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Research Article

Acute Triangulations of Trapezoids and Pentagons

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An acute triangulation of a polygon is a triangulation whose triangles have all their angles less than $\pi/2$. The number of triangles in a triangulation is called the size of it. In this paper, we investigate acute triangulations of trapezoids and convex pentagons and prove new results about such triangulations with minimum size. This completes and improves in some cases the results obtained in two papers of Yuan (2010).

1. Introduction and Preliminaries

A triangulation of a planar polygon is a finite set of nonoverlapping triangles covering the polygon in such a way that any two distinct triangles are either disjoint or intersect in a single common vertex or edge. An acute (resp., nonobtuse) triangulation of a polygon is a triangulation whose triangles have all their angles less (resp., not larger) than $\pi/2$. The number of triangles in a triangulation is called the size. Burago and Zalgaller [1] and, independently, Goldberg and Manheimer [2] proved that every obtuse triangle can be triangulated into seven acute triangles and this bound is the best possible. Cassidy and Lord [3] showed that every square can be triangulated into eight acute triangles and eight is the minimum number. This remains true for any rectangle as proved by Hangan et al. in [4]. Acute triangulations of trapezoids, quadrilaterals, and pentagons were investigated in [5–8]. Further information, historical notes, and problems about acute triangulations of polygons and surfaces can be found in the survey paper [9]. Let $\mathscr{K}$ denote a family of planar polygons, and for $K \in \mathscr{K}$, let $f(K)$ be the minimum size of an acute triangulation of K. Then, let $f(\mathscr{K})$ denote the maximum value of $f(K)$ for all $K \in \mathscr{K}$. The following results are known.

Theorem 1. *(i) Reference [7]: let $\mathscr{T}$ denote the family of all trapezoids, that is, quadrilaterals with at least one pair of parallel sides. Then, $f(\mathscr{T} \setminus \mathscr{R}) = 7$, where $\mathscr{R}$ is the family of all rectangles (also including squares).*

(ii) Reference [6]: let $\mathbb{Q}$ be the family of all quadrilaterals. Then, $f(\mathbb{Q}) = 10$.

(iii) Reference [5]: let $\mathbb{Q}_c$ be the family of all convex quadrilaterals. Then, $f(\mathbb{Q}_c) = 8$.

(iv) Reference [8]: let $\mathscr{P}$ denote the family of all planar pentagons. Then, $f(\mathscr{P}) \leq 54$.

In this paper, we discuss acute triangulations of trapezoids and convex pentagons and prove new results of such triangulations with minimum size. For example, we get the following characterization of the right trapezoids: *they are the only trapezoids needing exactly six triangles and one interior vertex for an acute triangulation of minimum size*. For the family of convex pentagons, we show that the bound stated in Theorem 1(iv) can be improved under some additional conditions.

Let Γ be a convex planar polygon. A vertex P of Γ is called an acute (resp., *right*) corner if the interior angle of Γ at P is less than (resp., equal to) $\pi/2$; otherwise, P is called an obtuse corner. Let $\mathbf{T}$ be an acute triangulation of Γ. A vertex or edge of $\mathbf{T}$ is called a boundary (resp., interior) vertex or edge if it lies on the boundary of Γ (resp., lies inside Γ). Let n, n_i, and n_∂ be the number of vertices, interior vertices, and boundary vertices of $\mathbf{T}$. Let m, m_i, and m_∂ be the number of edges, interior edges, and boundary edges of $\mathbf{T}$. Clearly, we have $n = n_i + n_\partial$, $m = m_i + m_\partial$, and $n_\partial = m_\partial$. For each vertex P in $\mathbf{T}$, the number of edges incident to P is called the degree of P, denoted by $\deg(P)$. Let ν_k denote the number of vertices in $\mathbf{T}$ of degree k. Let f denote the number of triangles in $\mathbf{T}$. The following lemma is easily verified (cf. [6, Lemma 1]).

Lemma 2. *Let* $\mathbf{T}$ *be an acute triangulation of a planar convex* n*-gon* Γ. *Then, one has*

(1) $f = m - n + 1 = n + n_i - 2 = m_i - n_i + 1$,

(2) $3f = 2m - m_\partial = 2m_i + n_\partial$,

(3) $\nu_1 = 0$; $\nu_2 + \nu_3 \le n$,

(4) $n_\partial = m_\partial \le f + \nu_2$; $4f + \nu_2 \ge 2m$,

(5) *If a vertex* P *of* $\mathbf{T}$ *is an interior vertex, then* $\deg(P) \ge 5$, *if* P *lies within a side of* Γ, *then* $\deg(P) \ge 4$, *and if* P *is an obtuse or right corner of* Γ, *then* $\deg(P) \ge 3$.

2. Characterizations of Trapezoids

2.1. Parallelograms. Let $ABCD$ be a parallelogram with acute corners at B and D and $|AB| \le |BC|$. If the diagonal AC divides the angles at A and C into acute angles, then $ABCD$ is triangulable with exactly two acute triangles. Otherwise, we have the following lemma that completes Theorem 2.1 in Section 2 of [7].

Lemma 3. *Let ABCD be a parallelogram with acute corners at B and D. Suppose that the diagonal AC does not divide the angle at A (or C) into acute angles. Then, ABCD is triangulable with four acute triangles, and this bound is the best possible.*

Proof. Let f be the smallest number of triangles in an acute triangulation $\mathbf{T}$ of the parallelogram $ABCD$. Some edge of $\mathbf{T}$ incident with A (resp., C) must meet the interior of $ABCD$. Denote AP to be such an edge, and suppose that P is an interior vertex of $ABCD$, that is, $n_i \ge 1$. Then, the vertices B and D have degree at least 2, the vertices A and C have degree at least 3, and the vertex P has degree at least 5. Summing up, we have $2m \ge 2 \times 2 + 2 \times 3 + 5 = 15$, hence $2m \ge 16$. Since $\nu_2 \le 2$, by Lemma 2(4), we get $4f + 2 \ge 16$, so $f \ge 4$. Assume now that $n_i = 0$. Then, P lies in the interior of the edge BC (or DC). Otherwise, if $P = C$, at least one of the vertices A and C must have degree ≥ 4 by hypothesis. This implies that there is a neighbor of it with degree ≥ 4, giving a contradiction as $n_i = 0$. Thus, $n_\partial = n$ and $m_i \ge 3$ as the degree of P is at least 4, and the degree of A (resp., C) is at least 3. By Lemma 2(1), we get $f = m_i - n_i + 1 \ge 4$. Now, for a parallelogram as in the statement, an acute triangulation with 4 triangles is given in [7, Section 2]. □

2.2. Trapezoids. Following [7], we say that a trapezoid is a quadrilateral with at least one pair of parallel sides. Let $ABCD$ be a trapezoid with parallel sides AD and BC with $|AD| < |BC|$. Let E (resp., F) be the orthogonal projection of A (resp., D) on the straight line ℓ_{BC} containing BC. Suppose that E is interior to BC and F exterior to BC. If the diagonal AC divides the angles at A and C into acute angles, then $ABCD$ is triangulable with exactly two acute triangles. Otherwise, we have the following lemma which can be proved in the same manner as Lemma 3.

Lemma 4. *Let ABCD be a trapezoid with acute corners at B and D, parallel sides AD and BC with an obtuse angle* $B\hat{A}C$, *and an acute angle* $C\hat{A}D$. *Then, ABCD is triangulable with four acute triangles, and this bound is the best possible.*

Let $ABCD$ be a trapezoid with two adjacent acute angles at B and C, parallel sides AD and BC, and, consequently, $|AD| < |BC|$. If there exists an interior point P on BC such that the triangles BAP, APD, and PDC are acute, then $ABCD$ is triangulable by 3 acute triangles. Otherwise, we have the following lemma.

Lemma 5. *Let ABCD be a trapezoid with two adjacent acute angles at B and C and parallel sides AD and BC. Suppose that there is no interior point P on BC such that BAP, APD, and PDC are acute triangles. Then, ABCD is triangulable with five acute triangles, and this bound is the best possible.*

Proof. Let f be the smallest number of triangles in an acute triangulation $\mathbf{T}$ of any trapezoid as in the statement. Some edge AP must meet the interior of $ABCD$. Suppose that P is an interior point of $ABCD$, that is, $n_i \ge 1$. Then, it immediately follows that $\deg(P) \ge 5$ and so $f \ge 5$. Suppose that there is no interior vertex in $ABCD$, that is, $n_i = 0$ and P is a boundary vertex. Some edge DQ must meet the interior of $ABCD$, and Q is also a boundary vertex. We can assume that $P \ne Q$ since at least one of the triangles BAP, APD, and PDC is not acute by hypothesis. This implies that $m_i \ge 4$. By Lemma 2(1), $f = m_i - n_i + 1 \ge 5$. Now, for a trapezoid as in the statement, an acute triangulation with 5 triangles is given in [7, Section 3]. □

The following result gives a characterization of the right trapezoids.

Proposition 6. *Every trapezoid with exactly two right angles is triangulable with six acute triangles and one interior vertex, and this bound is the best possible.*

Proof. Let f be the smallest number of triangles in an acute triangulation $\mathbf{T}$ of any trapezoid $ABCD$ with exactly two right corners at D and C. Let B be an acute corner, thus $|AB| \le |BC|$. Some edge AP must meet the interior of $ABCD$. Suppose that P is an interior vertex of $ABCD$ and $n_i = 1$. Then, P is an end vertex of five interior edges. At least one neighbour of P is interior to a side of the trapezoid and is therefore incident of a further interior edge. So, $m_i \ge 6$, $n_i = 1$ and $f = m_i - n_i + 1 \ge 6$. Suppose that P is an interior vertex and $n_i \ge 2$. Then, there are at least two interior vertices with degree ≥ 5, the degree of A (resp., C and D) is ≥ 3, the degree of B is ≥ 2, and there is at least one vertex of degree ≥ 4. Then, we have $2m \ge 3 \times 3 + 2 \times 5 + 4 + 2 \ge 25$, so $2m \ge 26$. Since $\nu_2 \le 1$ by Lemma 2(4), we get $4f + 1 \ge 26$, so $f > 6$. Suppose that $n_i = 0$ and $n_\partial = n$. Some edge of AP must meet the interior of $ABCD$. Then P cannot lie on the edge DC; otherwise, some edge incident to D must meet the interior of $ABCD$, and we get an interior vertex against the fact that $n_i = 0$. So, P must be in the interior of BC. If the angles at P are not right, we get a contradiction since $APCD$ is a trapezoid with only two right angles which admits an acute triangulation of at least size f. But this contrasts with the minimum size f of $\mathbf{T}$. If the angles at P are right, then $APCD$ is a square. But any acute triangulation of a square must have at least an interior vertex. So, we get again a contradiction as $n_i = 0$. Now, for a trapezoid as in the statement, an acute triangulation with 6 triangles and one interior vertex are given in [7, Section 3]. □

Lemma 7. *Let $ABCD$ be a trapezoid with acute corners at B and D, parallel sides AD and BC, and E and F both exterior to BC. Then, $ABCD$ is triangulable with seven acute triangles. This bound is the best possible among the acute triangulations of such a trapezoid which have at least one interior vertex.*

Proof. Let $\mathbf{T}$ be an acute triangulation with at least one interior vertex for a trapezoid as in the statement. Let f denote the size of $\mathbf{T}$. Suppose that $n_i = 2$. Then, there are two interior vertices P and S in $\mathbf{T}$, and at least two neighbours of P and/or S have degree at least 4. Since $m_i \geq 9$ and $n_i = 2$, Lemma 2(1) gives $f = m_i - n_i + 1 \geq 8$. Suppose that $n_i \geq 3$. Then, there are at least three interior vertices P, R, and S with degree ≥ 5. The degrees of A and C are ≥ 3, and those of B and D are ≥ 2. There are at least two neighbours of P, R and/or S with degree ≥ 4. Summing up, we have $2m \geq 3\times5+2\times3+2\times2+2\times4 = 33$, hence $2m \geq 34$. Since $\nu_2 \leq 2$, by Lemma 2(4), we get $4f + 2 \geq 34$, hence $f \geq 8$. So, we can assume that $n_i = 1$. The interior vertex P cannot be connected to all the vertices of $ABCD$. Otherwise, there is a neighbour of P which is interior to a side of the trapezoid. It is therefore adjacent to a further interior vertex of the trapezoid, that is, $n_i \geq 2$. This contradicts $n_i = 1$. If P is connected to exactly three vertices of the trapezoid, say A, B, and C, there are two neighbours of P which have degree ≥ 4. Further, at least one of the obtuse corners A and C must have degree ≥ 4. Then, we have $m_i \geq 7$, $m_\partial = n_\partial = n-1$ and $n+f-1 = m_i+m_\partial \geq 7+n-1$, hence $f \geq 7$. If P is joined to exactly two vertices of $ABCD$, then there are three neighbours of P with degree ≥ 4. The degree of the two vertices of $ABCD$ joined with P is ≥ 3, and the remaining two vertices have degree ≥ 2. Summing up, we get $2m \geq 5 + 2\times2 + 2\times3 + 3\times4 = 27$, hence $2m \geq 28$. Since $\nu_2 \leq 2$, by Lemma 2(4), we have $4f + 2 \geq 28$, hence $f \geq 7$. If P is joined to exactly one vertex of $ABCD$, then four neighbours of P have degree ≥ 4. Thus, $m_i \geq 7$, and, by Lemma 2(1), $f = m_i - n_i + 1 \geq 7$. Now, for a trapezoid as in statement, an acute triangulation with 7 acute triangles was described in [7, Section 3]. □

3. Acute Triangulations of Pentagons

The following proposition follows directly from the results proved in [5].

Proposition 8. *Every convex quadrilateral admits an acute triangulation of size at most eight, such that there are at most two new vertices introduced on each side.*

It was shown in [8, Lemma 3.1] that every pentagon with at least one acute corner can be triangulated into at most 32 acute triangles. Under the hypothesis of convexity, we have the following.

Proposition 9. *Every convex pentagon with at least one acute corner can be triangulated into at most 25 acute triangles.*

Proof. Let $\Gamma = ABCDE$ (in the anticlockwise order) be a convex pentagon with at least one acute corner, say A. We distinguish some cases.

Case 1. The triangle ABE is acute. By Proposition 8, the convex quadrilateral $BCDE$ has an acute triangulation with size ≤ 8 such that there are at most 2 side vertices on BE.

Subcase 1.1. There is no side vertex on BE. Then, Γ admits an acute triangulation with at most 9 triangles.

Subcase 1.2. There is precisely one side vertex, say P, on BE. By Lemma 4 of [6], since ABE is an acute triangle, for any point P on the side BE, there are two points R on AE and S on AB such that the line segments PR, RS, and PS divide ABE into four acute triangles. Then, we get an acute triangulation of Γ into at most 12 triangles.

Subcase 1.3. There are exactly two side vertices, say M and N, on BE. In this case, the convex quadrilateral $BCDE$ has an acute triangulation of size 7, as shown in [5]. Suppose that M is an interior point of EN. Let P be the point on AE such that PN is parallel to AB. Then, the triangle PNE is acute. By Lemma 4 of [6], the triangle PNE can be triangulated into four acute triangles with M as the only side vertex on EN and two new vertices R and S on the edges EP and NP, respectively. Let H and K be the orthogonal projections of S and N on the edge AB. The segments NK and SK divide the right trapezoid $HBNS$ into three right triangles. By [7, Section 3], there is an acute triangulation of the right trapezoid $AHSP$ of size 6 without new vertices introduced on the sides PS and HS. Then, we can slightly slide K and H in direction from B to A such that the triangles SHK, SKN, and BKN become acute. This gives an acute triangulation of Γ whose size is at most 20.

Case 2. The triangle ABE is nonacute, that is, E, for example, is a nonacute corner.

Subcase 2.1. There is no side vertex on BE. Then, there exists an acute triangle, say EBP, which belongs to an acute triangulation of $BCDE$ with size ≤ 8. Let H be the orthogonal projection of P on the side BE. By Theorem 2 of [6], since ABE is a triangle with nonacute corner E, for any point H on the side BE, there is an acute triangulation of ABE with size 7 such that H is the only side vertex lying on BE. Such an acute triangulation of ABE has new vertices R (resp., S and T) introduced on the side AE (resp., AB). Finally, we can slightly slide H away from EBP in direction perpendicular to HP such that the triangles EHP and BHP become acute. This gives an acute triangulation of Γ into at most 16 acute triangles.

Subcase 2.2. There is precisely one side vertex P on BE. As in the previous subcase, by Theorem 2 of [6], for any point P

on the side BE, there is an acute triangulation of the triangle ABE with size 7 such that P is the only side vertex lying on BE. This gives an acute triangulation of Γ into at most 15 acute triangles.

Subcase 2.3. There are exactly two side vertices, say M and N, on BE. In this case, $BCDE$ has an acute triangulation of size 7 by [5]. Suppose that M is an interior point of EN. Let P be the point on AE such that PN is parallel to AB. Then, the triangle ENP has a nonacute corner E. By Theorem 2 of [6], ENP can be triangulated into 7 acute triangles with M as the only side vertex on EN and new vertices R, respectively, S and T on the edges EP, respectively, NP. Let H, K, and L be the orthogonal projections of S, T, and N on the edge AB, respectively. The line segments NL, TL, TK, and SK divide the right trapezoid $HBNS$ into 5 right triangles. By [7, Section 3], there is an acute triangulation of the right trapezoid $AHSP$ of size 6 without new vertices introduced on the sides PS and HS. Then, we can slightly slide L, K, and H in direction from B to A such that the triangles BLN, TNL, KTL, KTS, and SHK become acute. This gives an acute triangulation of Γ whose size is at most 25. □

Corollary 10. *Every convex pentagon with at least two nonadjacent acute corners can be triangulated into at most 16 acute triangles.*

Proof. By the hypothesis and the results from [5], we can avoid subcases 1.3 and 2.3 in the above proof. The remaining cases give the requested bound. □

The following proposition follows directly from the results proved in [3, 4, 7].

Proposition 11. *Every trapezoid (resp., rectangle) admits an acute triangulation of size at most 7 (resp., 8) such that there are at most one new vertex introduced on each side.*

Proposition 12. *Let Γ be a convex pentagon which has at least one acute corner and two parallel sides, nonincident to it. Then, Γ can be triangulated into at most 14 acute triangles.*

Proof. Let $\Gamma = ABCDE$ (in the anticlockwise order) be a convex pentagon with at least one acute corner, say D, and two parallel sides AE and BC with $|AE| \leq |BC|$. We distinguish some cases.

Case 1. The triangle CDE is acute.

Subcase 1.1. Let H and K be the orthogonal projections of A and E, respectively, on the straight line ℓ_{BC} containing the edge BC. Suppose that H (resp., K) is interior (resp., exterior) of BC. By [7, Section 2], the trapezoid $ABCE$ admits an acute triangulation of size at most 4 such that there are no new vertices on the side CE. Then, Γ has an acute triangulation of size at most 5.

Subcase 1.2. Suppose that the above orthogonal projections H and K are interior to the edge BC. By [7, Section 2], the trapezoid $ABCE$ admits an acute triangulation of size at most 5 such that no new vertices are introduced on CE. Then, Γ has an acute triangulation of size at most 6.

Subcase 1.3. Suppose that $ABCE$ is a right trapezoid (this implies that the triangle CDE is acute). By [7, Section 3], $ABCE$ can be triangulated into 6 acute triangles such that there are no new vertices on CE. Then, Γ has an acute triangulation of size at most 7.

Subcase 1.4. Suppose that the above orthogonal projections H and K are exterior to BC. By [7, Section 3], $ABCE$ admits an acute triangulation of size at most 7 such that there is only one vertex, say P, on the side CE. By Lemma 4 of [6], for any point P in the side CE, there are two points R on DE and S on CD such that the line segments PR, PS, and RS divide CDE into 4 acute triangles. Then, Γ can be triangulated into at most 11 acute triangles.

Case 2. The triangle CDE is nonacute, that is, the corner E, for example, is nonacute.

Subcase 2.1. Let $ABCE$ be as in subcase 1.1. There is an acute triangle, say CEP, which belongs to the triangulation of size ≤ 4 of $ABCE$. Let R be the orthogonal projection of P on the edge CE. By Theorem 2 of [6], there is an acute triangulation of CDE with size 7 such that R is the only side vertex on CE. Then, Γ has an acute triangulation of size at most 12.

Subcase 2.2. Let $ABCE$ be as in subcase 1.2. Reasoning as in the previous subcase gives an acute triangulation of Γ with size ≤ 13.

Subcase 2.3. Let $ABCE$ be as in Subcase 1.4. By Theorem 2 of [6] the triangle CDE can be triangulated into at most 7 acute triangles such that the only side vertex on CE is P. This gives an acute triangulation of Γ with size ≤ 14. □

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