Shear deformable beams in contact with an elastic half-plane

F.O. Falope [1,2], L. Lanzoni [1,2] and E. Radi [3]
[1] *DIEF – Department of Engineering Enzo Ferrari", University of Modena e Reggio Emilia, 41121, Modena, Italy*
[2] *DESD Department of Economics, Science, and Law, University of Repubblic of San Marino, 47890*
[3] *DISMI – Department of Sciences and Methods for Engineering, Università degli Studi di Modena e Reggio Emilia, 42122, Reggio Emilia, Italy*

Summary: The present work deals with the contact problem of a Timoshenko beam bonded to an elastic semi-infinite substrate under different loading conditions. The analysis allows investigating the effects induced by shear compliance of the beam, the stress intensity factors ad the beam edges as well as the singular nature of the interfacial stresses.

Formulation of the Problem

Let to consider a shear deformable beam of length $2a$ with a cross section area $A = 1{\cdot}h$, in contact with a homogeneous elastic half-space. The cover beam element is subjected to static axial (N_1, N_2) and vertical (T_1, T_2) concentrated forces and couples (M_1, M_2) acting at the beam edges, as reported in Figure 1 for the case of symmetric external concentrated loads.

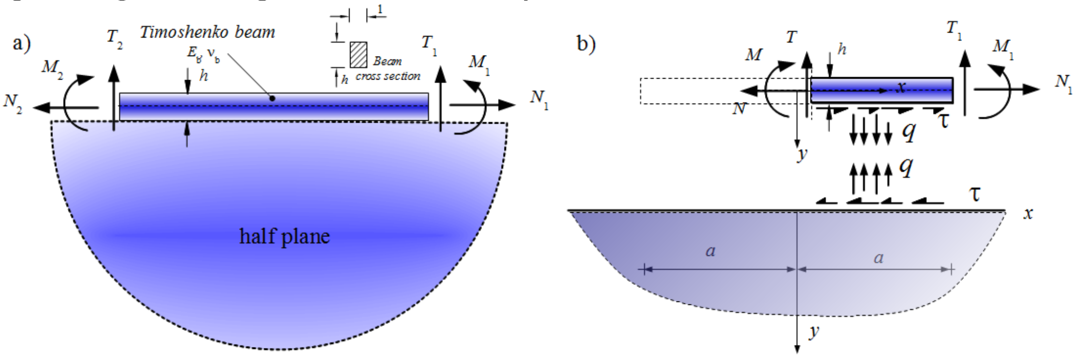

Figure 1: a) Timoshenko beam bonded to an elastic half plane subjected to external loads; b) shear and peel stresses acting within the contact region.

Then, the strains at the lower side of the beam, for $|x| < a$, read to,

$$u'_b(x)=\frac{N_1}{E_bA}+\frac{1}{E_bA}\int_x^a \tau(s)\,ds+\frac{h}{2E_bI}[M_1+T_1(a-x)]+\frac{h}{4E_bI}\int_x^a h\tau(s)+2q(s)(x-s)\,ds,$$

$$v'_b(x)=-\frac{M_1x}{E_bI}-\frac{T_1}{2E_bI}(2ax-x^2)-\frac{h}{2E_bI}x\int_x^a \tau(s)\,ds-\frac{h}{2E_bI}\int_0^x s\tau(s)\,ds+\frac{1}{2E_bI}\int_0^a s^2q(s)\,ds\ldots \tag{1}$$

where $-\frac{1}{2E_bI}\int_x^a (s-x)^2q(s)\,ds-\phi_p-\frac{\chi T_1}{G_bA}+\frac{\chi T_1}{G_bA}\int_x^a q(s)\,ds$, $E_b=E_0$ or $E_0/(1-v^2_b)$ denotes the Young modulus of the beam in plane stress or plain strain conditions, respectively, v_b is the Poisson ratio of the beam, I represents the moment of inertia of the beam cross section, G_b is the shear modulus of the beam, whereas χ denotes the dimensionless shear factor. In eq(1), $v_b(x)$ denotes the transverse deflection of the beam along the y axis, $u_b(x,y)$ is the axial displacement of the beam cross section at the interface, i.e. $u_b(x,y)|_{y=0}$, and φ_p denotes a constant of integration (i.e. the rotation of the beam cross section at x=0, positive if counterclockwise), to be determined.
The half-plane strains at the contact domain are known in closed form [2],

a) Corresponding author. Email: *federicooyedeji.falope@unimore.it*

$$u_s' = -\frac{2}{E_s \pi}\int_{-a}^{a}\frac{\tau(\xi)}{\xi - x}d\xi + \left(\frac{2}{E_s} - \frac{1}{2G_s}\right)q(x),$$
$$v_s' = -\frac{2}{E_s \pi}\int_{-a}^{a}\frac{q(\xi)}{\xi - x}d\xi - \left(\frac{2}{E_s} - \frac{1}{2G_s}\right)\tau(x), \qquad \text{for } |x| < a. \tag{2}$$

being E_s and v_s the Young modulus and the is the Poisson ratio of the half plane, respectively, and $G_s = E_s/2(1+\nu_s)$ its shear modulus.

The strain compatibility conditions between the beam and the half plane require:

$$u_s' = u_b' \qquad v_s' = v_b', \qquad \text{for } |x| < a. \tag{3}$$

System (3) cannot be solved in closed form. However, an approximate solution can be straightforwardly found by expanding the unknown shear and peeling stresses in series of Jacobi orthogonal polynomials, namely [3]

$$\tau(x) = (a+x)^s (a-x)^s \sum_n C_n P_n(x/a), \qquad q(x) = (a+x)^s (a-x)^s \sum_n D_n P_n(x/a), \tag{4}$$

being $P_n(x)$ the Jacobi polynomial of order n and the index s of the polynomials denotes the singular strength of the interfacial stresses at the end of the contact region, i.e. at $x = \pm a$. The solution of system (3) is imposed at selected collocation points, thus founding the coefficients C_n and D_n.

Results

As an example, the shear and peel stresses of a Timoshenko beam under axial loads are reported in Figure 2 for some values of the parameter $\gamma = E_s\chi/G_b$. The case of an Euler-Bernoulli beam can be recovered as the limiting case of a Timoshenko beam having a vanishing value of γ.

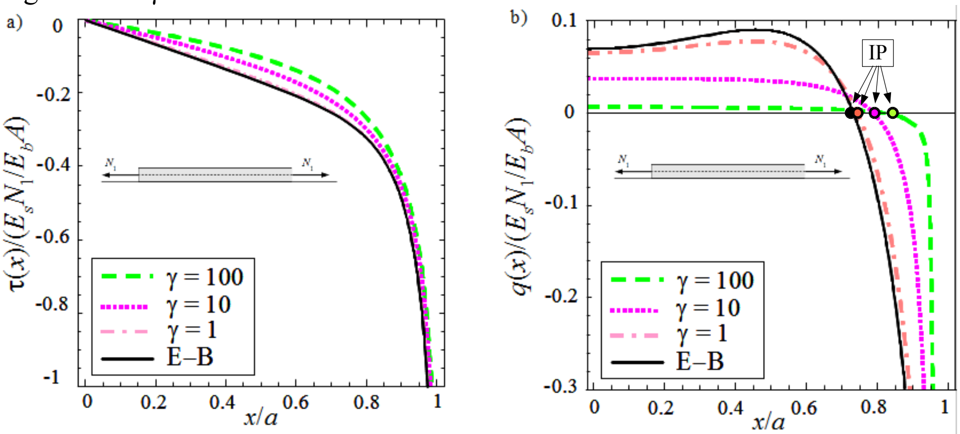

Figure 2. a) Dimensionless interfacial a) shear and b) peel stress of a Timoshenko beam subjected to two axial forces acting at the beam ends varying the parameter g.

Conclusion

The analysis of a Timoshenko beam in contact with an elastic half plane under static loads has been performed in the present work. The investigation allows to evaluate the effects induced by the shear compliance of the beam on the behaviour of the beam-substrate system. The special case of a membrane bonded to a half plane or a beam in frictionless contact with the underlying support can be retrieved by imposing only the condition $(3)_1$ or $(3)_2$, respectively. The inversion point (IP) position, the point where the peeling components inverts its sign, is moved closer to the beam middle section with the increase of shear deformation as well as the inverse of the beam slenderness.

References

[1] Cowper G.R : The shear coefficients in Timoshenko's beam theory. *ASME J. Appl. Mech* **13(3)**:335-340, 1966.
[2] Johnson K.L : Contact Mechanics. Cambridge University Press, 1985.
[3] Erdogan F., Gupta G. D., Cook, T. S: Numerical solution of singular integral equations. In "Mechanics of fracture" Vol.1: Methods of analysis and solutions of crack problems, 368-425, 1973.

[a] Corresponding author. Email: *federicooyedeji.falope@unimore.it*