)\right\},\\
QV_3^{\psi_1} &= \mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\left(-\int_{\tau_2<\tau_3}d\tau_2 d\tau_3\,\left(\tilde{\mathcal{L}}_B^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)+\mathcal{L}_F^{\psi_1}(\tau_2)\tilde{\mathcal{L}}_B^{\psi_1}(\tau_3)\right)\right.\right.\\
&\quad \left.\left.+i\int_{\tau_2<\tau_3<\tau_4}d\tau_2\,d\tau_3\,d\tau_4\,\mathcal{L}_F^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)\mathcal{L}_F^{\psi_1}(\tau_4)\right)\right\}.
\end{aligned}
\tag{4.25}
$$

These indeed match the terms in the expansion of $\mathcal{W}^{\psi_1}-\mathcal{W}^{1/4}$, demonstrating that to third order in the expansion the ψ_1-loop is cohomologically equivalent to the 1/4 BPS loop. However, as in the case of ABJM, we expect this relationship to hold to all orders, and the corresponding localization calculation to reduce to the known matrix model calculation of the 1/4 BPS loops [11].

4.2.2 The ψ_2-loop

The calculation for the ψ_2-loop, (3.21) found in section 3.1.2, proceeds very similarly. Defining

$$
\Lambda^{\psi_2}=\frac{i}{\sqrt{2}}\begin{pmatrix}0 & \bar{c}(\tau)q_{(1)}^{\bar{1}}\\ -c(\tau)\bar{q}_{(1)\bar{1}} & 0\end{pmatrix},
\tag{4.26}
$$

we see that it satisfies

$$
Q\Lambda^{\psi_2}=\mathcal{L}_F^{\psi_2}\,,\qquad 4\cos\tau(\Lambda^{\psi_2})^2=\tilde{\mathcal{L}}_B^{\psi_2}\,,
\tag{4.27}
$$

and

$$
Q\mathcal{L}_F^{\psi_2}=2\sqrt{2}\mathcal{D}_\tau\left[\cos\tau\begin{pmatrix}0 & \bar{c}(\tau)q_{(1)}^{\bar{1}}\\ -c(\tau)\bar{q}_{(1)\bar{1}} & 0\end{pmatrix}\right]=-4i\mathcal{D}_\tau\left[\cos\tau\,\Lambda^{\psi_2}\right].
\tag{4.28}
$$

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Equations (4.27) and (4.28) are the corresponding identities for the ψ_2-loop as (4.22) and (4.23) for the ψ_1-loop. As these are identical, it immediately follows that the action of the supercharge on

$$
\begin{aligned}
V_1^{\psi_2} &= -i\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\,\Lambda^{\psi_1}\right\},\\
V_2^{\psi_2} &= -\frac{1}{2}\,\mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\int_0^{2\pi}d\tau_2\int_{\tau_2}^{2\pi}d\tau_3\,\left(\Lambda^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)-\mathcal{L}_F^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)\right)\right\}\\
V_3^{\psi_2} &= \mathrm{tr}\,\mathcal{P}\left\{e^{-i\int_0^{2\pi}d\tau_1\,\mathcal{L}^{1/4}(\tau_1)}\left(-\int_0^{2\pi}d\tau_2\int_{\tau_2}^{2\pi}d\tau_3\left(\tilde{\mathcal{L}}_B^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)+\Lambda^{\psi_1}(\tau_2)\tilde{\mathcal{L}}_B^{\psi_1}(\tau_3)\right)\right.\right.\\
&\quad +i\int_{\tau_2<\tau_3<\tau_4}d\tau_2\,d\tau_3\,d\tau_4\,\left(\Lambda^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)\mathcal{L}_F^{\psi_1}(\tau_4)+\mathcal{L}_F^{\psi_1}(\tau_2)\Lambda^{\psi_1}(\tau_3)\mathcal{L}_F^{\psi_1}(\tau_4)\right.\\
&\quad \left.\left.\left.+\mathcal{L}_F^{\psi_1}(\tau_2)\mathcal{L}_F^{\psi_1}(\tau_3)\Lambda^{\psi_1}(\tau_4)\right)\right)\right\},
\end{aligned}
\tag{4.29}
$$

will exactly match the expansion of $\mathcal{W}^{\psi_2}-\mathcal{W}^{1/4}$, thus demonstrating the equivalence of the two loops with respect to the localizing supercharge.

5 Holographic description

The goal of this section is to find the M2-brane configurations which preserve half of the supersymmetries of the vacuum and are the holographic duals to the gauge theory operators discussed above.

We limit our attention to the case of circular quivers whose corresponding space-time metric is $AdS_4\times M_{p,q,k}$ [32], where the compact manifold is an orbifold of the 7-sphere

$$
M_{p,q,k}=\left(S^7/(\mathbb{Z}_p\oplus\mathbb{Z}_q)\right)/\mathbb{Z}_k\,,
\tag{5.1}
$$

with radius $R^6=2^5\pi^2Nkpq$ (in units of $\ell_s=1$). The gauge group rank N corresponds here to the number of M2-branes located at the orbifold point.

For the AdS_4 factor, we consider a metric with an AdS_2 foliation

$$
ds^2_{AdS_4}=du^2+\cosh^2u\,ds^2_{AdS_2}+\sinh^2u\,d\phi^2\,,
\tag{5.2}
$$

where for a time-like (straight) Wilson loop we take a Lorentzian AdS_2

$$
ds^2_{AdS_2}=d\rho^2-\cosh^2\rho\,dt^2\,,
\tag{5.3}
$$

and for the space-like circle we consider the Euclidean metric

$$
ds^2_{AdS_2}=d\rho^2+\sinh^2\rho\,d\psi^2\,.
\tag{5.4}
$$

The M2-brane world-volume will be along these AdS_2 coordinates in AdS_4, while on the internal space it will wrap an S^1 (i.e., the M-theory circle) and sit at a fixed point in the remaining coordinates, which is determined by supersymmetry, as we shall see.

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The orbifold action giving rise to the space $M_{p,q,k}$ is most easily described in terms of complex embedding coordinates z_i $(i = 1, \ldots, 4)$ subject to the constraint

$$\sum_{i=1}^{4} |z_i|^2 = 1\,, \tag{5.5}$$

and to the identifications [32]

$$(z_1, z_2, z_3, z_4) \sim (\omega_{kp}^m z_1, \omega_{kp}^m z_2, \omega_{kq}^{-m} z_3, \omega_{kq}^{-m} z_4)\,, \qquad \omega_r = e^{2\pi i/r}\,, \qquad m \in \mathbb{Z}\,. \tag{5.6}$$

This last line alone without the constraint (5.5) would define the manifold

$$\mathcal{M}_{p,q,k} = \left(\mathbb{C}^2/\mathbb{Z}_p \otimes \mathbb{C}^2/\mathbb{Z}_q\right)/\mathbb{Z}_k\,. \tag{5.7}$$

The $\mathrm{SU}(2)_A$ subgroup of the R-symmetry acts on $\mathbb{C}^2/\mathbb{Z}_p$ and $\mathrm{SU}(2)_B$ acts on $\mathbb{C}^2/\mathbb{Z}_q$.

The supersymmetries preserved by the Wilson loops (1.2) are doublets of the R-symmetry subgroup $\mathrm{SU}(2)_B$, which is left unbroken. We therefore expect that the M2-brane embedding will be at a fixed point of the $\mathrm{SU}(2)_B$ action. This is the tip of the $\mathbb{C}/\mathbb{Z}_q$ cone in $\mathcal{M}_{p,q,k}$. A similar story applies of course if we were to consider loops invariant under $\mathrm{SU}(2)_A$ with the two cones exchanged.

5.1 Parameterizations of the 7-sphere

To be explicit, we parametrize the S^7 by the coordinates [16]

$$\begin{aligned} z_1 &= \cos\alpha\, \cos\frac{\theta_1}{2}\, e^{i\xi_1}\,, \qquad & z_2 &= \cos\alpha\, \sin\frac{\theta_1}{2}\, e^{i\xi_2}\,, \\ z_3 &= \sin\alpha\, \cos\frac{\theta_2}{2}\, e^{i\xi_3}\,, \qquad & z_4 &= \sin\alpha\, \sin\frac{\theta_2}{2}\, e^{i\xi_4}\,. \end{aligned} \tag{5.8}$$

This choice induces the following metric

$$\begin{aligned} ds^2 = \frac{1}{4}\Bigg[4d\alpha^2 + \cos^2\alpha\left(d\theta_1^2 + 4\cos^2\frac{\theta_1}{2}\, d\xi_1^2 + 4\sin^2\frac{\theta_1}{2}\, d\xi_2^2\right) \\ + \sin^2\alpha\left(d\theta_2^2 + 4\cos^2\frac{\theta_2}{2}\, d\xi_3^2 + 4\sin^2\frac{\theta_2}{2}\, d\xi_4^2\right)\Bigg]\,, \end{aligned} \tag{5.9}$$

with appropriate ranges for the coordinates to cover the entire sphere once, namely $0 \leq \alpha \leq \pi/2$, $0 \leq \theta_{1,2} \leq \pi$ and $0 \leq \xi_i \leq 2\pi$.

To make the action of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ more transparent, we can relabel the ξ_i as

$$\xi_1 = \frac{\phi_1}{2} + x\,, \qquad \xi_2 = -\frac{\phi_1}{2} + x\,, \qquad \xi_3 = \frac{\phi_2}{2} + v\,, \qquad \xi_4 = -\frac{\phi_2}{2} + v\,, \tag{5.10}$$

which results in

$$ds^2_{S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)} = \frac{1}{4}\left[4d\alpha^2 + \cos^2\alpha\, ds^2_{S^3/\mathbb{Z}_p} + \sin^2\alpha\, ds^2_{S^3/\mathbb{Z}_q}\right]\,, \tag{5.11}$$

with

$$\begin{aligned} ds^2_{S^3/\mathbb{Z}_p} &= d\theta_1^2 + d\phi_1^2 + 4dx^2 + 4\cos\theta_1\, dx\, d\phi_1\,, \\ ds^2_{S^3/\mathbb{Z}_q} &= d\theta_2^2 + d\phi_2^2 + 4dv^2 + 4\cos\theta_2\, dv\, d\phi_2\,. \end{aligned} \tag{5.12}$$

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We are implicitly assuming that $k = 1$, so that $\mathbb{Z}_k$ is trivial. The action of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ then identify [32, 33]

$$(z_1, z_2, z_3, z_4) \sim (\omega_p^m z_1, \omega_p^m z_2, \omega_q^n z_3, \omega_q^n z_4)\,, \qquad m, n \in \mathbb{Z}\,, \tag{5.13}$$

and therefore impose the following periodicities on the angular coordinate x and v

$$x \sim x + \frac{2\pi m}{p}\,, \qquad v \sim v + \frac{2\pi n}{q}\,. \tag{5.14}$$

With these coordinates the action of $\mathrm{SU}(2)_A$ and $\mathrm{SU}(2)_B$ is manifest on the two S^3 orbifolds in (5.12).

The full metric of the space (before the $\mathbb{Z}_k$ orbifold) is given by [33]

$$ds^2 = R^2 \left(ds^2_{AdS_4} + 4 ds^2_{S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)} \right)\,. \tag{5.15}$$

The Killing spinor for this 11-dimensional geometry is derived in appendix B and is given by

$$\eta = \Psi\, \Xi\, \eta_0\,, \tag{5.16}$$

with

$$\begin{aligned} \Psi = {} & e^{-\frac{\alpha-\pi/2}{2}\Gamma_4\Gamma_\star}\, e^{\frac{\theta_1-\pi/2}{4}(\Gamma_{45}+\Gamma_{67})}\, e^{\frac{\theta_2-\pi/2}{4}(\Gamma_{9\#}-\Gamma_8\Gamma_\star)}\, e^{\frac{x}{2}(\Gamma_{46}+\Gamma_{57})} \\ & \times e^{\frac{v}{2}(\Gamma_{8\#}-\Gamma_9\Gamma_\star)}\, e^{-\frac{\phi_1}{4}(\Gamma_{47}+\Gamma_{56})}\, e^{-\frac{\phi_2}{4}(\Gamma_{89}-\Gamma_\#\Gamma_\star)}\,, \\ \Xi = {} & e^{\frac{u}{2}\Gamma_2\Gamma_\star} e^{\frac{\rho}{2}\Gamma_1\Gamma_\star} e^{\frac{t}{2}\Gamma_0\Gamma_\star} e^{\frac{\phi}{2}\Gamma_{23}}\,, \end{aligned} \tag{5.17}$$

where η_0 is a 32-component constant spinor and $\Gamma_\star \equiv \Gamma_{0123}$. The gamma matrices live in the tangent space and the indices $0, 1, \ldots, 9, \#$ are flat indices.

A second parameterization of the angles is useful to study the general case of $k \geq 2$ and is obtained by setting

$$\begin{aligned} \xi_1 &= \tfrac{1}{4}\left(-2\phi_1 + \chi + \zeta\right), & \xi_2 &= \tfrac{1}{4}\left(2\phi_1 + \chi + \zeta\right), \\ \xi_3 &= \tfrac{1}{4}\left(-2\phi_2 - \chi + \zeta\right), & \xi_4 &= \tfrac{1}{4}\left(2\phi_2 - \chi + \zeta\right), \end{aligned} \tag{5.18}$$

in the metric above. This results in [16]

$$\begin{aligned} ds^2_{M_{p,q,k}} = \frac{1}{4}\Big[& 4d\alpha^2 + \sin^2\alpha\left(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2\right) + \sin^2\alpha\left(d\theta_2^2 + \cos^2\theta_2\, d\phi_2^2\right) \\ & + \cos^2\alpha\, \sin^2\alpha\left(d\chi + \cos\theta_1\, d\phi_1 - \cos\theta_2\, d\phi_2\right)^2 + \frac{1}{4}\left(d\zeta + A\right)^2 \Big]\,, \end{aligned} \tag{5.19}$$

where

$$A = \cos 2\alpha\, d\chi + 2\cos^2\alpha\, \cos\theta_1\, d\phi_1 + 2\sin^2\alpha\, \cos\theta_2\, d\phi_2\,. \tag{5.20}$$

In these coordinates, the action of the $\mathbb{Z}_k$ orbifold is along the ζ direction, i.e.

$$\zeta \sim \zeta + \frac{2\pi m}{k}\,, \qquad m \in \mathbb{Z}\,. \tag{5.21}$$

The Killing spinor for these coordinates was obtained in [16] and is as in (5.16), with Ψ replaced by

$$\tilde{\Psi} = e^{\frac{\alpha}{2}(\Gamma_4\Gamma_\star - \Gamma_{7\#})} e^{\frac{\theta_1}{4}(\Gamma_5\Gamma_\star - \Gamma_{8\#})} e^{\frac{\theta_2}{4}(\Gamma_{79}+\Gamma_{46})} e^{-\frac{\xi_1}{2}\Gamma_\#\gamma_\star} e^{-\frac{\xi_2}{2}\Gamma_{58}} e^{-\frac{\xi_3}{2}\Gamma_{47}} e^{-\frac{\xi_4}{2}\Gamma_{69}}\,, \tag{5.22}$$

where the ξ_i are given by (5.18).

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5.2 Supersymmetry analysis

The task at this point is twofold. First, we want to check that the three orbifolds of $M_{p,q,r}$ preserve half of the supersymmetries of the vacuum, namely 16 supercharges. Secondly, we want to find an M2-brane embedding that is 1/2 BPS, breaking these 16 supersymmetries down to 8.

To address the first point, we consider the first parameterization, equations (5.11) and (5.17) above, and use the procedure employed in [16] for the ABJM case. This begins by choosing a basis for the gamma matrices such that

$$\Gamma_{0123456789\#} = 1\,, \tag{5.23}$$

and by writing the constant spinor η_0 in a basis which block-diagonalizes as follows

$$\Gamma_{46}\eta_0 = s_1\eta_0\,, \quad \Gamma_{57}\eta_0 = s_2\eta_0\,, \quad \Gamma_{8\#}\eta_0 = s_3\eta_0\,, \quad \Gamma_9\Gamma_\star\eta_0 = s_4\eta_0\,. \tag{5.24}$$

All s_i eigenvalues take values ± 1, see [16]. A translation in x corresponds to an action of the $\mathbb{Z}_p$ orbifold and rescales the Killing spinor by

$$\Psi\,\Xi\,\eta_0 \to e^{i(s_1+s_2)\delta/2}\Psi\,\Xi\,\eta_0\,, \tag{5.25}$$

as can be readily seen from (5.17). In order for this to be a symmetry of the Killing spinor (for a generic translation parameter δ), we are restricted to

$$(s_1, s_2, s_3, s_4) \in \{(+,-,\cdot\,,\cdot\,)\,,(-,+,\cdot\,,\cdot\,)\}\,, \tag{5.26}$$

where $\pm$ indicates ± 1 and $\cdot$ indicates an unrestricted value. The action of the $\mathbb{Z}_q$ orbifold (i.e., a translation in v) similarly gives

$$(s_1, s_2, s_3, s_4) \in \{(\cdot\,,\cdot\,,+,+)\,,(\cdot\,,\cdot\,,-,-)\}\,. \tag{5.27}$$

Now we notice that $\zeta \propto x + v$, so that the action of the $\mathbb{Z}_k$ orbifold rescales the Killing spinor as

$$\Psi\,\Xi\,\eta_0 \to e^{i(s_1+s_2+s_3-s_4)\delta/2}\Psi\,\Xi\,\eta_0\,. \tag{5.28}$$

This restricts to the following choices

$$\begin{aligned}(s_1, s_2, s_3, s_4) \in \{&(+,-,-,-)\,,(-,+,-,-)\,,(-,-,+,-)\,,\\ &(-,+,+,+)\,,(+,-,+,+)\,,(+,+,-,+)\}\,.\end{aligned} \tag{5.29}$$

Note that each set of s_i eigenvalues represents four supercharges. Thus, the $\mathbb{Z}_k$ orbifold by itself restricts us to 24 supercharges, i.e., the ABJM theory, as expected. Taking the intersection of the three sets of allowed eigenvalues, we are left with

$$(s_1, s_2, s_3, s_4) \in \{(+,-,-,-)\,,(-,+,-,-)\,,(-,+,+,+)\,,(+,-,+,+)\}\,, \tag{5.30}$$

namely 16 supercharges.

The supersymmetry analysis of the M2-brane embedding is most easily carried out using the second parametrization, equations (5.19) and (5.22) above. As anticipated earlier,

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the M2-brane is taken to be oriented along the AdS_2 directions in AdS_4 (while sitting at $u = 0$) and to wrap the S^1 parametrized by ζ. The world-volume coordinates are then given by (t, ρ, ζ), with the remaining coordinates on the sphere assuming constant values. The induced metric is thus simply given by

$$\frac{ds^2_{\text{ind}}}{R^2} = d\rho^2 - \cosh^2\rho\, dt^2 + \frac{1}{4} d\zeta^2 \,, \tag{5.31}$$

where $R^6 = 2^5\pi^2 Nkpq$, as seen above [32]. The amount of supersymmetry preserved by this embedding is obtained by considering the projection equation [34]

$$\Gamma\eta = \eta \,, \tag{5.32}$$

for the Killing spinor (5.22), with the projector being given by

$$\Gamma = \frac{1}{\sqrt{-\det g}} \partial_t x^\mu \partial_\rho x^\nu \partial_\zeta x^\sigma \gamma_\mu \gamma_\nu \gamma_\sigma = \Gamma_{01\#} \,. \tag{5.33}$$

This is formally identical to the ABJM case considered in [16], so that we can quote the results from that case. Setting therefore $\alpha = 0$ and $\theta_1 = 0$ the projection equation for ϵ reduces to an equation for ϵ_0 and the brane breaks half of the supercharges in (5.30). Selecting $\alpha = \pi/2$ and $\theta_2 = 0$ also gives 1/2 BPS embeddings, which are the loops invariant under $\text{SU}(2)_A$.

Unlike the ABJM case, in the present case there are also the $\mathbb{Z}_p$ and $\mathbb{Z}_q$ orbifolds. A crucial fact is that $\alpha = 0$, where our M2-brane is located, is a fixed point of the $\mathbb{Z}_q$ orbifold. In fact, we expect q different states at the singular point and find holographic duals for q different fundamental Wilson loops. This mirrors (part of) the degeneracy of 1/2 BPS loop operators we have uncovered in the gauge theory side. We expect there to be a single fundamental Wilson loop for every one of the q twisted hypermultiplets, represented by a dashed line in the graphs in section 2. It would be advantageous to use the type IIB description [33, 35–37], where this singularity is resolved, and which allows also to study linear quivers. See more in the discussion below.

The computation of the renormalized on-shell action for the M2-branes also follows from the ABJM theory case. The brane action (after renormalization) will be proportional to $R^3 \int d\zeta$. Using the explicit expression of the radius written above and that the range of ζ goes up to $1/kp$, the expectation value would then seem to be given by

$$\langle W \rangle \simeq e^{\pi\sqrt{\frac{2Nq}{kp}}} \,. \tag{5.34}$$

It is actually unlikely that this expression be valid for all the q different Wilson loops, as the answer should depend on the details of the resolution of the orbifold singularity (and fluxes, and fractional branes). This expression does in fact hold at least in the simplest case where the Wilson loops are all equivalent, so theories with $p = q$ and alternating $\pm k$ CS levels. Those in fact are orbifolds of the ABJM theory.

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6 Discussion

The structure of $\mathcal{N} = 4$ Chern-Simons-matter theories in three dimensions is very rich and the story of the 1/2 BPS Wilson loops in those theories is even richer and rather complicated.

We expect that a theory with p untwisted hypermultiplets and q twisted hypermultiplets possesses q independent 1/2 BPS Wilson loops (not accounting for possible representations) preserving our supercharges (1.2). There are p other Wilson loops preserving the supercharges with the underlined and overlined indices interchanged. They can be easily written explicitly, and one can also study them by looking at a quiver with the untwisted and twisted hypermultipets exchanged.

In the M-theory picture we found that the Wilson loops are indeed dual to M2-branes wraping a circle in $\left(S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)\right)/\mathbb{Z}_k$ which is at a fixed point of the $\mathbb{Z}_q$ orbifold and of circumference proportional to $1/kp$. We expect that there are in fact q different degenerate states at the singular point, representing the holographic duals of the q Wilson loops in fundamental representations. To resolve the singularity one can study the type IIB duals of these theories [33, 35–37]. These also allow to study linear quivers, while M-theory is only a good description for the circular quivers. It is also easier to specify the different ranks of the gauge groups and the location of the $k = 0$ nodes in the IIB language. The holographic duals of the Wilson loops will be fundamental strings at q specific points on one of the boundaries of the strip/annulus which appears in the metric. This mirrors the brane construction of these theories [33, 37] and the possible ways to add fundamental strings to them [12].[4]

On the field theory side the story is more convoluted. In the simple case of alternating levels we found $2q$ possible superconnections. A pair for each pair of vector multiplets connected by a hypermultiplet, coupling to the fermion $\psi^+_{\underline{1}}$ or to $\psi^-_{\underline{2}}$. Each of those superconnections is invariant under supersymmetry up to a total derivative, that cancels when we consider the entire Wilson loop. We expect the degeneracy between those Wilson loops to be lifted by quantum corrections.

This is a phenomenon that has so far not afflicted the study of BPS Wilson loops, but is in fact common in other settings. The Wilson loop is a composite operator made of the fields at arbitrary points along the curve and whose interactions could lead to quantum effects. The variation performed here is classical and does not take into account possible interactions between the fields. If there is a unique operator carrying some set of quantum numbers (in the case of the Wilson loop it is the contour and representation), then the classical calculation should suffice. In our case we found a pair of operators with the same quantum numbers that are classically BPS. As a simple analog, consider scalar operators in $\mathcal{N} = 4$ SYM (in $3d$ or $4d$). We can take two complex scalar fields Z and X in the vector multiplet. $\mathrm{Tr}(Z^J)$ and $\mathrm{Tr}(X^J)$ are both 1/2 BPS operators, but $\mathrm{Tr}(Z^2X^2)$ is not. The first two are the only operators of classical dimension J carrying J units of charge, so they must be BPS. The fields Z and X are each annihilated by half of the supercharges, but there is another operator $\mathrm{Tr}(ZXZX)$ which has the same classical dimension and charge as

[4] We thank B. Assel and J. Estes for explaining some of these points to us.

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$\mathrm{Tr}(Z^2X^2)$, so the classical analysis is not enough and indeed only one linear combination of these is BPS once one-loop effects are included.

We expect the same to be true in the case of the Wilson loops and only one combination (presumably the sum of the loop made of $\mathcal{L}^{\psi_1}$ and the loop made of $\mathcal{L}^{\psi_2}$) to be BPS. This story should hold also for the case of the circle and also for the equivalence to the 1/4 BPS loop in section 4. The matrix model will only calculate the expectation value of the correct linear combination.

One possible way to check this is to calculate the expectation value of the circular Wilson loops in perturbation theory. We expect that the Wilson loops with either $\mathcal{L}^{\psi_1}$ or $\mathcal{L}^{\psi_2}$ to not be BPS and therefore to suffer from UV divergences. Only the correct linear combination will be finite and equal to the result of the matrix model calculation.[5] This calculation should follow along the lines of the perturbative calculations of the Wilson loops in ABJM theory [38–40].

Further complications arise when one considers quivers with nodes with vanishing CS levels, as studied in section 2.3. In the case studied there, we found five possible superconnections. Their variation does not vanish and is not a total derivative, but we argued that the remainder cancels in every expectation value. Four of the connections involve a pair of nodes and the last one has a richer structure involving all three nodes (one of them doubled). This last one is a superconnection in $U(N_1+N_3|2N_2)$. Indeed it would seem natural to also combine the other connections into four possible pairings of block-diagonal $U(N_1|N_2)\oplus U(N_2|N_3)\subset U(N_1+N_3|2N_2)$ superconnections. The resulting Wilson loop are classified by representations of $U(N_1+N_3|2N_2)$ and we again expect only one linear combinations to be BPS. It is not clear actually why a Wilson loop involving only a pair of nodes would not be BPS. We leave this question for the future.

It may also be interesting to look for 1/2 BPS loops in the non-linear sigma model of [24] which arise from integrating out all the fields of the $k=0$ nodes.

We have shown that the 1/2 BPS Wilson loops are cohomologically equivalent to certain 1/4 BPS Wilson loops which can be reduced using localization to simple observables in the Chern-Simons-matter matrix models [11]. We leave the evaluation of these matrix integrals to future work.

In fact, one way to address the open questions is to look at the structure of the Wilson loops in the matrix model. In ABJM theory one can calculate the expectation value of either the 1/6 BPS Wilson loop or the 1/2 BPS one [28–31]. But the calculation of the latter is significantly easier. One can try to evaluate possible Wilson loops in the matrix models for arbitrary $\mathcal{N}=4$ CS-matter theories. For theories with equal ranks there is an elegant Fermi-gas formulation of the partition functions of those theories [41], which is given to all orders in the $1/N$ expansion by an Airy function dependent on two parameters (one of which was found explicitly for all $\mathcal{N}=4$ circular quivers in [42]). Our expectation is that in the alternating CS level case, the simple Wilson loops will be in representations of $U(N_I|N_{I+1})$, with a vanishing level $U(N_I+N_{I+2}|2N_{I+1})$, with two vanishing levels

[5] The Wilson loop found in [25] is a linear combination of loops on *all* the nodes of the quiver, and all of them are made of the appropriate $\mathcal{L}^{\psi_i}$, so we don't expect it to be BPS.

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$U(N_I+2N_{I+2}|2N_{I+1}+N_{I+3})$, etc. Note though, that in the case considered in section 2.4, the 1/2 BPS loops are simply in representations of $U(N_{I+1})$. In fact, it is not clear that the matrix model will see a distinction between the q loops we constructed in this paper and the p other 1/2 BPS loops preserving the set of supersymmetries invariant under $\mathrm{SU}(2)_A$. In fact, we would expect both type of operators to give simple observables in the matrix model.

Another approach would be to try to derive the form of the Wilson loops by Higgsing part of the gauge groups and finding the resulting operators as was done for the ABJM theory in [43].

It would be interesting to also study the transformation of these loops under mirror symmetry, where we would expect them to become vortex loop operators [44]. This was studied in other $3d$ theories in [12, 45]. Note also that in the case of the ABJM theory there were 1/3 BPS vortices preserving 8 supercharges [44]. An analog Wilson loop was never found. Possibly a linear combination of two 1/2 BPS Wilson loops with different couplings (the analogs of $\psi_{\underline{1}}$ and $\psi_{\underline{2}}$) should be considered the appropriate dual.

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A Notation and conventions

We consider the loop in $\mathbb{R}^3$. Spinor indices are raised and lowered as

$$\psi^a = \epsilon^{ab}\psi_b\,, \qquad \psi_a = \epsilon_{ab}\psi^b\,, \qquad \epsilon^{+-} = -\epsilon_{+-} = 1, \tag{A.1}$$

and we employ the gamma matrix basis

$$(\gamma^\mu)^a{}_b = \{\sigma^3,\, \sigma^1,\, -\sigma^2\}, \tag{A.2}$$

satisfying $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho$ (with $\epsilon^{123} = 1$).

As in [22], we use the epsilon symbol to raise and lower the indices of the supersymmetry parameters via $\xi^{\bar{A}\underline{B}} = \epsilon^{\bar{A}\bar{C}}\epsilon^{\underline{B}\underline{D}}\xi_{\bar{C}\underline{D}}$, with $\epsilon^{12} = \epsilon_{12} = 1$ for both overlined and underlined indices. In the main text, we shall always write the supersymmetry parameters with lowered indices.

In this gamma matrix basis, the supersymmetry transformations are given by [22]

$$\delta q^{\bar{A}}_{(I)} = i\sqrt{2}(\xi^{\bar{A}\underline{B}})^a(\psi_{(I)\underline{B}})_a\,, \qquad \delta\bar{q}_{(I)\bar{A}} = i\sqrt{2}(\xi_{\bar{A}\underline{B}})^a(\bar{\psi}^{\underline{B}}_{(I)})_a\,, \tag{A.3a}$$

$$\begin{aligned}\delta(\psi_{(I)\underline{A}})^a = &\sqrt{2}(\not{D})^a{}_b q^{\bar{B}}_{(I)}(\xi_{\bar{B}\underline{A}})^b - \frac{\sqrt{2}s_I}{k}(\xi_{\bar{B}\underline{A}})^a \left[\nu_{(I)} q^{\bar{B}}_{(I)} - q^{\bar{B}}_{(I)}\tilde{\nu}_{(I)}\right] \\ &+ \left(\sqrt{2}(\xi_{\bar{C}\underline{B}})^a(\varphi_{(I)})^{\underline{B}}{}_{\underline{A}} q^{\bar{C}}_{(I)}\right)_{k_I=0} - \left(\frac{2\sqrt{2}s_I}{k}(\xi_{\bar{C}\underline{B}})^a(\tilde{\mu}_{(I-1)})^{\underline{B}}{}_{\underline{A}} q^{\bar{C}}_{(I)}\right)_{k_I\neq 0} \\ &- \left(\sqrt{2}(\xi_{\bar{C}\underline{B}})^a q^{\bar{C}}_{(I)}(\varphi_{(I+1)})^{\underline{B}}{}_{\underline{A}}\right)_{k_{I+1}=0} \\ &+ \left(\frac{2\sqrt{2}s_I}{k}(\xi_{\bar{C}\underline{B}})^a q^{\bar{C}}_{(I)}(\mu_{(I+1)})^{\underline{B}}{}_{\underline{A}}\right)_{k_{I+1}\neq 0},\end{aligned} \tag{A.3b}$$

$$\begin{aligned}\delta(\bar{\psi}^{\underline{A}}_{(I)})_a = &\sqrt{2}(\not{D})_{ab}\bar{q}^{\bar{B}}_{(I)}(\xi^{\bar{B}\underline{A}})^b + \frac{\sqrt{2}s_I}{k}(\xi^{\bar{B}\underline{A}})_a \left[\tilde{\nu}_{(I)}\bar{q}_{(I)\bar{B}} - \bar{q}_{(I)\bar{B}}\nu_{(I)}\right] \\ &+ \left(\sqrt{2}(\xi^{\bar{C}\underline{B}})_a\bar{q}_{(I)\bar{C}}(\varphi_{(I)})^{\underline{A}}{}_{\underline{B}}\right)_{k_I=0} - \left(\frac{2\sqrt{2}s_I}{k}(\xi^{\bar{C}\underline{B}})_a\bar{q}_{(I)\bar{C}}(\tilde{\mu}_{(I-1)})^{\underline{A}}{}_{\underline{B}}\right)_{k_I\neq 0} \\ &- \left(\sqrt{2}(\xi^{\bar{C}\underline{B}})_a(\varphi_{(I+1)})^{\underline{A}}{}_{\underline{B}}\bar{q}_{(I)\bar{C}}\right)_{k_{I+1}=0} \\ &+ \left(\frac{2\sqrt{2}s_I}{k}(\xi^{\bar{C}\underline{B}})_a(\mu_{(I+1)})^{\underline{A}}{}_{\underline{B}}\bar{q}_{(I)\bar{C}}\right)_{k_{I+1}\neq 0},\end{aligned} \tag{A.3c}$$

$$\begin{aligned}\delta A_{(I)\mu} = &-\left(\frac{s_I}{k}(\xi_{\bar{A}\underline{B}})^a(\gamma_\mu)_{ab}\left[(j^{\bar{A}\underline{B}}_{(I)})^b - (\tilde{j}^{\underline{B}\bar{A}}_{(I-1)})^b\right]\right)_{k_I\neq 0} \\ &-\left((\xi_{\bar{A}\underline{B}})^a(\gamma_\mu)_{ab}\left((\lambda^{\bar{A}\underline{B}}_{(I)})^b + \frac{s_I}{2k}\left[(j^{\bar{A}\underline{B}}_{(I)})^b + (\tilde{j}^{\bar{A}\underline{B}}_{(I-1)})_b\right]\right)\right)_{k_I=0},\end{aligned} \tag{A.3d}$$

$$\delta(\varphi_{(I)})^{\underline{A}}{}_{\underline{B}} = 2i(\xi_{\bar{C}\underline{B}})^a(\lambda^{\bar{C}\underline{A}}_{(I)})_a - \delta^{\underline{A}}{}_{\underline{B}}\, i(\xi_{\bar{C}\underline{D}})^a(\lambda^{\bar{C}\underline{D}}_{(I)})_a\,, \tag{A.3e}$$

where for the hypermultiplet fields the label (I) indicates the link between the I^{th} and $(I+1)^{\text{st}}$ node and $s_I = 1$ for a hypermultiplet. The bracketed terms vanish unless the condition in the subscript is satisfied. To obtain the transformations for the twisted multiplets, we replace the overlined $\mathrm{SU}(2)_A$ indices for underlined $\mathrm{SU}(2)_B$ indices and choose $s_I = -1$.

The superconformal transformations of the fields are found by replacing $\xi^a \to x^\mu(\gamma_\mu)^a{}_b\eta^b$, with the exception of the fermions. The variation of the fermions with respect to the superconformal supercharges are given by

$$\begin{aligned}\delta(\psi_{(I)\underline{A}})^a = &\sqrt{2}(\not{D})^a{}_b q^{\bar{B}}_{(I)} x^\mu(\gamma_\mu)^b{}_c(\eta_{\bar{B}\underline{A}})^c + \sqrt{2}q^{\bar{B}}_{(I)}\eta^a_{\bar{B}\underline{A}} \\ &- \frac{\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{B}\underline{A}})^b\left[(\nu_{(I)})^{\bar{C}}{}_{\bar{C}}q^{\bar{B}}_{(I)} - q^{\bar{B}}_{(I)}(\tilde{\nu}_{(I)})^{\bar{C}}{}_{\bar{C}}\right] \\ &+ \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{C}\underline{B}})^b(\varphi_{(I)})^{\underline{B}}{}_{\underline{A}}q^{\bar{C}}_{(I)}\right)_{k_I=0} \\ &- \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{C}\underline{B}})^b(\tilde{\mu}_{(I-1)})^{\underline{B}}{}_{\underline{A}}q^{\bar{C}}_{(I)}\right)_{k_I\neq 0} \\ &- \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{C}\underline{B}})^b q^{\bar{C}}_{(I)}(\varphi_{(I+1)})^{\underline{B}}{}_{\underline{A}}\right)_{k_{I+1}=0} \\ &+ \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b(\eta_{\bar{C}\underline{B}})^b q^{\bar{C}}_{(I)}(\mu_{(I+1)})^{\underline{B}}{}_{\underline{A}}\right)_{k_{I+1}\neq 0},\end{aligned} \tag{A.4}$$

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$$
\begin{aligned}
\delta(\bar{\psi}^{\underline{A}}_{(I)})_a = &\sqrt{2}(\not{D})_{ab}\bar{q}^{\bar{B}}_{(I)}x^\mu(\gamma_\mu)^b{}_c\eta^{\bar{B}\underline{A}})^c + \sqrt{2}\bar{q}_{(I)\bar{B}}\eta_a^{\bar{B}\underline{A}} \\
&+ \frac{\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{B}\underline{A}})^b\left[(\tilde{\nu}_{(I)})^{\bar{C}}{}_{\bar{C}}\bar{q}_{(I)\bar{B}} - \bar{q}_{(I)\bar{B}}(\nu_{(I)})^{\bar{C}}{}_{\bar{C}}\right] \\
&+ \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}\underline{B}})^b\bar{q}_{(I)\bar{C}}(\varphi_{(I)})^{\underline{A}}{}_{\underline{B}}\right)_{k_I=0} \\
&- \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}\underline{B}})^b\bar{q}_{(I)\bar{C}}(\tilde{\mu}_{(I-1)})^{\underline{A}}{}_{\underline{B}}\right)_{k_I\neq 0} \\
&- \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}\underline{B}})^b(\varphi_{(I+1)})^{\underline{A}}{}_{\underline{B}}\bar{q}_{(I)\bar{C}}\right)_{k_{I+1}=0} \\
&+ \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}\underline{B}})^b(\mu_{(I+1)})^{\underline{A}}{}_{\underline{B}}\bar{q}_{(I)\bar{C}}\right)_{k_{I+1}\neq 0}.
\end{aligned}
\tag{A.5}
$$

B Derivation of the Killing spinor (5.17)

In this appendix we work out the explicit expression for the Killing spinor associated to the first parameterization of the space employed in the main text, namely equations (5.11) and (5.15).

The AdS part is standard. For example, for a time-like line the vielbeins can be chosen as follows

$$
e^0 = R\cosh u\,\cosh\rho\,dt\,,\quad e^1 = R\cosh u\,d\rho\,,\quad e^2 = R\,du\,,\quad e^3 = R\sinh u\,d\phi\,, \tag{B.1}
$$

and the relevant non-vanishing components of the spin connection are given by

$$
\omega^{01} = \sinh\rho\,dt\,,\quad \omega^{02} = \sinh u\,\cosh\rho\,dt\,,\quad \omega^{12} = \sinh u\,d\rho\,,\quad \omega^{32} = \cosh u\,d\phi\,, \tag{B.2}
$$

The indices $0, 1, \ldots, 9, \#$ are on the tangent space and in the following we denote them by the letters from the beginning of the Latin alphabet $a, b, \ldots$. We can take the background 4-form field-strength to be proportional to the volume form of AdS_4

$$
H_{(4)} = \kappa\, e^0\wedge e^1\wedge e^2\wedge e^3 \;= \kappa R^4\cosh^2 u\,\cosh\rho\,\sinh u\,dt\wedge d\rho\wedge du\wedge d\phi\,, \tag{B.3}
$$

with κ a constant to be determined. The Killing spinor equation is given by the variation of the gravitino and reads [46]

$$
D_\mu\eta + \frac{1}{288}\left(\gamma_{\mu\nu\rho\sigma\tau}H^{\nu\rho\sigma\tau} - 8H_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}\right)\eta = 0\,, \tag{B.4}
$$

where $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}$ and $\gamma_\mu = e_\mu^a\Gamma_a$ (lower case γ's are in the curved space, while upper case Γ's are in the tangent space). Now we distinguish between the AdS and the sphere directions. Using the expression for the 4-form we arrive at

$$
\begin{aligned}
D_\mu\eta &= -\frac{\kappa}{6}\gamma_\mu\Gamma_\star\eta\,, \qquad \mu \text{ along } AdS_4\,, \\
D_\mu\eta &= \frac{\kappa}{12}\gamma_\mu\Gamma_\star\eta\,, \qquad \mu \text{ along } S^7\,,
\end{aligned}
\tag{B.5}
$$

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with $\Gamma_\star \equiv \Gamma_{0123}$. To find the solution it is useful to use the following identities. If $[X,Y]=2Z$ and $[X,Z]=-2Y$ then

$$e^{\frac{1}{2}\theta X} Y = (\cos\theta\, Y + \sin\theta\, Z)\, e^{\frac{1}{2}\theta X}\,. \tag{B.6}$$

If, on the other hand, $[X,Y]=\pm 2Z$ and $[X,Z]=\pm 2Y$ then

$$e^{\frac{1}{2}\theta X} Y = (\cosh\theta\, Y \pm \sinh\theta\, Z)\, e^{\frac{1}{2}\theta X}\,. \tag{B.7}$$

The AdS_4 part of the Killing spinor is given by

$$e^{\frac{u}{2}\Gamma_2\Gamma_\star} e^{\frac{\rho}{2}\Gamma_1\Gamma_\star} e^{\frac{t}{2}\Gamma_0\Gamma_\star} e^{\frac{\phi}{2}\Gamma_{23}} \eta_0\,. \tag{B.8}$$

It can be readily checked that it must be $\kappa = -3/R$.

On the other hand, the vielbeins for the 7-sphere (5.11) are given by

$$\begin{aligned} & e^4 = 2R\, d\alpha\,, \\ & e^5 = -R\cos\alpha\, d\theta_1\,, \quad e^6 = -R\cos\alpha\,(2dx + \cos\theta_1\, d\phi_1)\,, \quad e^7 = R\cos\alpha\, \sin\theta_1\, d\phi_1\,, \\ & e^8 = R\sin\alpha\, d\theta_2\,, \quad e^9 = R\sin\alpha\,(2dv + \cos\theta_2\, d\phi_2)\,, \quad e^\# = -R\sin\alpha\, \sin\theta_2\, d\phi_2\,, \end{aligned} \tag{B.9}$$

and the non-vanishing components of the spin connection are

$$\begin{aligned} & \omega^{54} = \frac{1}{2}\sin\alpha\, d\theta_1\,, \quad \omega^{64} = \sin\alpha\left(dx + \frac{1}{2}\cos\theta_1\, d\phi_1\right)\,, \quad \omega^{74} = -\frac{1}{2}\sin\alpha\, \sin\theta_1\, d\phi_1\,, \\ & \omega^{48} = -\frac{1}{2}\cos\alpha\, d\theta_2\,, \quad \omega^{49} = -\cos\alpha\left(dv + \frac{1}{2}\cos\theta_2\, d\phi_2\right)\,, \quad \omega^{4\#} = \frac{1}{2}\cos\alpha\, \sin\theta_2\, d\phi_2\,, \\ & \omega^{65} = -\frac{1}{2}\sin\theta_1\, d\phi_1\,, \quad \omega^{75} = dx - \frac{1}{2}\cos\theta_1\, d\phi_1\,, \quad \omega^{76} = \frac{1}{2}d\theta_1\,, \\ & \omega^{98} = -\frac{1}{2}\sin\theta_2\, d\phi_2\,, \quad \omega^{\#8} = dv - \frac{1}{2}\cos\theta_2\, d\phi_2\,, \quad \omega^{\#9} = \frac{1}{2}d\theta_2\,. \end{aligned} \tag{B.10}$$

The Killing spinor equations for the sphere components are

$$\begin{aligned} \partial_\alpha\eta &= -\frac{1}{2}\Gamma_4\Gamma_\star\eta\,, \\ \partial_{\theta_1}\eta &= \frac{1}{4}(\sin\alpha\,\Gamma_{45} + \Gamma_{67} + \cos\alpha\,\Gamma_5\Gamma_\star)\,\eta\,, \\ \partial_x\eta &= \frac{1}{2}(\sin\alpha\,\Gamma_{46} + \Gamma_{57} + \cos\alpha\,\Gamma_6\Gamma_\star)\,\eta\,, \\ \partial_{\phi_1}\eta &= -\frac{1}{4}\Big(\sin\alpha\,(\cos\theta_1\,\Gamma_{64} - \sin\theta_1\,\Gamma_{74}) + \sin\theta_1\,\Gamma_{56} + \cos\theta_1\,\Gamma_{57} \\ &\qquad - \cos\alpha\,(\cos\theta_1\Gamma_6\Gamma_\star - \sin\theta_1\,\Gamma_7\Gamma_\star)\Big)\eta\,, \\ \partial_{\theta_2}\eta &= -\frac{1}{4}(\cos\alpha\,\Gamma_{84} + \Gamma_{9\#} + \sin\alpha\,\Gamma_8\Gamma_\star)\,\eta\,, \\ \partial_v\eta &= -\frac{1}{2}(\cos\alpha\,\Gamma_{94} + \Gamma_{8\#} + \sin\alpha\,\Gamma_9\Gamma_\star)\,\eta\,, \\ \partial_{\phi_2}\eta &= -\frac{1}{4}\Big(-\cos\alpha\,(\cos\theta_2\,\Gamma_{49} - \sin\theta_2\,\Gamma_{4\#}) + \sin\theta_2\,\Gamma_{89} + \cos\theta_2\,\Gamma_{8\#} \\ &\qquad + \sin\alpha\,(\cos\theta_2\Gamma_9\Gamma_\star - \sin\theta_2\,\Gamma_\#\Gamma_\star)\Big)\eta\,. \end{aligned} \tag{B.11}$$

With a little bit of algebra it can be checked that $\eta = \Psi\,\eta_0$, with Ψ given by (5.17), is indeed a solution of these equations.

References

[1] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[2] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002] [INSPIRE].

[3] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001] [INSPIRE].

[4] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191] [INSPIRE].

[5] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055] [INSPIRE].

[6] N. Drukker and D.J. Gross, *An Exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[7] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [arXiv:0909.0945] [INSPIRE].

[8] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, *JHEP* **02** (2010) 057 [arXiv:0909.1105] [INSPIRE].

[9] D. Gaiotto, G.W. Moore and A. Neitzke, *Framed BPS States*, *Adv. Theor. Math. Phys.* **17** (2013) 241 [arXiv:1006.0146] [INSPIRE].

[10] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-Matter theories*, *JHEP* **08** (2007) 056 [arXiv:0704.3740] [INSPIRE].

[11] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089 [arXiv:0909.4559] [INSPIRE].

[12] B. Assel and J. Gomis, *Mirror Symmetry And Loop Operators*, arXiv:1506.01718 [INSPIRE].

[13] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [INSPIRE].

[14] O. Aharony, O. Bergman and D.L. Jafferis, *Fractional M2-branes*, *JHEP* **11** (2008) 043 [arXiv:0807.4924] [INSPIRE].

[15] B. Chen and J.-B. Wu, *Supersymmetric Wilson Loops in $N = 6$ Super Chern-Simons-matter theory*, *Nucl. Phys.* **B 825** (2010) 38 [arXiv:0809.2863] [INSPIRE].

[16] N. Drukker, J. Plefka and D. Young, *Wilson loops in 3-dimensional $N = 6$ supersymmetric Chern-Simons Theory and their string theory duals*, *JHEP* **11** (2008) 019 [arXiv:0809.2787] [INSPIRE].

JHEP10(2015)140

[17] S.-J. Rey, T. Suyama and S. Yamaguchi, *Wilson Loops in Superconformal Chern-Simons Theory and Fundamental Strings in Anti-de Sitter Supergravity Dual*, *JHEP* **03** (2009) 127 [arXiv:0809.3786] [INSPIRE].

[18] N. Drukker and D. Trancanelli, *A Supermatrix model for $N = 6$ super Chern-Simons-matter theory*, *JHEP* **02** (2010) 058 [arXiv:0912.3006] [INSPIRE].

[19] B. Chen, J.-B. Wu and M.-Q. Zhu, *Holographical Description of BPS Wilson Loops in Flavored ABJM Theory*, *JHEP* **12** (2014) 143 [arXiv:1410.2311] [INSPIRE].

[20] D. Gaiotto and E. Witten, *Janus Configurations, Chern-Simons Couplings, And The theta-Angle in $N = 4$ Super Yang-Mills Theory*, *JHEP* **06** (2010) 097 [arXiv:0804.2907] [INSPIRE].

[21] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *N=4 Superconformal Chern-Simons Theories with Hyper and Twisted Hyper Multiplets*, *JHEP* **07** (2008) 091 [arXiv:0805.3662] [INSPIRE].

[22] Y. Imamura and K. Kimura, *N=4 Chern-Simons theories with auxiliary vector multiplets*, *JHEP* **10** (2008) 040 [arXiv:0807.2144] [INSPIRE].

[23] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds*, *JHEP* **09** (2008) 002 [arXiv:0806.4977] [INSPIRE].

[24] E. Koh, S. Lee and S. Lee, *Topological Chern-Simons σ-model*, *JHEP* **09** (2009) 122 [arXiv:0907.1641] [INSPIRE].

[25] H. Ouyang, J.-B. Wu and J.-j. Zhang, *Supersymmetric Wilson loops in $N = 4$ super Chern-Simons-matter theory*, arXiv:1506.06192 [INSPIRE].

[26] V. Cardinali, L. Griguolo, G. Martelloni and D. Seminara, *New supersymmetric Wilson loops in ABJ(M) theories*, *Phys. Lett.* **B 718** (2012) 615 [arXiv:1209.4032] [INSPIRE].

[27] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *$\mathcal{N} = 8$ Superconformal Chern-Simons Theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242] [INSPIRE].

[28] M. Mariño and P. Putrov, *Exact Results in ABJM Theory from Topological Strings*, *JHEP* **06** (2010) 011 [arXiv:0912.3074] [INSPIRE].

[29] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [arXiv:1007.3837] [INSPIRE].

[30] A. Klemm, M. Mariño, M. Schiereck and M. Soroush, *Aharony-Bergman-Jafferis-Maldacena Wilson loops in the Fermi gas approach*, *Z. Naturforsch.* **A 68** (2013) 178 [arXiv:1207.0611] [INSPIRE].

[31] Y. Hatsuda, M. Honda, S. Moriyama and K. Okuyama, *ABJM Wilson Loops in Arbitrary Representations*, *JHEP* **10** (2013) 168 [arXiv:1306.4297] [INSPIRE].

[32] Y. Imamura and S. Yokoyama, *N=4 Chern-Simons theories and wrapped M-branes in their gravity duals*, *Prog. Theor. Phys.* **121** (2009) 915 [arXiv:0812.1331] [INSPIRE].

[33] B. Assel, C. Bachas, J. Estes and J. Gomis, *IIB Duals of $D = 3$ $\mathcal{N} = 4$ Circular Quivers*, *JHEP* **12** (2012) 044 [arXiv:1210.2590] [INSPIRE].

[34] K. Skenderis and M. Taylor, *Branes in AdS and p p wave space-times*, *JHEP* **06** (2002) 025 [hep-th/0204054] [INSPIRE].

[35] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS Type IIB interface solutions. I. Local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [arXiv:0705.0022] [INSPIRE].

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[36] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS Type IIB interface solutions. II. Flux solutions and multi-Janus*, *JHEP* **06** (2007) 022 [arXiv:0705.0024] [INSPIRE].

[37] B. Assel, C. Bachas, J. Estes and J. Gomis, *Holographic Duals of $D = 3$ $\mathcal{N} = 4$ Superconformal Field Theories*, *JHEP* **08** (2011) 087 [arXiv:1106.4253] [INSPIRE].

[38] M.S. Bianchi, G. Giribet, M. Leoni and S. Penati, *1/2 BPS Wilson loop in $N = 6$ superconformal Chern-Simons theory at two loops*, *Phys. Rev.* **D 88** (2013) 026009 [arXiv:1303.6939] [INSPIRE].

[39] M.S. Bianchi, G. Giribet, M. Leoni and S. Penati, *The 1/2 BPS Wilson loop in ABJ(M) at two loops: The details*, *JHEP* **10** (2013) 085 [arXiv:1307.0786] [INSPIRE].

[40] L. Griguolo, G. Martelloni, M. Poggi and D. Seminara, *Perturbative evaluation of circular 1/2 BPS Wilson loops in $N = 6$ Super Chern-Simons theories*, *JHEP* **09** (2013) 157 [arXiv:1307.0787] [INSPIRE].

[41] M. Mariño and P. Putrov, *ABJM theory as a Fermi gas*, *J. Stat. Mech.* **1203** (2012) P03001 [arXiv:1110.4066] [INSPIRE].

[42] S. Moriyama and T. Nosaka, *Partition Functions of Superconformal Chern-Simons Theories from Fermi Gas Approach*, *JHEP* **11** (2014) 164 [arXiv:1407.4268] [INSPIRE].

[43] K.-M. Lee and S. Lee, *1/2-BPS Wilson Loops and Vortices in ABJM Model*, *JHEP* **09** (2010) 004 [arXiv:1006.5589] [INSPIRE].

[44] N. Drukker, J. Gomis and D. Young, *Vortex Loop Operators, M2-branes and Holography*, *JHEP* **03** (2009) 004 [arXiv:0810.4344] [INSPIRE].

[45] N. Drukker, T. Okuda and F. Passerini, *Exact results for vortex loop operators in 3d supersymmetric theories*, *JHEP* **07** (2014) 137 [arXiv:1211.3409] [INSPIRE].

[46] H. Lü, C.N. Pope and J. Rahmfeld, *A Construction of Killing spinors on S^n*, *J. Math. Phys.* **40** (1999) 4518 [hep-th/9805151] [INSPIRE].

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