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BOUNDING FUNCTION APPROACH FOR IMPULSIVE DIRICHLET PROBLEMS WITH UPPER-CARATHÉODORY RIGHT-HAND SIDE

MARTINA PAVLAČKOVÁ, VALENTINA TADDEI

Abstract. In this article, we prove the existence and localization of solutions for a vector impulsive Dirichlet problem with multivalued upper-Carathéodory right-hand side. The result is obtained by combining the continuation principle with a bound sets technique. The main theorem is illustrated by an application to the forced pendulum equation with viscous damping term and dry friction coefficient.

1. Introduction

Given an upper-Carathéodory multivalued mapping $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$, we consider the multivalued vector Dirichlet problem

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \tag{1.1}$$

$$x(T) = x(0) = 0. \tag{1.2}$$

Moreover, let a finite number of points $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and real $n \times n$ matrices A_i, B_i, $i = 1, \dots, p$, be given.

In this article, we study the solvability of the boundary-value problem (1.1)-(1.2), in the presence of the impulse conditions

$$x(t_i^+) = A_i x(t_i), \quad i = 1, \dots, p, \tag{1.3}$$

$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \dots, p, \tag{1.4}$$

where $\lim_{t\to a^+} x(t) = x(a^+)$.

By a solution of (1.1)-(1.4) we mean a function $x \in PAC^1([0,T], \mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.1)–(1.4).

Boundary value problems with impulses have attracted lots of interest because of their applications in many areas such as: aircraft control, drug administration, biotechnology and population dynamics, where processes are characterized by the fact that the model parameters are subject to short term perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics, impulses may correspond to abrupt

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changes of prices. Impulsive differential equations and inclusions are adequate apparatus for modeling such processes and phenomena. The theory of single valued impulsive problems is widely developed and presents in many cases direct analogies with the results for problems without impulses (see, e.g., [11, 12, 24, 30]). The theory dealing with multivalued impulsive problems arises e.g. from single valued problems with discontinuous right-hand sides, problems with inaccurately known right-hand sides or from control theory. This field has not been so deeply studied and the results have been obtained in particular for the first-order problems and using fixed point theorems or upper and lower-solutions methods; for the overview of known results, we recommend the monographs [13, 21] and the references therein. Few results were obtained for Dirichlet impulsive problems using topological or variational approaches in cases when right-hand sides do not dependent on the first derivative or when the impulses depend only on the first derivative (see [1, 15, 16, 18, 25, 29]).

In this paper, not only the existence but also the localization of solutions for the impulsive multivalued Dirichlet problem (1.1)-(1.4) are obtained by means of bound sets technique. The bound sets approach was introduced in the single valued case by Gaines and Mawhin [20] for obtaining the existence of solutions of first and second order differential equations. This technique was applied for multivalued Dirichlet, Floquet or two-point problems without impulses in [4]-[9], [28, 32]. The existence and localization result presented in Theorem 4.1 below will be obtained by combining the bound sets approach with the continuation principle developed in Section 2.

This article is organized as follows. In the second section, we recall suitable definitions and statements which will be used in the sequel. Section 3 is devoted to the study of bound sets and Liapunov-like bounding functions for impulsive Dirichlet problems. At first, we consider C^1-bounding functions with locally Lipschitzian gradients. Consequently, it is shown how conditions ensuring the existence of bound set become in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with the continuation principle and an existence and localization result is obtained in this way for the impulsive Dirichlet problem (1.1)-(1.4). Section 5 deals with an application to the forced pendulum equation with viscous damping term and dry friction coefficient.

2. Preliminaries

We start with the notation used in this article. Let (X, d) be a metric space and $A \subset X$. By $\overline{A}$, $\operatorname{int} A$ and ∂A, we mean the closure, interior and boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X : \exists a \in A : d(x, a) < \varepsilon\}$, hence $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a continuous function $r : X \to A$ satisfying $r(x) = x$ for every $x \in A$; this function is called a retraction.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we denote the set of functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J. In the sequel, the norm of a real $n \times n$ matrix will be denoted by $\|\cdot\|$ and the norm in $L^1(J, \mathbb{R})$ by the symbol $\|\cdot\|_1$.

Let $PAC^1([0,T],\mathbb{R}^n)$ be the space of functions $x:[0,T]\to\mathbb{R}^n$ such that

$$x(t)=\begin{cases} x_{[0]}(t), & \text{for } t\in[0,t_1],\\ x_{[1]}(t), & \text{for } t\in(t_1,t_2],\\ \dots \\ x_{[p]}(t), & \text{for } t\in(t_p,T],\end{cases}$$

where $x_{[0]}\in AC^1([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in AC^1((t_i,t_{i+1}],\mathbb{R}^n)$, $x(t_i^+)=\lim_{t\to t_i^+}x(t)\in\mathbb{R}$ and $\dot{x}(t_i^+)=\lim_{t\to t_i^+}\dot{x}(t)\in\mathbb{R}$, for $i=1,\dots,p$. The space $PAC^1([0,T],\mathbb{R}^n)$ equipped with the norm

$$\|x\|_E:=\sup_{t\in[0,T]}|x(t)|+\sup_{t\in[0,T]}|\dot{x}(t)|, \tag{2.1}$$

is denoted by $(E,\|\cdot\|_E)$. In a similar way, we can define the spaces $PC([0,T],\mathbb{R}^n)$ and $PC^1([0,T],\mathbb{R}^n)$ as the spaces of functions $x:[0,T]\to\mathbb{R}^n$ satisfying the previous definition with $x_{[0]}\in C([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in C((t_i,t_{i+1}],\mathbb{R}^n)$, and with $x_{[0]}\in C^1([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in C^1((t_i,t_{i+1}],\mathbb{R}^n)$, for $i=1,\dots,p$, respectively. The space $PC^1([0,T],\mathbb{R}^n)$ with the norm defined in (2.1) is a Banach space (see [27, page 128]). A compactness result for subsets of $PC^1([0,T],\mathbb{R}^n)$ will be needed. So we recall that a family $\mathcal{F}\subset PC([0,T],\mathbb{R}^n)$ is left equicontinuous (see [27]) if for every $\epsilon>0$ and $x\in[0,T]$ there exists $\delta>0$ such that, for every $f\in\mathcal{F}$,

$$|f(x)-f(y)|<\epsilon,\quad \text{for all } y\in(x-\delta,x]$$

and

$$|f(x^+)-f(y)|<\epsilon,\ \text{for all }\ y\in(x,x+\delta).$$

In the sequel, we use a generalized Ascoli-Arzelà theorem whose prove is given in [27, Theorem 2], in a slightly different case, i.e. when the real valued functions are discontinuous from the left and are just continuous in each interval $[t_i,t_{i+1})$.

Proposition 2.1. *A family $\mathcal{F}\subset PC^1([0,T],\mathbb{R}^n)$ is compact if and only if it is bounded, left equicontinuous and the set $\{f':f\in\mathcal{F}\}$ is left equicontinuous.*

We also need the following definitions and notion for multivalued mappings. We say that F is a *multivalued mapping* from X to Y (written $F:X\multimap Y$), if, for every $x\in X$, a nonempty subset $F(x)$ of Y is given. We associate to F its graph Γ_F, i.e. the subset of $X\times Y$ defined by

$$\Gamma_F:=\{(x,y)\in X\times Y\mid y\in F(x)\}.$$

The single valued function $f:X\to Y$ is called a *selection* of F if $\Gamma_f\subset\Gamma_F$, i.e. if $f(x)\in F(x)$, for every $x\in X$.

A multivalued mapping $F:X\multimap Y$ is called *upper semi-continuous* (abbreviated, u.s.c.) if, for each open set $U\subset Y$, the set $\{x\in X:F(x)\subset U\}$ is open in X. A multivalued mapping $F:X\multimap Y$ is called *compact* if the set $F(X)=\cup_{x\in X}F(x)$ is contained in a compact subset of Y. Let us note that every u.s.c. mapping with closed values has a closed graph and that every compact multivalued mapping with closed graph is u.s.c.

Let Y be a metric space and $(\Omega,\mathcal{U},\mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F:\Omega\multimap Y$ is called *measurable* if $\{\omega\in\Omega:F(\omega)\subset V\}\in\mathcal{U}$, for each open set $V\subset Y$.

We say that the mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for a.a. $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We shall use the following selection result, which was proved in [14, Proposition 6] in a quite general setting for a continuous function q. Its proof can be easily extended to piecewise continuous functions, so we omit it here.

Proposition 2.2. *Let $J \subset \mathbb{R}$ be a compact interval and $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that for every $r > 0$ there exists an integrable function $\mu_r : J \to [0, \infty)$ satisfying $|y| \le \mu_r(t)$, for every $(t, x) \in J \times \mathbb{R}^m$, with $|x| \le r$, and every $y \in F(t, x)$. Then the composition $F(t, q(t))$ admits, for every $q \in PC(J, \mathbb{R}^m)$, a measurable selection.*

Let $X \cap Y \neq \emptyset$ and $F : X \multimap Y$. We say that a point $x \in X \cap Y$ is a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F is denoted by $Fix(F)$, i.e.

$$Fix(F) := \{x \in X : x \in F(x)\}.$$

The following proposition will be applied for obtaining the existence of solutions to boundary value problems. It follows from a result in [2, 3].

Proposition 2.3. *Let X be a retract of a Banach space Y, and let $\mathfrak{T} : X \times [0, 1] \multimap Y$ be a compact u.s.c. mapping with convex values such that $\mathfrak{T}(X, 0) \subset X$ and that $Fix(\mathfrak{T}(x, \lambda)) \cap \partial X = \emptyset$, for every $\lambda \in [0, 1)$. Then $\mathfrak{T}(\cdot, 1)$ has a fixed point.*

We also need the following modification of the continuation principle developed in [10] for problems on arbitrary, possibly non-compact, intervals. The differences between the presented result and the one in [10] consist in replacement of the non-compact interval by the compact one which simplify the last, so called transversality condition, and in replacement of the space $AC^1_{loc}([0, T], \mathbb{R}^n)$ by the space E defined above. For the completeness, the proof of this modified result is given here.

Proposition 2.4. *Let us consider the boundary-value problem*

$$\begin{gathered} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ x \in S, \end{gathered} \tag{2.2}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of E. Let $H : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [0, T] \times \mathbb{R}^{2n}. \tag{2.3}$$

Assume that

(i) *there exists a retract Q of $PC^1([0, T], \mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\begin{gathered} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [0, T], \\ x \in S_1 \end{gathered} \tag{2.4}$$

has, for each $(q, \lambda) \in Q \times [0, 1]$, a non-empty and convex set of solutions $\mathfrak{T}(q, \lambda)$;

(ii) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t)(1 + |x(t)| + |\dot{x}(t)|),$$

for a.a. $t \in [0, T]$, and for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$;

(iii) $\mathfrak{T}(Q \times \{0\}) \subset Q$;
(iv) *there exist constants* $M_0 \geq 0$, $M_1 \geq 0$ *such that* $|x(0)| \leq M_0$ *and* $|\dot{x}(0)| \leq M_1$, *for all* $x \in \mathfrak{T}(Q \times [0,1])$;
(v) *the solution map* $\mathfrak{T}(\cdot, \lambda)$ *has no fixed points on the boundary* ∂Q *of* Q, *for every* $\lambda \in [0, 1)$.

Then (2.2) *has a solution in* $S_1 \cap Q$.

Proof. Let us apply Proposition 2.3, where $X = Q$ is a retract of the Banach space $Y = PC^1([0,T], \mathbb{R}^n)$. First of all, notice that if there exists $q \in \partial Q$ such that $\mathfrak{T}(q, 1) = q$, then the result is proven. Otherwise, we get that $\mathfrak{T}(Q \times [0, 1]) \cap \partial Q = \emptyset$, according to assumption (v). Moreover, it follows from conditions (i) and (iii), that $\mathfrak{T}$ has convex values and that $\mathfrak{T}(Q, 0) \subset Q$.

Let us now show that $\mathfrak{T}$ has a closed graph. Let $\{(q_k, \lambda_k, x_k)\} \subset \Gamma_{\mathfrak{T}}$ such that $(q_k, \lambda_k, x_k) \to (q, \lambda, x)$, $(q, \lambda) \in Q \times [0, 1]$ be arbitrary. Then, since $x_k \in S_1$, $x_k \to x$ and S_1 is closed, it holds that $x \in S_1$. Moreover, x_k is a solution of (2.4), and so, according to Proposition 2.2, we get the existence of $h_k \in H(\cdot, x_k(\cdot), \dot{x}_k(\cdot), q_k(\cdot), \dot{q}_k(\cdot), \lambda_k)$ such that $\dot{x}_k(t_{i+1}) - \dot{x}_k(t) = \int_t^{t_{i+1}} h_k(s)ds$, for every $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. The convergence of $\{x_k\}$ implies its boundedness in $PC^1([0,T], \mathbb{R}^n)$, and therefore, we get from (ii) that $|h_k(t)| \leq \alpha(t)(1 + M)$, for some $M > 0$, every $k \in \mathbb{N}$ and a.a. $t \in [0,T]$. This implies that $\{h_k\}$ is bounded in $L^1([0,T], \mathbb{R}^n)$, and so it has a weakly convergent subsequence, for the sake of simplicity still denoted as the sequence, which converges to a function h. In particular, $\int_t^{t_{i+1}} h_k(s)\, ds \to \int_t^{t_{i+1}} h(s)\, ds$, for every $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. Hence,

$$\dot{x}(t_{i+1}) - \dot{x}(t) = \lim_{k \to \infty} [\dot{x}_k(t_{i+1}) - \dot{x}_k(t)] = \lim_{k \to \infty} \int_t^{t_{i+1}} h_k(s)\, ds = \int_t^{t_{i+1}} h(s)\, ds,$$

for $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. Therefore, there exists $\ddot{x}(t) = h(t)$, for a.a. $t \in [0,T]$. It remains to prove that $h \in H(\cdot, x(\cdot), \dot{x}(\cdot), q(\cdot), \dot{q}(\cdot), \lambda)$. Since H is upper-Carathéodory, there exists, for every $\epsilon > 0$ and a.a. $t \in [0,T]$, a positive number δ such that, if $|(c, d, e, f, g) - (q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)| \leq \delta$, then

$$H(t, c, d, e, f, g) \subset H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda) + B_0^\epsilon.$$

Recalling that the convergence in $PC^1([0,T], \mathbb{R}^n)$ of q_k to q and x_k to x implies the pointwise convergence of both sequences and of the sequences of their derivatives to the same limits, we get that, for every $t \in [0,T]$ and $\delta > 0$, there exists $\overline{k}$ such that, for $k \geq \overline{k}$, $|(q_k(t), \dot{q}_k(t), x_k(t), \dot{x}_k(t), \lambda_k) - (q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)| \leq \delta$. Therefore, for every $\epsilon > 0$ and a.a. $t \in [0,T]$, there exists $\overline{k}$ such that, if $k \geq \overline{k}$, then

$$h_k(t) \in H(t, q_k(t), \dot{q}_k(t), x_k(t), \dot{x}_k(t), \lambda_k) \subset H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda) + B_0^\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get that $h(t) \in H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)$, for a.a. $t \in [0,T]$, i.e. that $\mathfrak{T}$ has a closed graph. Recalling that a compact mapping with closed graph is u.s.c. and has compact values, it remains only to prove that $\mathfrak{T}$ is compact. According to Proposition 2.1, we need to prove that $\mathfrak{T}(Q \times [0, 1])$ is bounded, left equicontinuous, and has left equicontinuous set of derivatives.

Let $x \in \mathfrak{T}(q, \lambda)$. Then there exists $h \in H(\cdot, x(\cdot), \dot{x}(\cdot), q(\cdot), \dot{q}(\cdot), \lambda)$ such that, for every $\overline{t}, \tilde{t} \in (t_i, t_{i+1}]$, with $\overline{t} > \tilde{t}$, and $i = 0, \dots, p$,

$$\dot{x}(\overline{t}) = \dot{x}(\tilde{t}) + \int_{\tilde{t}}^{\overline{t}} h(s)\, ds, \tag{2.5}$$

and consequently, according to Fubini's theorem,

$$\begin{aligned} x(\bar{t}) &= x(\tilde{t}) + \dot{x}(\tilde{t})(\bar{t}-\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} \int_{\tilde{t}}^{r} h(s)\,ds\,dr \\ &= x(\tilde{t}) + \dot{x}(\tilde{t})(\bar{t}-\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} (\bar{t}-s)h(s)\,ds. \end{aligned} \tag{2.6}$$

According to (ii) and (iv), for every $t \in [0, t_1]$, it holds that

$$|x(t)| + |\dot{x}(t)| \le M_0 + M_1(t_1+1) + (t_1+1)\int_0^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds.$$

Therefore, if we denote by $\beta_1 := M_0 + M_1(t_1+1) + (t_1+1)\int_0^{t_1} \alpha(s)ds$, we obtain by Gronwall's lemma that

$$|x(t)| + |\dot{x}(t)| \le \beta_1 + \beta_1(t_1+1)\int_0^{t_1} \alpha(s)e^{(t_1+1)\int_s^{t_1}\alpha(r)\,dr}\,ds := C_1.$$

Take now $t \in (t_1, t_2]$. Reasoning as above we obtain

$$\begin{aligned} &|x(t)| + |\dot{x}(t)| \\ &\le |x(t_1^+)| + |\dot{x}(t_1^+)|(t_2+1) + (t_2-t_1+1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds \\ &\le \|A_1\|\cdot|x(t_1)| + \|B_1\|\cdot|\dot{x}(t_1)|(t_2+1) \\ &\quad + (t_2-t_1+1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds \\ &\le \max\{\|A_1\|, \|B_1\|(t_2+1)\}C_1 + (t_2-t_1+1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds. \end{aligned}$$

Hence, denoted by $\beta_2 := \max\{\|A_1\|, \|B_1\|(t_2+1)\}C_1 + (t_2-t_1+1)\int_{t_1}^{t_2}\alpha(s)\,ds$, we obtain that

$$|x(t)| + |\dot{x}(t)| \le \beta_2 + \beta_2(t_2-t_1+1)\int_{t_1}^{t_2} \alpha(s)e^{(t_2-t_1+1)\int_s^{t_2}\alpha(r)dr}\,ds := C_2.$$

Iterating we obtain the existence of $D > 0$ such that $|x(t)| + |\dot{x}(t)| \le D$, for every $t \in [0, T]$, i.e. we obtain that $\mathfrak{T}(Q \times [0,1])$ is bounded in $PC^1([0,T], \mathbb{R}^n)$.

Moreover, it follows from (2.5) and (2.6) that, that for every $\bar{t}, \tilde{t} \in (t_i, t_{i+1}]$ with $\bar{t} > \tilde{t}$ and $i = 0, \dots, p$,

$$|\dot{x}(\bar{t}) - \dot{x}(\tilde{t})| = \Big|\int_{\tilde{t}}^{\bar{t}} h(s)ds\Big| \le (1+D)\int_{\tilde{t}}^{\bar{t}} \alpha(s)ds,$$

$$|x(\bar{t}) - x(\tilde{t})| \le D|\bar{t}-\tilde{t}| + (1+D)\int_{\tilde{t}}^{\bar{t}} (\bar{t}-s)\alpha(s)ds.$$

Thus, if $t \ne t_1, \dots, t_p$, one can take δ sufficiently small such that $(t-\delta, t+\delta) \cap \{t_1, \dots, t_p\} = \emptyset$ and conclude (from the absolute continuity of the Lebesgue integral) that the functions x and $\dot{x}$ are equicontinuous at t. The left equicontinuity can be deduced similarly for $t \in \{t_1, \dots, t_p\}$.

So, we have proved that $\mathfrak{T}(Q \times [0,1))$ is compact, and hence, it follows from Proposition 2.3, that there exists a fixed point of $\mathfrak{T}(\cdot, 1)$ in $S_1 \cap Q$. $\square$

The continuation principle described in Proposition 2.4 requires in particular that any of corresponding problems does not have solutions tangent to the boundary of a given set Q of candidate solutions. In Section 4, we will ensure that the candidate solutions are not tangent to the boundary of Q by means of Hartman-type conditions (see Section 3) and by means of the following result based on Nagumo conditions (see [31, Lemma 2.1] and [23, Lemma 5.1]).

Proposition 2.5. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous and non-decreasing function, with*

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{2.7}$$

and let R be a positive constant. Then there exists a positive constant

$$B = \psi^{-1}(\psi(2R) + 2R) \tag{2.8}$$

such that if $x \in PC^1([0,T], \mathbb{R}^n)$ is such that $|\ddot{x}(t)| \le \psi(|\dot{x}(t)|)$, for a.a. $t \in [0,T]$, and $|x(t)| \le R$, for every $t \in [0,T]$, then it holds that $|\dot{x}(t)| \le B$, for every $t \in [0,T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [7]) that the statement holds also for piecewise continuously differentiable functions.

3. Bound sets theory for impulsive Dirichlet problems

The direct verification of transversality condition (v) in Proposition 2.4 is quite complicated. Therefore, we now introduce a Liapunov-like function V, usually called *bounding function*, which can guarantee this condition.

Let $K \subset \mathbb{R}^n$ be a nonempty, open set with $0 \in K$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

(H1) $V|_{\partial K} = 0$,
(H2) $V(x) \le 0$, for all $x \in \overline{K}$.

Definition 3.1. A set K is called a *bound set for the impulsive Dirichlet problem* (1.1)-(1.4) if every solution x of (1.1)-(1.4) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0,T]$.

Remark 3.2. Note that the existence of a bound set K for problem (1.1)-(1.4) does not guarantee the existence of a solution for (1.1) -(1.4). It only ensures that if there would exist a solution laying in $\overline{K}$, then this solution would not touch the boundary of K at any point, i.e. it would lay in $\operatorname{int} K$.

At first, the sufficient conditions for the existence of a bound set for the impulsive Dirichlet problem (1.1)-(1.4) in the general case will be shown in Proposition 3.3 below. Afterwards, the regularity assumptions on the bounding function V will be made more strict and the practically applicable version of Proposition 3.3 will be obtained (see Corollary 3.5 below).

Proposition 3.3. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1, \dots, p$, be real $n \times n$ matrices such that $A_i \partial K = \partial K$, for all $i = 1, \dots, p$.*

Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions (H1) *and* (H2)*. Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, the condition*

$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{3.1}$$

holds for all $w \in F(t,x,v)$, and that

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0, \tag{3.2}$$

for all $i = 1, \dots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$ with $\langle \nabla V(x), v\rangle \neq 0$.

Then K is a bound set for the impulsive Dirichlet problem (1.1)-(1.4)*.*

Proof. We assume, by a contradiction, that K is not a bound set for the Dirichlet problem (1.1)-(1.4), i.e. that there exist a solution $x : [0,T] \to \overline{K}$ of (1.1)-(1.4) and $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. The point t^* must lay in $(0,T)$, according to the boundary condition (1.2) and the fact that $0 \in K$.

Let us define a function $g : [0,T] \to \mathbb{R}$ by the formula $g(t) := V(x(t))$. According to the properties of x and V, $g \in PC^1([0,T], \mathbb{R})$ and $g(t) \le 0$ for all t. Since $g(t^*) = 0$, the point t^* is a local maximum point for g. Therefore, if $t^* \notin \{t_1, \dots, t_p\}$, $\dot g(t^*) = 0$. Let us now prove that $\dot g(t^*) = 0$ also when $t^* = t_{i+1}$, for some $i = 0, \dots, p-1$. By a contradiction, suppose that

$$0 < \dot g(t_{i+1}) = \langle \nabla V(x(t_{i+1})), \dot x(t_{i+1})\rangle. \tag{3.3}$$

Notice that also $A_{i+1}x(t_{i+1}) \in \partial K$, and hence $g(t_{i+1}^+) = g(A_{i+1}x(t_{i+1})) = 0$. According to condition (3.2), there exist two functions $a(h)$ and $b(h)$, with $a(h) \to 0, b(h) \to 0$ when $h \to 0$, such that

$$\begin{aligned}
\dot g(t_{i+1}^+) &= \lim_{h\to 0^+} \frac{V(x(t_{i+1}+h)) - V(x(t_{i+1}^+))}{h} \\
&= \lim_{h\to 0^+} \frac{V(x(t_{i+1}^+) + \dot x(t_{i+1}^+)h + a(h)h) - V(x(t_{i+1}^+))}{h} \\
&= \lim_{h\to 0^+} \frac{\langle \nabla V(x(t_{i+1}^+), \dot x(t_{i+1}^+) + a(h)\rangle h + b(h)h}{h} \\
&= \langle \nabla V(x(t_{i+1}^+)), \dot x(t_{i+1}^+)\rangle \\
&= \langle \nabla V(A_{i+1}x(t_{i+1})), B_{i+1}\dot x(t_{i+1})\rangle > 0.
\end{aligned}$$

Thus, for $t > t_{i+1}$ sufficiently close to t_{i+1}, we get that $0 \ge g(t) > g(t_{i+1}^+) = 0$, a contradiction. Therefore, $\dot g(t^*) = 0$ also in the case when $t^* = t_{i+1}$.

Since ∇V is locally Lipschitzian, there exist a bounded set $U \subset \mathbb{R}^n$ with $x(t^*) \in U$ and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. The continuity of x in $(t_i, t_{i+1}]$ then yields the existence of $\delta > 0$, $\delta < t^* - t_i$, such that $x(t) \in U \cap N_\varepsilon(\partial K)$, for each $t \in [t^* - \delta, t^*]$. Since $\dot g(t) = \langle \nabla V(x(t)), \dot x(t)\rangle$, where $\nabla V(x(t))$ is locally Lipschitzian and $\dot x(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, there exists $\ddot g \in L^1([t^* - \delta, t^*], \mathbb{R})$. Moreover, there exists a point $t^{**} \in (t^* - \delta, t^*)$, such that $\dot g(t^{**}) \ge 0$, because t^* is a local maximum point. Consequently,

$$0 \ge -\dot g(t^{**}) = \dot g(t^*) - \dot g(t^{**}) = \int_{t^{**}}^{t^*} \ddot g(s)\, ds. \tag{3.4}$$

Let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then there exist two functions $a(h)$ and $b(h)$, with $a(h) \to 0$, $b(h) \to 0$ when $h \to 0$, such that, for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h[\ddot{x}(t) + a(h)], \tag{3.5}$$

$$x(t+h) = x(t) + h[\dot{x}(t) + b(h)]. \tag{3.6}$$

Consequently,

$$\begin{aligned}
&\ddot{g}(t) \\
&= \lim_{h\to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \limsup_{h\to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t+h)), \dot{x}(t+h)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h[\dot{x}(t) + b(h)]), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\geq \limsup_{h\to 0^-} \Big[\frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \Big] \\
&= \limsup_{h\to 0^-} \Big[\frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| + \langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle \Big].
\end{aligned}$$

Since $\langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \to 0$ as $h \to 0$ and since assumption (3.1) holds,

$$\ddot{g}(t) \geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} > 0,$$

which leads to a contradiction with inequality (3.4). □

Definition 3.4. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying (H1), (H2), (3.1), and (3.2) is called a *bounding function* for (1.1)-(1.4).

When the bounding function V is of class C^2, condition (3.1) can be rewritten in terms of gradients and Hessian matrices.

Corollary 3.5. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1, \dots, p$, be real $n \times n$ matrices such that $A_i \partial K = \partial K$, for all $i = 1, \dots, p$.*

Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions (H1), (H2), *and* (3.2). *Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, the condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{3.7}$$

holds for all $w \in F(t,x,v)$. Then K is a bound set for problem (1.1)-(1.4).

Proof. The statement follows immediately from the fact that if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, $v \in \mathbb{R}^n$ and $w \in F(t,x,v)$, there exists

$$\lim_{h\to 0} \frac{\langle \nabla V(x + hv), v + hw\rangle - \langle \nabla V(x), v\rangle}{h} = \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle.$$

□

Remark 3.6. In conditions (3.1), (3.2) and (3.7), the element v plays the role of the first derivative of the solution x. If $x(t) \in \overline{K}$, for every $t \in J$, then, according to Proposition 2.5 and the fact that $R = \max\{|c| : c \in \overline{K}\} \in \mathbb{R}$, it holds that $|\dot{x}(t)| \le B$, for every $t \in J$, where B is defined by (2.8). Hence, it is sufficient to require conditions (3.1), (3.2) and (3.7) in Proposition 3.3 and Corollary 3.5 only for all $v \in \mathbb{R}^n$ with $|v| \le B$ and not for all $v \in \mathbb{R}^n$.

4. Existence and localization results for Dirichlet problems

In this section,we study (1.1)-(1.4) by combining the continuation principle in Proposition 2.4 with bound sets results developed in the previous section. After rewriting (1.1)-(1.4) in the abstract form (2.2), we will be able to verify all conditions in Proposition 2.4.

Theorem 4.1. *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider (1.1)-(1.4), where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and A_i, B_i, $i = 1, \dots, p$, are real $n \times n$ matrices with $A_i \partial K = \partial K$, for all $i = 1, \dots, p$. Moreover, assume that*

(i) *there exists a function $\beta : [0,\infty) \to [0,\infty)$ continuous and non-decreasing satisfying*
$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty \tag{4.1}$$
such that
$$|F(t,c,d)| \le \beta(|d|), \tag{4.2}$$
for a.a. $t \in [0,T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*
$$\begin{gathered} \ddot{x}(t) = 0, \quad \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), \quad i = 1, \dots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \dots, p, \end{gathered} \tag{4.3}$$
has only the trivial solution;

(iii) *there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions (H1) and (H2);*

(iv) *there exists $\varepsilon > 0$ such that, for all $\lambda \in (0,1), x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$, the condition*
$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{4.4}$$
holds for all $w \in \lambda F(t,x,v)$;

(v) *for all $i = 1, \dots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$ and $\langle \nabla V(x), v\rangle \ne 0$, it holds*
$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0.$$

Then (1.1)-(1.4) has a solution $x(\cdot)$ such that $x(t) \in K$, for all $t \in [0,T]$.

Proof. For every $c \in \overline{K}$, it holds that $|c| \le R$. According to Proposition 2.5, for every $x \in PC^1([0,T],\mathbb{R}^n)$ with $|\ddot{x}(t)| \le \beta(|\dot{x}(t)|)$, for a.a. $t \in [0,T]$, and $x(t) \in \overline{K}$, for every $t \in [0,T]$, it holds $|\dot{x}(t)| \le B$, for every $t \in [0,T]$, with B defined by

$$B = \beta^{-1}(\beta(2R) + 2R).$$

Define

$$Q := \{q \in PC^1([0,T],\mathbb{R}^n) : q(t) \in \overline{K},\ |\dot{q}(t)| \le 2B, \quad \text{for all } t \in [0,T]\}, \tag{4.5}$$

$S = S_1 = Q$ and $H(t,c,d,e,f,\lambda) = \lambda F(t,e,f)$. Thus the associated problem (2.4) is the fully linearized problem

$$\begin{gathered} \ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), \quad \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i),\ i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\dots,p. \end{gathered} \tag{4.6}$$

For each $(q,\lambda) \in Q \times [0,1]$, let $\mathfrak{T}(q,\lambda)$ be the solution set of (4.6). Now we check that all the assumptions of Proposition 2.4 are satisfied.

Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0,T],\mathbb{R}^n)$.

Condition (ii) follows from assumption (i) and the fact that

$$\begin{aligned} |H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda| &= \lambda |F(t,q(t),\dot{q}(t))| \le \beta(|\dot{q}(t)|) \le \beta(2B) \\ &\le \beta(2B)(1 + |x(t)| + |\dot{x}(t)|), \end{aligned}$$

for every $\lambda \in [0,1], q \in Q, x \in \mathfrak{T}(q,\lambda)$. In particular $|F(t,e,f)| \le \beta(r)$ for every $(t,e,f) \in J \times \mathbb{R}^{2n}$ with $|f| \le r$.

Let $q \in Q$ and let f_q be a measurable selection of $F(\cdot,q(\cdot),\dot{q}(\cdot))$, whose existence is guaranteed applying Proposition 2.2 with $\mu_r(t) \equiv \beta(r)$. Then, for any $\lambda \in [0,1]$, λf_q is a measurable selection of $\lambda F(\cdot,q(\cdot),\dot{q}(\cdot))$. Let us consider the corresponding single valued linear problem with linear impulses

$$\begin{gathered} \ddot{x}(t) = \lambda f_q(t),\ \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i),\ i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\dots,p. \end{gathered} \tag{4.7}$$

First of all, let us prove that problem (4.7) has a unique solution $x_{\lambda f_q}$. If we denote

$$C := \begin{cases} B_1(T - t_1) & \text{if } p = 1 \\ \prod_{l=1}^p B_l(T - t_p) + \prod_{k=1}^p A_k t_1 \\ + \sum_{j=2}^p \prod_{k=j}^p A_k \prod_{l=1}^{j-1} B_l(t_j - t_{j-1}) & \text{if } p \ge 2, \end{cases} \tag{4.8}$$

it is easy to prove that the initial problem

$$\begin{gathered} \ddot{x}(t) = 0, \quad \text{for a.a. } t \in [0,T], \\ x(0) = 0, \\ x(t_i^+) = A_i x(t_i), \quad i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1,\dots,p \end{gathered}$$

has infinitely many solutions,

$$x_0(t)=\begin{cases}\dot x_0(0)t & \text{if } t\in[0,t_1],\\ B_1\dot x_0(0)(t-t_1) & \text{if } t\in(t_1,t_2]\\ \Big[\prod_{l=1}^i B_l(t-t_i)+\prod_{k=1}^i A_k t_1\\ +\sum_{j=2}^i\prod_{k=j}^i A_k\prod_{l=1}^{j-1}B_l(t_j-t_{j-1})\Big]\dot x_0(0)\\ \quad\text{if } t\in(t_i,t_{i+1}],\ 2\le i\le p+1\end{cases}$$

with $\dot x_0(0)\in\mathbb{R}^n$. Since $x_0(T)=0$ if and only if $C\dot x_0(0)=0$, condition (ii) holds if and only if C is regular. Then, for every $\lambda\in[0,1],\ q\in Q$ and every measurable selection f_q of $F(\cdot,q(\cdot)\dot q(\cdot))$, (4.7) has a unique solution,

$$x_{\lambda f_q}(t)=\begin{cases}\dot x_{\lambda f_q}(0)t+\int_0^t(t-\tau)f_q(\tau)d\tau & \text{if } t\in[0,t_1],\\ B_1\dot x_{\lambda f_q}(0)(t-t_1)+\int_{t_1}^t(t-\tau)f_q(\tau)d\tau\\ +B_1(t-t_1)\int_0^{t_1}f_q(\tau)d\tau & \text{if } t\in(t_1,t_2]\\ \prod_{l=1}^i B_l\dot x_{\lambda f_q}(0)(t-t_i)+\int_{t_i}^t(t-\tau)f_q(\tau)d\tau\\ +\sum_{r=1}^i\prod_{l=r}^i B_l(t-t_i)\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau+\prod_{k=1}^i A_k\dot x_{\lambda f_q}(0)t_1\\ +\prod_{k=1}^i A_k\int_0^{t_1}(t_1-\tau)f_q(\tau)d\tau\\ +\sum_{j=2}^i\prod_{k=j}^i A_k\Big[\prod_{l=1}^{j-1}B_l\dot x_{\lambda f_q}(0)(t_j-t_{j-1})\\ +\int_{t_{j-1}}^{t_j}(t_j-\tau)f_q(\tau)d\tau\\ +\sum_{r=1}^{k-1}\prod_{l=r}^{k-1}B_l(t_j-t_{j-1})\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\Big]\\ \quad\text{if } t\in(t_i,t_{i+1}],\ 2\le i\le p+1\end{cases}$$

with

$$\dot x_{\lambda f_q}(0)=-C^{-1}\Big(\int_{t_1}^T(T-\tau)f_q(\tau)d\tau+B_1(T-t_1)\int_0^{t_1}f_q(\tau)d\tau\Big) \tag{4.9}$$

if $p=1$, and

$$\begin{aligned}\dot x_{\lambda f_q}(0)=-C^{-1}\Big(&\int_{t_p}^T(T-\tau)f_q(\tau)d\tau+\sum_{r=1}^p\prod_{l=r}^p B_l(T-t_p)\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\\ &+\prod_{k=1}^p A_k\int_0^{t_1}(t_1-\tau)f_q(\tau)d\tau+\sum_{j=2}^p\prod_{k=j}^p A_k\Big[\int_{t_{j-1}}^{t_j}(t_j-\tau)f_q(\tau)d\tau\\ &+\sum_{r=1}^{k-1}\prod_{l=r}^{k-1}B_l(t_j-t_{j-1})\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\Big]\Big)\end{aligned} \tag{4.10}$$

if $p\ge 2$. Therefore

$$\mathfrak{T}(q,\lambda)=\{x_{\lambda f_q}: f_q \text{ is a selection of } F(\cdot,q(\cdot),\dot q(\cdot))\}\ne\emptyset.$$

Given $x_1,x_2\in\mathfrak{T}(q,\lambda)$, there exist measurable selections f_q^1,f_q^2 of $F(\cdot,q(\cdot),\dot q(\cdot))$ such that $x_1=x_{\lambda f_q^1}$ and $x_2=x_{\lambda f_q^2}$. Since the right-hand side F has convex values, it holds that, for any $c\in[0,1], cf_q^1+(1-c)f_q^2$ is a measurable selection of $F(\cdot,q(\cdot),\dot q(\cdot))$ as well. The linearity of both the equation and of the impulses yields that $cx_1+(1-c)x_2=x_{cf_q^1+(1-c)f_q^2}$, i.e. that the set of solutions of problem

(4.6) is convex, for each $(q,\lambda)\in Q\times[0,1]$. Therefore, assumptions (i) and (ii) in Proposition 2.4 are satisfied.

Condition (iii) follows immediately from the fact that $0\in K$ and that, for $\lambda=0$, the associated problem has only the trivial solution, see assumption (ii).

Let $x_{\lambda f_q}$ be the solution of (4.7). Then $|x_{\lambda f_q}(0)|=0$. Moreover, according to assumption (i) and formulas (4.9) and (4.10),

$$|\dot x_{\lambda f_q}(0)|\le \|C^{-1}\|\Big[\beta(2B)\frac{1}{2}T^2+T^2\|B_1\|\beta(2B)\Big]=T^2\|C^{-1}\|\cdot\beta(2B)\Big[\frac{1}{2}+\|B_1\|\Big]$$

if $p=1$ and

$$\begin{aligned}|\dot x_{\lambda f_q}(0)|\le \|C^{-1}\|\Big[&\frac{1}{2}T^2\beta(2B)+T^2\prod_{l=1}^{p}\|B_l\|\cdot\beta(2B)\\&+\frac{1}{2}T^2\prod_{k=1}^{p}\|A_k\|\beta(2B)+T^2\prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\|\cdot\beta(2B)\Big]\\=T^2\|C^{-1}\|\cdot\beta(2B)\Big[&\frac{1}{2}+\prod_{l=1}^{p}\|B_l\|\\&+\prod_{k=1}^{p}\|A_k\|+\prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\|\Big]\end{aligned}$$

if $p\ge 2$. Therefore there exists a constant M_1 such that $|\dot x(0)|\le M_1$, for all solutions x of (4.6). Hence, condition (iv) in Proposition 2.4 is satisfied.

Let us assume that $q_*\in Q$ is, for some $\lambda\in[0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot,\lambda)$. We will show now that q_* can not lay in ∂Q.

At first, let us investigate the case when $\lambda=0$. Then (4.6) transforms into (4.3) which has only the trivial solution. Therefore, for $\lambda=0$, it holds that $q_*\equiv 0$ which lays in $Int\ Q$. Hence, if $\lambda=0$, condition (v) in Proposition 2.4 is satisfied.

Secondly, let us assume that $\lambda\in(0,1)$. If q_* belongs to ∂Q, then there exists $t_0\in[0,T]$ such that $q_*(t_0)\in\partial K$ or $|\dot q_*(t_0)|=2B$. Since, for a.a. $t\in[0,T]$, we have

$$|\ddot q_*(t)|=\lambda|F(t,q_*(t),\dot q_*(t))|\le\beta(|\dot q_*(t)|)$$

and $|q_*(t)|\le R$, for every $t\in[0,T]$, Proposition 2.5 implies that $|\dot q_*(t)|\le B<2B$, for every $t\in[0,T]$. Hence, $q(t_0)\in\partial K$, which is impossible, since, according to Remark 3.6, hypotheses (iii), (iv) and (v) guarantee that K is a bound set for (4.6), i.e. that $q_*(t)\in K$, for all $t\in[0,T]$. Thus $q_*\in Int\ Q$.

Therefore, condition (v) from Proposition 2.4 is satisfied, for all $\lambda\in[0,1]$, which completes the proof. $\square$

Remark 4.2. An easy example of impulses conditions guaranteeing assumption (ii) in Theorem 4.1 are the antiperiodic impulses, i.e. $A_i=B_i=-I$, for every $i=1,\dots,p$. It follows from the proof of Theorem 4.1 that for the fulfilment of assumption (ii), it is sufficient to prove the regularity of the matrix C defined in (4.8). For $p=1$, $C=(t_1-T)I$ which is obviously regular. Let us show that C is regular also when $p\ge 2$. If p is even, then $\prod_{k=j}^{p}(-I)\prod_{l=1}^{j-1}(-I)=\prod_{l=1}^{p}(-I)=I$. Hence

$$C=[T-t_p+t_1+\sum_{j=2}^{p}(t_j-t_{j-1})]I=TI$$

which is regular. It can be shown that a similar reasoning holds also in the case when p is odd.

Remark 4.3. When V is of class C^2, then, according to Corollary 3.5, condition (iv) in Theorem 4.1 is equivalent to requiring that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R)+2R)$,

$$\langle HV(x)v, v\rangle + \lambda\langle \nabla V(x), w\rangle > 0, \quad \text{for every } \lambda \in (0,1) \text{ and } w \in F(t,x,v). \tag{4.11}$$

Since the function $g(\lambda) = \lambda\langle \nabla V(x), w\rangle$ is monotone, (4.11) is then equivalent to the following two conditions

$$\langle HV(x)v, v\rangle \ge 0 \quad \text{and} \quad \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle \ge 0 \tag{4.12}$$

that do not depend on λ.

5. Application to the forced pendulum equation

Let us consider the forced (mathematical) pendulum equation with viscous damping and dry friction terms

$$\ddot{x} + e\dot{x} + b\sin x + f\operatorname{sgn}\dot{x} = h(t), \quad \text{for a.a. } t \in [0,\pi], \tag{5.1}$$

with antiperiodic impulses and Dirichlet boundary conditions

$$x(t_i^+) = -x(t_i), \quad i = 1,\dots,p, \tag{5.2}$$

$$\dot{x}(t_i^+) = -\dot{x}(t_i), \quad i = 1,\dots,p, \tag{5.3}$$

$$x(0) = x(\pi) = 0, \tag{5.4}$$

where e, b and f are real constants and $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = \pi$, $p \in \mathbb{N}$. The function $h : [0,\pi] \to \mathbb{R}$ plays the role of the forcing term and we assume that $h \in L^\infty([0,\pi],\mathbb{R})$.

The study of the pendulum equation (i.e. the case $b > 0$, $e = f = 0$) dates back to a century ago (see [22]), when it was shown that it is worth to consider Dirichlet boundary conditions since the symmetry of the equation implies that such solutions are related to periodic solutions. The mathematical pendulum equation (i.e. the case $b < 0$, $e = f = 0$) was considered for the first time in [19]. More recently, the pendulum equation was generalized introducing a non-zero viscous damping coefficient e or a non-zero friction coefficient f (see [5, 26] for more details about this topic). Let us mention also the paper [17], where an impulse problem is considered in the case $e = f = 0$.

Because the function $\operatorname{sgn} y$ is discontinuous at $y = 0$, we should consider Filippov solutions of (5.1) which can be identified as Carathéodory solutions of the inclusion

$$\ddot{x} + e\dot{x} + b\sin x \in h(t) - f\operatorname{Sgn}\dot{x}, \tag{5.5}$$

where

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0,\\ [-1,1], & \text{for } y = 0,\\ 1, & \text{for } y > 0.\end{cases}$$

Let us now consider the Dirichlet multivalued problem (5.5), (5.4) with impulse conditions (5.2), (5.3) and let us check that all the assumptions of Theorem 4.1 are satisfied.

To verify condition (i), let us define the continuous and non-decreasing function

$$\beta(d) = \|h\|_\infty + |e||d| + |b| + |f|, \quad \text{for all } d \in \mathbb{R}.$$

The function β obviously satisfies (4.1) and $F(t,c,d) = h(t) - ed - b\sin c - f\,\mathrm{Sgn}\,d$ satisfies (4.2), for all $t \in [0,\pi]$ and all $c,d \in \mathbb{R}$.

Assumption (ii) holds as well since, according to Remark 4.2, the associated homogeneous problem has only the trivial solution.

For verifying condition (iii), consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-k,k)$ with $k \in (0,\frac{\pi}{2}]$ which will be specified later and the C^2-function $V(x) = \frac{1}{2}(x^2 - k^2)$ that trivially satisfies conditions (H1) and (H2).

To check condition (4.4) (which takes in our case the form (4.12), according to Corollary 3.5 and Remark 4.3), since $\langle HV(x)v, v\rangle = v^2$ is obviously non-negative, it is sufficient to verify that

$$\begin{aligned} &v^2 + x\big(h(t) - ev - b\sin x - f\,\mathrm{Sgn}\,v\big) \\ &= v^2 - exv + xh(t) - bx\sin x - fx\,\mathrm{Sgn}\,v \subset (0,\infty), \end{aligned} \tag{5.6}$$

for every $t \in (0,\pi), v \in \mathbb{R}$ and $x \in \mathbb{R}$ with $k - \varepsilon \le |x| \le k$.

(1) If $x = k$, then (5.6) becomes

$$v^2 - ekv + kh(t) - bk\sin k - fk\,\mathrm{Sgn}\,v \subset (0,\infty), \tag{5.7}$$

for every $t \in (0,\pi)$ and $v \in \mathbb{R}$. Since $k > 0$,

$$kh(t) \ge k \inf_{t\in(0,\pi)} h(t), \text{ for all } t \in (0,\pi),$$

and so condition (5.7) holds if

$$v^2 - ekv + k \inf_{t\in(0,\pi)} h(t) - bk\sin k - fk\,\mathrm{Sgn}\,v \subset (0,\infty), \forall v \in \mathbb{R}. \tag{5.8}$$

(a) If $v = 0$, then (5.8) takes the form

$$k \inf_{t\in(0,\pi)} h(t) - bk\sin k - fks > 0$$

for every $s \in [-1,1]$. This is equivalent to

$$\inf_{t\in(0,\pi)} h(t) > b\sin k + |f|, \tag{5.9}$$

since $\max_{s\in[-1,1]} fs = |f|$.

(b) If $v > 0$, then (5.8) takes the form

$$v^2 - ekv + k \inf_{t\in(0,\pi)} h(t) - bk\sin k - fk > 0. \tag{5.10}$$

If we define the function $g : [0,\infty) \to \mathbb{R}$ by $g(v) = v^2 - ekv + k\inf_{t\in(0,\pi)} h(t) - bk\sin k - fk$, then $g(0) > 0$, according to (5.9), and the minimum of g is achieved at the point $\bar{v} = \frac{ek}{2}$. Therefore, the inequality (5.10) holds if (5.9) is satisfied in case of $e \le 0$ and if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + f, \quad \text{for } e > 0.$$

Summing up, inequality (5.10) holds if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + f.$$

(c) If $v < 0$, then (5.8) takes the form

$$v^2 - ekv + k \inf_{t\in(0,\pi)} h(t) - bk\sin k + fk > 0. \tag{5.11}$$

In the same way as before, it is possible to obtain that (5.11) holds if

$$\begin{gathered} \inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k - f, \quad \text{for } e < 0, \\ \inf_{t\in(0,\pi)} h(t) > b\sin k - f, \text{ for } e \geq 0. \end{gathered}$$

Summing up, (5.6) holds, for $x = k$, if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + |f|. \tag{5.12}$$

(2) If $x = -k$, then (5.6) becomes

$$v^2 + ekv - kh(t) - bk\sin k + fk\operatorname{Sgn} v \subset (0,\infty), \quad \text{for every } t \in (0,\pi) \text{ and } v \in \mathbb{R}$$

and analogously as in the case $x = k$, we obtain that (5.6) holds for $x = -k$ if

$$\sup_{t\in(0,\pi)} h(t) < -\frac{e^2k}{4} - b\sin k - |f|. \tag{5.13}$$

Therefore, (5.6) holds, for all $t \in (0,\pi), v \in \mathbb{R}$ and $x \in \mathbb{R}$ with $k - \varepsilon \leq |x| \leq k$, for some $\varepsilon > 0$ sufficiently small, (due to the continuity and the inequalities (5.12) and (5.13)) if

$$\frac{e^2k}{4} + b\sin k + |f| < \inf_{t\in(0,\pi)} h(t) \leq \sup_{t\in(0,\pi)} h(t) < -\frac{e^2k}{4} - b\sin k - |f|,$$

which, in particular, implies that $\frac{e^2k}{4} + b\sin k + |f| < 0$.

Since $\nabla V(x) = \dot{V}(x) = x$ and $HV(x) = \ddot{V}(x) = 1$, for all $x \in \mathbb{R}$, condition (v) trivially holds.

In conclusion, assuming that $k \in (0,\pi/2]$ is such that

$$\frac{e^2k}{4} + b\sin k + |f| < 0 \tag{5.14}$$

and that

$$|h(t)| < -\frac{e^2k}{4} - b\sin k - |f|, \text{ for all } t \in (0,\pi),$$

then all the assumptions of Theorem 4.1 are satisfied, and problem (5.1) admits a solution laying in $[-k,k]$. We stress that such solution is not trivial, according to the presence of the forcing term. Notice moreover that condition (5.14) is consistent, since it never holds for small k, and therefore (5.6) is not satisfied in the whole corresponding set $\overline{K}$ but only in some neighborhood of its boundary, as required.

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Martina Pavlačková
Dept. of Math. Analysis and Appl. of Mathematics, Fac. of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic
E-mail address: `martina.pavlackova@upol.cz`

Valentina Taddei
Dept. of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, Via G. Amendola, 2 - pad. Morselli, I-42122 Reggio Emilia, Italy
E-mail address: `valentina.taddei@unimore.it`