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Non-compact quantum spin chains as integrable stochastic particle processes

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Abstract

In this paper we discuss a family of models of particle and energy diffusion on a one-dimensional lattice, related to those studied previously in [1], [2] and [3] in the context of KPZ universality class. We show that they may be mapped onto an integrable $\mathfrak{sl}(2)$ Heisenberg spin chain whose Hamiltonian density in the bulk has been already studied in the AdS/CFT and the integrable system literature.

Using the quantum inverse scattering method, we study various new aspects, in particular we identify boundary terms, modeling reservoirs in non-equilibrium statistical mechanics models, for which the spin chain (and thus also the stochastic process) continues to be integrable. We also show how the construction of a "dual model" of probability theory is possible and useful.

The fluctuating hydrodynamics of our stochastic model corresponds to the semiclassical evolution of a string that derives from correlation functions of local gauge invariant operators of $\mathcal{N} = 4$ super Yang-Mills theory (SYM), in imaginary-time. As any stochastic system, it has a supersymmetric completion that encodes for the thermal equilibrium theorems: we show that in this case it is equivalent to the $\mathfrak{sl}(2|1)$ superstring that has been derived directly from $\mathcal{N} = 4$ SYM.

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1 Introduction

1.1 The setting

Stochastic systems may be described by a linear operator – called here the Hamiltonian $\mathcal{H}$ – that describes the infinitesimal evolution of the probability distribution. This operator has to guarantee probability conservation, and obviously be such that transition rates are real and positive. In many cases, such as the overdamped Langevin equation, $\mathcal{H}$ is, when written in some appropriate base, Hermitean. When this happens, we have a direct connection between the original stochastic system and a quantum one: the former evolves with Euclidean time ($e^{-t\mathcal{H}}$) and the latter with "real" time ($e^{-it\mathcal{H}}$). In particular, if we compute the sum over periodic trajectories of period β of the stochastic model, we are in fact calculating the partition function of the "quantum" system with temperature $1/\beta$. Establishing a link between a stochastic and a quantum model is illuminating, and allows to exchange techniques between two different fields. It sometimes happens that a *single* Hamiltonian has more than one basis in which it may be interpreted as a stochastic process. The picture described by these *dual* models may be quite different: for example one may describe the transport of a continuous quantity, interpreted as energy, while its dual the transport of discrete particles. We shall meet this situation here.

An example in point is the Symmetric Exclusion Process (SEP) in which particles move randomly to the right or to the left of a one-dimensional grid but are not allowed to superpose. It is well known that the evolution operator may be mapped onto a one-dimensional ferromagnetic chain $\mathcal{H} = \sum_i \left[2\sigma_0^{[i]}\sigma_0^{[i+1]} + \sigma_+^{[i]}\sigma_-^{[i+1]} + \sigma_-^{[i]}\sigma_+^{[i+1]}\right]$ of spins one-half, satisfying the $\mathfrak{su}(2)$ algebra:

$$[\sigma_0, \sigma_\pm] = \pm\sigma_\pm, \qquad [\sigma_+, \sigma_-] = 2\sigma_0\,. \tag{1.1}$$

Another popular system is the Kipnis-Marchioro-Presutti (KMP) model [4], where each pair of neighboring sites exchange randomly their energies. It was realized years ago [5, 6] that it may be put in direct relation with a chain of $\mathfrak{sl}(2)$ 'spins' $\mathcal{H} = \sum_i \left[2S_0^{[i]}S_0^{[i+1]} - S_+^{[i]}S_-^{[i+1]} - S_-^{[i]}S_+^{[i+1]}\right]$ satisfying at each site of the spin chain the commutation relations

$$[S_0, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = -2S_0\,. \tag{1.2}$$

This mapping immediately explained why many formulas for the KMP model could be obtained directly as an analytic continuation of the SEP ones: the $\mathfrak{sl}(2)$ algebra may be though of as "negative spin" representations of the $\mathfrak{su}(2)$ algebra. The relation has however a disappointing side: although the spin one-half $\mathfrak{su}(2)$ chain is integrable [7], the $\mathfrak{sl}(2)$ one associated with KMP is not.

Non-compact *integrable* spin chains were studied in theoretical physics, in relation to high energy QCD [8–10], $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM) [11–13] and the AdS/CFT dual string theory limit considered in [14–16]. However, so far their interpretation as stochastic processes has not been studied. In this paper we address precisely this question.

1.2 Models and relation to previous literature

We shall be interested in the relation between integrable non-compact spin chains and interacting particle systems, especially in the context of *non-equilibrium statistical mechanics*. Thus we shall consider the role of boundary stochastic processes, which amounts to including boundary terms in the spin chains. For the particle/energy process, the boundary processes model infinite reservoirs that fix the chemical potentials/temperatures at the boundaries. When their values are different, a non-equilibrium steady state sets in, with a non-zero current. It turns out that the class of *boundary-driven models* that we introduce have a bulk part that coincides with the *symmetric* version of several particle processes that have been previously studied on the infinite line $\mathbb{Z}$ as microscopic models for the Kardar-Parisi-Zhang (KPZ) universality class [17–19].

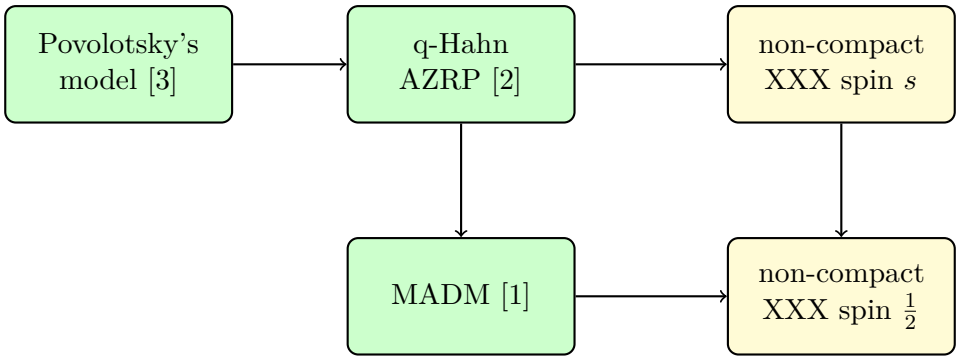

Figure 1: Relation between the stochastic hopping models in green boxes and the spin chains in yellow boxes. The arrows indicate that the process can be obtained as a limit.

We start with the case of spin $s = \frac{1}{2}$ for which we introduce an *open* integrable Hamiltonian (Section 2.1). The bulk part of this Hamiltonian was first studied by N. Beisert in [13]; the closed spin chain also appears in weakly-coupled $\mathcal{N} = 4$ super Yang-Mills theory in the so-called $\mathfrak{sl}(2)$ sector. The boundary interaction we consider is novel and, remarkably, preserves integrability. Seen as a stochastic operator, the open Hamiltonian is – in an appropriate base choice – the probability evolution operator for an interacting particle system. We show (Section 2.2) that this process happens to be the boundary-driven version of the Multi-particle Asymmetric Diffusion Model (MADM) of Sasamoto-Wadati [1].

Furthermore, again for spin $s = \frac{1}{2}$, we show that from the open Hamiltonian one can also get boundary driven processes taking values in the continuum, that can be used to model energy transport. This is achieved by taking a scaling limit that leads to integral operators. In this way one obtains a Markov process in the form of a Lévy process (Section 2.3). For this process we have not been able to find a reference in the literature. The bulk part is however similar, yet different, from the expression given in [20], which indeed is not stochastic.

We also discuss briefly the case of general spin s (Section 2.4), although we do not address the boundary processes in this paper. The integrable Hamiltonian of the higher spin models can be obtained from the expression in terms of integral operators [20]. Seen as a stochastic operator the bulk Hamiltonian density is related to the q-Hahn Asymmetric Zero Range Process (AZRP) of Barraquand and Corwin [2], which in turn generalizes the q-Hahn Totally Asymmetric Zero Range Process (TAZRP) introduced in [3] by allowing jumps in both directions. Algebraically, these models and their multi species generalizations can be described using the stochastic R-matrix [21]. In contrast to the stochastic R-matrix approach, where the particle process is described in terms of two commuting Hamiltonians which generate left and right moving particles separately, we find that the standard nearest-neighbor Hamiltonian of the non-compact spin chain yields immediately a process of particles hopping to the left and the right.

We summarize the connections between integrable interacting particle systems and integrable non-compact spin chains in Figure 1.

1.3 Informal description of the main results

For the benefit of the reader we summarize here the main results of this paper:

(i) We study continuous time stochastic processes that arise from rational non-compact spin chains. More precisely we introduce a class of boundary-driven processes that maps to the integrable non-compact *open* Heisenberg XXX spin chains. The Hamiltonian is thus made by a *bulk part*, which is the sum of nearest neighbors terms, and by a *boundary part*, which involves

only the first and last spin of the chain. For spin $s = \frac{1}{2}$ we identify boundary terms that preserve integrability.

(ii) It is well known that integrable non-compact spin chains can be described in the framework of the *quantum inverse scattering method*. This allows to determine the family of commuting operators in terms of transfer matrices and gives access to the spectrum, eigenvectors and observables via the algebraic Bethe ansatz, separation of variables or functional methods. Further, knowing the algebraic structure of the bulk model one can construct integrable boundary models following Sklyanin [22]. We exemplify this procedure in the case of the rational non-compact spin $\frac{1}{2}$ $\mathfrak{sl}(2)$ chain (Section 3). We deduce certain integrable stochastic boundary conditions from the most general K-matrix that we derive from the boundary Yang-Baxter equation. Furthermore we comment on the application of the algebraic Bethe ansatz to this model.

(iii) We show that our boundary-driven systems admits a *dual process* (Section 4). Again, for a proof we restrict to the particle process of spin $\frac{1}{2}$, however the result is more general and in particular it applies to general spin s, as well as to the Levy process obtained in the scaling limit. The dual process has two absorbing states, thus the non-equilibrium steady state of the original system is encoded in the *absorption probabilities of dual particles*. We illustrate this by analyzing the n-point correlation functions in the stationary state.

(iv) We investigate the connection of our models with $\mathcal{N} = 4$ super Yang-Mills theory in the setting of a closed chain. The limit of *fluctuating hydrodynamics* of the $\mathfrak{sl}(2)$ chain, describing coarse-grained properties of the stochastic system, turns out to be nothing but the semiclassical string equation, in Euclidean time. Indeed, both derivations of infrared properties [14, 15, 23] are virtually identical, and have been independently made with coherent states (in the stochastic case more rigorous constructions of the theory of large deviations around hydrodynamic limit exist [24]). One may ask why a stochastic model would appear in the SYM context: part of the answer is probably the underlying supersymmetry. We describe this in Section 5, where we shall also show how *supersymmetry* arises in this stochastic model.

2 Non-compact spin chains as integrable stochastic process

2.1 The case of spin $s = 1/2$

In the following we study aspects of the non-compact $\mathfrak{sl}(2)$ spin $\frac{1}{2}$ Heisenberg spin chain in the setting of open boundaries. The nearest neighbor Hamiltonian of the model is of the form

$$\mathcal{H} = \mathcal{H}_1 + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + \mathcal{H}_N \,. \tag{2.1}$$

We focus on spin chains with infinite-dimensional discrete series representation with lowest weight such that the representation at each site of the spin chain is given via

$$S_+|m\rangle = (m+1)|m+1\rangle \,, \quad S_-|m\rangle = m|m-1\rangle \,, \quad S_0|m\rangle = (m+\frac{1}{2})|m\rangle \,, \tag{2.2}$$

Here the generators of the $\mathfrak{sl}(2)$ algebra at each site of the spin chain satisfy the commutation relations (1.2) and $|0\rangle$ denotes the lowest weight state. Then the action of the Hamiltonian (2.1) on quantum space V of the spin chain

$$V = |m_1\rangle \otimes \ldots \otimes |m_N\rangle \,, \tag{2.3}$$

is defined as follows. The Hamiltonian density $\mathcal{H}_{i,i+1}$ for the spin $\frac{1}{2}$ chain acts on two sites and can be written as

$$\begin{aligned}\mathcal{H}_{i,i+1}|m_i\rangle\otimes|m_{i+1}\rangle = (h(m_i)+h(m_{i+1}))\,|m_i\rangle\otimes|m_{i+1}\rangle - \sum_{k=1}^{m_i}\frac{1}{k}|m_i-k\rangle\otimes|m_{i+1}+k\rangle \\ - \sum_{k=1}^{m_{i+1}}\frac{1}{k}|m_i+k\rangle\otimes|m_{i+1}-k\rangle,\end{aligned} \tag{2.4}$$

with the harmonic numbers

$$h(m) = \sum_{k=1}^{m}\frac{1}{k}\,. \tag{2.5}$$

This particular form of the Hamiltonian was studied in the context of $\mathcal{N}=4$ super Yang-Mills theory in [13]. The closed chain is known to be integrable and can be written in the standard form, cf. [25], in terms of the digamma function $\psi(x)$ defined as the logarithmic derivative of the Gamma function $\Gamma(x)$ as

$$\mathcal{H}_{i,i+1} = 2(\psi(\mathbb{S}_{i,i+1})-\psi(1))\,. \tag{2.6}$$

Here, the operator $\mathbb{S}_{i,i+1}$ is related to the two-site Casimir acting on two sites i and $i+1$ via $C_{i,i+1} = (S^{[i]}+S^{[i+1]})^2 = \mathbb{S}_{i,i+1}(\mathbb{S}_{i,i+1}-1)$. [1] The eigenvalues of $\mathbb{S}$ are determined from the irreducible tensor product decomposition

$$\left[\frac{1}{2}\right]\otimes\left[\frac{1}{2}\right] = \bigoplus_{j=0}^{\infty}\left[\Lambda_j\right], \tag{2.7}$$

where $\left[\frac{1}{2}\right]$ denotes the spin $\frac{1}{2}$ representation and $\Lambda_j = 1+j$. The action of the operator $\mathbb{S}$ is diagonal on each term on the right hand side of (2.25). Its eigenvalues are degenerate due to the $\mathfrak{sl}(2)$ invariance and simply given by Λ_j. Thus (2.6) immediately yields the eigenvalues of the harmonic action (2.4).

In addition to the bulk terms we introduce the integrable boundary terms

$$\mathcal{H}_i|m_i\rangle = \left(h(m_i)+\sum_{k=1}^{\infty}\frac{\beta_i^k}{k}\right)|m_i\rangle - \sum_{k=1}^{m_i}\frac{1}{k}|m_i-k\rangle - \sum_{k=1}^{\infty}\frac{\beta_i^k}{k}|m_i+k\rangle, \tag{2.8}$$

where $i=1,N$ and $0<\beta_1,\beta_N<1$. To our knowledge these types of boundary conditions have not been considered so far.

The Hamiltonian (2.1) defined above is stochastic because its matrix elements outside of the diagonal are non-positive and the sum over its columns vanishes. In other words, $-\mathcal{H}^t$ (i.e. the negative of the transposed of the Hamiltonian) is the generator of a continuous time Markov process $\{\mathbf{M}(t),\, t\geq 0\}$, representing a *system of interacting particles*. Here $\mathbf{M}(t) = (M_1(t), M_2(t),\ldots,M_N(t))$ and the ith component $M_i(t)$ denotes the number of particles at site $i\in\{1,\ldots,N\}$ at time $t\geq 0$. In the bulk particles jumps symmetrically to their nearest neighbour sites, so that $k\in\mathbb{N}$ particles (if available) moves to the left or to the right a rate $\varphi(k)=1/k$. At the boundary site "1", $k\in\mathbb{N}$ particles are created at rate $\varphi^+_{\beta_1}(k)=\beta_1^k/k$, where $\beta_1\in(0,1)$ is parameter, and $k\in\mathbb{N}$ particles (if available) are removed at a rate $\varphi^-(k)=1/k$. A similar process, now with a parameter $\beta_N\in(0,1)$, occurs at the boundary site N. See Figure 2 for a pictorial representation.

One can easily show that the particular case $\beta_1=\beta_N=\beta$ describes the *equilibrium* set-up with a Boltzmann-Gibbs invariant measure. More precisely the product measure with marginal the Geometric distribution of parameter $0<\beta<1$, i.e. the law

$$\mathbb{P}(\mathbf{M}(t)=\mathbf{m}) = \prod_{i=1}^{N}\beta^{m_i}(1-\beta), \tag{2.9}$$

[1] A formula for the Hamiltonian density in terms of the the two site Casimir $C_{i,i+1}$ has been discussed in [25, 26].

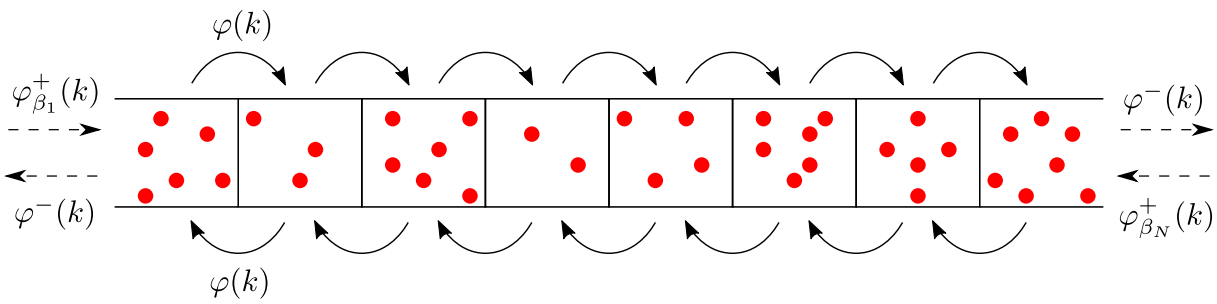

Figure 2: Stochastic hopping model emerging from the non-compact spin $\frac{1}{2}$ Heisenberg spin chain for the case of length $N = 8$. Here the ith site corresponds to the ith box. The rates $\varphi(k)$ with $i = 1, \ldots, 7$ are determined from the integrable nearest-neighbor bulk Hamiltonian while $\varphi^{\pm}(k)$ emerge from the boundary terms.

where $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{N}_0^N$ denotes a particle configuration, is reversible (and thus stationary) for the Markov process defined by the Hamiltonian (2.1). The generic case $\beta_1 \neq \beta_N$ yields instead a boundary driven *non-equilibrium* interacting particle system, whose invariant measure is unknown.

2.2 Rational limit of Sasamoto-Wadati model

The Multiparticle Asymmetric Diffusion Model (MADM) was introduced in [1] by Sasamoto and Wadati. In the following we show that the bulk part of the Hamiltonian of the non-compact spin $\frac{1}{2}$ in (2.4) can be obtained from the rational limit of the MADM.

First we note that the MADM can be written in terms of nearest neighbor Hamiltonian densities as

$$\mathcal{H}^{\mathrm{SW}} = \sum_{k \in \mathbb{Z}} \mathcal{H}^{\mathrm{SW}}_{k,k+1} \tag{2.10}$$

The components of the Hamiltonian density for a given number of excitations M then read

$$\left(\mathcal{H}^{\mathrm{SW}}_{k,k+1}\right)_{ij} = q \left(\sum_{\substack{l=1 \\ l \neq i}}^{M+1} \frac{q^{i-l}}{[|i-l|]_q} \delta_{i,j} - \frac{q^{j-i}}{[|i-j|]_q} \delta_{i \neq j} \right) \tag{2.11}$$

where we use the conventions $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and $i, j = 1, 2, \ldots, M+1$. Taking $q \to 1$ we immediately obtain

$$\lim_{q \to 1} \left(\mathcal{H}^{\mathrm{SW}}_{k,k+1}\right)_{ij} = \sum_{\substack{l=1 \\ l \neq i}}^{M} \frac{1}{|i-l|} \delta_{i,j} - \frac{1}{|i-j|} \delta_{i \neq j} \tag{2.12}$$

which can be identified with the Hamiltonian density in (2.4) after noting that

$$\sum_{\substack{k=1 \\ k \neq i}}^{M+1} \frac{1}{|i-k|} = h(M-i+1) + h(i-1)\,. \tag{2.13}$$

Interestingly, the Hamiltonian density in (2.11) can be identified with the trigonometric version of the non-compact spin $\frac{1}{2}$ Heisenberg chain [27].

2.3 Scaling limit leading to integral form

The Hamiltonian introduced in the Section 2.1 admits an integral representation that somehow naturally generalizes the stochastic evolution in a discrete space to a random dynamics in a continuum space. This is obtained by considering the non-negative real variable $x_i \in [0,\infty)$ defined by the scaling limit $x_i = \lim_{M\to\infty} x_i^{(M)}$ where $x_i^{(M)} = \frac{m_i}{M}$ is the fraction of particles at site i. Furthermore to have a meaningful limit of the Hamiltonian boundary terms one needs to scale the parameters as $\beta_i^{(M)} = 1 - \frac{\lambda_i}{M}$ with $\lambda_i \in (0,\infty)$, $i = 1, N$. With this procedure (details of computation are reported in Appendix A) one moves from the generator $-\mathcal{H}^t$, with $\mathcal{H}$ given in (2.1), to the generator $\mathcal{L}$ defined by

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N, \tag{2.14}$$

with bulk term

$$\begin{aligned} \mathcal{L}_{i,i+1} f(x_i, x_{i+1}) &= \int_0^{x_i} \frac{d\alpha}{\alpha} \Big\{ f(x_i - \alpha, x_{i+1} + \alpha) - f(x_i, x_{i+1}) \Big\} \\ &+ \int_0^{x_{i+1}} \frac{d\alpha}{\alpha} \Big\{ f(x_i + \alpha, x_{i+1} - \alpha) - f(x_i, x_{i+1}) \Big\}, \end{aligned} \tag{2.15}$$

and boundary terms

$$\begin{aligned} \mathcal{L}_i f(x_i) &= \int_0^{x_i} \frac{d\alpha}{\alpha} \Big\{ f(x_i - \alpha) - f(x_i) \Big\} \\ &+ \int_0^{\infty} d\alpha \frac{e^{-\lambda_i \alpha}}{\alpha} \Big\{ f(x_i + \alpha) - f(x_i) \Big\}, \end{aligned} \tag{2.16}$$

where $i = 1, N$ and $0 < \lambda_1, \lambda_N < \infty$. As the Markov generator $-\mathcal{H}^t$ is associated to an interacting particle systems, the operator in (2.14) is also the generator of a Markov process $\{\mathbf{X}(t),\, t \geq 0\}$ that turns out to be a *Lévy process*. Here $\mathbf{X}(t) = (X_1(t), \ldots, X_N(t))$ and the ith component $X_i(t)$ denotes the amount of a non-negative quantity (e.g. mass, energy, ...) at site $i \in \{1, \ldots, N\}$ at time $t \geq 0$. The process $\{\mathbf{X}(t),\, t \geq 0\}$ is a pure-jump process described as follows. In the bulk, jumps of size $[\alpha, \alpha + d\alpha)$, that move mass across the edge $(i, i+1)$ symmetrically, occur as Poisson process with intensity $\frac{d\alpha}{\alpha}$. At the left boundary (site 1), negative jumps decreasing the mass by an amount in the range $[\alpha, \alpha + d\alpha)$ occur with intensity $\frac{d\alpha}{\alpha}$, and positive jumps increasing the mass by the same amount occur with intensity $\frac{d\alpha}{\alpha} e^{-\lambda_1 \alpha}$ with $\lambda_1 \in (0,\infty)$. Similar jumps occur at the right boundary (site N), now with a parameter $\lambda_N \in (0,\infty)$.

One can show that in the particular case $\lambda_1 = \lambda_N = \lambda$ the product measure with marginal the Exponential distribution, i.e. the measure with density

$$p(x_1, \ldots, x_N) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i} 1_{\{x_i > 0\}}, \tag{2.17}$$

is reversible. This can be proved by showing that the generator $\mathcal{L}$ in (2.14) is self-adjoint in the Hilbert space $L_2(p(\cdot), dx)$. The generic case $\lambda_1 \neq \lambda_N$ yields a boundary driven *non-equilibrium* Lévy process.

The bulk generator in (2.15) reminds of the bulk Hamiltonian considered by Derkachov in [20] which reads

$$\begin{aligned} H_{i,i+1} f(x_i, x_{i+1}) &= \int_0^{\infty} \frac{d\alpha}{\alpha} \Big\{ f((1-\alpha)x_i + \alpha x_{i+1}, x_{i+1}) - f(x_i, x_{i+1}) \Big\} \\ &+ \int_0^{\infty} \frac{d\alpha}{\alpha} \Big\{ f(\alpha x_i + (1-\alpha) x_{i+1}, x_{i+1}) - f(x_i, x_{i+1}) \Big\}. \end{aligned} \tag{2.18}$$

This indeed corresponds to a non-stochastic representation of the Hamiltonian (2.6). The relation between (2.16) and (2.18) is that, defining

$$f_1(x_i, x_{i+1}) = x_i^{m_i} x_{i+1}^{m_{i+1}} \qquad \text{and} \qquad f_2(x_i, x_{i+1}) = \frac{x_i^{m_i}}{(m_i)!} \frac{x_{i+1}^{m_{i+1}}}{(m_{i+1})!}, \tag{2.19}$$

one has

$$H_{i,i+1} f_1(x_i, x_{i+1}) = \mathcal{L}_{i,i+1} f_2(x_i, x_{i+1}). \tag{2.20}$$

2.4 General spin and relation to q-Hahn zero range process

The Hamiltonian for the general spin s non-compact Heisenberg chain can be obtained from the integral formula given in [20] by acting on polynomials. The local action reads

$$\begin{aligned} \mathcal{H}^{(s)}_{i,i+1} |m_i\rangle \otimes |m_{i+1}\rangle = & \left(h^{(s)}(m_i) + h^{(s)}(m_{i+1}) \right) |m_i\rangle \otimes |m_{i+1}\rangle \\ & - \sum_{k=1}^{m_i} \frac{1}{k} \frac{\Gamma(m_i+1)\Gamma(m_i-k+2s)}{\Gamma(m_i-k+1)\Gamma(m_i+2s)} |m_i - k\rangle \otimes |m_{i+1}+k\rangle \\ & - \sum_{k=1}^{m_{i+1}} \frac{1}{k} \frac{\Gamma(m_{i+1}+1)\Gamma(m_{i+1}-k+2s)}{\Gamma(m_{i+1}-k+1)\Gamma(m_{i+1}+2s)} |m_i + k\rangle \otimes |m_{i+1}-k\rangle \end{aligned} \tag{2.21}$$

with

$$h^{(s)}(m) = \sum_{k=1}^{m} \frac{1}{k+2s-1} \,. \tag{2.22}$$

Here the action of the $\mathfrak{sl}(2)$ generators (1.2) on each site of the spin chain is given by

$$S_+|m\rangle = (m+2s)|m+1\rangle \,, \quad S_-|m\rangle = m|m-1\rangle \,, \quad S_0|m\rangle = (m+s)|m\rangle \tag{2.23}$$

The Hamiltonian density in (2.21) can be written in the standard form as

$$\mathcal{H}_{i,i+1} = 2(\psi(\mathbb{S}_{i,i+1}) - \psi(2s)) \,, \tag{2.24}$$

which is a generalization of (2.6). The action of $\mathbb{S}_{i,i+1}$ is determined via the tensor product decomposition for general spin s

$$[s] \otimes [s] = \bigoplus_{j=0}^{\infty} [\Lambda_j] \tag{2.25}$$

where $[s]$ denotes the representation with representation label s and $\Lambda_j = 2s + j$. If we set $s = \frac{1}{2}$ we recover (2.7).

The Hamiltonian density in (2.21) is stochastic. To our knowledge, the process with Hamiltonian density (2.21) has not been considered so far, not even on the closed chain. It is not difficult to check that for the closed chain the particle process $\{\mathbf{M}^{(s)}(t)\,,\, t \geq 0\}$ with Hamiltonian

$$\mathcal{H}^{(s)} = \sum_{i=1}^{N} \mathcal{H}^{(s)}_{i,i+1} \quad \text{with} \quad \mathcal{H}^{(s)}_{N,N+1} = \mathcal{H}^{(s)}_{N,1} \tag{2.26}$$

has a invariant measure given by a product of Negative Binomial distribution with parameters $0 < \beta < 1$ and $2s > 0$. Namely, the law

$$\mathbb{P}(\mathbf{M}^{(s)}(t) = \mathbf{m}) = \prod_{i=1}^{N} \frac{\beta^{m_i}}{m_i!} \frac{\Gamma(m_i+2s)}{\Gamma(2s)} (1-\beta)^{2s} \tag{2.27}$$

is reversible (and thus stationary). For $s = 1/2$ we recover (2.9).

Remarkably, the non-compact spin s Hamiltonian density in (2.21) is related to to the class of zero range models studied by Povolotsky in [3] in the totally asymmetric context, and further extended by Barraquand and Corwin in [2] to the partially asymmetric case. More precisely we find that the transition rates

$$\varphi_{\mu,\nu,\gamma}(m|n) = \mu^m \frac{(\nu/\mu;\gamma)_m(\mu;\gamma)_{n-m}}{(\nu;\gamma)_n} \frac{(\gamma;\gamma)_n}{(\gamma;\gamma)_{n-m}(\gamma;\gamma)_m}\,, \tag{2.28}$$

where $m = 0, 1, \ldots, n$ and $(a;\gamma)_m = \prod_{j=0}^{m-1}(1 - a\gamma^j)$, reduce to the ones in (2.21) in the limit

$$\lim_{s'\to s} \frac{1}{2(s-s')} \lim_{\gamma\to 1} \varphi_{\gamma^{2s},\gamma^{2s'},\gamma}(m|n) = -\frac{1}{m}\frac{\Gamma(n+1)\Gamma(n-m+2s)}{\Gamma(n-m+1)\Gamma(n+2s)}\,. \tag{2.29}$$

The rational case studied here and the results for the trigonometric case in [27] suggest that the transfer matrix constructed from the stochastic R-matrix for $U_q(A_1^{(1)})$ for the continuous time Markov chain in [21] is essentially Baxter's Q-operator of the non-compact spin-s XXZ chain in the closed setting. This would explain why there are two special points in [21] which yield a left and a right moving process respectively. Baxter's TQ equation then combines the two Hamiltonians into the one of the non-compact XXZ spin chain. In the rational limit with $s = \frac{1}{2}$ the aforementioned relation between the local charges of the Q-operator and the transfer matrix can be deduced from the studies in [28, Appendix C] based on the oscillator construction of Q-operators [29–31], see also [32, 33] for the Q-operator construction of supersymmetric spin chains including the one that underlies $\mathcal{N} = 4$ SYM at weak coupling.

3 Quantum inverse scattering method

In this section we describe the spin chain defined by the nearest neighbor Hamiltonian (2.1) within the quantum inverse scattering method for boundary integrable models [22]. We construct the fundamental transfer matrix with the infinite-dimensional spin $s = \frac{1}{2}$ representation in the auxiliary space and the transfer matrix with the two-dimensional fundamental representation in the auxiliary space for the most general, non-diagonal, boundary conditions. The corresponding K-matrices for the case of the fundamental transfer matrix appear to be new. They are expressed in terms of generators of $\mathfrak{sl}(2)$. While the fundamental transfer matrix yields the Hamiltonian, the transfer matrix with two-dimensional auxiliary space is commonly used in the framework of the algebraic Bethe ansatz. After specifying to appropriate boundary conditions we extract the Hamiltonian (2.1) from the fundamental transfer matrix and thus find that it is integrable. This was known for the closed chain, see in particular [13]. The open chain has been studied for the case of diagonal (identity) boundary conditions in [34, 35] and for a triangular case in [36]. Finally, we discuss how the Hamiltonian (2.1) can be diagonalised using the algebraic Bethe ansatz.

3.1 Construction of the transfer matrix with two-dimensional auxiliary space

In order to construct the fundamental transfer matrix which contains the information about the Hamiltonian we first define the transfer matrix with the two-dimensional representation in the auxiliary space. It can be defined as the trace

$$T_\square(x) = \mathrm{tr}_\square\, K(x)U(x)\,, \tag{3.1}$$

where $U(x)$ denotes the double-row monodromy

$$U(x) = L_1(x)L_2(x)\cdots L_N(x)\hat{K}(x)L_N(x)\cdots L_2(x)L_1(x)\,. \tag{3.2}$$

Here we introduced the spectral parameter $x \in \mathbb{C}$ and the Lax matrix

$$L_i(x) = \begin{pmatrix} x + \frac{1}{2} + S_0^{[i]} & -S_-^{[i]} \\ S_+^{[i]} & x + \frac{1}{2} - S_0^{[i]} \end{pmatrix} \tag{3.3}$$

with the generators $S_\pm^{[i]}$ and $S_0^{[i]}$ acting on spin chain site i and the most general 2×2 K-matrices

$$K(x) = \begin{pmatrix} p_1 + p_2(x+1) & p_3(x+1) \\ p_4(x+1) & p_1 - p_2(x+1) \end{pmatrix}, \qquad \hat{K}(x) = \begin{pmatrix} q_1 + xq_2 & xq_3 \\ xq_4 & q_1 - xq_2 \end{pmatrix}, \tag{3.4}$$

see [37]. Both of the K-matrices depend on four complex variables p_i and q_i respectively with $i = 1, \ldots, 4$. We note that one degree of freedom can be absorbed into an overall normalisation. It will however be convenient for us to keep it. The transfer matrix defined in this way commutes at different values of the spectral parameter

$$[T_\square(x), T_\square(y)] = 0\,. \tag{3.5}$$

This can be verified using the standard R-matrix $R(z) = z + P$ where $P = \sum_{a,b=1}^{2} e_{ab} \otimes e_{ba}$, with $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$, denotes the permutation matrix. The commutativity (3.5) then follows from the Yang-Baxter equation

$$R(x-y)\,(L(x) \otimes I)\,(I \otimes L(y)) = (I \otimes L(y))\,(L(x) \otimes I)\,R(x-y)\,, \tag{3.6}$$

the boundary Yang-Baxter equation

$$R(x-y)\left(\hat{K}(x) \otimes I\right) R(x+y)\left(I \otimes \hat{K}(y)\right) = \left(I \otimes \hat{K}(y)\right) R(x+y)\left(\hat{K}(x) \otimes I\right) R(x-y)\,, \tag{3.7}$$

and its corresponding version for K. Here I denotes the 2×2 identity matrix. See [22] for further details.

In the next subsection we construct the fundamental transfer matrix which contains the physical information about the spin chain and commutes with the transfer matrix $T_\square$.

3.2 Fundamental transfer matrix

For the fundamental transfer matrix the representation of the auxiliary space coincides with the one at a single site of the quantum space, cf. (2.2). It can be defined in analogy to $T_\square$ via

$$T(z) = \mathrm{tr}_a\, \mathcal{K}(z)\mathcal{U}(z)\,. \tag{3.8}$$

Here the trace is taken in the infinite-dimensional auxiliary space a and the double-row monodromy

$$\mathcal{U}(z) = \mathcal{R}_1(z)\mathcal{R}_2(z)\cdots\mathcal{R}_N(z)\hat{\mathcal{K}}(z)\mathcal{R}_N(z)\cdots\mathcal{R}_2(z)\mathcal{R}_1(z)\,. \tag{3.9}$$

Here $\mathcal{R}_i$ acts non-trivially in space i of the quantum space and the auxiliary space a which is suppressed in our notation. We stress again, that the difference between $T_\square$ and T is that $T_\square$ is constructed as the trace of a two-dimensional auxiliary space while one of T is infinite-dimensional.

The R-matrix $\mathcal{R}(z)$ in (3.9) is well known, see [38] as well as [25]. It can be written in terms of Γ-functions as

$$\mathcal{R}(x) = (-1)^{\mathbb{S}} \frac{\Gamma(1-x)}{\Gamma(1+x)} \frac{\Gamma(\mathbb{S}+x)}{\Gamma(\mathbb{S}-x)}\,. \tag{3.10}$$

where the operator $\mathbb{S}$ is the same that appeared in the Hamiltonian (2.6). Thus the eigenvalues of the R-matrix (3.10) can easily be obtained for any irreducible representation of the tensor product decomposition (2.25) setting $s = \frac{1}{2}$. Here the normalisation in (3.10) ensures that $\mathcal{R}(0) = (-1)^{\mathbb{S}}$ which can be interpreted as the permutation operator.

The R-matrix (3.10) satisfied the Yang-Baxter equation

$$\mathcal{R}(x-y)\,(L(x)\otimes I)\,(I\otimes L(y)) = (I\otimes L(y))\,(L(x)\otimes I)\,\mathcal{R}(x-y)\,. \tag{3.11}$$

Further, demanding that the fundamental transfer matrix (3.8) commutes with the transfer matrix (3.1), i.e.

$$[T(x), T_{\square}(y)] = 0 \tag{3.12}$$

we obtain the boundary Yang-Baxter equations for $\mathcal{K}$ and $\hat{\mathcal{K}}$ for the left and right boundary respectively. They involve the K-matrices (3.4) and read

$$L(x-y)\hat{\mathcal{K}}(x)L(x+y)\hat{K}(y) = \hat{K}(y)L(x+y)\hat{\mathcal{K}}(x)L(x-y) \tag{3.13}$$

and

$$L(y-x)\mathcal{K}(x)L(-x-y-2)K(y) = K(y)L(-x-y-2)\mathcal{K}(x)L(y-x) \tag{3.14}$$

Here we note that the Lax matrix (3.3) satisfies the unitarity relations

$$L(x)L(-x) = -x^2\,, \qquad L^t(x)L^t(-x-2) = -(x+1)^2\,, \tag{3.15}$$

where L^t denotes the Lax matrix transposed in the 2×2 auxiliary space.

An expression in terms of $\mathfrak{sl}(2)$ generators is unknown for the off-diagonal case, see [39] for the diagonal case which has been studied in the context of Q-operators [39, 40]. In the next subsection, we will derive the solutions $\mathcal{K}$ and $\hat{\mathcal{K}}$ of the boundary Yang-Baxter equations (3.13) and (3.14).

3.3 General solution to the boundary Yang-Baxter equation

In the following we first obtain the solution $\hat{\mathcal{K}}$ to the boundary Yang-Baxter equation (3.13). As we will see the solution $\mathcal{K}$ then follows from $\hat{\mathcal{K}}$.

First we note that the boundary Yang-Baxter equation (3.13) is equivalent to the conditions

$$\Big[\{L(x), Q\}\,, \hat{\mathcal{K}}(x)\Big] = 0\,, \tag{3.16}$$

and

$$2q_1[L(x), \hat{\mathcal{K}}(x)] = \Big[Q, L(x)\hat{\mathcal{K}}(x)L(x)\Big]\,. \tag{3.17}$$

Here we defined the matrix

$$Q = \begin{pmatrix} q_2 & q_3 \\ q_4 & -q_2 \end{pmatrix}. \tag{3.18}$$

One finds that the first equation (3.16) is equivalent to the condition

$$\Big[2q_2S_0 + q_3S_+ - q_4S_-, \hat{\mathcal{K}}(x)\Big] = 0 \tag{3.19}$$

and the second (3.17) can be written as

$$\hat{\mathcal{K}}(x)\,(2q_1L(-x) + L(x)QL(-x)) = (2q_1L(-x) + L(-x)QL(x))\,\hat{\mathcal{K}}(x) \tag{3.20}$$

using the unitarity relation (3.15). In order to find a solution to these equations we note that the operator in (3.19) can be diagonalised via

$$e^{\alpha S_-}e^{-\beta S_+}\,[2q_2S_0 + q_3S_+ - q_4S_-]\,e^{\beta S_+}e^{-\alpha S_-} = \gamma S_0 \tag{3.21}$$

after introducing the parametrisation

$$q_1 = \delta\,, \quad q_2 = \frac{1}{2}(1+2\alpha\beta)\gamma\,, \quad q_3 = -(1+\alpha\beta)\beta\gamma\,, \qquad q_4 = \alpha\gamma\,. \tag{3.22}$$

This motivates the ansatz for the K-matrix

$$\hat{\mathcal{K}}(x) = e^{\beta S_+}\, e^{-\alpha S_-}\, \hat{\mathcal{K}}_0(S_0;x)\, e^{\alpha S_-}\, e^{-\beta S_+}\,, \tag{3.23}$$

where the middle term $\hat{\mathcal{K}}_0(S_0;x)$ only depends on the generator S_0 and not on $S_\pm$. Substituting this ansatz into (3.20) we obtain the difference equation

$$\frac{\hat{\mathcal{K}}_0(S_0;x)}{\hat{\mathcal{K}}_0(S_0-1;x)} = \frac{(S_0-\frac{1}{2}+x)\gamma+2\delta}{(S_0-\frac{1}{2}-x)\gamma+2\delta}\,. \tag{3.24}$$

This equation is solved by the fraction of Γ-functions

$$\hat{\mathcal{K}}_0(S_0;x) = \frac{\Gamma(1+2\frac{\delta}{\gamma}-x)}{\Gamma(1+2\frac{\delta}{\gamma}+x)}\frac{\Gamma\left(\frac{1}{2}+S_0+2\frac{\delta}{\gamma}+x\right)}{\Gamma\left(\frac{1}{2}+S_0+2\frac{\delta}{\gamma}-x\right)} \tag{3.25}$$

up to a function periodic in S_0. The normalisation is chosen such that $\hat{\mathcal{K}}(0)=I$. To our knowledge the solution (3.23) with (3.25) is new.

The K-matrix for the other boundary can be obtained from the relation

$$\mathcal{K}(x) = \frac{1}{\hat{\mathcal{K}}(x+1)}\,, \tag{3.26}$$

and renaming the parameters appearing. We find that it can be written explicitly as

$$\mathcal{K}(x) = e^{\beta' S_+}\, e^{-\alpha' S_-}\, \mathcal{K}_0(S_0;x)\, e^{\alpha' S_-}\, e^{-\beta' S_+} \tag{3.27}$$

with

$$\mathcal{K}_0(S_0;x) = \frac{\Gamma(2+2\frac{\delta'}{\gamma'}+x)}{\Gamma(2\frac{\delta'}{\gamma'}-x)}\frac{\Gamma\left(-\frac{1}{2}+S_0+2\frac{\delta'}{\gamma'}-x\right)}{\Gamma\left(\frac{3}{2}+S_0+2\frac{\delta'}{\gamma'}+x\right)} \tag{3.28}$$

Here we used the same parametrisation as in (3.22) for the variables p_i, i.e.

$$p_1=\delta'\,,\quad p_2=\frac{1}{2}(1+2\alpha'\beta')\gamma'\,,\quad p_3=-(1+\alpha'\beta')\beta'\gamma'\,,\qquad p_4=\alpha'\gamma'\,. \tag{3.29}$$

Finally we note that the K-operators $\hat{\mathcal{K}}$ and $\mathcal{K}$ only depend on the ratio δ/γ and δ'/γ'. In the following we will be interested in the case where $\delta=\delta'=0$ and $\gamma,\gamma'\neq 0$. The latter will appear as an overall factor in the K-matrices (3.4), c.f. (3.48), and thus can be set to $\gamma=\gamma'=1$.

In the next subsection we derive the Hamiltonian (2.1) from the fundamental transfer matrix (3.8) for a special choice of boundary parameters (3.22) and (3.29).

3.4 Derivation of the stochastic boundary terms

In this section we compute the Hamiltonian by taking the logarithmic derivative of the fundamental transfer matrix at $x=0$, see [22]. The latter can be written in terms of the R-matrices and K-matrices in (3.8) as

$$\frac{\partial}{\partial x}\ln T(x)\big|_{x=0} = \frac{\mathrm{tr}_a\,\mathcal{K}_a'(0)}{\mathrm{tr}_a\,\mathcal{K}_a(0)} + 2\frac{\mathrm{tr}_a\,\mathcal{K}_a(0)\mathcal{H}_{a,1}}{\mathrm{tr}_a\,\mathcal{K}_a(0)} + \frac{\hat{\mathcal{K}}_N'(0)}{\hat{\mathcal{K}}_N(0)} + 2\sum_{k=1}^{N-1}\frac{\partial}{\partial x}\ln\mathcal{R}_{k,k+1}(x)\big|_{x=0}\,, \tag{3.30}$$

where the subscript a denotes an infinite-dimensional auxilliary space and $\hat{\mathcal{K}}_N$ the K-matrix at the N-th site of the spin chain while $\mathcal{K}_a$ the K-matrix acting non-trivially in the auxiliary space.

In the following we will show that the logarithmic derivative at $x=0$ in (3.30) coincides with the Hamiltonian (2.1) up to a constant, i.e.

$$\mathcal{H} = \frac{1}{2}\left(1+\frac{\partial}{\partial x}\ln T(x)\big|_{x=0}\right)\,, \tag{3.31}$$

when imposing the conditions

$$\alpha = \frac{1}{1-\beta} = \sum_{k=0}^{\infty} \beta^k \,, \qquad \alpha' = \frac{1}{1-\beta'} = \sum_{k=0}^{\infty} \beta'^k \,, \qquad \delta = \delta' = 0 \,, \tag{3.32}$$

where $0 < \beta, \beta' < 1$. Here, the constant term in (3.31) does not spoil the commutativity with the fundamental transfer matrix T and $T_\square$. The logarithmic derivative of the R-matrix at the permutation point is straightforward to compute and yields the Hamiltonian density in (2.6), i.e.

$$\mathcal{H}_{i,i+1} = \frac{\partial}{\partial x} \ln \mathcal{R}_{i,i+1}(x)\big|_{x=0} = 2(\psi(\mathbb{S}_{i,i+1}) - \psi(1)) \tag{3.33}$$

We stress again that it was argued in [13] that this expression is equivalent to the bulk action (2.4). The identification of the boundary terms $\mathcal{H}_1$ and $\mathcal{H}_N$ is given in the following two subsections.

3.4.1 The right boundary

First we note that that the logarithmic derivative of the K-matrix for $\delta = 0$ at $x = 0$ can be written as

$$\frac{\hat{\mathcal{K}}'(0)}{2\hat{\mathcal{K}}(0)} = e^{\beta S_+}\, e^{-\alpha S_-}\, \psi(S_0 + \frac{1}{2})\, e^{\alpha S_-}\, e^{-\beta S_+} - \psi(1) \,. \tag{3.34}$$

To derive the action of this operator on a state $|m_N\rangle$ we evaluate the matrix elements

$$\mathcal{O} = e^{\beta S_+}\, e^{-\alpha S_-}\, \psi(S_0 + \frac{1}{2})\, e^{\alpha S_-}\, e^{-\beta S_+} \,. \tag{3.35}$$

This can be done by inserting two identity operators into $\mathcal{O}$ as follows. We find

$$\begin{aligned} \langle k|\mathcal{O}|l\rangle &= \sum_{m_1,m_2=0}^{\infty} \langle k|e^{\beta S_+}\,|m_1\rangle\langle m_1|e^{-\alpha S_-}\psi\left(S_0+\frac{1}{2}\right)e^{\alpha S_-}\,|m_2\rangle\langle m_2|e^{-\beta S_+}|l\rangle \\ &= \sum_{m_1,m_2=0}^{\infty} \binom{k}{m_1}\binom{m_2}{l}\beta^{k-m_1}(-\beta)^{m_2-l}\langle m_1|e^{-\alpha S_-}\psi\left(S_0+\frac{1}{2}\right)e^{\alpha S_-}\,|m_2\rangle \,, \end{aligned} \tag{3.36}$$

where we have used the relation

$$\langle m|e^{\gamma S_+}|l\rangle = \gamma^{m-l}\binom{m}{l} \,. \tag{3.37}$$

Next we note that the last part in (3.36) yields

$$\langle m_1|e^{-\alpha S_-}\psi\left(S_0+\frac{1}{2}\right)e^{\alpha S_-}\,|m_2\rangle = \psi(m_1+1)\delta_{m_1,m_2} - \sum_{r=1}^{m_2}\frac{\alpha^r}{r}\delta_{m_1,m_2-r} \,. \tag{3.38}$$

Now we can evaluate the matrix elements of $\mathcal{O}$ in (3.36) by substituting (3.38). The diagonal term in (3.38) yields

$$\sum_{m=0}^{\infty}\binom{k}{m}\binom{m}{l}\beta^{k-m}(-\beta)^{m-l}\psi(m+1) = \begin{cases} 0, & \text{for} \quad k<l \\ \psi(k+1), & \text{for} \quad k=l \\ -\frac{1}{k-l}\beta^{k-l} & \text{for} \quad k>l \end{cases} \,. \tag{3.39}$$

To evaluate the part that emerges from the non diagonal term in (3.38) we identify

$$\alpha = \frac{1}{1-\beta} = \sum_{q=0}^{\infty}\beta^q \tag{3.40}$$

with $0 < \beta < 1$. Then we find

$$-\sum_{m_1=0}^{\infty}\sum_{m_2=m_1+1}^{\infty}\binom{k}{m_1}\binom{m_2}{l}\beta^{k-m_1}(-\beta)^{m_2-l}\frac{\alpha^{m_2-m_1}}{m_2-m_1} = \begin{cases} -\frac{1}{l-k}, & \text{for} \quad k<l \\ \sum_{m=1}^{\infty}\frac{\beta^m}{m}, & \text{for} \quad k=l \\ 0 \quad \text{for} \quad k>l \end{cases} . \tag{3.41}$$

Combining the results in (3.39) and (3.41) we obtain the matrix elements of $\mathcal{O}$ in (3.36). We finally find that for the identification in (3.32) the operator in (3.34) acting on a state $|m_N\rangle$ can be written as

$$\frac{\hat{\mathcal{K}}'_N(0)}{2\hat{\mathcal{K}}_N(0)}|m_N\rangle = \left(h(m_N)+\sum_{k=1}^{\infty}\frac{\beta^k}{k}\right)|m_N\rangle - \sum_{k=1}^{m_N}\frac{1}{k}|m_N-k\rangle - \sum_{k=1}^{\infty}\frac{\beta^k}{k}|m_N+k\rangle\,. \tag{3.42}$$

Further details can be found in Appendix B.1 and B.2.

3.4.2 The left boundary

The computation of the matrix elements for the left boundary turns out to be more simple than for the right boundary. First noting that

$$\mathrm{tr}_a\,\mathcal{K}_a(x) = \frac{1+x}{1+2x} \tag{3.43}$$

we find that

$$\mathrm{tr}_a\,\mathcal{K}_a(0) = 1\,, \qquad \mathrm{tr}_a\,\mathcal{K}'_a(0) = -1\,. \tag{3.44}$$

This yields in particular the first term of the logarithmic derivative in (3.30). The non-trivial part is to compute the matrix elements of $\mathrm{tr}_a\,\mathcal{K}_a(0)\mathcal{H}_{a,1}$. First we evaluate the matrix elements of $\mathcal{K}_a(0)$. The middle part becomes a projector on the Fock vacuum. We find that

$$\langle k|\mathcal{K}_a(0)|l\rangle = \beta'^k\sum_{m=l}^{\infty}\binom{m}{l}\alpha'^m(-\beta')^{m-l} = \beta'^k(1-\beta')\,. \tag{3.45}$$

using (B.6). Thus we have

$$\mathrm{tr}_a\,\mathcal{K}_a(0)\mathcal{H}_{a,1} = \sum_{m,n=0}^{\infty}\langle m|\mathcal{K}_a(0)|n\rangle\langle n|\mathcal{H}_{a,1}|m\rangle = (1-\beta')\sum_{m,n=0}^{\infty}\beta'^m\langle n|\mathcal{H}_{a,1}|m\rangle\,. \tag{3.46}$$

where all bra's and ket's are in the auxiliary space a. Then acting on a state $|m_1\rangle$ in the first space 1 we get

$$\mathrm{tr}_a\,\mathcal{K}_a(0)\mathcal{H}_{a,1}|m_1\rangle = \left(\sum_{k=1}^{\infty}\frac{\beta'^k}{k}+h(m_1)\right)|m_1\rangle - \sum_{k=1}^{m_1}\frac{1}{k}|m_1-k\rangle - \sum_{k=1}^{\infty}\frac{\beta'^k}{k}|m_1+k\rangle\,. \tag{3.47}$$

We have thus evaluated all terms in (3.30) that are needed to compute the Hamiltonian via (3.31), cf. (3.42), (3.44) and (3.47). Finally we obtain the Hamiltonian action proposed in (2.1) with the bulk part (2.4) and the boundary terms (2.8) setting $\beta = \beta_N$ and $\beta' = \beta_1$. We conclude that the Hamiltonian is integrable. The corresponding transfer matrices T and $T_\square$ are obtained from (3.8) and (3.1) after identifying the parameters via (3.32).

3.5 Symmetries and Bethe ansätze

In this section we discuss the symmetries of the spin chain transfer matrix (3.1) and how the algebraic Bethe ansatz can be applied. We find that for the equilibrium case the K-matrices can be diagonalised and that the general case it can be brought to a triangular form by a similarity transformation in the quantum space.

The transfer matrix $T_\square$ which is commonly used to for the algebraic Bethe ansatz. The K-matrices (3.4) are naively of the form

$$\hat{K}(x) = \frac{x\gamma}{1-\beta}\begin{pmatrix} \frac{(\beta+1)}{2} & -\beta \\ 1 & -\frac{(\beta+1)}{2} \end{pmatrix}, \qquad K(x) = \frac{(x+1)\gamma'}{1-\beta'}\begin{pmatrix} \frac{(\beta'+1)}{2} & -\beta' \\ 1 & -\frac{(\beta'+1)}{2} \end{pmatrix} \tag{3.48}$$

under the identifications (3.22), (3.29) and (3.32).

3.5.1 Equilibrium

In the case of equilibrium, i.e. $\beta = \beta'$, we find that the K-matrices in the corresponding transfer matrices can be diagonalised by a similarity transformation which can be absorbed in the quantum space.

We note that the K-matrices in (3.48) obey the relations

$$\hat{K}'(x) = S_\beta^{-1}\hat{K}(x)S_\beta = \frac{x\gamma}{2}\sigma_3\,, \qquad K'(x) = S_{\beta'}^{-1}K(x)S_{\beta'} = \frac{(z+1)\gamma}{2}\sigma_3 \tag{3.49}$$

with the Pauli matrix $\sigma_3 = \mathrm{diag}(+1,-1)$. The similarity transformation reads

$$S_\beta = \begin{pmatrix} 1 & -\beta \\ \frac{1}{\beta-1} & \frac{1}{1-\beta} \end{pmatrix}. \tag{3.50}$$

We further find that the Lax matrix enjoys the property

$$S_\beta L(x) S_\beta^{-1} = e^{\beta S_+} e^{\frac{1}{\beta-1}S_-} L(x) e^{-\frac{1}{\beta-1}S_-} e^{-\beta S_+} \tag{3.51}$$

where we used the relations

$$e^{\omega S_\pm} S_0 e^{-\omega S_\pm} = S_0 \mp \omega S_\pm\,, \qquad e^{\omega S_\pm} S_\mp e^{-\omega S_\pm} = S_\mp \mp 2\omega S_0 + \omega^2 S_\pm\,. \tag{3.52}$$

As a consequence we can rewrite the transfer matrix (3.8) for $\beta = \beta'$ as

$$T_\square(z) = \mathcal{S}_\beta T'_\square(z) \mathcal{S}_\beta^{-1}\,, \qquad \text{with} \qquad \mathcal{S}_\beta = e^{\beta S_+} e^{\frac{1}{\beta-1}S_-} \otimes \ldots \otimes e^{\beta S_+} e^{\frac{1}{\beta-1}S_-} \tag{3.53}$$

The transfer matrix T' has diagonal boundary conditions and we expect that the standard algebraic Bethe ansatz [22] can be applied taking $|0\rangle \otimes \ldots \otimes |0\rangle$ as a reference state.

3.5.2 Non-equilibrium

For general parameters β and β', the transfer matrix $T_\square$ in (3.1) the K-matrices (3.48) can be brought to a triangular form.

This can be seen as follows. First we note that the K-matrices (3.48) satisfy

$$\hat{K}''(x) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \hat{K}(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -x\gamma \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{\beta-1} & -\frac{1}{2} \end{pmatrix} \tag{3.54}$$

and

$$K''(x) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} K(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -(x+1)\gamma' \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{\beta'-1} & -\frac{1}{2} \end{pmatrix} \tag{3.55}$$

The change of the Lax matrices under this transformation can be absorbed into the quantum space. We have

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} L(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{S_+} L(x) e^{-S_+} \tag{3.56}$$

using (3.52). As a consequence we can write the transfer matrix in (3.1) as

$$T_\square(x) = S T''_\square(x) S^{-1}\,, \qquad \text{with} \qquad S = e^{S_+} \otimes \ldots \otimes e^{S_+} \tag{3.57}$$

The algebraic Bethe ansatz for these types of transfer matrices with triangular K-matrices has been studied in [41, 42]. Again $|0\rangle \otimes \ldots \otimes |0\rangle$ can be taken as a reference state to diagonalise T''.

Alternatively we can bring the K-matrices to an upper triangular form via

$$\hat{K}'''(x) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \hat{K}(x) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = x\gamma \begin{pmatrix} \frac{1}{2} & -\frac{\beta}{1-\beta} \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{3.58}$$

and

$$K'''(x) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} K(x) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (x+1)\gamma' \begin{pmatrix} \frac{1}{2} & -\frac{\beta'}{1-\beta'} \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{3.59}$$

The change of the Lax matrices under this transformation can be absorbed into the quantum space. We have

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} L(x) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = e^{-S_-} L(x) e^{S_-} \tag{3.60}$$

using (3.52). As a consequence we can write the transfer matrix in (3.1) as

$$T_\square(x) = S T'''_\square(x) S^{-1}\,, \qquad \text{with} \qquad S = e^{-S_-} \otimes \ldots \otimes e^{-S_-} \tag{3.61}$$

Finally we remark that as the K-matrices can be brought to a triangular form we expect the Baxter equation to be of the standard form, i.e. to coincide with the one of the spin chain with diagonal boundary conditions. We plan to elaborate on this elsewhere.

4 Duality

In this Section we show that the continuos-time Markov process defined by the Hamiltonian (2.1) has a *dual process* that allows to express the correlation functions in terms of finitely many particles. To introduce such a process we need to consider the enlarged quantum space $\widetilde{V}$ which is defined as the $N+2$ fold tensor product

$$\widetilde{V} = |\ell_0\rangle \otimes |\ell_1\rangle \otimes \ldots \otimes |\ell_N\rangle \otimes \ldots \otimes |\ell_{N+1}\rangle. \tag{4.1}$$

Then the dual Hamiltonian is defined by

$$\widetilde{H} = \widetilde{\mathcal{H}}_{0,1} + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + \widetilde{\mathcal{H}}_{N,N+1}, \tag{4.2}$$

where the bulk part is given by (2.1), while the boundary terms reads

$$\widetilde{\mathcal{H}}_{0,1}|\ell_0\rangle \otimes |\ell_1\rangle = h(\ell_1)|\ell_0\rangle \otimes |\ell_1\rangle - \sum_{k=1}^{\ell_1} \frac{1}{k} |\ell_0 + k\rangle \otimes |\ell_1 - k\rangle, \tag{4.3}$$

and

$$\widetilde{\mathcal{H}}_{N,N+1}|\ell_N\rangle \otimes |\ell_{N+1}\rangle = h(\ell_N)|\ell_N\rangle \otimes |\ell_{N+1}\rangle - \sum_{k=1}^{\ell_N} \frac{1}{k}|\ell_N - k\rangle \otimes |\ell_{N+1} + k\rangle. \tag{4.4}$$

Thus, in the dual process, particles moves as in the original system while they are in the bulk; when they reach the boundary sites they can be absorbed in *two extra sites*, called 0 and $N+1$, where they remain forever. As a consequence, in the long-time limit the dual process voids the chain: eventually all particles are absorbed either at the extra left site or at the extra right site. In Section 4.1 we give a precise formulation of duality, whose proof is then found in Section 4.2. Some consequences of duality are proved in Section 4.3.

4.1 Duality function

For a configuration $\boldsymbol{m} = (m_1 \dots, m_N) \in \mathbb{N}^N$ and a dual configuration $\boldsymbol{\ell} = (\ell_0, \ell_1 \dots, \ell_N, \ell_{N+1}) \in \mathbb{N}^{N+2}$, we define the duality function

$$D(\boldsymbol{m}, \boldsymbol{\ell}) = \rho_a^{\ell_0} \cdot \Big[\prod_{i=1}^{N} \binom{m_i}{\ell_i} \Big] \cdot \rho_b^{\ell_{N+1}} \tag{4.5}$$

where

$$\rho_a = \frac{\beta_1}{1-\beta_1}, \qquad \rho_b = \frac{\beta_N}{1-\beta_N}.$$

We denote by $\{\mathbf{M}(t),\, t \geq 0\}$ the original process defined by the Hamiltonian (2.1) and by $\{\mathbf{L}(t),\, t \geq 0\}$ the dual process with Hamiltonian (4.2). Here $\mathbf{M}(t) = (M_1(t), \dots, M_N(t))$ and $\mathbf{L}(t) = (L_0(t), L_1(t), \dots, L_N(t), L_{N+1}(t))$. Note that the dual process has two additional components. Duality is then expressed by the following equivalence of expectation values

$$\mathbb{E}_{\mathbf{m}}\Big[D(\mathbf{M}(t), \boldsymbol{\ell})\Big] = \widetilde{\mathbb{E}}_{\boldsymbol{\ell}}\Big[D(\mathbf{m}, \mathbf{L}(t))\Big], \tag{4.6}$$

where $\mathbb{E}_{\mathbf{m}}$ denotes expectation w.r.t. the original process started from the configuration $\mathbf{m}$, and $\widetilde{\mathbb{E}}_{\boldsymbol{\ell}}$ denotes expectation w.r.t. the dual process started from the configuration $\boldsymbol{\ell}$. More explicitly, duality amounts to

$$\sum_{\mathbf{m}' \in \mathbb{N}^N} D(\mathbf{m}', \boldsymbol{\ell})\, \mathbb{P}(\mathbf{M}(t) = \mathbf{m}' \mid \mathbf{M}(0) = \mathbf{m}) = \sum_{\boldsymbol{\ell}' \in \mathbb{N}^{N+2}} D(\mathbf{m}, \boldsymbol{\ell}')\, \mathbb{P}(\mathbf{L}(t) = \boldsymbol{\ell}' \mid \mathbf{L}(0) = \boldsymbol{\ell}). \tag{4.7}$$

4.2 Proof of duality

In bra-ket notation, the duality relation (4.6) reads

$$\sum_{\mathbf{m}' \in \mathbb{N}^N} \langle \mathbf{m}'|D|\boldsymbol{\ell}\rangle \langle \mathbf{m}'|e^{-t\mathcal{H}}|\mathbf{m}\rangle = \sum_{\boldsymbol{\ell}' \in \mathbb{N}^{N+2}} \langle \mathbf{m}|D|\boldsymbol{\ell}'\rangle \langle \boldsymbol{\ell}'|e^{-t\widetilde{\mathcal{H}}}|\boldsymbol{\ell}\rangle, \tag{4.8}$$

where D is the matrix whose elements are the duality function in (4.5)

$$\langle \mathbf{m}'|D|\boldsymbol{\ell}\rangle = D(\mathbf{m}', \boldsymbol{\ell}) \tag{4.9}$$

and $e^{-t\mathcal{H}}$ is the semigroup of the original process whose elements $\langle \mathbf{m}'|e^{-t\mathcal{H}}|\mathbf{m}\rangle$ give the probability that the original process $\{\mathbf{M}(t),\, t \geq 0\}$ is at configuration $\mathbf{m}'$ at time t, having started from the configuration $\mathbf{m}$ at time 0. Similarly $e^{-t\widetilde{\mathcal{H}}}$ is the semigroup of the dual process $\{\mathbf{L}(t),\, t \geq 0\}$.

Using in the left hand side of (4.8) the equality

$$\langle \mathbf{m}'|e^{-t\mathcal{H}}|\mathbf{m}\rangle = \langle \mathbf{m}|e^{-t\mathcal{H}^t}|\mathbf{m}'\rangle, \tag{4.10}$$

where $\mathcal{H}^t$ denotes the transpose of $\mathcal{H}$, and using the resolutions of the identity

$$1 = \sum_{\mathbf{m}' \in \mathbb{N}^N} |\mathbf{m}'\rangle\langle\mathbf{m}'| = \sum_{\boldsymbol{\ell}' \in \mathbb{N}^{N+2}} |\boldsymbol{\ell}'\rangle\langle\boldsymbol{\ell}'|, \tag{4.11}$$

the duality relation (4.6) is rewritten as

$$\langle\mathbf{m}|e^{-t\mathcal{H}^t} D|\boldsymbol{\ell}\rangle = \langle\mathbf{m}|De^{-t\widetilde{\mathcal{H}}}|\boldsymbol{\ell}\rangle. \tag{4.12}$$

To prove this it is clearly enough to show that

$$\langle\mathbf{m}|\mathcal{H}^t D|\boldsymbol{\ell}\rangle = \langle\mathbf{m}|D\widetilde{\mathcal{H}}|\boldsymbol{\ell}\rangle. \tag{4.13}$$

Furthermore, to establish (4.13), considering the additive form of the original Hamiltonian (2.1) and the dual Hamiltonian (4.2), it is enough to prove the single edge (self-)duality

$$\langle\mathbf{m}|\mathcal{H}^t_{i,i+1} D|\boldsymbol{\ell}\rangle = \langle\mathbf{m}|D\mathcal{H}_{i,i+1}|\boldsymbol{\ell}\rangle, \tag{4.14}$$

and the boundary dualities

$$\langle\mathbf{m}|\mathcal{H}^t_1 D|\boldsymbol{\ell}\rangle = \langle\mathbf{m}|D\widetilde{\mathcal{H}}_{0,1}|\boldsymbol{\ell}\rangle, \tag{4.15}$$

$$\langle\mathbf{m}|\mathcal{H}^t_N D|\boldsymbol{\ell}\rangle = \langle\mathbf{m}|D\widetilde{\mathcal{H}}_{N,N+1}|\boldsymbol{\ell}\rangle. \tag{4.16}$$

These will be shown in the next two sections.

4.2.1 Bulk duality

We start by proving (4.14). To this aim we express the duality function in terms of the generators (2.2) of the $\mathfrak{sl}(2)$ algebra. We recall (3.37) by which we may write

$$\binom{m_i}{\ell_i} = \langle m_i|e^{S_+^{[i]}}|\ell_i\rangle. \tag{4.17}$$

Thus, inserting the previous expressions into (4.5) we find

$$D(\boldsymbol{m},\boldsymbol{\ell}) = \rho_a^{\ell_0} \cdot \Big[\prod_{i=1}^N \langle m_i|e^{S_+^{[i]}}|\ell_i\rangle\Big] \cdot \rho_b^{\ell_{N+1}} \tag{4.18}$$

Notice that the duality function has a product structure. As a consequence, since $\mathcal{H}_{i,i+1}$ only acts on $|\ell_i\rangle \otimes |\ell_{i+1}\rangle$, we find that (4.14) is equivalent to

$$\mathcal{H}^t_{i,i+1} e^{S_+^{[i]}} e^{S_+^{[i+1]}} = e^{S_+^{[i]}} e^{S_+^{[i+1]}} \mathcal{H}_{i,i+1}. \tag{4.19}$$

We observe that the Hamiltonian density is symmetric, i.e. $\mathcal{H}^t_{i,i+1} = \mathcal{H}_{i,i+1}$. Moreover, due to the $\mathfrak{sl}(2)$ symmetry of the Hamiltonian density (cf. (2.6)), we have

$$[\mathcal{H}_{i,i+1}, S_+^{[i]} + S_+^{[i+1]}] = 0. \tag{4.20}$$

This can also be verified by a simple explicit computation that just uses the definition (2.4) of the Hamiltonian density and the definition (2.2) of the creation operator. As a result, (4.20) implies (4.19), which in turn proves (4.14).

We remark that the duality relation (4.14) explicitly reads:

$$\sum_{k=1}^{m_i}\frac{1}{k}\Big\{\binom{m_i-k}{\ell_i}\binom{m_{i+1}+k}{\ell_{i+1}}-\binom{m_i}{\ell_i}\binom{m_{i+1}}{\ell_{i+1}}\Big\}+$$
$$\sum_{k=1}^{m_{i+1}}\frac{1}{k}\Big\{\binom{m_i+k}{\ell_i}\binom{m_{i+1}-k}{\ell_{i+1}}-\binom{m_i}{\ell_i}\binom{m_{i+1}}{\ell_{i+1}}\Big\}=$$
$$\sum_{k=1}^{\ell_i}\frac{1}{k}\Big\{\binom{m_i}{\ell_i-k}\binom{m_{i+1}}{\ell_{i+1}+k}-\binom{m_i}{\ell_i}\binom{m_{i+1}}{\ell_{i+1}}\Big\}+$$
$$\sum_{k=1}^{\ell_{i+1}}\frac{1}{k}\Big\{\binom{m_i}{\ell_i+k}\binom{m_{i+1}}{\ell_{i+1}-k}-\binom{m_i}{\ell_i}\binom{m_{i+1}}{\ell_{i+1}}\Big\}. \tag{4.21}$$

A direct proof of this relation seems very difficult to obtain. Changing perspective, we see that the $\mathfrak{sl}(2)$ symmetry can be used to deduce the non trivial identity (4.21).

4.2.2 Boundary duality

We now prove the right boundary duality (4.16). Since the action of $\mathcal{H}_N$ and $\widetilde{\mathcal{H}}_{N,N+1}$ is local, (4.16) is equivalent to

$$\sum_{k=1}^{m_1}\frac{1}{k}\Big\{\binom{m_N-k}{\ell_N}-\binom{m_N}{\ell_1}\Big\}+\sum_{k=1}^{\infty}\frac{\beta_N^k}{k}\Big\{\binom{m_N+k}{\ell_N}-\binom{m_N}{\ell_1}\Big\}=$$
$$\sum_{k=1}^{\ell_1}\frac{1}{k}\Big\{\rho_b^k\binom{m_N}{\ell_N-k}-\binom{m_N}{\ell_N}\Big\}. \tag{4.22}$$

One way to show that the relation in (4.22) holds is using the boundary Yang-Baxter equation. We have discussed in Section 3.5.2 how the K-matrices (3.48) can be brought to an upper triangular form by a similarity transformation, cf. (3.58) and (3.59). Inserting the similarity transformation in (3.58) into the boundary Yang-Baxter equation (3.13) under the identification of the parameters (3.32) yields

$$L(x-y)e^{S_-}\hat{\mathcal{K}}(x)e^{-S_-}L(x+y)\hat{K}'''(y)=\hat{K}'''(y)L(x+y)e^{S_-}\hat{\mathcal{K}}(x)e^{-S_-}L(x-y)\,. \tag{4.23}$$

Alternatively we can solve again the boundary Yang-Baxter equation with K'' in the two-dimensional space and read off the solution from (3.23) by specifying the parameters appropriately. In this way we derive the relation

$$e^{S_-}\hat{\mathcal{K}}(x)e^{-S_-}=f(x)e^{\frac{\beta}{1-\beta}S_+}\frac{\Gamma(1-x)}{\Gamma(1+x)}\frac{\Gamma(\frac{1}{2}+S_0+x)}{\Gamma(\frac{1}{2}+S_0-x)}e^{-\frac{\beta}{1-\beta}S_+}\,. \tag{4.24}$$

Here $f(x)$ is a normalisation that cannot be determined by the boundary Yang-Baxter equation, cf. (3.25). We fix the normalisation in (4.24) by comparing both sides for $\beta=0$ assuming that $f(x)$ does not depend on β. We find that $f(x)=1$. The relation (4.24) for the original K-matrix $\hat{\mathcal{K}}(x)$ and the triangular K-matrix $e^{S_-}\hat{\mathcal{K}}(x)e^{-S_-}$ then immediately yields a relation for the boundary term of the Hamiltonian. By taking the logarithmic derivative at $x=0$, we find that the Hamiltonian $\mathcal{H}_N$ enjoys the property

$$e^{S_-^{[N]}}\mathcal{H}_Ne^{-S_-^{[N]}}=e^{\frac{\beta_N}{1-\beta_N}S_+^{[N]}}\psi(S_0^{[N]}+\frac{1}{2})e^{-\frac{\beta_N}{1-\beta_N}S_+^{[N]}}-\psi(1)\,. \tag{4.25}$$

Thus, introducing the boundary operator $\widetilde{\mathcal{H}}_N$

$$\widetilde{\mathcal{H}}_N=e^{-\frac{\beta_N}{1-\beta_N}S_-^{[N]}}\psi(S_0^{[N]}+\frac{1}{2})e^{\frac{\beta_N}{1-\beta_N}S_-^{[N]}}-\psi(1)\,, \tag{4.26}$$

we obtain the identity

$$e^{S_-^{[N]}}\mathcal{H}_N e^{-S_-^{[N]}} = \left(\widetilde{\mathcal{H}}_N\right)^t . \tag{4.27}$$

In other words, e^{-S_-} is the intertwiner between $\mathcal{H}_N$ and $\left(\widetilde{\mathcal{H}}_N\right)^t$. Now it is straightforward to show (4.22) using the relation (4.27). Indeed, from the computation done in Section 3.4.1, we find

$$\begin{aligned}\left(\langle \ell_N|e^{S_-^{[N]}}\mathcal{H}_N|m_N\rangle\right)^t &= \left(\langle \ell_N|e^{S_-^{[N]}}\left(\left(h(m_N)+\sum_{k=1}^{\infty}\frac{\beta_N^k}{k}\right)|m_N\rangle - \sum_{k=1}^{m_N}\frac{1}{k}|m_N-k\rangle - \sum_{k=1}^{\infty}\frac{\beta_N^k}{k}|m_N+k\rangle\right)\right)^t \\ &= \sum_{k=1}^{m_N}\frac{1}{k}\left\{\binom{m_N}{\ell_N} - \binom{m_N-k}{\ell_N}\right\} + \sum_{k=1}^{\infty}\frac{\beta_N^k}{k}\left\{\binom{m_N}{\ell_N} - \binom{m_N+k}{\ell_N}\right\},\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\left(\langle \ell_N|\left(\widetilde{\mathcal{H}}_N\right)^t e^{S_-^{[N]}}|m_N\rangle\right)^t &= \langle m_N|e^{S_+^{[N]}}\widetilde{\mathcal{H}}_N|\ell_N\rangle = \langle m_N|e^{S_+^{[N]}}\left(h(\ell_N)|\ell_N\rangle - \sum_{k=1}^{\ell_N}\frac{1}{k}|\ell_N-k\rangle\right) = \\ &= \sum_{k=1}^{\ell_N}\frac{1}{k}\left\{\binom{m_N}{\ell_N} - \rho_a^k\binom{m_N}{\ell_N-k}\right\},\end{aligned} \tag{4.29}$$

with $\rho_a = \beta_N/(1-\beta_N)$. The identity (4.27) shows the relation (4.22) and thus the right boundary duality (4.16). The proof of the left boundary duality (4.15) is similar and is omitted.

4.3 Correlation functions

In this section we show that duality allows to study the correlation functions of the boundary driven process in terms of the dynamics of n dual walkers.

For $1 \le i_1 \le i_2 \le \ldots \le i_n \le N$, the n-point correlation functions of $\{\mathbf{M}(t)\,,\, t \ge 0\}$ are defined by

$$\mathbb{E}[M_{i_1}(t)M_{i_2}(t)\ldots M_{i_n}(t)] = \sum_{\mathbf{m}\in\mathbb{N}^N} m_{i_1}m_{i_2}\ldots m_{i_n}\mu_t(\mathbf{m}) \tag{4.30}$$

where $\mu_t(\mathbf{m}) = \mathbb{P}(\mathbf{M}(t) = \mathbf{m})$ is the law of the process at time $t \ge 0$. In particular we shall be interested in the n-point correlation functions in the stationary invariant state of the dynamics that is reached in the limit of very large times, i.e.

$$\mathbb{E}[M_{i_1}M_{i_2}\ldots M_{i_n}] = \lim_{t\to\infty}\mathbb{E}[M_{i_1}(t)M_{i_2}(t)\ldots M_{i_n}(t)] \tag{4.31}$$

The product $M_{i_1}(t)M_{i_2}(t)\ldots M_{i_n}(t)$ can in general be obtained from the duality function in (4.5) evaluated in special dual configurations. For instance, if the indexes are all different, i.e. $1 \le i_1 < i_2 < \ldots < i_n \le N$, then, choosing the special dual configuration $\boldsymbol{\ell}^\star$ defined by

$$\boldsymbol{\ell}_i^\star = \begin{cases} 1 & \text{if } x \in \{i_1,\ldots,i_n\}, \\ 0 & \text{otherwise.}\end{cases} \tag{4.32}$$

it follows from (4.5) that

$$D(\mathbf{M}(\mathbf{t}), \boldsymbol{\ell}^\star) = M_{i_1}(t)M_{i_2}(t)\ldots M_{i_n}(t). \tag{4.33}$$

As a consequence of the duality relation (4.6), one can express the n-point correlation functions of the original process defined by the Hamiltonian (2.1) in terms of the dynamics of the dual

process (defined by the Hamiltonian (4.2)) initialized with $n \in \mathbb{N}$ dual particles. Thus, by duality, we can study the original systems by looking at a dual system with *finitely many particles*. In particular, since the sites 0 and $N+1$ are absorbing for the dual particles, we can express the expectation of the duality function in the stationary state of the original system in terms of the *absorption probabilities* of the dual walkers.

To show this we take the limit $t \to \infty$ in (4.6). We first consider the left hand side of (4.6), where the evolution concerns the process $\{\mathbf{M}(t)\,,\, t \geq 0\}$. In (4.6), the initial configuration of the process is chosen to be $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{N}^N$, thus the initial distribution is $|\mu_0\rangle = |\mathbf{m}\rangle$. The law of the process at time $t \geq 0$ is described by the ket $|\mu_t\rangle$, that encodes the time-dependent distribution $\mu_t(\mathbf{m})$ as

$$|\mu_t\rangle = \sum_{\mathbf{m}\in\mathbb{N}^N} \mu_t(\mathbf{m})|\mathbf{m}\rangle. \tag{4.34}$$

and solves the master equation, which in bra-ket notation is written as the Schrödinger equation with Euclidean time

$$\frac{d}{dt}|\mu_t\rangle = -\mathcal{H}|\mu_t\rangle. \tag{4.35}$$

In the limit of large times, regardless of the choice of the initial configuration $\mathbf{m}$, the process will approach its non-equilibrium steady state, i.e.

$$\lim_{t\to\infty} e^{-t\mathcal{H}}|\mathbf{m}\rangle = |\mu\rangle \tag{4.36}$$

where $|\mu\rangle$ is the stationary solution of (4.35), i.e. it solves $\mathcal{H}|\mu\rangle = 0$. As a consequence, for left hand side of (4.6) we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_{\mathbf{m}}\Big[D(\mathbf{M}(t), \boldsymbol{\ell})\Big] &= \lim_{t\to\infty} \sum_{\mathbf{m}'\in\mathbb{N}^N} \langle \mathbf{m}'|D|\boldsymbol{\ell}\rangle\langle \mathbf{m}'|e^{-t\mathcal{H}}|\mathbf{m}\rangle \\
&= \sum_{\mathbf{m}'\in\mathbb{N}^N} \langle \mathbf{m}'|D|\boldsymbol{\ell}\rangle\langle \mathbf{m}'|\mu\rangle \\
&= \sum_{\mathbf{m}'\in\mathbb{N}^N} D(\mathbf{m}', \boldsymbol{\ell})\mu(\mathbf{m}').
\end{aligned} \tag{4.37}$$

We now move to the right hand side of (4.6), where the evolution concerns the dual process $\{\mathbf{L}(t)\,, t \geq 0\}$. We recall that the dual process *conserves the total number of particles* and it has two absorbing sites. Thus, if initially the dual process is started from the measure $|\widetilde{\mu}_0\rangle = |\boldsymbol{\ell}\rangle$ concentrated on the initial configuration $\boldsymbol{\ell} = (\ell_0, \ell_1, \ldots, \ell_N, \ell_{N+1}) \in \mathbb{N}^{N+2}$, in the long time limit it will approach its stationary distribution

$$\lim_{t\to\infty} e^{-t\widetilde{\mathcal{H}}}|\boldsymbol{\ell}\rangle = |\widetilde{\mu}^{(\boldsymbol{\ell})}\rangle. \tag{4.38}$$

At variance with the original process, the stationary state of the dual process will depend on the initial configuration $\boldsymbol{\ell}$, more precisely it will depend on the total number of particles $|\boldsymbol{\ell}| = \sum_{i=0}^{N+1} \ell_i$. Furthermore, given the properties of the dual dynamics, the stationary state will concentrate on the subset of configurations

$$\Omega_{\boldsymbol{\ell},N} = \{\boldsymbol{\ell}' \in \mathbb{N}^{N+2} \;:\; \boldsymbol{\ell}' = (l'_0, 0, \ldots, 0, \ell'_{N+1}),\;\; \ell'_0 + \ell'_{N+1} = |\boldsymbol{\ell}|\}. \tag{4.39}$$

Thus

$$|\widetilde{\mu}^{(\boldsymbol{\ell})}\rangle = \sum_{\boldsymbol{\ell}'\in\Omega_{\boldsymbol{\ell},N}} \widetilde{\mu}^{(\boldsymbol{\ell})}(\boldsymbol{\ell}')|\boldsymbol{\ell}'\rangle. \tag{4.40}$$

Equivalently we can write

$$|\widetilde{\mu}^{(\boldsymbol{\ell})}\rangle = \sum_{\ell'_0,\ell'_{N+1}}^{*} q_{\boldsymbol{\ell}}(\ell'_0, \ell'_{N+1})\, |\ell'_0, 0, \ldots, 0, \ell'_{N+1}\rangle. \tag{4.41}$$

where $\sum^*$ denotes the sum restricted to $(\ell'_0, \ell'_{N+1}) \in \mathbb{N}^2$ such that $\ell'_0 + \ell'_{N+1} = |\boldsymbol{\ell}|$ and $q_{\boldsymbol{\ell}}(\ell'_0, \ell'_{N+1})$ denotes the probability that, being the dual particles initially placed as prescribed by the configuration $\boldsymbol{\ell}$, eventually ℓ'_0 of them are absorbed in 0 and the remaining $\ell'_{N+1} = |\boldsymbol{\ell}| - \ell'_0$ are absorbed in $N+1$.

If we use all this in the right hand side of (4.6) we find

$$\begin{aligned}
\lim_{t\to\infty} \widetilde{\mathbb{E}}_{\boldsymbol{\ell}}\Big[D(\mathbf{m}, \mathbf{L}(t))\Big] &= \lim_{t\to\infty} \sum_{\boldsymbol{\ell}' \in \mathbb{N}^{N+2}} \langle \mathbf{m}|D|\boldsymbol{\ell}'\rangle\langle \boldsymbol{\ell}'|e^{-t\widetilde{\mathcal{H}}}|\boldsymbol{\ell}\rangle \\
&= \sum_{\boldsymbol{\ell}' \in \mathbb{N}^{N+2}} \langle \mathbf{m}|D|\boldsymbol{\ell}'\rangle\langle \boldsymbol{\ell}'|\widetilde{\mu}^{(\boldsymbol{\ell})}\rangle \\
&= \sum_{(\ell'_0, \ell'_{N+1}) \in \mathbb{N}^2}^{*} \rho_a^{\ell'_0}\, \rho_b^{\ell'_{N+1}}\, q_{\boldsymbol{\ell}}(\ell'_0, \ell'_{N+1}),
\end{aligned} \tag{4.42}$$

where in the last equality it has been used that

$$\langle m_1, \ldots, m_N|D|\ell'_0, 0, \ldots, 0, \ell'_{N+1}\rangle = \rho_a^{\ell'_0}\, \rho_b^{\ell'_{N+1}}, \tag{4.43}$$

which follows from the definition of the duality function (4.5).

Taking the limit $t \to \infty$ in (4.6) and using (4.37) and (4.42) we arrive to the main result of this section:

$$\lim_{t\to\infty} \mathbb{E}_{\mathbf{m}}[D(\mathbf{M}(t), \boldsymbol{\ell})] = \sum_{(\ell'_0, \ell'_{N+1}) \in \mathbb{N}^2}^{*} \rho_a^{\ell'_0}\, \rho_b^{\ell'_{N+1}}\, q_{\boldsymbol{\ell}}(\ell'_0, \ell'_{N+1}). \tag{4.44}$$

We close by showing the applications of this formula to the one and two points correlation function.

One point correlation function. We take $n = 1$ in (4.30) and put $i_1 = i$. Considering the dual configurations $\boldsymbol{\delta}_i$ that have one particle at site i and zero elsewhere, i.e.

$$(\boldsymbol{\delta}_i)_k = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \tag{4.45}$$

from (4.5) we have that

$$\begin{aligned}
&D(\mathbf{M}(t), \boldsymbol{\delta}_0) = \rho_a \\
&D(\mathbf{M}(t), \boldsymbol{\delta}_i) = M_i(t) \qquad i = 1, \ldots, N \\
&D(\mathbf{M}(t), \boldsymbol{\delta}_{N+1}) = \rho_b
\end{aligned} \tag{4.46}$$

For the average number of particles at site i, being the process started from the measure μ, we have

$$\begin{aligned}
\mathbb{E}(M_i(t)) &= \int d\mu(\mathbf{m}) \mathbb{E}_{\mathbf{m}}[D(\mathbf{M}(t), \boldsymbol{\delta}_i)] \\
&= \int d\mu(\mathbf{m}) \widetilde{\mathbb{E}}_{\boldsymbol{\delta}_\mathbf{i}}[D(\mathbf{m}, \boldsymbol{\delta}_i(t))]
\end{aligned} \tag{4.47}$$

where in the last equality we used duality (4.6). Since the dual process has one particle only, it follows from the dual Hamiltonian (4.2) that this particle moves as a continuous time symmetric random walk $\{I(t)\,,\, t \geq 0\}$ started at $i \in \{1, \ldots, N\}$, jumping at rate 1 on $\{0, 1, \ldots, N, N+1\}$ and being absorbed at $\{0\}$ and at $\{N+1\}$. Thus

$$\mathbb{E}(M_i(t)) = \int d\mu(\mathbf{m}) \mathbb{E}_{\boldsymbol{\delta}_\mathbf{i}}[D(\mathbf{m}, \boldsymbol{\delta}_{I(t)})]. \tag{4.48}$$

By taking the limit $t \to \infty$ of the above expression, formula (4.44) allows to compute the profile in the non-equilibrium stationary state:

$$\mathbb{E}(M_i) \;=\; \lim_{t\to\infty} \mathbb{E}(M_i(t)) = \rho_a\, q_{\boldsymbol{\delta}_i}(1,0) + \rho_b\, q_{\boldsymbol{\delta}_i}(0,1). \tag{4.49}$$

The absorption probabilities of the simple symmetric random walk $\{I(t)\,,\, t \geq 0\}$ are easily computed thus implying

$$\begin{aligned} \mathbb{E}(M_i) &= \rho_a\Big(1 - \frac{i}{N+1}\Big) + \rho_b\Big(\frac{i}{N+1}\Big) \\ &= \rho_a + \frac{\rho_b - \rho_a}{N+1} i. \end{aligned} \tag{4.50}$$

Two point correlation function. We take $n = 2$ in (4.30), put $i_1 = i$ and $i_2 = j$. To analyze the two point function $\mathbb{E}[M_i(t)M_j(t)]$ with $i \neq j$ we need to run the dual process with two particles initially at sites i and j. We can read from the dual Hamiltonian (4.2) that they move as two random walks $\{(I(t), J(t))\,,\, t \geq 0\}$ on $\{0, \ldots, N+1\}^2$ that are absorbed at 0 and at $N+1$ and evolve via the generator

$$\begin{aligned} L^{dual,2} f(i,j) &= \Big[f(i-1,j) + f(i+1,j) + f(i,j-1) + f(i,j+1) - 4f(i,j)\Big]\mathbf{1}_{\{i\neq j\}} \\ &+ \Big[(f(i-1,i) - f(i,i)) + \frac{1}{2}(f(i-1,i-1) - f(i,i)) \\ &\quad +(f(i,i+1) - f(i,i)) + \frac{1}{2}(f(i+1,i+1) - f(i,i))\Big]\mathbf{1}_{\{i=j\}} \end{aligned}$$

with

$$L^{dual,2} f(0,0) = L^{dual,2} f(0,N+1) = L^{dual,2} f(N+1,0) = L^{dual,2} f(N+1,N+1) = 0.$$

Duality (4.6) yields

$$\mathbb{E}(M_i(t)M_j(t)) \;=\; \int d\mu(\mathbf{m}) \mathbb{E}_{\boldsymbol{\delta}_\mathbf{i}+\boldsymbol{\delta}_\mathbf{j}}[D(\mathbf{m}, \boldsymbol{\delta}_{I(t)} + \boldsymbol{\delta}_{J(t)})]. \tag{4.51}$$

Formula (4.44) tell that the two point functions in the stationary state can be written in terms of the absorption probabilities of the dual walkers as

$$\begin{aligned} \mathbb{E}(M_i M_j) &= \lim_{t\to\infty} \mathbb{E}[M_i(t)M_j(t)] \\ &= \rho_a^2\, q_{\boldsymbol{\delta}_i+\boldsymbol{\delta}_j}(2,0) + \rho_a\rho_b\, q_{\boldsymbol{\delta}_i+\boldsymbol{\delta}_j}(1,1) + \rho_b^2\, q_{\boldsymbol{\delta}_i+\boldsymbol{\delta}_j}(0,2). \end{aligned} \tag{4.52}$$

It would be interesting to investigate the application of Bethe ansatz to compute the absorption probabilities.

4.4 Other dualities

The duality results discussed so far can be generalized into several directions. Firstly, the process $\{\mathbf{L}(t)\,,\, t \geq 0\}$ with Hamiltonian (4.2), besides being the dual process of $\{\mathbf{M}(t)\,,\, t \geq 0\}$ with Hamiltonian (2.1), is also the dual process of $\{\mathbf{X}(t)\,,\, t \geq 0\}$ with generator (2.14). It can be verified with an explicit computations that the duality functions now read

$$D(\mathbf{x}, \boldsymbol{\ell}) = \lambda_1^{\ell_0} \cdot \Big[\prod_{i=1}^{N} \frac{{x_i}^{\ell_i}}{\ell_i!}\Big] \cdot \lambda_N^{\ell_{N+1}} \tag{4.53}$$

This is ax example of the situation – that was alluded to in the introduction – where a transport model of a continuous quantity, interpreted as energy, has a dual that instead transport

discrete particles. Algebraically, this is a consequence of a change of representation for the same abstract Hamiltonian, that we identified in (2.6). The results on the correlations described in Section 4.3 hold true mutatis mutandis, now for the n-point correlation function of the energies $\mathbb{E}[X_{i_1}(t)X_{i_2}(t)\dots X_{i_n}(t)]$.

A second generalization is the one to higher spin values. We discuss here only the closed periodic chain. Similarly to what has been done in Section 4.2, one can show that the process $\{\mathbf{M}^{(s)}(t)\,,\, t\geq 0\}$ with Hamiltonian (2.26) is self-dual with self-duality function

$$D(\mathbf{m},\boldsymbol{\ell}) = \prod_{i=1}^{N} \frac{m_i!}{(m_i-\ell_i)!}\frac{\Gamma(2s)}{\Gamma(\ell_i+2s)}. \tag{4.54}$$

Finally, we may also expect that orthogonal duality functions arise when considering unitary equivalent representations of the $\mathfrak{sl}(2)$ algebra, as it happens in the case of the spin chain related to the KMP model [43–45].

5 Connection with $\mathcal{N}=4$ SYM

5.1 Semiclassical string and fluctuating hydrodynamics

As mentioned in the introduction, the imaginary and real time versions of the same $\mathfrak{sl}(2)$ chain are the stochastic and the quantum models, respectively. The latter is derived from expectation values of local operators in $\mathcal{N}=4$ SYM theory [11–13], and its long wavelength limit is described by a string spinning on $AdS_5\times S^5$ [15, 16], see also [46–48] for an overview. One may obtain in a similar manner the coarse-grained dynamics of the stochastic model – its Fluctuating Hydrodynamics – and it turns out to be the Euclidean counterpart of the string equation.

Fluctuating Hydrodynamics is a stochastic process, and as such has an associated supersymmetry which encodes, through its Ward-Takahashi identities, probability conservation and the equilibrium theorems: time-translational invariance and fluctuation-dissipation relations [49, 50]. The supersymmetric extension of the stochastic chain is easy to construct directly by exponentiating a Jacobian with fermions. We show here that, interestingly, this supersymmetric chain is equivalent to the $\mathfrak{sl}(2|1)$ superstring obtainable directly from $\mathcal{N}=4$ SYM, which has been discussed in the literature [51]. In other words, we follow the 'stochastic quantization' procedure (see Ref. [52] for an overview of this) to obtain the chain and its susy completion.

Here we follow [14, 15] and [23]. The derivation of the string limit of the quantum system and of the hydrodynamic limit of the stochastic system are the same, for real and Euclidean times, respectively. We choose the following definition for the coherent state:

$$|z\rangle \propto e^{zS_+}\,|0\rangle \tag{5.1}$$

where $|0\rangle$ is in the spin s representation and is such that $S_0|0\rangle = s|0\rangle$. We have the expectation values (the so-called Q-symbols of the operators)

$$\begin{aligned}
\langle z|S_+|z\rangle &= 2s\frac{\bar z}{1-z\bar z} &\equiv\quad & s\,n_+\\
\langle z|S_-|z\rangle &= 2s\frac{z}{1-z\bar z} &\equiv\quad & s\,n_-\\
\langle z|S_0|z\rangle &= s\frac{1+z\bar z}{1-z\bar z} &\equiv\quad & s\,n_0
\end{aligned} \tag{5.2}$$

where $\vec n$ spans the hyperboloid:

$$\vec n^2 = n_0^2-n_1^2-n_2^2 = 1, \qquad n_0>0\ .$$

The semiclassical limit is achieved, even for a single site, in the large 'spin' (s) limit. In this limit, operators may be replaced by their expectations (the Q symbols), and commutators by

Poisson brackets. Here (and in references [14, 15, 23]) we are not quite interested in the large spin s limit, but in a coarse-grained (long wavelength) description. Making the implicit assumption that the hydrodynamic limit does not depend on the spin, one first does a semiclassical large-s approximation and then a long-wavelength approximation. In the probabilistic literature there are, however, direct derivations of fluctuating hydrodynamics without assuming large spins [24]. In the semiclassical limit, commutators become Poisson brackets

$$\{n_0, n_\pm\} = \pm n_\pm \quad ; \quad \{n_+, n_-\} = -2n_0 \tag{5.3}$$

Let us also recall the action of the spin operators in the coherent state representation:

$$\begin{aligned}
\langle z|S_+|\psi\rangle &= \left[2s\bar{z} + \bar{z}^2\frac{\partial}{\partial\bar{z}}\right]\langle z|\psi\rangle = s\left[2\hat{\rho} + \hat{\rho}^2\rho\right]\langle z|\psi\rangle \\
\langle z|S_-|\psi\rangle &= \frac{\partial}{\partial\bar{z}}\langle z|\psi\rangle = s\,\rho\langle z|\psi\rangle \\
\langle z|S_0|\psi\rangle &= \left[\bar{z}\frac{\partial}{\partial\bar{z}} + s\right]\langle z|\psi\rangle = s\left[\hat{\rho}\rho + 1\right]\langle z|\psi\rangle
\end{aligned} \tag{5.4}$$

yielding the Bargmann-Fock representation. Here we have defined

$$\rho = \frac{1}{s}\frac{\partial}{\partial\bar{z}} \quad ; \quad \bar{z} = \hat{\rho} \;\Rightarrow\; [\rho, \hat{\rho}] = \frac{1}{s} \tag{5.5}$$

Again, in the semiclassical limit we replace commutator by Poisson brackets $s\,[\ ,\ ] \to \{\ ,\ \}$

$$\begin{aligned}
n^+ &= 2\hat{\rho} + \hat{\rho}^2\rho \\
n^- &= \rho \\
n^0 &= \hat{\rho}\rho + 1
\end{aligned} \tag{5.6}$$

where here and in what follows $(\rho, \hat{\rho})$ are c-numbers, and (5.3) becomes $\{\rho, \hat{\rho}\} = 1$.

We have two alternatives for describing the model in the semiclassical limit: through the $(n_\pm, n_0)$ or through $(\rho, \hat{\rho})$ on each site, for the full chain through the variables:

$$\begin{aligned}
\{n_0(x), n_\pm(y)\} &= \pm n_\pm(x)\,\delta(x-y) \quad ; \quad \{n_+(x), n_-(y)\} = -2n_0\,\delta(x-y) \\
\{\rho(x), \hat{\rho}(y)\} &= \delta(x-y)
\end{aligned} \tag{5.7}$$

The Hamiltonian of the chain has a simple (Q-symbol) representation in the coherent state basis $|\vec{n}\rangle$ in [15, 16]:

$$\langle\vec{n}^k\vec{n}^{k+1}|\,H^{k\,k+1}\,|\vec{n}^k\vec{n}^{k+1}\rangle = \log\left(1 - \frac{\left(\vec{n}^k - \vec{n}^{k+1}\right)^2}{4}\right) . \tag{5.8}$$

Here

$$\vec{n}^k \cdot \vec{n}^{k+1} = n_0^k n_0^{k+1} - n_+^k n_-^{k+1} - n_-^k n_+^{k+1} .$$

In the hydrodynamic limit of large k and tending to the continuum the Hamiltonian reads

$$\mathcal{H} = -\frac{1}{2}\left[\partial_x n_+ \cdot \partial_x n_- - (\partial_x n_0)^2\right] \tag{5.9}$$

The same result in real time is found by considering semiclassical strings spinning on $AdS_5 \times S^5$ [15].

In Euclidean time we obtain the hydrodynamic limit of the stochastic chains (see [23], where this has been done for the hydrodynamic limit of the Kipnis-Marchioro-Presutti chain, but the result is the same). To see this in detail, we go to the $(\rho, \hat{\rho})$ variables (5.6):

$$\begin{aligned}
\mathcal{H}(\vec{n}) &= -\frac{1}{2}\int dx\,\{\partial_x n_+ \partial_x n_- - (\partial_x n_0)^2\} = -\frac{1}{2}\int dx\,\{\partial_x(\hat{\rho}\rho + 2\hat{\rho})\,\partial_x\rho - [\partial_x(\hat{\rho}\rho)]^2\} \\
&= \int dx\;\{\partial_x\hat{\rho}\partial_x\rho - \frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\}
\end{aligned} \tag{5.10}$$

5.2 Stochastic dequantization

The path integral may then be written as:

$$\int D\rho D\hat{\rho}\; e^{N\int dxdt\,\{-\hat{\rho}\dot{\rho}-\partial_x\hat{\rho}\partial_x\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\}} = \int D\rho D\xi D\hat{\rho}\; e^{N\int dxdt\,\{-\hat{\rho}\dot{\rho}-\partial_x\hat{\rho}\partial_x\rho+\rho(\partial_x\hat{\rho})\xi(x,t)-\frac{1}{2}\xi(x,t)^2\}}$$
$$= \int D\rho D\xi\;\delta\left\{\dot{\rho}(x,t)-\partial_x[\partial_x\rho(x,t)-\rho\xi(x,t)]\right\}\; e^{-\frac{1}{2}\int dtdx\;\xi^2(x,t)} \tag{5.11}$$

where we have used Gaussian (Hubbard-Stratonovich) decoupling with the field $\xi(x,t)$, integration by parts, and finally integrated over $\hat{\rho}$. The meaning of (5.11) is clear, we have a stochastic system satisfying

$$\mathrm{Eq}\left[\rho(x,t)\right] = \dot{\rho}(x,t)-\partial_x[\partial_x\rho(x,t)-\rho(x,t)\xi(x,t)] = 0 \tag{5.12}$$

with Gaussian noise $\xi(x,t)$, white in space and in time. This is a hydrodynamics with fluctuating current $J=-\partial_x\rho(x,t)+\rho(x,t)\xi(x,t)$. We have been careless about time-discretization of $\xi(x,t)$: this is abundantly discussed in the literature (see [50]) of multiplicative noise.

The equations of motion for Euclidean (imaginary) time given by (5.10) are the Freidlin-Wenzel / WKB [24] description of the coarse-grained dynamics of the stochastic model (5.12), with $\rho(x,t)$ the density field and $\hat{\rho}(x,t)$ the 'response fields' familiar [53] in stochastic systems:

$$\dot{n}_a(x,t) = \{n_a, \mathcal{H}(\vec{n})\} \tag{5.13}$$

and

$$\dot{\rho}(x,t) = \{\rho, \mathcal{H}(\rho,\hat{\rho})\} \quad ; \quad \dot{\hat{\rho}}(x,t) = \{\hat{\rho}, \mathcal{H}(\rho,\hat{\rho})\} \tag{5.14}$$

Note that if ρ and $\hat{\rho}$ are real, the equation (5.13) implies complex components for the n_0, n_1, n_2, and vice-versa (in this sense, the equation for ρ, $\hat{\rho}$ is an instanton equation for the quantum model). The semiclassical real-time equations are the same, with $t\to i\tau$.

5.3 Stochastic requantization and superstring

As is well known [49], a stochastic system can be promoted into a system having a BRST symmetry (that guarantees probability conservation), plus, *only* when the system satisfies detailed balance, an extra supersymmetry with the interpretation of thermodynamic equilibrium – so that for example, Fluctuation Dissipation relations appear as Ward identities. For multiplicative noise this has been less discussed, see however Refs [50]. The construction is as follows: we retrace our steps starting from (5.12), but this time exponentiating the Jacobian term with Grassmann variables [52]:

$$\begin{aligned}
1 &= \int D\rho D\xi\;\delta\left\{\dot{\rho}(x,t)-\partial_x[\partial_x\rho(x,t)-\rho\xi(x,t)]\right\}\det\left|\frac{\delta\mathrm{Eq}\left[\rho(x,t)\right]}{\delta\rho(x',t')}\right|\; e^{-\frac{1}{2}\int dtdx\;\xi^2(x,t)} \\
&= \int D\rho D\hat{\rho} D\xi\; D\eta\; D\bar{\eta}\; e^{N\int dtdx\,\left\{-\hat{\rho}(x,t)\dot{\rho}(x,t)+\hat{\rho}(x,t)\partial_x[\partial_x\rho(x,t)-\rho\xi(x,t)]-\frac{1}{2}\xi^2(x,t)\right\}} \\
&\qquad e^{-N\int dtdxdt'dx'\;\bar{\eta}(t,x)\frac{\delta\mathrm{Eq}_{[\rho(x,t)]}}{\delta\rho(x',t')}\eta(x',t')} \\
&= \int D\rho D\hat{\rho} D\xi\; D\eta\; D\bar{\eta}\; e^{N\int dtdx\,\left\{-\hat{\rho}(x,t)\dot{\rho}(x,t)-\bar{\eta}(x,t)\dot{\eta}(x,t)+\hat{\rho}(x,t)\partial_x^2\rho(x,t)\right\}} \\
&\qquad e^{N\int dtdx\{[\rho(x,t)\partial_x\hat{\rho}+\partial_x\bar{\eta}(t,x)\eta(x,t)]\xi(x,t)+\partial_x^2\bar{\eta}(t,x)\eta(x,t)-\frac{1}{2}\xi^2(x,t)\}}
\end{aligned} \tag{5.15}$$

This corresponds to a system with fermions, with the total number of them $\langle\int dx\;\bar{\eta}(x)\eta(x)\rangle$ a conserved quantity. We may now integrate away the noise variable ξ to obtain:

$$\begin{aligned}
e^{-NS} &= \int D\rho D\hat{\rho}\; D\eta\; D\bar{\eta}\; e^{N\int dtdx\,\left\{-\hat{\rho}\;\dot{\rho}-\bar{\eta}\;\dot{\eta}-\hat{\rho}\;\partial_x^2\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2+\rho\;\partial_x\hat{\rho}\;\partial_x\bar{\eta}\;\eta-\partial_x^2\bar{\eta}\;\eta\right\}} \\
&= \int D\rho D\hat{\rho}\; D\eta\; D\bar{\eta}\; e^{N\int dtdx\,\left\{-\dot{\hat{\rho}}\;\rho-\bar{\eta}\;\dot{\eta}-\partial_x^2\hat{\rho}\;\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2+\rho\partial_x\hat{\rho}\;\partial_x\bar{\eta}\;\eta+\partial_x\bar{\eta}\;\partial_x\eta\right\}}
\end{aligned} \tag{5.16}$$

where we have integrated by parts in time and space, valid for a trace (periodic in time) and a closed chain. The restriction to the zero-fermion subspace of (5.16) yields back (5.11): this is the usual relation between supersymmetric quantum mechanics and stochastic processes [49]. One has to be careful about the time-discretization of the stochastic process – Ito, Stratonovich and other conventions, here it is encapsulated in the actual value equal-time expectations $\langle\bar{\eta}(x,t)\eta(x,t)\rangle$ (see discussion in [50]).

Let us write an alternative expression for the action in (5.16). We make the change of variables $\rho(x,t)\to\rho(x,t)(1-\frac{1}{2}\bar{\eta}(x,t)\eta(x,t))$

$$
\begin{aligned}
S &= \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\bar{\eta}\ \dot{\eta}+\partial_x^2\hat{\rho}\rho-\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2-\rho\partial_x\hat{\rho}\ \partial_x\bar{\eta}\ \eta-\partial_x\bar{\eta}\ \partial_x\eta\right\}\\
&\to \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\bar{\eta}\ \dot{\eta}-\frac{1}{2}\dot{\hat{\rho}}\rho\ \bar{\eta}\ \eta+\partial_x^2\hat{\rho}\rho\ \left(1-\frac{1}{2}\bar{\eta}\ \eta\right)\right.\\
&\qquad \left.-\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\ (1-\bar{\eta}\ \eta)-\rho\partial_x\hat{\rho}\ \partial_x\bar{\eta}\ \eta-\partial_x\bar{\eta}\ \partial_x\eta\right\}\\
&= \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\bar{\eta}\ \left(\partial_t-\frac{1}{2}\dot{\hat{\rho}}\rho\right)\ \eta-\left[\partial_x\hat{\rho}\partial_x\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\right]\ \left(1-\frac{1}{2}\bar{\eta}\ \eta\right)+\frac{1}{2}\partial_x\hat{\rho}\rho\ \partial_x(\bar{\eta}\ \eta)\right.\\
&\qquad \left.+\frac{1}{4}\rho^2(\partial_x\hat{\rho})^2\ \bar{\eta}\ \eta-\rho\partial_x\hat{\rho}\ \partial_x\bar{\eta}\ \eta-\partial_x\bar{\eta}\ \partial_x\eta\right\}\\
&= \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\bar{\eta}\ \left(\partial_t-\frac{1}{2}\dot{\hat{\rho}}\rho\right)\eta-\left[\partial_x\hat{\rho}\partial_x\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\right]\ \left(1-\frac{1}{2}\bar{\eta}\ \eta\right)\right.\\
&\qquad \left.+\frac{1}{4}\rho^2(\partial_x\hat{\rho})^2\ \bar{\eta}\ \eta+\frac{1}{2}\partial_x\hat{\rho}\rho\ \partial_x(\bar{\eta}\ \eta)-\rho\partial_x\hat{\rho}\partial_x\bar{\eta}\ \eta-\partial_x\bar{\eta}\ \partial_x\eta\right\}\\
&= \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\bar{\eta}\ \left(\partial_t-\frac{1}{2}\dot{\hat{\rho}}\rho\right)\eta-\left[\partial_x\hat{\rho}\partial_x\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\right]\ \left(1-\frac{1}{2}\bar{\eta}\ \eta\right)\right.\\
&\qquad \left.-\left(\partial_x+\frac{1}{2}\rho\partial_x\hat{\rho}\right)\bar{\eta}\ \left(\partial_x-\frac{1}{2}\rho\partial_x\hat{\rho}\right)\eta\right\}\\
&= \int dtdx\ \left\{\dot{\hat{\rho}}\rho+\left[\partial_x\hat{\rho}\partial_x\rho+\frac{1}{2}\rho^2(\partial_x\hat{\rho})^2\right]\ \left(1-\frac{1}{2}\bar{\eta}\ \eta\right)+\bar{\eta}\ D_t^-\eta+D_x^+\bar{\eta}\ D_x^-\eta\right\}
\end{aligned}
\tag{5.17}
$$

where $D_\sigma^\pm\equiv\partial_\sigma\pm\frac{1}{2}\rho\partial_\sigma\hat{\rho}$.

Written in this way, one may compare directly with the expression given in Ref. [51] for the $\mathfrak{sl}(2|1)$ closed superstring. [2]

6 Conclusion and outlook

In this article we have studied the stochastic particle processes that arise from the non-compact Heisenberg chain. We explicitly gave the corresponding hopping rates for a spin chain with spin s at each site and showed that they can be recovered from the hopping rates in [1], [2] and [3]. Further we introduced integrable boundary conditions that were derived from the boundary Yang-Baxter equation using a general solution for the off-diagonal K-matrix. We studied the duality properties of this chain and exemplified how correlation functions in the stationary state can be computed. In the final section we show that the fluctuating hydrodynamics of the system without boundaries corresponds to the semiclassical evolution of a spinning string which naturally arises from the AdS/CFT dictionary.

The identification of the stochastic process with the non-compact spin chain allows to study the Markov process using powerful integrability tools. In particular one may expect that the steady state of the boundary model can be obtained from some sort of Bethe ansatz, see also

[2] This may be seen directly by making the change of variables $\bar{\eta}\to-2\bar{\xi}$, $\eta\to\xi$ for fermions, $\hat{\rho}\to2\tilde{\rho}$ for the response field and $S\to2S$ for the action. Note that our ρ is not theirs, which is the 'radial' coordinate of the hyperboloid.

[54] where the relation between the QISM approach and the matrix product ansatz has been discussed. Duality has told us that instead of computing the steady state we can compute the absorption probabilities of the dual process. This seems very similar to the Bethe ansatz which we have shown is equivalent to the one of a spin chain with triangular K-matrices. The precise relation for this apparent connection between duality and Bethe ansatz deserves some further investigation.

The fact that we are obtaining quantum systems that yield, in imaginary times, the generators of stochastic dynamics, may partially be argued from the fact that the spin chains obtained from $\mathcal{N} = 4$ super Yang-Mills theory are naturally the bosonic parts of some larger supersymmetric Hamiltonian. Indeed, in order to be stochastic and to have a stationary limit, the generator has to have a positive semi-definite spectrum, a fact that comes naturally from supersymmetry. This same supersymmetry guarantees that the equilibrium theorems for the dynamics of a system in contact with a stochastic bath hold. The only thing that in the end one needs to check is the construction of one *or more* bases in which the matrix elements of the Hamiltonian are negative on the diagonal and positive elsewhere, so that the probability rates are positive. Given what we have said, the temptation is great to conjecture that the bosonic part of $\mathcal{N} = 4$ super Yang-Mills theory may be mapped (at least in the planar approximation) to a stochastic system, and that the fermions come from "stochastic quantization".

We expect that the models studied in this article lift to the trigonometric/asymmetric case. Here the hopping rates of the stochastic process in the bulk are the ones of a q-Hahn antisymmetric process [2] and can be identified with non-compact XXZ spin chain [27]. The spin $\frac{1}{2}$ case then yields the MADM [1]. It would be interesting to derive the K-matrix in this case and study the process with reservoirs. The string picture may be connected to a deformation of $AdS_5 \times S^5$.

Finally it would be interesting to study the relation to the stochastic R-matrix approach in [21] and the stochastic K-matrix that has been recently obtained in [55] in connection to Baxter Q-operators as discussed at the end of Section 2.4.

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A Taylor expansion

A.1 Derivation (2.15)

The action of the bulk generator $-\mathcal{H}^t_{i,i+1}$ on a function of the rescaled variables $(x_i^{(M)}, x_{i+1}^{(M)})$ reads

$$\begin{aligned} -\mathcal{H}^t_{i,i+1} f(x_i^{(M)}, x_{i+1}^{(M)}) &= \sum_{k=1}^{x_i M} \frac{1}{k}\Big[f\Big(x_i^{(M)} - \frac{k}{M}, x_{i+1}^{(M)} + \frac{k}{M}\Big) - f\Big(x_i^{(M)}, x_{i+1}^{(M)}\Big)\Big] \\ &+ \sum_{k=1}^{x_i M} \frac{1}{k}\Big[f\Big(x_i^{(M)} + \frac{k}{M}, x_{i+1}^{(M)} - \frac{k}{M}\Big) - f\Big(x_i^{(M)}, x_{i+1}^{(M)}\Big)\Big]. \end{aligned} \tag{A.1}$$

We only consider the scaling limit of the right jumps, the proof is analogous for the left jumps. One one hand we notice that, by a bivariate Taylor expansion, we may write

$$\begin{aligned} &\sum_{k=1}^{x_i M} \frac{1}{k}\Big[f\Big(x_i^{(M)} - \frac{k}{M}, x_{i+1}^{(M)} + \frac{k}{M}\Big) - f\Big(x_i^{(M)}, x_{i+1}^{(M)}\Big)\Big] = \\ &\sum_{k=1}^{x_i M} \frac{1}{k} \sum_{a=1}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \Big[\Big(\frac{\partial}{\partial x_i}\Big)^b \Big(\frac{\partial}{\partial x_{i+1}}\Big)^{a-b} f\Big(x_i^{(M)}, x_{i+1}^{(M)}\Big)\Big]\Big(-\frac{k}{M}\Big)^b \Big(\frac{k}{M}\Big)^{a-b}. \end{aligned} \tag{A.2}$$

This implies

$$\begin{aligned} &\lim_{M\to\infty} \sum_{k=1}^{x_i M} \frac{1}{k}\Big[f\Big(x_i^{(M)} - \frac{k}{M}, x_{i+1}^{(M)} + \frac{k}{M}\Big) - f\Big(x_i^{(M)}, x_{i+1}^{(M)}\Big)\Big] = \\ &\sum_{a=1}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \Big[\Big(\frac{\partial}{\partial x_i}\Big)^b \Big(\frac{\partial}{\partial x_{i+1}}\Big)^{a-b} f(x_i, x_{i+1})\Big](-1)^b \int_0^{x_i} \alpha^{a-1} d\alpha \end{aligned} \tag{A.3}$$

where the convergence of the Riemann sum

$$\sum_{k=1}^{x_i M} \frac{1}{k}\Big(\frac{k}{M}\Big)^a \longrightarrow \int_0^{x_i} \alpha^{a-1} d\alpha \qquad \text{as} \quad M \to \infty \tag{A.4}$$

has been exploited. On the other hand we may also write

$$\begin{aligned} &\int_0^{x_i} \frac{d\alpha}{\alpha}\Big\{f(x_i - \alpha, x_{i+1} + \alpha) - f(x_i, x_{i+1})\Big\} = \\ &\int_0^{x_i} \frac{d\alpha}{\alpha} \sum_{a=1}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \Big(\frac{\partial}{\partial x_i}\Big)^b \Big(\frac{\partial}{\partial x_{i+1}}\Big)^{a-b} f(x_i, x_{i+1})(-\alpha)^b (\alpha)^{a-b} = \\ &\int_0^{x_i} \frac{d\alpha}{\alpha} \sum_{a=1}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \Big(\frac{\partial}{\partial x_i}\Big)^b \Big(\frac{\partial}{\partial x_{i+1}}\Big)^{a-b} f(x_i, x_{i+1})(-1)^b \alpha^a. \end{aligned} \tag{A.5}$$

The right hand sides of (A.3) and (A.5) do coincide, thus concluding the proof.

A.2 Derivation (2.16)

The action of the bulk generator $-\mathcal{H}^t_1$ on a function of the rescaled variables $x_1^{(M)}$ reads

$$\begin{aligned} -\mathcal{H}^t_1 f(x_1^{(M)}) &= \sum_{k=1}^{x_1 M} \frac{1}{k}\Big[f\Big(x_1^{(M)} - \frac{k}{M}\Big) - f\Big(x_1^{(M)}\Big)\Big] \\ &+ \sum_{k=1}^{\infty} \frac{[\beta_1^{(M)}]^k}{k}\Big[f\Big(x_1^{(M)} + \frac{k}{M}\Big) - f\Big(x_1^{(M)}\Big)\Big]. \end{aligned} \tag{A.6}$$

By Taylor expansion, we may write

$$
\begin{aligned}
-\mathcal{H}_1^t f(x_1^{(M)}) &= \sum_{k=1}^{x_1 M} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}\Big(x_1^{(M)}\Big)\Big(-\frac{k}{M}\Big)^n \\
&+ \sum_{k=1}^{\infty} \frac{[\beta_1^{(M)}]^k}{k} \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}\Big(x_1^{(M)}\Big)\Big(\frac{k}{M}\Big)^n. \qquad \text{(A.7)}
\end{aligned}
$$

This yields in the limit

$$
\begin{aligned}
-\lim_{M\to\infty} \mathcal{H}_1^t f(x_1^{(M)}) &= \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_1)(-1)^n \int_0^{x_1} \alpha^{n-1} d\alpha \\
&+ \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_1) \int_0^{\infty} \alpha^{n-1} e^{-\lambda_1 \alpha} d\alpha. \qquad \text{(A.8)}
\end{aligned}
$$

where, besides (A.4), it has been used the convergence

$$
\sum_{k=1}^{\infty} \frac{[\beta_1^{(M)}]^k}{k}\Big(\frac{k}{M}\Big)^n = \sum_{k=1}^{\infty} \frac{[1-\frac{\lambda_1}{M}]^k}{k}\Big(\frac{k}{M}\Big)^n \longrightarrow \int_0^{\infty} \alpha^{n-1} e^{-\lambda_1 \alpha} d\alpha, \qquad \text{as} \quad M \to \infty. \qquad \text{(A.9)}
$$

Clearly, the right hand sides of (A.8) coincides with the right hand side of (2.16).

B Infinite sums

B.1 Derivation (3.39)

It is straightforward to see that (3.39) is true for $k \le l$. In the case $k > l$ the sum reduces to

$$
\sum_{m=l}^{k} \binom{k}{m}\binom{m}{l} \beta^{k-m}(-\beta)^{m-l}\psi(m+1) = \sum_{m=l}^{k}\sum_{r=1}^{m} \binom{k}{m}\binom{m}{l} \frac{(-1)^{m-l}\beta^{k-l}}{m+1-r} = -\frac{\beta^{k-l}}{k-l} \qquad \text{(B.1)}
$$

where we first used

$$
\psi(m+1) = \psi(1) + \sum_{r=1}^{m} \frac{1}{m+1-r} \qquad \text{(B.2)}
$$

and exchanged the sums in the second step.

B.2 Derivation (3.41)

After noting that the first sum truncates and shifting the second sum we rewrite (3.41) as

$$
\begin{aligned}
&\sum_{m_2=1}^{\infty} \frac{(-1)^{l+m_2}}{m_2} \alpha^{m_2}\beta^{k-l+m_2} \binom{m_2}{l} {}_2F_1(-k, m_2+1; -l+m_2+1; 1) = \\
&\qquad = \sum_{m_2=1}^{\infty} (-1)^{l+m_2} \alpha^{m_2}\beta^{k-l+m_2} \frac{\Gamma(m_2)\Gamma(k-l)}{\Gamma(l+1)\Gamma(-l)\Gamma(k-l+m_2+1)} \qquad \text{(B.3)}
\end{aligned}
$$

The case $k > l$ in (3.41) is straightforward, the denominator is finite while the numerator diverges. For $k \le l$ one has to be more careful. One gets

$$
\begin{aligned}
&\sum_{m_2=1}^{\infty} (-1)^{l+m_2} \alpha^{m_2}\beta^{k-l+m_2} \frac{\Gamma(m_2)\Gamma(k-l)}{\Gamma(l+1)\Gamma(-l)\Gamma(k-l+m_2+1)} \\
&\qquad = \frac{(-1)^{k+l}}{(l-k)!} \sum_{m_2=1}^{\infty} (-1)^{m_2} \alpha^{m_2}\beta^{k-l+m_2} \frac{\Gamma(m_2)}{\Gamma(k-l+m_2+1)} \qquad \text{(B.4)}
\end{aligned}
$$

To show the remaining relations we substitute

$$\alpha = \frac{1}{1-\beta} = \sum_{p=0}^{\infty} \beta^p \tag{B.5}$$

Using the binomial series

$$\alpha^{m_2} = \frac{1}{(1-\beta)^{m_2}} = \sum_{n=0}^{\infty} \frac{\Gamma(m_2+n)}{\Gamma(m_2)\Gamma(1+n)} \beta^n \tag{B.6}$$

we get for $k = l$

$$\begin{aligned} \sum_{m_2=1}^{\infty} \frac{(-1)^{m_2}\alpha^{m_2}\beta^{m_2}}{m_2} &= \sum_{n=0}^{\infty}\sum_{m_2=1}^{\infty} \frac{(-1)^{m_2}\beta^{n+m_2}(m_2+n-1)!}{n!m_2!} \\ &= \sum_{q=1}^{\infty}\sum_{m_2=1}^{q} \frac{(-1)^{m_2}\beta^{q}(q-1)!}{(q-m_2)!m_2!} = -\sum_{q=1}^{\infty}\frac{\beta^q}{q} \end{aligned} \tag{B.7}$$

For $k < l$ we get

$$\begin{aligned} \frac{(-1)^{k+l}\beta^{k-l}}{(l-k)!} \sum_{m_2=l-k}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m_2}\beta^{n+m_2}(m_2+n-1)!}{n!(k-l+m_2)!} &= \frac{(-1)^{k+l}\beta^{k-l}}{(l-k)!} \sum_{m_2=l-k}^{\infty}\sum_{q=m_2}^{\infty} \frac{(-1)^{m_2}\beta^{q}(q-1)!}{(q-m_2)!(k-l+m_2)!} \\ &= \frac{(-1)^{k+l}\beta^{k-l}}{(l-k)!} \sum_{q=l-k}^{\infty}\sum_{m_2=l-k}^{q} \frac{(-1)^{m_2}\beta^{q}(q-1)!}{(q-m_2)!(k-l+m_2)!} \end{aligned} \tag{B.8}$$

Finally noting that

$$\sum_{m=0}^{p} \frac{(-1)^m}{(p-m)!m!} = \delta_{p,0} \tag{B.9}$$

we obtain the final result.

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