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ON THE EDGE-WAVE ON A THIN ELASTIC PLATE SUPPORTED BY AN ELASTIC HALF-SPACE

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Abstract. *In this contribution, we consider edge-wave propagating in a thin elastic semi-infinite plate which is bilaterally supported by a homogenenous isotropic elastic half-space. The problem is formulated in terms of a eigenproblem constituted by a system of five linear PDEs in the plate transverse displacement and in the scalar and vector elastic potentials subject to mixed boundary conditions accounting for plate-fundation displacement continuity under the plate and zero normal stress outside. Zero tangential stress is envisaged throughout. The problem could be reduced to an inhomogenenous Wiener-Hopf functional equation in terms of the half-space surface displacement and of the plate-to-fundation contact pressure only. The kernel function is analyzed and the Rayleigh wave speed is obtained together with a novel dispersion equation. Finally, kernel factorization is performed.*

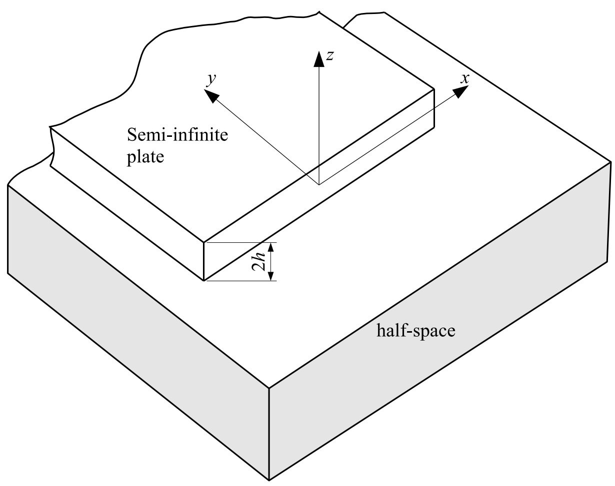

Figure 1: A semi-infinite isotropic elastic plate occupying the region $\{(x, y, z) : y > 0, z = 0\}$, supported by an isotropic elastic half-space

1 INTRODUCTION

Edge-wave propagation in thin elastic structures is credited with a long history of investigation, dating back to 1960 and the work by Konenkov [8] and going through a fascinating record of discovery and re-discovery [14]. This problem follows on a large body of literature on elastic flexural waves, see the survey paper [11]. In recent times, [1] investigated the existence of edge flexural waves in a plate partially submerged in a inviscid fluid. In [7] dispersion relation and asymptotic models for flexural edge waves propagating in a supported elastic plate are given. In [5], edge waves in an elastically supported plate are considered taking into account a non-local response of the foundation. It is worth observing that resonant excitation of edge waves may play a significant role in promoting crack propagation in supported elastic plates [15, 13]. In this communication, we extend the analysis to a plate resting on an elastic half-space and thus consider the sophisticated interaction occurring with the elastic plate. The solution of this problem may help cast some light onto the existence condition for and the shape of edge waves propagating in elastic structures resting on foundations endowed with a complex non-local character. Results may be useful for modeling edge effects in flexible composite structures [6], in living tissues embedded in a compliant elastic matrix [2] or in thin film laid on top of an elastic substrate [10, 4].

2 PROBLEM FORMULATION

Let us consider a semi-infinite Kirchhoff-Love elastic plate of thickness $2h$ that is bilaterally supported by an elastic half-space and occupies the region of three-dimensional Euclidean space $z \leq 0$ (Fig.1). The governing equation for the plate transverse displacement w (positive along z) reads [3]

$$D\nabla_2^4 w + 2\rho_p h w_{,tt} = \sigma_{zz}, \quad \text{in } y \geq 0, z = 0, \tag{1}$$

where $\triangle_2 w = w_{,xx} + w_{,yy}$ is the 2D Laplace operator and $\nabla_2^4 = \triangle_2\triangle_2$ is the bi-harmonic operator, $D = 2E_p h^3/(3(1-\nu_p^2))$ and ρ_p are the plate flexural rigidity and mass density, while E_p and ν_p are Young modulus and Poisson ratio, respectively. It is noted that $w = w(x, y, t)$ and the specification $z = 0$ in (1) only applies to $\sigma_{zz} = \sigma_{zz}(x, y, z, t)$. Hereinafter, a subscript comma denotes partial differentiation with respect to the specified variable, e.g. $w_{,tt} = \partial^2 w/\partial t^2$. The plate is acted upon by the contact pressure σ_{zz} exerted by the half-space. Since the half-space is homogeneous and isotropic, its constitutive behavior is expressed through the Lamé elastic constants $\lambda = \nu E/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$ as

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I}, \tag{2}$$

wherein the linear deformation tensor $\boldsymbol{\epsilon}$ is related to the displacement field in the half-space $\boldsymbol{u} = [u_x\, u_y\, u_z]$ through

$$\boldsymbol{\epsilon} = \operatorname{Sym}\operatorname{grad}\boldsymbol{u} = \tfrac{1}{2}\left(\operatorname{grad}\boldsymbol{u} + \operatorname{grad}^T\boldsymbol{u}\right) \tag{3}$$

and $\boldsymbol{I}$ is the rank-2 identity tensor. Hereinafter, a superscript T denotes transposition while tr is the trace operator, i.e. the operator which yields the first tensor invariant. We observe that $\operatorname{grad}\boldsymbol{u}$ is a rank-2 tensor which, component-wise, reads

$$(\operatorname{grad}\boldsymbol{u})_{ij} = u_{i,j}, \quad i,j \in \{x,y,z\}, \tag{4}$$

so that, adopting Einstein's summation convention over twice repeated indices, we have

$$\operatorname{tr}\boldsymbol{\epsilon} = \operatorname{tr}\operatorname{grad}\boldsymbol{u} = \operatorname{div}\boldsymbol{u} = u_{i,i}. \tag{5}$$

Through Eqs.(2) and (5), we can write

$$\sigma_{zz} = 2\mu u_{z,z} + \lambda\operatorname{div}\boldsymbol{u}, \quad \text{at } z = 0. \tag{6}$$

The displacement field in the half-space $\boldsymbol{u}$ is related to the elastic potentials (whose dimension is length squared, which we denote as ℓ^2) through the Helmholtz decomposition [12]

$$\boldsymbol{u} = \operatorname{grad}\phi + \operatorname{curl}\boldsymbol{\psi}, \tag{7}$$

wherein the vector potential is defined up to a gauge normalization condition, usually

$$\operatorname{div}\boldsymbol{\psi} = 0. \tag{8}$$

Component-wise Eqs.(7) and (8) read, respectively,

$$u_i = \phi_{,i} + e_{ijk}\psi_{k,j}, \quad i \in \{x,y,z\} \quad \text{and} \quad \psi_{j,j} = 0. \tag{9}$$

Here, the symbol e_{ijk} stands for the rank-3 permutation tensor (which is the only skew-symmetric rank-3 tensor normalized to 1, see [9]). In particular, it is

$$u_z = \phi_{,z} + \psi_{y,x} - \psi_{x,y} \tag{10}$$

such that Eq.(6) may be rewritten as

$$\sigma_{zz} = 2\mu\left(\phi_{,z} + \psi_{y,x} - \psi_{x,y}\right)_{,z} + \lambda\nabla_3^2\phi, \quad \text{at } z = 0, \tag{11}$$

for, in light of Eqs.(5) and (7), it is

$$\operatorname{tr}\boldsymbol{\epsilon} = \operatorname{div}\boldsymbol{u} = \triangle_3\phi \tag{12}$$

and $\triangle_3$ is the 3D Laplace operator.

Outside the contact region, the contact pressure vanishes, namely

$$\sigma_{zz} = 0, \quad \text{at } y < 0, z = 0, \tag{13}$$

while, in the absence of friction, the tangential load is zero throughout the half-space surface

$$\sigma_{xz} = \sigma_{yz} = 0, \quad \text{at } z = 0. \tag{14}$$

These equations may be rewritten in terms of elastic potentials as

$$\mu \left(2\phi_{,xz} + \psi_{y,xx} + \psi_{z,yz} - \psi_{x,xy} - \psi_{y,zz}\right) = 0, \qquad z = 0 \tag{15a}$$

$$\mu \left(2\phi_{,yz} + \psi_{y,xy} + \psi_{x,zz} - \psi_{z,xz} - \psi_{x,yy}\right) = 0, \qquad z = 0. \tag{15b}$$

As well known, perfect bilateral contact imposes the geometric condition

$$u_z = w, \quad \text{at } y \geq 0, z = 0, \tag{16}$$

which may be recast in terms of the elastic potentials through Eq.(10) as

$$w = \phi_{,z} - \psi_{x,y} + \psi_{y,x}, \quad \text{at } y \geq 0, z = 0. \tag{17}$$

Consideration of the plate boundary $y = 0$ being load-free gives

$$w_{,yy} + \nu_p w_{,xx} = 0, \quad \text{and} \quad w_{,yyy} + (2 - \nu_p) w_{,xxy} = 0, \quad \text{at } y = 0. \tag{18}$$

In the absence of body forces, the equilibrium equation for the half-space $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ may be recast entirely in terms of elastic potentials, thus giving the pair of wave equations [3, Chap.5]

$$\triangle_3 \phi - \frac{1}{c_1^2} \phi_{,tt} = 0, \quad \text{in } z \leq 0, \tag{19}$$

and

$$\triangle_3 \psi_i - \frac{1}{c_2^2} \psi_{i,tt} = 0, \quad i \in \{x, y, z\}, \quad \text{in } z \leq 0. \tag{20}$$

The constants appearing in Eqs.(19,20) are the dilatational and the rotational wave speeds, respectively, i.e.

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} > c_2 = \sqrt{\frac{\mu}{\rho}},$$

whose ratio is let as

$$\theta_1 = \frac{c_1}{c_2} = \sqrt{2 + \tfrac{\lambda}{\mu}} = \sqrt{2(1-\nu)/(1-2\nu)} > 2/\sqrt{3}. \tag{21}$$

3 FUNCTIONAL EQUATIONS

3.1 Travelling-wave solution

Let us consider a travelling-wave solution in the form

$$\phi(x, y, z, t) = \Phi(y, z) \exp \imath(kx + \omega t), \tag{22a}$$

$$\psi_i(x, y, z, t) = \Psi_i(y, z) \exp \imath(kx + \omega t), \tag{22b}$$

$$\sigma_{zz}(x, y, z, t) = \Sigma_{zz}(y, z) \exp \imath(kx + \omega t), \tag{22c}$$

$$w(x, y, t) = \zeta(y) \exp \imath(kx + \omega t), \quad y \geq 0, \tag{22d}$$

having let the wavenumber k and the angular frequency ω. Hereinafter, $\imath$ denotes the imaginary unit, i.e. $\imath^2 = -1$. We observe that these travelling-wave solutions propagate along x and their propagation direction is really immaterial. Here, propagation in the negative direction is chosen, without loss of generality, to go along with the analyticity strip of Fig.2. We further let the half-space surface displacement

$$u_z(x, y, 0, t) = \eta(y) \exp \imath(kx + \omega t) \tag{23}$$

and, in light of the perfect contact condition (16), it is $\eta(y) = \zeta(y)$ for $y \geq 0$. Since interest is set on edge disturbances, we require the decay conditions on the half-space surface and away from the plate edge

$$\{\Phi, \Psi_i, \eta, \zeta\} \to 0, \quad \text{at } z = 0 \text{ and as } y \to \infty, \tag{24}$$

and along the half-space depth

$$\{\Phi, \Psi_i\} \to 0, \quad \text{as } z \to -\infty. \tag{25}$$

Eq.(10), together with the travelling-wave assumptions (22), gives the half-space surface displacement

$$\eta(y) = \Phi_{,z} - \Psi_{x,y} + \imath k \Psi_y, \quad \text{at } z = 0. \tag{26}$$

Similarly, Eq.(19) provides the amplitude of the scalar elastic potential

$$\left(\omega^2 - c_1^2 k^2\right) \Phi + c_1^2 \left(\Phi_{,yy} + \Phi_{,zz}\right) = 0, \tag{27}$$

while Eqs.(20) lend the vector potential

$$\left(\omega^2 - c_2^2 k^2\right) \Psi_i + c_2^2 \left(\Psi_{i,yy} + \Psi_{i,zz}\right) = 0 \quad i \in \{x, y, z\}. \tag{28}$$

The gauge condition (8) turns into

$$\imath k \Psi_x + \Psi_{y,y} + \Psi_{z,z} = 0, \tag{29}$$

while the zero tangential loading conditions (15) become

$$2\imath k \Phi_{,z} - k^2 \Psi_y + \Psi_{z,yz} - \imath k \Psi_{x,y} - \Psi_{y,zz} = 0, \qquad \text{at } z = 0, \tag{30a}$$

$$2\Phi_{,yz} + \imath k \Psi_{y,y} + \Psi_{x,zz} - \imath k \Psi_{z,z} - \Psi_{x,yy} = 0, \qquad \text{at } z = 0. \tag{30b}$$

The plate equation (1) now reads

$$D\left[k^4 \zeta - 2k^2 \zeta'' + \zeta''''\right] - 2\rho_p h \omega^2 \zeta = \Sigma_{zz}(y, 0), \tag{31}$$

it being understood that prime denotes differentiation with respect to the single dependent variable y and that, according to Eq.(11),

$$\Sigma_{zz} = 2\mu \left(\Phi_{,z} - \Psi_{x,y} + \imath k \Psi_y\right)_{,z} + \lambda \left(-k^2 \Phi + \Phi_{,yy} + \Phi_{,zz}\right). \tag{32}$$

In similar fashion, the plate BCs (18) become

$$\zeta'' - \nu_p k^2 \zeta = 0, \quad \text{and} \quad \zeta''' - (2 - \nu_p) k^2 \zeta' = 0, \quad \text{at } y = 0. \tag{33}$$

Finally, perfect contact implies the mixed conditions,

$$\zeta = \eta, \qquad \text{for } y \geq 0, \tag{34a}$$
$$\Sigma_{zz}(y,0) = 0, \qquad \text{for } y < 0, \tag{34b}$$

respectively coming from Eqs.(16) and (13). Summing up, determination of the plate edge disturbances requires solving a set of four homogeneous PDEs and one inhomogeneous ODE, respectively Eqs.(27,28) and Eq.(31), in five unknown functions, namely the elastic potentials Φ, Ψ_i and the plate displacement ζ. This system of differential equations is supplemented by the gauge condition (29) and it is subject to the decaying conditions (24,25), to the boundary conditions (30,33) and to the mixed conditions (34). On the overall, this system may be seen as expressing an eigenvalue problem, for the load on the plate is expressed through the elastic potentials.

3.2 Analysis in the frequency domain

We introduce the (bilateral or full-range) Fourier transform along y in the usual manner [16] and denote it by a superposed hat. Accordingly, the transform of $\Phi(y,z)$ is defined as

$$\hat{\Phi}(s,z) = \int_{-\infty}^{\infty} \Phi(y,z)\exp(\imath ys)\mathrm{d}y,$$

and the inversion relation holds

$$\Phi(y,z) = (2\pi)^{-1}\int_{-\infty}^{\infty} \hat{\Phi}(s,z)\exp(-\imath ys)\mathrm{d}s, \tag{35}$$

integration being performed along the real axis. The corresponding unilateral (or half-range) Fourier transforms are defined in likewise manner

$$\hat{\Phi}^{+}(s,z) = \int_{0}^{\infty} \Phi(y,z)\exp(\imath ys)\mathrm{d}y,$$

for the plus transform, and

$$\hat{\Phi}^{-}(s,z) = \int_{-\infty}^{0} \Phi(y,z)\exp(\imath ys)\mathrm{d}y,$$

for the minus transform. The important connection holds which relates full to half-range transforms

$$\hat{\Phi}(s,z) = \hat{\Phi}^{+}(s,z) + \hat{\Phi}^{-}(s,z). \tag{36}$$

Let us define

$$\Lambda_0(s) = s^2 + k^2, \quad \Lambda_{1,2}(s) = \Lambda_0(s) - k_{1,2}^2, \tag{37}$$

with

$$k_{1,2} = \frac{\omega}{c_{1,2}}.$$

Taking the Fourier transform of Eqs.(27) and (28) along y lends a system of 4 uncoupled constant coefficient ODEs which can be immediately solved giving

$$\hat{\Phi}(s,z) = a(s)\exp\left[\sqrt{\Lambda_1(s)}z\right], \tag{38a}$$
$$\hat{\Psi}_i(s,z) = b_i(s)\exp\left[\sqrt{\Lambda_2(s)}z\right], \tag{38b}$$

being $i \in \{x, y, z\}$ and having incorporated the depth-wise decay conditions (25). To this aim, the square root in Eqs.(38) is made definite by choosing the Riemann sheet which possesses positive real part on a strip about the real axis. This is achieved through taking a branch cut for the square root that is described by

$$\Im\left(\Lambda_{1,2}(s)\right) = 0 \quad \text{and} \quad \Re\left(\Lambda_{1,2}(s)\right) \leq 0, \tag{39}$$

where $\Re(s)$ and $\Im(s)$ indicate the real and the imaginary part of s, respectively. Denoting with $\beta_{1,2}$ the branch points for the square roots, i.e. $\Lambda_{1,2}(\beta_{1,2}) = 0$, we have

$$\beta_{1,2} = \sqrt{k^2 - k_{1,2}^2},$$

and

$$\Lambda_{1,2}(s) = (s - \imath\beta_{1,2})(s + \imath\beta_{1,2}).$$

Under the restriction [1]

$$k > k_2 > k_1 > 0, \tag{40}$$

branch points are real and positive, i.e. $0 < \beta_2 < \beta_1$, and it appears that Eqs.(39) locate the branch cuts on the imaginary axis, extending from $\imath\beta_{1,2}$ to $-\imath\beta_{1,2}$ through $\imath\infty$. Conversely, when $k < k_{1,2}$, the corresponding branch point becomes imaginary-valued and branch cuts reach the real axis.

In a similar fashion, taking the full-range Fourier transform along y of Eqs.(29,30) and plugging in the solutions (38), yields a system of 3 linear algebraic equations in the 4 unknown a, b_x, b_y, b_z, that are functions of s. This linear system has determinant

$$\Delta = -\sqrt{\Lambda_2(s)}\left(\Lambda_0(s) + \Lambda_2(s)\right)\left(\Lambda_0(s) - \Lambda_2(s)\right),$$

which, apart from the special case $\omega = 0$, vanishes at

$$s = \pm\imath\beta_2, \quad \text{and} \quad s_\Delta = \pm\imath\sqrt{k^2 - \tfrac{1}{2}k_2^2}. \tag{41}$$

Indeed, in the special case $\omega = 0$, we get $k_1 = k_2 = 0$, $\Lambda_1(s) = \Lambda_2(s) = \Lambda_0(s)$ and only two equations are now linearly independent so thus their solution reads

$$\begin{aligned} b_y(s) &= k^{-1}\left(\imath\sqrt{\Lambda_0(s)}a(s) - sb_x(s)\right), \\ b_z(s) &= -k^{-1}\left(sa(s) + \imath\sqrt{\Lambda_0(s)}b_x(s)\right). \end{aligned}$$

In the general case, this linear system can be solved to express three unknowns in terms of the fourth, say

$$b_x(s) = 2\imath s\frac{\sqrt{\Lambda_1(s)}}{s^2 + k^2 + \Lambda_2(s)}a(s), \tag{42a}$$

$$b_y(s) = 2\imath k\frac{\sqrt{\Lambda_1(s)}}{s^2 + k^2 + \Lambda_2(s)}a(s), \tag{42b}$$

$$b_z(s) = 0. \tag{42c}$$

We take the plus Fourier transform of the plate equation (31) and add to it the minus transform of Eq.(34b), namely

$$\hat{\Sigma}^{-}_{zz}(s,0) = 0, \tag{43}$$

whence, making use of the connection (36), we get

$$p(s)\,\hat{\zeta}^{+} + BT(s) = D^{-1}\hat{\Sigma}_{zz}(s,0), \tag{44}$$

having let the polynomial function

$$p(s) = (s^2+k^2)^2 - k_0^4,$$

where $k_0 \geq 0$ is the plate in-vacuum edge-wave wavenumber

$$k_0 = \sqrt[4]{\frac{2h\rho_p}{D}\omega^2}.$$

Here $BT(s)$ is a boundary term (with dimension ℓ^{-2})

$$BT(s) = -\zeta''' + \imath s\zeta'' + \zeta'\left(2k^2+s^2\right) - \imath s\zeta\left(2k^2+s^2\right), \quad \text{at } y=0, \tag{45}$$

assuming that, in view of the decay conditions (24),

$$\lim_{y\to\infty}\left[\zeta''' - \imath s\zeta'' - \zeta'\left(2k^2+s^2\right) + \imath s\zeta\left(2k^2+s^2\right)\right]\exp(\imath sy) = 0.$$

Taking advantage of the plate BCs (33) we can write

$$BT(s) = E_e(s)\,\zeta'(0) + E_o(s)\,\zeta(0),$$

having let the even and odd entire functions

$$E_e(s) = \nu_p k^2 + s^2, \quad E_o(s) = -\imath s\left[k^2(2-\nu_p)+s^2\right].$$

We shall agree to denote by $s_{1,3} = \pm\imath\sqrt{k^2-k_0^2}$ and $s_{2,4} = \pm\imath\sqrt{k_0^2+k^2}$ the roots of the polynomial $p(s)$, i.e. $p(s) = \prod_{i=1}^{4}(s-s_i)$. For the sake of definiteness, we assume $k > k_0$ so that s_1 is purely imaginary and it is located in the upper complex half-plane. Obviously, it is $0 < \Im(s_1) < \Im(s_2)$.

The full-range Fourier transform of the vertical stress (32) reads

$$\hat{\Sigma}_{zz} = 2\mu\left(\hat{\Phi}_{,z} + \imath s\hat{\Psi}_x + \imath k\hat{\Psi}_y\right)_{,z} - \lambda\left(\Lambda_0(s)\hat{\Phi} - \hat{\Phi}_{,zz}\right),$$

whence, on the surface $z=0$ and through Eqs.(38,42), we get

$$\hat{\Sigma}_{zz}(s,0) = \Big\{2\mu\left(\Lambda_1(s) - 2\sqrt{\Lambda_1(s)\Lambda_2(s)}\frac{\Lambda_0(s)}{\Lambda_0(s)+\Lambda_2(s)}\right) - \lambda\left[\Lambda_0(s)-\Lambda_1(s)\right]\Big\}a(s). \tag{46}$$

Taking the full-range Fourier transform of the surface displacement (26) gives

$$\hat{\eta} = \hat{\Phi}_{,z} + \imath s\hat{\Psi}_x + \imath k\hat{\Psi}_y, \quad \text{at } z=0$$

which, upon substituting the solution (38) and making use of the connections (42), can be solved for $a(s)$

$$a(s) = -\frac{\Lambda_0(s) + \Lambda_2(s)}{(\Lambda_2(s) - \Lambda_0(s))\sqrt{\Lambda_1(s)}}\hat{\eta}, \tag{47}$$

which has a pole at $s = \beta_1$. Thus, Eq.(46) becomes

$$\hat{\Sigma}_{zz}(s,0) = \mu\left(\Lambda_2(s) - \Lambda_0(s)\right)^{-1} f_1(s)\hat{\eta}, \tag{48}$$

where the function $f_1(s)$ has been let as

$$f_1(s) = \left(2\sqrt{\Lambda_1(s)} + \frac{\lambda}{\mu}\frac{\Lambda_1(s) - \Lambda_0(s)}{\sqrt{\Lambda_1(s)}}\right)(\Lambda_2(s) + \Lambda_0(s)) - 4\Lambda_0(s)\sqrt{\Lambda_2(s)}, \tag{49}$$

and $\lambda/\mu = \theta_1^2 - 2 = 2\nu/(1-2\nu)$. In light of Eqs.(43) and (36), Eq.(48) provides a connection between the plus and the minus transforms, namely

$$\hat{\eta}^+ = \frac{\Lambda_2(s) - \Lambda_0(s)}{\mu f_1(s)}\hat{\Sigma}_{zz}^+(s,0) - \hat{\eta}^-, \tag{50}$$

which, substituted in Eq.(44) and in view of the plus Fourier transform of the perfect contact condition (34a), gives

$$\hat{\eta}^- + \frac{K(s)}{Dp(s)}\hat{\Sigma}_{zz}^+(s,0) = \frac{BT(s)}{p(s)}. \tag{51}$$

This is the Wiener-Hopf functional equation of the second kind [16] that needs be solved. Here, the dimensionless kernel function $K(s)$ is introduced as

$$K(s) = 1 + \delta h\frac{p(s)}{f_1(s)}, \tag{52}$$

where δh is a characteristic length analogous to ℓ_s of [2], relating the plate stiffness to the half-space's through the wavenumber k_2, namely

$$\delta = \frac{Dk_2^2}{\mu h} = \frac{4(1+\nu)E_p}{3(1-\nu_p^2)E}h^2k_2^2.$$

We observe that k_0 may be easily rewritten in terms of δ

$$k_0 = \sqrt[4]{\frac{2\rho_p\rho}{\delta\mu^2}}\omega$$

and that, as $|s| \to +\infty$,

$$f_1(s) = \frac{\delta h}{c}|s| + O(|s|^{-1}), \quad \text{and} \quad K(s) = c|s|^3 + O(|s|),$$

having let the constant

$$c = -\frac{\delta h}{2k_1^2\left(1 + \frac{\lambda}{\mu}\right)}.$$

4 ANALYSIS AND FACTORIZATION OF THE KERNEL FUNCTION

We begin by considering the poles of the kernel $K(s)$, which are the roots of the equation

$$f_1(s) = 0. \tag{53}$$

Indeed, Eq.(48) shows that at such roots there is no contact pressure, i.e. the half-plane and the plate are entirely decoupled. Therefore, for them, we expect to retrieve the classical solution for the edge-wave of a free plate. To find real and imaginary roots of Eq.(53), we let $s^2+k^2 = k_2^2\xi^{-1}$ and then solve for $0 < \xi < 1$ the resulting Rayleigh equation [3, Eq.(6.1.86)]

$$R(\xi) = 0, \tag{54}$$

where

$$R(\xi) = (2-\xi)^2 - 4\sqrt{1-\theta_1^{-2}\xi}\sqrt{1-\xi}.$$

As well known, besides the trivial solution $\xi = 0$, this equation always admits the positive order-one root $\xi_R < 1$. Besides, the root ξ_R is monotonic increasing with θ_1 and

$$\sup \xi_R = \xi_{R0} = 0.912622,$$

where ξ_{R0} is the single positive real root of the polynomial equation $(1-\xi/2)^4 - (1-\xi) = 0$, which is obtained letting $\theta_1 \to +\infty$ in Eq.(54). Once ξ_R has been determined, it is easy to find the roots $s = \pm s_p$ and we agree to locate $s_p = \sqrt{k_2^2\xi_R^{-1} - k^2}$ in the upper complex half-plane. Let us put $k_R = k_2\xi_R^{-1/2}$, then clearly $k \gtrless k_R$ implies that Eq.(53) admits purely imaginary/real roots, which we denote by $s_p = \imath y_p$, $y_p = \sqrt{k^2 - k_R^2}$ and $s_p = x_p$, respectively. We shall restrict the analysis to the former situation, named the subsonic regime, although the latter case may be formally considered through letting $y_p = -\imath x_p$ (the minus sign is introduced to preserve the plus/minus properties of the functions). Indeed, the roots $\pm s_p$ are poles for the kernel function $K(s)$ and when they are located on the real axis a supersonic regime is dealt with. Recalling the connection among angular frequency, wavenumber and speed $\omega = kc$, we have $\xi = c^2/c_2^2$ and we can then give a simple physical meaning to ξ_R and k_R, which are the Rayleigh wave speed ratio c_R/c_2 and the Rayleigh wavenumber, respectively. We conclude that the Rayleigh wave solution marks the turning point between the subsonic and the supersonic regime and it occurs at $s_p = 0$, when the pair of imaginary poles of $K(s)$ meet at the origin of the complex plane.

In a similar fashion, we look for real and purely imaginary roots of the equation

$$f_1(s) + \delta h p(s) = 0,$$

which demands solving for ξ the Rayleigh-like equation

$$R(\xi) = -\delta h k_2 P(\xi), \tag{55}$$

having let the monotonic increasing function

$$P(\xi) = \xi^{-1/2}(1-\theta_0^{-4}\xi^2)\sqrt{1-\theta_1^{-2}\xi}, \quad \theta_0 = k_2/k_0. \tag{56}$$

We first observe that the solution of Eq.(55) coincides with the solution of (54) only in the trivial case $\delta h \to 0$. Here, three scenarios are possible, according to the values of the parameters θ_0, θ_1 and $\delta h k_2$, namely

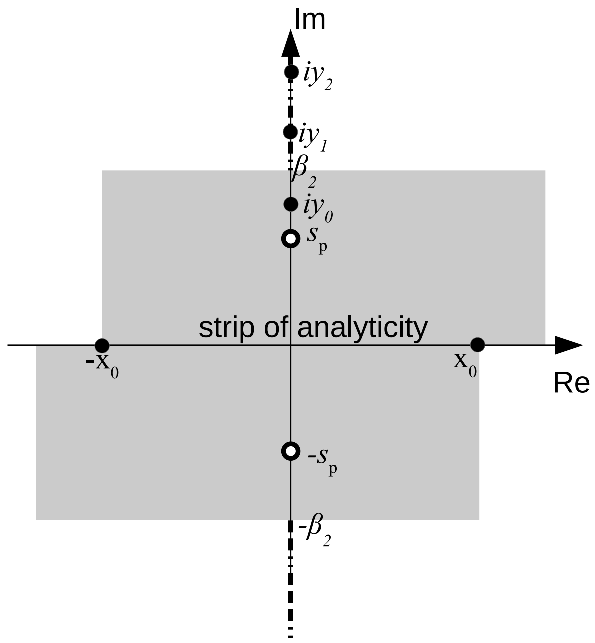

Figure 2: Zeros and poles of the kernel function $K(s)$ and analyticity strip $\mathcal{S}$

1. Eq.(55) admits two real roots, $0 < \xi_1 \leq \xi_2 \leq 1$, which correspond to a pair of real roots $s = \pm x_0 = \pm\sqrt{k_2^2 \xi_1^{-1} - k^2}$ and to a pair of purely imaginary roots, $s = \pm \imath y_0 = \pm \imath \sqrt{k^2 - k_2^2 \xi_2^{-1}}$;
2. Eq.(55) admits a single real root of multiplicity two, which correspond to a pair of real roots $s = \pm x_0$ of order two;
3. Eq.(55) admits four complex roots $s = \pm s_0, \pm s_0^*$.

Let us assume, for the sake of definiteness, that scenario 1 holds and define

$$F(s) = K(s) \frac{s^2 - s_p^2}{c(s^2 - x_0^2)(s^2 + y_0^2)\sqrt{s^2 + \beta_2^2}}. \tag{57}$$

This definition may be easily extended in the general case, for in scenario 2, we need only take $y_0 = \imath x_0$, while, in scenario 3, we substitute s_0 for x_0 and s_0^* for $\imath y_0$. The function $F(s)$ is even, as its functional dependence on s is only through s^2, it is dimensionless and defined in such a way that

$$\lim_{|s| \to +\infty} F(s) = 1.$$

The function $F(s)$ is analytic in the complex plane apart from the poles $\pm \imath \beta_2$ from which two branch cuts issue, denoted by $\mathcal{C}^{\pm}$, which sit on the imaginary axis and extend to $\pm \imath \infty$. Indeed, the function $F(s)$ is obtained depriving $K(s)$ from its poles and zeros in its region of analyticity (Fig.2). Looking at the definitions (52,49), it appears that $F(s)$ is real-valued whenever s sits on the real axis, in light of the restriction (40), and likewise so on the imaginary axis, $s = \imath y$, provided that $|y| < \beta_2$.

The function $F(s)$ may be split into the product of two complex-valued functions through logarithmic decomposition, i.e.

$$F(s) = F^+(s) F^-(s), \tag{58}$$

where $F^{\pm}(s)$ are analytic in the whole complex plane apart from $\mathcal{C}^{\mp}$, respectively. Indeed, they may be initially defined as

$$F^{+}(s) = \exp G(-s), \quad F^{-}(s) = F^{+}(-s), \tag{59}$$

where

$$G(s) = \frac{1}{2\pi\imath} \int_{-\infty-\imath\eta}^{+\infty-\imath\eta} \frac{\ln F(z)}{s-z} \mathrm{d}z, \quad 0 < \eta < \beta_2.$$

According to this definition, the functions $F^{\pm}(s)$ share the common analyticity strip $\mathcal{S} = \{s \in \mathbb{C} : |\Im(s)| < \beta_2\}$. Successively, the integration path may be deformed around the lower branch cut $\mathcal{C}^{-}$ to warrant the above-stated analyticity region

$$G(s) = -\frac{1}{2\pi} \int_{\beta_2}^{+\infty} \lim_{\epsilon \to 0} \frac{\ln F(-\imath t + \epsilon) - \ln F(-\imath t - \epsilon)}{s + \imath t} \mathrm{d}t.$$

We can then define the dimensionless functions

$$K^{\pm}(s) = \sqrt{c} \frac{(s \mp x_0)(s \pm \imath y_0)\sqrt{s \pm \imath\beta_2}}{s \pm s_p} F^{\pm}(s) \exp\left(\mp\imath 3\pi/4\right),$$

analytic in the relevant complex half-plane. The exponential factor warrants that $K^{-}(s) = K^{+}(-s)$. It is easy to determine the asymptotic behavior of $K^{\pm}(s)$ (the coefficient c is kept for dimensionality reasons)

$$K^{\pm}(s) \sim \sqrt{c} \exp\left(\mp\imath 3\pi/4\right) s^{3/2}, \quad \text{as } |s| \to +\infty.$$

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