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ANALYSIS OF STRESS SINGULARITIES IN THIN COATINGS BONDED TO A SEMI-INFINITE ELASTIC SUBSTRATE

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ABSTRACT

In the present work, the interfacial stress arising between thin structures bonded to a 2D elastic substrate has been investigated. Chebyshev polynomials have been adopted to approximate the shear stress occurring across the contact region. Perfect adhesion has been assumed among the coating structures and the underlying substrate, leading to a singular integral equation which has been reduced to an algebraic system. Thin bonded structures having several geometric configurations under different load conditions have been considered. In particular, the stress concentration in the neighbourhood of the coating edges and around the points of load application has been evaluated in detail.

Keywords: interfacial stress, stress singularity, Chebyshev polynomials, integral equation.

1. Introduction

A variety of devices employed in high-tech industries, mainly in microelectronics, electrochemistry, semiconductors and optical electronics, involves thin films. Micro-electro-mechanical systems (MEMS), biomedical components, chemical reactors, integrated circuit, solar cells, flat panel displays and protection systems are typically based on thin film coatings technology. In particular, flip-chip microprocessor packages, high power tunable microwave devices, disk resonators, bulk ceramics-based phase shifters, coplanar-plate capacitors and varactors represent some modern systems involving coatings deposit and thin film layers (Gevorgian, 2010).

In most engineering applications, thin films and coatings have not a primary structural function; nonetheless, the most part of thin film-based devices are often subject to high stress levels and, in turn, to damaging phenomena like fractures and excessive permanent deformations which can compromise the usefulness and functionality of this kind of components (e.g. Freund, 2000; Hsueh, 2002). Thus, the design of thin film-substrate systems requires the accurate knowledge of the mechanical interaction between the bonded structures and the underlying substrate, with particular reference to the interfacial stress arising across the contact region (Freund et al., 2003; Shen, 2010). In particular, the proper evaluation of the local effects, like stress and strain localizations, becomes essential in order to predict the functionality and performances of structures coated by thin films.

The contact problem of thin elastic structures like ribs, bars and rods welded to an elastic substrate was largely investigated by many researchers. In particular, the work of Melan (1932) is one of the first studies devoted to evaluate the mechanical stiffener-substrate interaction. Melan analyzed the contact problem of an infinite bar loaded by a longitudinal force applied at its midpoint and bonded to an infinite or semi-infinite plate. He obtained a close form solution for the interfacial shear stress and the axial force acting in the bar, founding a singularity of the interfacial stress of logarithmic kind in the neighbouring of the point of load application. The problem of an infinite flat sheet axially loaded at the boundary and stiffened by a finite cover was considered also by Reissner (1940). Buell (1948) studied a semi-infinite stiffener bonded to a semi-infinite plate through a proper series expansion of the Airy stress function for the cover. Koiter (1955) solved the problem of the load diffusion in an infinite elastic layer stiffened by a semi-infinite sheet axially loaded at one end by using the Green function of the infinite or semi-infinite elastic space. He solved the singular integral equation of Prandtl type governing the problem by using Mellin transformation and Laplace transforms, founding an interfacial stress characterized by a square-root singularity in the neighbourhood of the sheet edge. Brown (1957) studied the same problem considering various load conditions by using complex potentials. Rybakov (1977) analyzed the frictionless contact problem of an infinite elastic rod periodically riveted in some points to a semi-infinite plate by adopting a complex potentials representation for the displacement field and the internal forces in the stiffener. In this reference, the forces acting in the rivets are determined by solving a system of linear algebraic equations. Later, the same author solved the problem by taking into account a fracture occurring in the stiffener (Rybakov, 1982).

The study of the coating-substrate system was extended to bonded structures of finite dimension also. For instance, Benscoter (1949) studied the contact problem of a finite strip welded to an infinite shell by using the stress field of an elastic half plane loaded by a horizontal force acting at its free surface. He evaluated the magnitude of the axial force in the cover by solving numerically a singular integral equation. Arutiunian (1968) solved the problem adopting a series representation for the unknown interfacial stress, solving the integral equation by means of contour integrals in complex domain. Later, a simpler method was proposed by Morar and Popov (1971) and Erdogan and Gupta (1971), who solved the problem by using series of Chebyshev polynomials to approximate the interfacial shear stress.

Successively, various methods have been proposed to evaluate the stress and strain field in single or multi-coated systems. For example, Hu (1979) adopted a finite difference technique to evaluate the stress transferred to the surface of an elastic half plane by a bonded film. A proper refinement of the grid elements was used by the author in order to correctly estimate the interfacial stress near the film edge. Conventional Finite Element (FE) programs were used by Djabella et al. (1993) to evaluate contact stress in multi-layered systems subject to a specified pressure distribution acting on a portion of the boundary. Jain et al. (1995) investigated the stress field in an elastic layer induced by a bonded strip-like film via FE models, founding the distribution of the shear and normal stress in the substrate and the axial stress in the overlying film. A numerical analysis based on a proper Boundary Element (BE) method has been used by Takahashi et al. (1997) to investigate the singular nature of the interfacial stress in film-substrate systems subject to thermoelastic strain. The authors analyzed the interfacial stress distribution of a coating made of different superconducting ceramics bonded to a ceramic or metallic substrate.

By means of the elementary theory of bent plates, Zhang et al., (2005) evaluated the stress distribution in multilayered systems under thermal strain, comparing the results with the numerical solution provided by FE simulations. However, the authors recognized that their model can not be used to properly estimate the strength of interfacial stress singularities.

In the present paper, some analytical models based on the use of orthogonal Chebyshev polynomials are proposed to study the contact problem of thin structures bonded to a 2D elastic substrate under several loading conditions. Perfect adhesion is supposed between the bonded structure and the underlying half plane. This condition leads to a singular integral equation which can be reduced to an infinite system of linear algebraic equations. The solution of the algebraic system allows the evaluation of the displacement and stress field of the system, with particular reference to the strength of stress singularities in the neighbouring of the points of load application and at the edges of the coating. The mechanical interaction between the bonded structure and the substrate is found governing by a rigidity parameter involving the axial stiffness of the cover and the mechanical parameters of the underlying substrate.

The paper is organized as follows. The problem of a coating bonded to the substrate and subject to thermal load or, similarly, two opposite axial forces acting at the coating ends is considered in Section 2. The film-substrate interaction is discussed varying the rigidity parameter of the system. The comparison with the Koiter solution in terms of strength of stress singularity is given also. A coating with variable thickness under thermal load is solved in Section 3. Interfacial stress, axial displacement and axial force in the coating are reported for a given geometric configuration. The contact problem of a coating subject to a single axial force applied at one edge is solved and discussed in Section 4 for several values of the rigidity parameter, founding a large agreement with respect the strength of stress singularities predicted by the Koiter solution. Section 5 concerns the study of a film partially detached from the underlying half space. For a certain location of the interior detachment, a comparison with the results obtained by solving the contact problem of a perfectly bonded film is reported. The interfacial stress singularities in the neighbourhood of the ends of the contact region have been investigated. Finally, the problem of a coating subject to an interior axial force is treated in Section 6. A comparison between the obtained results with the Melan solution is given and discussed.

2. Film subject to two opposite axial forces or a uniform thermal variation

The case of an elastic film of total length $2a$ and thickness δ welded to an elastic half-plane and subject to a uniform thermal variation or, similarly, loaded by two opposite axial forces applied at its ends, as reported in Fig. 2.1, is considered in the present section. A Cartesian coordinate system (O, x, y) placed at the interface between the film and the underlying half plane is adopted, with the origin located at the midpoint of the film. Linear elastic and isotropic behaviour is assumed for both the half plane and coating. Generalized plane stress or plane strain condition can be considered; nonetheless, in the present study, a plane strain regime is assumed. In that follows, E_0, ν_0 and α_0 represent the Young modulus, the Poisson ratio and the coefficient of thermal expansion of the film, respectively, whereas E and ν are the Young modulus and Poisson ratio of the half plane.

The contact region $[-a, a]$ equals the length of the film. The thickness δ of the coating is assumed very thin, making it possible to neglect its flexural stiffness and, in turn, to ignore the effects produced by the vertical component of stress. Thus, only interfacial shear stress $\tau(x)$ arises across the contact region.

The strain of the coating $u_{0,x}(x)$ takes the form:

$$u_{0,x}(x, 0) = \frac{1-\nu_0^2}{E_0\delta}\int_x^a \tau(\xi)\, d\xi - \Delta\varepsilon, \qquad \text{for } |x| \le a, \tag{2.1}$$

where $\Delta\varepsilon = (1 + \nu_0)\, \alpha_0\, \Delta T$ or $\Delta\varepsilon = -F(1 - \nu_0^2)/(E_0\, \delta)$ represents the constant component of horizontal strain of the coating due to thermal load or two opposite axial forces, respectively.

The Green function (Cerruti solution) for a homogeneous half space loaded by a tangential force F_x acting on its free surface is known in closed form (Kachanov et al., 2003):

$$u(x) = -F_x \frac{2(1-\nu^2)}{E\,\pi} \ln|x|, \quad v(x) = -F_x \frac{(1-2\nu)(1+\nu)}{2E} \operatorname{sign}(x). \tag{2.2}$$

By using expression $(2.2)_1$, the following expression for the horizontal strain of the half plane $u_{,x}(x)$ is obtained by superposition (Johnson, 1985; Barber, 2002):

$$u_{,x}(x) = -\frac{2(1-\nu^2)}{E\,\pi}\int_{-a}^{a}\frac{\tau(\xi)}{\xi - x}\,\mathrm{d}\xi\,. \tag{2.3}$$

The assumption of perfect adhesion between the coating and the half space leads to the following condition:

$$u_{0,x}(x) = u_{,x}(x), \qquad \text{for } |x| \leq a, \tag{2.4}$$

whereas the equilibrium condition of the free surface of the half plain leads to the following boundary condition:

$$\tau(x) = 0, \qquad \text{for } |x| > a. \tag{2.5}$$

The substitution of the expressions (2.1), (2.3) in equation (2.4) yields the following singular integral equation with Cauchy kernel for the unknown interfacial shear stress:

$$\frac{1-\nu_0^2}{E_0\delta}\int_{x}^{a}\tau(\xi)\,\mathrm{d}\xi + \frac{2(1-\nu^2)}{E\,\pi}\int_{-a}^{a}\frac{\tau(\xi)}{\xi - x}\,\mathrm{d}\xi = \Delta\varepsilon\,, \qquad \text{for } |x| \leq a. \tag{2.6}$$

In Arutiunian (1968), equation (2.6) is solved through a technique based upon contour integrals in complex domain, as reported in Muskhelishvili (1952), by assuming the unknown function $\tau(\xi)$ in power series.

In the following, a simpler approach is adopted by approximating the contact shear stress through series of orthogonal Chebyshev polynomials displaying a square root singularity at the film edges. The same procedure was recently proposed by Villaggio (2003) to investigate the brittle detachment of a stiffener bonded to an elastic half plane, founding the critical load that causes the crack propagation by adopting the Griffith criterion for fracture.

Thus, the following expression for the interfacial shear stress is adopted:

$$\tau(x) = \frac{E}{\sqrt{1-(x/a)^2}}\sum_{n=1,3,5}^{\infty} C_n\,T_n(x/a)\,, \qquad \text{for } |x| \leq a, \tag{2.7}$$

$T_n(x/a)$ being the orthogonal Chebyshev polynomials of first kind of order n.

Note that equation (2.7) satisfies the equilibrium condition of the film along x axis:

$$\int_{-a}^{a} \tau(x)\,\mathrm{d}x = 0, \tag{2.8}$$

holding the relation (A1) (see Appendix).

Then, by introducing (2.7) in the equation (2.6), after the replacement $t = x/a$, one obtains:

$$\sum_{n=1}^{\infty} C_n \left\{ 2\,(1-\nu^2) \; + \; \frac{\gamma}{n}\sqrt{1-t^2} \right\} U_{n-1}(t) = \Delta\varepsilon, \quad \text{for } |t| \le 1, \tag{2.9}$$

where $U_n(t)$ indicate the Chebyshev polynomials of second kind, and $\gamma = \dfrac{a\,E(1-\nu_0^2)}{E_0\,\delta}$. The dimensionless parameter γ can be interpreted as the stiffness of the coating with respect that of the half plane and, as showed in the following, it will govern the mechanical response of the film-substrate system.

In order to obtain equation (2.9), identities (A2)-(A3) reported in Appendix have been used.

Equation (2.9) can be solved by applying the Bubnov-Galerkin method, which consists to integrate the equation in the [−1, 1] domain, after multiplication by $U_{n-1}(t)$ (Grigolyuk and Tolkachev, 1987). Then, by using the properties of Chebyshev polynomials, equation (2.9) reduces to the following infinite system of algebraic equations for the unknown coefficients C_n:

$$\sum_{n=1,3,}^{\infty} C_n \left\{ \frac{4}{\pi}\,(1-\nu^2)\,B_{nm} + \frac{\gamma}{m}\,\delta_{nm} \right\} = \frac{2\,\Delta\varepsilon}{\pi} A_m\,, \qquad m = 1, 3, 5, \ldots, \infty, \tag{2.10}$$

δ_{nm} being the Kronecker delta, and the expressions of coefficients B_{nm} and A_m have been reported in Appendix (see equations (A4) and (A5), respectively).

The series in (2.10) is truncated to N terms, ad the system is solved finding the coefficients C_n ($n = 1, 3, \ldots, N-1$). The surface displacements for the half plane can be evaluated by superposition, integrating the Green functions for a distribution of tangential forces:

$$u(t) = -2a\,(1-\nu^2)\sum_{n=1,3,}^{N} \frac{C_n}{n} T_n(t)\,, \tag{2.11}$$

$$v(t) = a\,(1-2\nu)(1+\nu)\sum_{n=1,3,}^{N} \frac{C_n}{n} U_{n-1}(t)\,\sqrt{1-t^2}\,, \tag{2.12}$$

where relation (A6) has been used (see Appendix).

The longitudinal stress in the coating $\sigma_0(x)$ is obtained by integrating the contact shear stress in the region $[x, a]$. The boundary condition of vanishing tractions at the half plane surface along the y direction requires $\sigma_{yy}(x, 0) = 0$. By using the constitutive relations it follows that the horizontal stress on the half plane surface $\sigma_{xx}(x, 0)$ can be expressed as a function of the horizontal component of the elastic strain only, namely $\sigma_{xx}(x,0) = [E/(1-\nu^2)]\,\varepsilon_0^{e}(x)$. So ($|t| \leq 1$):

$$\sigma_0(t) = E_0\,\gamma \sum_{n=1,3}^{N} \frac{C_n}{n} U_{n-1}(t)\,\sqrt{1-t^2}\,, \qquad \sigma_{xx}(t, 0) = -2\,E \sum_{n=1,3}^{N} C_n\,U_{n-1}(t)\,. \tag{2.13}$$

As predicted by the equation (2.7), the shear stress exhibits a square root singularity at the edges of the film. The singular behaviour of the interfacial stress at the film ends ($x = \pm a$) can be properly investigated by means of the shear stress singularity factor K_{II}:

$$K_{\mathrm{II}}(\pm a) = \lim_{x \to \pm a} \sqrt{2\pi(a \mp x)}\;\tau(x) = \pm E\sqrt{a\,\pi}\sum_{n=1,3}^{N} C_n\,. \tag{2.14}$$

2.1 Results

In the present section, the mechanical response of a coating-substrate system subject to a thermal load $-\Delta T$ is discussed. A number of terms equals to 40 has been considered in the series expansion, leading to an accurate evaluation of the mechanical response of the system. The dimensionless shear stress $\tau(x)/(E\,\Delta\varepsilon)$ and axial displacement $u(x)/(a\,\Delta\varepsilon)$ are shown in

Fig. 3.2a) and b), respectively, for different values of the parameter γ.

As γ increases, both interfacial shear stress and horizontal displacement decrease, due to the diminishing of the coating stiffness and, in turn, the effects transmitted to the half plane. In fact, γ is proportional to the ratio E/E_0, and it can be interpreted as a measure of compliance of the cover with respect to that of the half space. Moreover, for high values of γ, the effects of the coating are significant at the edges of the coating only; conversely, for small values of γ, the contact shear stress and the axial displacement become considerable near the middle of the cover also. It should be noted that the relative displacement among the edges of the coating depends on coefficient C_1 only, namely:

$$[u(a)-u(-a)]/a = \gamma\pi C_1/2-2\Delta\varepsilon. \tag{2.15}$$

Moreover, it is worth noting that in the limiting case of $\gamma \to 0$ (very stiff cover), the horizontal displacement tends to display an almost linear trend, and the coating behaves like a truss subjected to thermal load only, since in this case the coating is much stiffer than the substrate which has no sensible constraint effects on the coating. This is confirmed by the asymptotic value of the unknown coefficients of system (2.10):

$$C_1 = \Delta\varepsilon/2, \quad \text{and} \quad C_3, C_5, \ldots, C_N = 0, \qquad \text{for } \gamma \to 0. \tag{2.16}$$

Thus, from eq. (2.11), the total axial displacement of the coating $u(a)-u(-a)$ turns out to be $2a(1-\nu^2)\Delta\varepsilon$, as expected for a rigid film subject to an axial uniform strain $\Delta\varepsilon$, whereas the interfacial shear stress can be approximate as

$$\tau(x)/(E\Delta\varepsilon) = \frac{1}{2\sqrt{1-(x/a)^2}}, \quad \text{for } \gamma \to 0. \tag{2.17}$$

This behaviour is reflected in Figs. 2.3a-b also, where the dimensionless normal stress in the coating $\sigma_0(t)/(E_0\Delta\varepsilon)$ and the half plane surface $\sigma_{xx}(t, 0)/(E\Delta\varepsilon)$ is reported. It should be noted that, differently to the shear stress, the horizontal component of normal stress $\sigma_{xx}(x, 0)$ takes finite values in each point of the contact region.

The normalized shear stress singularity factors $K_{II}(a)/(E\,\Delta\varepsilon\,a^{0.5})$ are reported in Fig. 2.4a varying the parameter γ. As γ increases, the normalized K_{II} factors decrease due to the fact that the magnitude of the shear stress decreases, as showed by Fig. 2.2a. Conversely, for stiff coatings, the K_{II} factor increases, and for the limiting case of an inextensible film one finds:

$$K_{II}(\pm a) = \pm E\sqrt{a\,\pi}\;\frac{\Delta\varepsilon}{2}, \qquad \text{for } \gamma \to 0, \tag{2.18}$$

or, in dimensionless form, $K_{II}(\pm a)/(E\,a^{0.5}\Delta\varepsilon) = \pi^{0.5}/2 \cong 0.886227$.

As reported previously, the loading case of a thermal variation is similar to that of two opposite axial forces applied at the ends. This load condition resembles the situation of the problem of Koiter, who solved the contact problem of a semi-infinite rod bonded to a semi-infinite plate and loaded at the edge by an axial force F, founding the following expression for the axial stress $N(x)$ in the coating (Grigolyuk and Tolkachev, 1987, p.150):

$$\frac{N(x)}{F} = \left[1 - \sqrt{\frac{2\gamma x}{\pi}}\left(1 - \gamma x\left(0.25425 - 0.1061\ \mathrm{Log}(\gamma x)\right) + \gamma^2 x^2\left(0.008627 - \ldots\right)\right)\right], \tag{2.19}$$

where x denotes the longitudinal coordinate of the coating starting form the point of load application and the width of the rod-substrate system is taken unitary. By differentiating eq. (2.19) with respect x, the expression for the shear stress is obtained, from which, accordingly to (2.14), one finds:

$$K_{II}^*(\pm a) = E\,\Delta\varepsilon\sqrt{a/\gamma}\,/(1-\nu_0^2). \tag{2.20}$$

The strength of interfacial stress predicted by Koiter, in dimensionless form, i.e. $K_{II}^*(\pm a)/(E\,\Delta\varepsilon\,a^{0.5}/(1-\nu_0^2))$, is reported in Fig. 2.4a. As predicted by eq. (2.20), the singularity strength of the interfacial stress is monotonic decreasing with γ. As shown, a large agreement between the stress singularity factors $K_{II}(\pm a)$ and $K_{II}^*(\pm a)$ is found, confirming the square root singular behaviour of the Koiter solution. Thus, the Koiter solution can be used as a valid approximation to assess interfacial stress concentrations in coating-substrate systems provided that, approximately, $\gamma > 10$. Moreover, it is worth noting that, differently to the stress field in the neighbouring of an interfacial crack tip predicted by linear elastic fracture mechanics, here no oscillations of the shear and normal stress are found.

3. Film with variable thickness under thermal load

A coating of length $2a$ having variable thickness and bonded to a homogeneous isotropic half plane, as reported in Figure 3.1, is considered in the present section. On assumes that a uniform thermal variation ΔT acts on the system. The case of other loading conditions, like two opposite axial forces applied at the ends of the coating, can easily solved by following the same procedure here reported.

Similarly to a coating with uniform thickness, the total axial strain of the coating $u_{0,x}(x)$ under plain strain condition assumes the form:

$$u_{0,x}(x, 0) = \frac{1-\nu_0^2}{E_0\ \delta(x)} \int_x^a \tau(\xi)\ d\xi\ - \Delta\varepsilon, \qquad \text{for } |x| \le a_l + a_r, \tag{3.1}$$

where $\delta(x) = \delta_l$ for $-a \le x < a_l - a_r$ and $\delta(x) = \delta_r$ for $a_l - a_r \le x \le a$.

The horizontal strain for the half plane $u_{,x}(x)$ is given in Eq. (2.3).

The interfacial shear stress can be expressed as

$$\tau(x) = \frac{E}{\sqrt{1-(x/a)^2}} \sum_{n=1,2,}^{\infty} C_n\, T_n(x/a), \qquad \text{for } |x| \le a, \tag{3.2}$$

Note that, differently from eq. (2.7), expression (3.2) contains even terms also, due to the fact that symmetry condition with respect the y axis is lost because the variation of coating thickness. By following the same procedure reported in the previous section and making use of the expression (3.2), the compatibility condition (2.4) across the contact region leads to the following equation:

$$\sum_{n=1,2,}^{\infty} C_n \left\{ 2\,(1-\nu^2)\ +\ \frac{\gamma(t)}{n}\sqrt{1-t^2} \right\} U_{n-1}(t) = \Delta\varepsilon\,, \quad \text{for } |t| \le 1, \tag{3.3}$$

being $t = x/a$, and $\gamma(t) = \gamma_l = \dfrac{E\,a\,(1-\nu_0^2)}{E_0\delta_l}$ for $-1 \le t < (a_l - a_r)/a$, and $\gamma(t) = \gamma_r = \dfrac{E\,a\,(1-\nu_0^2)}{E_0\delta_r}$

for $(a_l - a_r)/a \le t \le 1$.

By applying the Bubnov-Galerkin method to Eq. (3.3) one finds:

$$\sum_{n=1,2,}^{\infty} C_n \left\{ 2\,(1-\nu^2)\,B_{nm} + \frac{\gamma_l}{n}\,D_{l\,nm} + \frac{\gamma_r}{n}\,D_{r\,nm} \right\} = \Delta\varepsilon\,A_m\,, \quad m = 1, 2, 3, \ldots \tag{3.4}$$

where $\beta = \dfrac{a_l - a_r}{a_l + a_r}$ and expressions of D_{lnm}, $D_{r\,mn}$ have been reported in equations (A7) of the Appendix.

The algebraic system (3.4) is solved for the unknown coefficients C_n, leading to the evaluation of the interfacial stress and, in turn, the stress and displacement fields of the system.

The horizontal components of stress and displacement of the coating and the half plane surface can be evaluated through equations (2.11-2.13), provided that also the even terms are included in the series.

Similarly to expression (2.14), the K_{II} factors are evaluated as follows:

$$K_{\mathrm{II}}(a) = E\sqrt{a\,\pi}\ \sum_{n=1,2,}^{N} C_n\,; \qquad K_{\mathrm{II}}(-a) = E\sqrt{a\,\pi}\ \sum_{n=1,2,}^{N} (-1)^n C_n\,. \tag{3.5}$$

It should be noted that, for $\beta = \pm 1$, the solution of the contact problem for a coating of uniform thickness with $\gamma = \gamma_l$ or $\gamma = \gamma_r$, respectively, is exactly retrieved.

3.1 Results

The results reported in the present section are related to a coating having variable thickness for which $\beta = 0$ and $\gamma_l = 10$. The dimensionless shear stress and the axial displacement are reported in Figs. 3.2a-b, respectively, for several values of γ_r. As reported in the figure, the shear stress drops in correspondence of thickness variation, except for $\gamma_r = 10$, since that for this value of γ_r a coating with uniform thickness is retrieved. The jump exhibited by the shear stress is due to the fact that the horizontal strain and, in turn, the normal stress in the film is discontinuous in the point of thickness variation, as showed by Fig. 3.3a. The peak of the shear stress assumes finite values, and its magnitude increases (in modulus) increasing the thickness variation of the film, as showed by Fig. 3.4. As expected, for large values of γ_r, the axial displacement is nearly linear in the stiffer part of the coating, whereas it assumes a non monotonic trend in the left side of the system, as showed in Fig. 3.2b. Note that the point of zero displacement moves toward the midpoint of the stiffer part of the film. The K_{II} factors at the edges of the film are reported in Fig. 3.3b varying γ_r. Similarly to the case of a coating of uniform thickness, a monotonic decreasing trend of the $K_{\mathrm{II}}(\pm a)$ factors with γ_r is found. As expected, the K_{II} factors calculated at the left edge of the film are not sensibly affected by variations of γ_r; in particular, for $\gamma_r > 20$ an almost constant value of $K_{\mathrm{II}}(-a)$ is obtained. Note that $K_{\mathrm{II}}(a) = K_{\mathrm{II}}(-a)$ for $\gamma_r = 10$.

4. Film subject to an axial force applied to a film end

In the present section, the contact problem of an elastic film bonded to a half-plane and loaded by an axial force applied at one edge, as reported in Fig. 4.1, is considered.
The horizontal component of strain of the coating $u_{0,x}(x)$ and half plane surface are given by equations (2.1) and (2.3), respectively, where $\Delta\varepsilon = 0$.

The interfacial shear stress is represented in the form:

$$\tau(x) = \frac{E}{\sqrt{1-(x/a)^2}} \sum_{n=0,1,2,}^{\infty} C_n\, T_n(x/a), \qquad \text{for } |x| \le a. \tag{4.1}$$

The equilibrium condition of the coating along the axial direction can be written as:

$$\int_{-a}^{a} \tau(x)\, \mathrm{d}x = F, \tag{4.2}$$

leading to the evaluation of coefficient C_0:

$$C_0 = \frac{F}{E\, a\, \pi} = \frac{F}{\gamma\, \pi\, E_0 \delta}. \tag{4.3}$$

The compatibility condition $u_{0,x}(x) = u_{,x}(x)$ inside the contact region $|x| \le a$ leads to the following equation ($t = x/a$)

$$\sum_{n=1,\,2,}^{\infty} C_n \left\{ 2\,(1-\nu^2) + \frac{\gamma}{n}\sqrt{1-t^2} \right\} U_{n-1}(t) = -\gamma\, \mathrm{C}_0 \ \mathrm{arcos}(t), \qquad \text{for } |t| \le 1, \tag{4.4}$$

which, after some algebraic manipulations, reads

$$\sum_{\mathrm{n}=1,2,}^{N} C_n \left\{ 2\,(1-\nu^2)\, B_{nm} + \frac{\gamma}{m}\frac{\pi}{2}\ \delta_{nm} \right\} = -\frac{\pi}{m}(-1)^{m+1}, \qquad m = 1, 2, 3, \ldots, N, \tag{4.5}$$

where the series has been truncated to the N_{th} term.

The horizontal and vertical displacements of the half plane surface take the form:

$$u(t) = -a\, 2\,(1-\nu^2) \left[C_0\ \mathrm{Log}\,(2) + \sum_{n=1,2,}^{N} \frac{C_n}{n} T_n(t) \right], \qquad \text{for } |t| \le 1, \tag{4.6}$$

$$v(t) = a\,(1-2\nu)(1+\nu) \left[C_0\ \mathrm{arcos}(t) + \sum_{n=1,2,}^{N} \frac{C_n}{n} U_{n-1}(t)\, \sqrt{1-t^2} \right], \qquad \text{for } |t| \le 1, \tag{4.7}$$

The horizontal component of stress in the coating is evaluated through the following expression ($|t| \le 1$):

$$\sigma_0(t) = E_0\, \gamma \left[C_0\ \mathrm{arcos}(t) + \sum_{n=1,2,}^{N} \frac{C_n}{n} U_{n-1}(t)\, \sqrt{1-t^2} \right], \tag{4.8}$$

whereas the horizontal stress in the half plane surface is given by expression $(2.13)_2$.

The K_{II} factors are evaluated through expressions (2.14), where the series start from $n = 0$, taking into account the coefficient C_0 also.

For the limiting case of very stiff coatings one finds

$$C_1, C_2, \ldots, C_N, \to 0, \quad \text{for } \gamma \to 0. \tag{4.9}$$

It follows that

$$K_{II}(\pm a) = \pm E\sqrt{a\,\pi}\; C_0, \qquad \text{for } \gamma \to 0, \tag{4.10}$$

or, in dimensionless form, $K_{II}(\pm a)/(F/a^{0.5}) = \pi^{-0.5} \cong 0.564190$.

4.1 Results

Some results related to the contact problem of a coating subject to an axial force applied at its left edge are reported in the present section. The interfacial shear stress is showed in Fig. 4.2a in dimensionless form, namely $\tau(x)/(F/a)$, for different values of γ. For very stiff coatings, the shear tractions are reasonably approximated by the symmetric law

$$\tau(x)/(F/a) = \frac{1}{\pi\sqrt{1-(x/a)^2}}, \quad \text{for } \gamma \to 0. \tag{4.11}$$

It is worth noting that, in the neighbouring of the point of load application, the shear stress increases as γ increases. Conversely, at the opposite edge of the coating, the shear stress increases as γ decreases. This behaviour is reflected in Fig. 4.3b, where the K_{II} factors at the film edges are reported varying γ. As shown in the figure, the strength of interfacial stress singularity at the loaded edge is monotonic increasing with γ, and its trend largely agree with that predicted by Koiter also for small values of γ. Note that, in order to plot the K_{II}* factors provided by the Koiter solution, a proper normalization of the stress singularity factors has bees adopted in Fig. 4.3b (cf. Tullini N., Tralli A., Lanzoni L., Interfacial shear stress analysis of bar and thin film bonded to 2D elastic substrate using a coupled FE-BIE method. Submitted for publication):

$$K_{II}^*(-a)/(F/\sqrt{a}) = \sqrt{\gamma}\,. \tag{4.12}$$

As shown, the free edge of the film is characterized by K_{II} factors monotonic decreasing with γ. The axial displacements in dimensionless form, i.e. $u(x)/(aF/(E_0\delta))$, are reported in Fig. 4.2b and are quite similar to those of a coating subject to thermal load (see Section 2.1). The axial displacement between the edges of the coating can be evaluated as:

$$[u(a)-u(-a)] = a\,\gamma\,\pi(C_0 - 0.5C_1). \tag{4.13}$$

Thus, it follows that $[u(a)-u(-a)]/a$ tends to unity for $\gamma \to 0$, as found for a bar axially loaded by and end force of magnitude F and subject to an uniform distribution of tractions along its length equilibrating the force F.

The dimensionless normal stress in the coating $\sigma_0(x)/(F/\delta)$ is reported in Figure 4.3a. As expected, an almost linear trend is found for a rigid film. Conversely, as γ increases, the stress tends to localize near the point of load application, and it rapidly diminishes going toward the free edge of the film.

5. Detached film under thermal load

An elastic thin film having total length $2a$ and constant thickness δ, subject to a uniform thermal variation $-\Delta T$ (or, similarly, two opposite axial forces, see Section 2) and partially detached from the underlying half plane, is studied in the present section (see Fig. 5.1).
The system can be analyzed by considering two separate coatings, named "l" and "r" in Fig. 5.2, having length $2a_l$ and $2a_r$, respectively, and subjected to thermal load and an unknown axial force F acting at its ends, transmitted by the detached portion of the coating (cf. Wang et

al., 2000). Two unknown distributions of interfacial shear stress, $\tau_l(\xi)$ and $\tau_r(\xi)$ occur across the contact region. The y axis of the reference system is placed at the midpoint of the internal debonding whose length equals $2a_d$.

The condition of perfect adhesion between the coatings and the half plane, together with the compatibility condition of the longitudinal displacement of the coating and the half plane in the detached region leads to the following equations:

$$u_{,x}(x) = u_{0l,x}(x), \quad \text{for } -2a_l - a_d \le x \le -a_d\,; \tag{5.1a}$$

$$u_{,x}(x) = u_{0r,x}(x), \quad \text{for } a_d \le x < a_d + 2a_r\,; \tag{5.1b}$$

$$u(a_d) - u(-a_d) = u_{0r}(a_d) - u_{0l}(-a_d); \tag{5.1c}$$

where $u_{,x}(x)$ denotes the axial strain of the half plain, $u_{0l,x}(x)$ and $u_{0r,x}(x)$ represent the axial strain of the coatings "l" and "r", $u_{0r}(a_d)$ and $u_{0l}(-a_d)$ are the horizontal displacements at the right and left end of the coatings and $u(a_d)$–$u(-a_d)$ is the relative displacement of the corresponding points of the half plane.

As usual, the interfacial stresses are assumed as

$$\tau_l(x) = \frac{E}{\sqrt{1-\left(\dfrac{x+a_l+a_d}{a_l}\right)^2}} \sum_{n=0,1,2,}^{\infty} C_{ln}\, T_n\left(\frac{x+a_l+a_d}{a_l}\right), \qquad \text{for } -2a_l - a_d \le x \le -a_d, \tag{5.2a}$$

$$\tau_r(x) = \frac{E}{\sqrt{1-\left(\dfrac{x-a_d-a_r}{a_r}\right)^2}} \sum_{n=0,1,2,}^{\infty} C_{rn}\, T_n\left(\frac{x-a_d-a_r}{a_r}\right), \qquad \text{for } a_d \le x < a_d + 2a_r. \tag{5.2b}$$

The equilibrium condition of the two bonded portion of the coating along the x axis leads the determination of the coefficients C_{l0} and C_{r0}:

$$C_{l0} = \frac{F}{E\, a_l\, \pi}; \quad C_{r0} = \frac{F}{E\, a_r\, \pi}. \tag{5.3}$$

By introducing expressions (5.2) in Eqs. (5.1a)-(5.1c), the following equations are found:

$$\sum_{j=1,2,}^{\infty} C_{lj}\left\{2\,(1-\nu^2)\;+\;\frac{\gamma_l}{j}\sqrt{1-t_l^{\,2}}\right\}U_{j-1}(t_l)+2\,(1-\nu^2)\sum_{j=0,1,}^{\infty} C_{rj}\,\frac{\left(t_r+\sqrt{t_r^{\,2}-1}\right)^j}{\sqrt{t_r^{\,2}-1}}= \tag{5.4a}$$
$$=\gamma_l\,C_{l0}\left(\pi-\arccos(t_l)\right)-\Delta\varepsilon,\quad \text{for }\ |t_l|\le 1;$$

$$-\sum_{j=1,2,}^{\infty} C_{rj}\left\{2\,(1-\nu^2)\;+\;\frac{\gamma_r}{j}\sqrt{1-t_r^{\,2}}\right\}U_{j-1}(t_r)\;-2\,(1-\nu^2)\sum_{j=0,1,}^{\infty} C_{lj}\,\frac{\left(t_l-\sqrt{t_l^{\,2}-1}\right)^j}{\sqrt{t_l^{\,2}-1}}= \tag{5.4b}$$
$$=\gamma_r\,C_{r0}\arccos(t_r)-\Delta\varepsilon,\quad \text{for }\ |t_r|\le 1;$$

$$2\,(1-\nu^2)\left(C_{l0}A_{l0}+\sum_{j=1,2,}^{\infty} C_{lj}A_{lj}-C_{r0}A_{r0}-\sum_{j=1,2,}^{\infty} C_{rj}A_{rj}\right)= \tag{5.4c}$$
$$=2\,a_d\,\,(1-\nu_0^2)\,F\,/(E_0\delta)-2\,a_d\,\,\Delta\varepsilon,\quad \text{for }\ |t_l|\le 1;$$

where the result (A3) has been used, and $t_l=\dfrac{x+a_l+a_d}{a_l}$, $t_r=\dfrac{x-a_d-a_r}{a_r}$, $\gamma_l=\dfrac{E\,a_l\,(1-\nu_0^2)}{E_0\delta}$,

$\gamma_r=\dfrac{E\,a_r\,(1-\nu_0^2)}{E_0\delta}$. The coefficients A_{l0}, A_{lj}, A_{r0} and A_{rj} are reported in expressions (A8).

If the series expansions are truncated to the N_{th} term and Eqs. (5.4a)-(5.4b) are imposed in N collocation points inside the regions $-2a_l-a_d\le x\le -a_d$ and $a_d\le x<a_d+2a_r$, respectively, an algebraic system of $2N+1$ equations is obtained, which can be solved for the unknowns coefficients C_{lj}, C_{rj} ($j = 1, 2,\ldots, N$) and the force F acting at the ends of the detached portion of the coating. As known, the optimal collocation points t_{lk}, t_{rk} are the roots of the Chebyshev polynomial of first kind of order N, namely:

$$t_{lk},\,t_{rk}\;=\;\cos\left[\frac{(2k-1)}{2\,N}\pi\right],\qquad \text{for }\ k=1,2,\ldots,N. \tag{5.5}$$

5.1 Results

Figs. 5.3a-b show the dimensionless interfacial stress $\tau(x)/(E\Delta\varepsilon)$ and the axial displacement $u(x)\,/(a\,\Delta\varepsilon)$ of a coating having $\gamma = 20$, detached between $0.3a$ and $0.4a$, i.e. for $a_l = 1.6\,a$ and

$a_r = a_d = 0.2\ a$. The interfacial stress is similar to that of a perfectly bonded film, except in the neighbourhood of the detached zone, where it is unbounded. This similarity occurs also for the axial displacement, as showed in Fig. 5.3b. It should be noted that a small difference with respect the case of a perfectly bonded film occurs in correspondence of the detached region, where the axial displacement of the detached coating exhibits a linear trend. The axial displacement in the coating has a continuous slope, since the axial force and the axial strain are continuous even though the presence of a detached zone. The dimensionless axial stress $\sigma_0(x)/(E_0\,\Delta\varepsilon)$ is almost coincident with that of the bonded film in correspondence of the bonded region, as reported in Fig. 5.4a, whereas it equals the constant value F/δ in the detached portion of the coating. The dimensionless stress singularity factors $K_{\mathrm{II}}\,/(E\sqrt{a}\,\Delta\varepsilon)$ at the edges of the film and at the ends of the detached region are reported in Fig. 5.4b varying γ. Similarly to the case of a perfectly bonded film loaded by thermal variation, the K_{II} factors at the film edges are monotonic decreasing with γ, and are well approximated by the Koiter solution. The K_{II} factors in the neighbourhood of the film ends, namely $x = -2a_l - a_d$ and $x = a_d{+}2a_r$ are almost coincident. At the ends of the internal debonding the dimensionless K_{II} factors assume lower values, and for $x = a_d$ the strength of interfacial stress singularity is greater than that evaluated at $x = -a_d$. This is due to the fact that the portion of the coating at the right side of the detached region is stiffer than the other portion, thus exhibiting larger values of K_{II}, accordingly to the trend of the K_{II} factors showed in Fig. 2.4 for a perfectly bonded film.

6. Film subject to an interior axial force

In the present section, a coating subject to an interior axial force is considered, as reported in Fig. 6.1.

Similarly to the procedure utilized to study a partially detached film, the coating can be divided in two portions having length $2a_l$ and $2a_r$, respectively, subjected to the axial forces F_1 and $F-F_1$ applied at its ends, as shown in Fig. 6.2.

Instead of condition (5.1c), here the adjunctive compatibility condition consists to impose the same relative displacements at the limits of the contact region between the half plane and the film, namely:

$$u(2a_r) - u(-2a_l) = u_{0l}(0) - u_{0l}(-2a_l) + u_{0r}(2a_r) - u_{0r}(0); \tag{6.1}$$

with obvious significance of symbols $u(x)$, $u_{0l}(x)$, $u_{0r}(x)$.

The procedure is analogous to that reported in the previous section to solve the contact problem of a partially detached film, thus it has been omitted in the present section.

In the neighbourhood of the point of load application the obtained results have been compared with the Melan solution, here reported for unitary width of the film-substrate system (Grigolyuk and Tolkachev 1987, p.142):

$$\frac{N(x)}{F} = \frac{1}{\pi}\left[\mathrm{si}\left(\frac{\gamma x}{2}\right)\cos\left(\frac{\gamma x}{2}\right) - \mathrm{ci}\left(\frac{\gamma x}{2}\right)\sin\left(\frac{\gamma x}{2}\right)\right], \tag{6.2}$$

$$\frac{\tau(x)}{F} = -\frac{\gamma}{2}\left[\mathrm{ci}\left(\frac{\gamma x}{2}\right)\cos\left(\frac{\gamma x}{2}\right) + \mathrm{si}\left(\frac{\gamma x}{2}\right)\sin\left(\frac{\gamma x}{2}\right)\right], \tag{6.3}$$

where $N(x)$ and $\tau(x)$ denote the axial force in the film and the interfacial shear stress, and $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the sine and cosine integral function, respectively.

6.1 Results

The results reported in the present section refer to a coating axially loaded at the midpoint, i.e. for $a_l = a_r = 0.5\, a$. The interfacial shear traction $\tau(x)/(F/a)$ and the axial displacement $u(x)$ $/(aF/(E_0\, \delta))$ of the film are reported in dimensionless form in Figs. 6.3a-6.3b for $\gamma = 1, 10, 100$. As expected, for low values of γ, the Melan solution agrees with the obtained results in the region close to the point of load application, whereas for $\gamma \geq 100$ the Melan solution practically holds for all points of the coating.

As depicted in Fig. 6.4a, as γ increases, the shear stress tends to concentrate in the neighbouring of the point of load application, whereas it tends to diminish at the edges of the film, similarly to the case of a coating subject to an axial force acting at one end (see Fig. 4.2a). Moreover, for a very rigid film, an almost bilinear trend is assumed by the axial displacement, which decreases increasing γ. The dimensionless axial stress in the film $\sigma_0(x)/(F/\delta)$ is reported in Fig. 6.5a and, similarly to Fig. 4.3a, it rapidly vanishes going toward the film edges. The K_{II} factors in the neighbourhood of the film edge ($x = a$) are reported in Fig. 6.5b varying γ for two load conditions: the solid line refers to an axial force applied at the midpoint ($x = 0$), whereas the dashed curve is related to an axial force acting at the free edge ($x = -a$). Accordingly to eq. (5.10), both curves start from 0.564190. As expected, the dashed curve stays below the solid one. This confirms that an axial force acting at the midpoint of the film produces larger effects at the film edge than the same force applied to the opposite edge of the coating. As expected, this difference diminishes increasing γ, as showed by Fig. 7.5b, and for large values of γ, almost the same $K_{\mathrm{II}}(a)$ factors are expected for these two loading conditions.

7. Conclusions

The contact problem of thin coating structures bonded to a homogeneous elastic half plane has been considered in the present paper. The singular interfacial stress arising across the contact region has been approximated via series expansion of orthogonal Chebyshev polynomials. This allows to reduce the compatibility condition of strains among the coating structure and the underlying substrate to a linear algebraic system. The rigidity parameter γ involving the geometric dimension of the coating and mechanical parameters of the half plane and the bonded structure is found governing the film-substrate mechanical interaction. The contact problem of thin structures having several geometric configurations under different loading conditions has been examined. A coating subject to a uniform thermal variation has been studied first, founding analogies with the load case of two opposite axial forces acting at the film ends. Moreover, a coating having non uniform thickness subject to thermal load has been examined. The load case of a horizontal force applied at one edge of the film has been studied also. The obtained results in terms of stress singularity factors have been compared with those predicted by the Koiter solution, founding a large agreement between them. In particular, the Koiter solution appears adequate to predict stress concentrations in the neighbourhood of the film edges approximately for $\gamma > 10$. A coating partially detached from the half plane under thermal load has been considered, founding at the coating edges almost the same values of stress singularities of a bonded coating. Conversely, in the neighbourhood of the limits of the detached region, lower values of the stress intensity factors have been found. The contact problem of a coating loaded by an internal axial force has been solved also, and the results have been compared with those predicted by the Melan solution. In particular, for this load condition, the Melan solution is retrieved for almost the point of the coating provided that the parameter γ is large enough to resemble an infinite strip, namely for $\gamma \geq 100$. Actually, the analytical results agree well with respect the Melan solution also for lower values of γ in the neighbouring of the point of load application.

Appendix

Some formulas and results used in the main text are reported in the following.

The equilibrium condition (2.8) is satisfied by the shear stress (2.7) since that:

$$\int_{-1}^{1} \frac{T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = \begin{cases} \pi, & \text{if } n=0, \\ 0, & \text{if } n \neq 0. \end{cases} \tag{A1}$$

Eq. (2.9) has bee obtained by using the following identities:

$$\int_{t}^{1} \frac{T_n(\xi)}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi = \begin{cases} \arccos(t), & \text{for } n=0, \\ \dfrac{1}{n}\, U_{n-1}(t)\,\sqrt{1-t^2}, & \text{for } n \neq 0, \end{cases} \tag{A2}$$

$$\int_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\,\mathrm{d}t = \begin{cases} 0, & \text{for } n=0,\ |x|<1, \\ \pi\, U_{n-1}(x), & \text{for } n \neq 0,\ |x|<1, \\ -\pi\left(x-\mathrm{sign}(x)\sqrt{x^2-1}\right)^n \Big/ \left(\mathrm{sign}(x)\sqrt{x^2-1}\right), & \text{for } n \geq 0,\ |x|>1. \end{cases} \tag{A3}$$

The coefficients B_{nm} and A_m in eq. (2.10) assume the form:

$$B_{nm} = B_{mn} = \sum_{k=1,2,}^{\min\{n,m\}} \frac{1+\cos\left[(n-m)\pi\right]}{n-m+2k-1}, \qquad \text{for } m = 1, 3, 5, \ldots, \infty, \tag{A4}$$

$$A_m = \frac{1-\cos(m\pi)}{m}, \qquad \text{for } m = 1, 3, 5, \ldots, \infty. \tag{A5}$$

The expression (2.11) for the axial displacement has been obtained by using the result:

$$\int_{-1}^{1} \frac{T_n(t)}{\sqrt{1-t^2}} \ln|x-t|\,\mathrm{d}t = \begin{cases} -\pi\,\ln(2), & \text{for } n=0, \\ -\dfrac{\pi}{n}\, T_n(t), & \text{for } n>0. \end{cases} \tag{A6}$$

The expression of the coefficients $D_{l\,nm}$, $D_{r\,nm}$ used in eqs. (3.4) are:

$$D_{l\,nm} = D_{l\,mn} = \int_{-1}^{\beta} \sin(n\vartheta)\,\sin(m\vartheta)\,\mathrm{d}\vartheta; \qquad D_{r\,mn} = \begin{cases} -D_{l\,mn}, & \text{if } n \neq m, \\ \dfrac{\pi}{2} - D_{l\,mn}, & \text{if } n = m. \end{cases} \tag{A7}$$

The coefficients A_{l0}, A_{lj}, A_{r0} and A_{rj} used in eq. (5.4c) are:

$$
\begin{aligned}
&A_{l0} = a_l \left(\mathrm{Log}(a_l) - 2\ \mathrm{Log}(\sqrt{a_d} + \sqrt{a_d + a_l}) \right);\\
&A_{lj} = a_l \left[-1 + \left(\frac{2a_d + a_l}{\mathrm{a}_l} - \sqrt{-1 + \left(\frac{2a_d + a_l}{a_l} \right)^2} \right)^j \right] \Big/ j;\\
&A_{r0} = -a_r \left(\mathrm{Log}(a_r) - 2\ \mathrm{Log}(\sqrt{a_d} + \sqrt{a_d + a_r}) \right);\\
&A_{rj} = \mathrm{a}_r \left[(-1)^j - \left(-\frac{2a_d + a_r}{a_r} + \sqrt{-1 + \left(\frac{2a_d + a_r}{a_r} \right)^2} \right)^j \right] \Big/ j.
\end{aligned}
\tag{A8}
$$

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Figure Captions

Figure 2.1. Film bonded to a half plane and subject to a) uniform thermal load or b) two opposite axial forces.

Figure 2.2. Film subject to thermal variation $-\Delta T$: dimensionless a) interfacial shear stress and (b) axial displacement of the coating for different values of γ. Symmetry holds for negative x coordinate.

Figure 2.3. Film subject to thermal variation $-\Delta T$: dimensionless normal stress a) in the coating and b) at the half plane surface for different values of γ. Symmetry holds for negative x coordinate.

Figure 2.4. Film subject to a thermal variation $-\Delta T$: normalized shear stress singularity factors K_{II} in the neighbourhoods of the film ends varying γ and comparison with K_{II}* factors provided by the Koiter solution.

Figure 3.1. Coating having variable thickness under thermal load.

Figure 3.2. Coating having variable thickness subject to thermal load $-\Delta T$: dimensionless a) interfacial shear stress and b) axial displacement of the coating for $\beta = 0$, $\gamma_l = 10$ and different values of γ_r.

Figure 3.3. Coating having variable thickness subject to thermal load $-\Delta T$: dimensionless a) normal stress in the coating for some values of γ_r and (b) K_{II} factors at the film edges for $\beta = 0$, $\gamma_l = 10$ and different values of γ_r.

Figure 6.2. Sketch for studying the contact problem of a film bonded to a half plane and subject to an interior axial force.

Figure 6.3. Film subject to an internal axial force at the midpoint: dimensionless a) interfacial shear stress and comparison with Melan solution, b) axial displacement of the coating for some values of γ. Symmetry holds for negative x coordinate.

Figure 6.4. Film subject to an internal axial force at the midpoint: dimensionless a) axial force in the film and comparison with Melan solution for some values of γ (symmetry holds for negative x coordinate), b) K_{II} factors at $x = a$ loaded by an axial force acting at $x = 0$ and $x = -a$ varying γ.