Chapter 1
A Wiener-Hopf system of equations in the steady-state propagation of a rectilinear crack in an infinite elastic plate

A. Nobili, E. Radi and L. Lanzoni

1.1 Introduction

Let us consider a Kirchhoff-Love infinite plate with thickness h, supported by a Winkler elastic foundation (Fig.1.1). The plate has a linear crack and a Cartesian reference frame is introduced such that the crack corresponds to the negative part of the x_1-axis. Besides, the crack is linearly propagating at a constant speed v (steady-state propagation) and the origin of the Cartesian reference frame is attached to and moves along with the crack tip. The governing equation for the transverse displacement of the plate w reads

$$D\triangle\triangle w + kw = -\rho h \partial_{tt} w + q, \tag{1.1}$$

being $\triangle = \partial_{x_1 x_1} + \partial_{x_2 x_2}$ the Laplace operator in two dimensions, q the transverse distributed load per unit area, D the plate bending stiffness, k the Winkler modulus and ρ the mass density per unit volume [6]. We let $w = w(x_1 - vt, x_2)$ and Eq.(1.1) may be rewritten as

$$\triangle\triangle w + \kappa^{-2}\partial_{x_1 x_1} w + \lambda^{-4} w = \frac{q}{D}, \tag{1.2}$$

having let the *characteristic lengths*

A. Nobili
Università degli Studi di Modena e Reggio Emilia, Modena, Italy,
e-mail: andrea.nobili@unimore.it

E. Radi
Università degli Studi di Modena e Reggio Emilia, Modena, Italy,
e-mail: enrico.radi@unimore.it

L. Lanzoni
Università degli Studi di San Marino, San Marino,
e-mail: luca.lanzoni@unirsm.sm

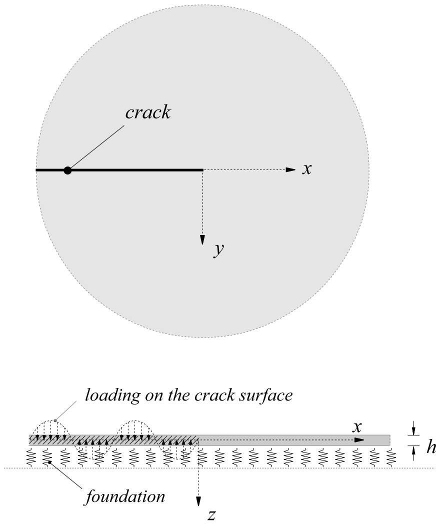

Fig. 1.1 Cracked Kirchhoff plate resting on a Winkler elastic foundation. The crack is propagating in the x direction with speed v

$$\lambda = \sqrt[4]{\frac{D}{k}}, \quad \text{and} \quad \kappa = \sqrt{\frac{D}{\rho h v^2}},$$

together with the positive dimensionless ratio

$$\eta = \lambda/\kappa.$$

We rescale the co-ordinate axes $(x,y) = \lambda^{-1}(x_1,x_2)$ and take $q \equiv 0$, with no loss of generality. Then, Eq.(1.2) becomes

$$\hat{\triangle}\hat{\triangle}w + \eta^2 \partial_{xx} w + w = 0, \tag{1.3}$$

where $\hat{\triangle} = \partial_{xx} + \partial_{yy}$ is the Laplacian operator in the dimensionless co-ordinates x, y. The special case $\eta = 0$ corresponds to the static problem, whose solution is given in [1] and extended in [7]. The Fourier transform of $w(x,y)$ along x is defined on the real axis in the usual way

$$\mathscr{F}[w](s,y) = \overline{w}(s,y) \doteq \int_{-\infty}^{+\infty} w(x,y)\exp(isx)\mathrm{d}x$$

along with the inverse transform

$$\mathscr{F}^{-1}[\overline{w}](x,y) = w(x,y) \doteq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{w}(s,y)\exp(-isx)\mathrm{d}s.$$

In the same fashion, the unilateral (or generalized) transforms are introduced:

$$\mathscr{F}^+[w](s,y) = \overline{w}^+(s,y) \doteq \int_0^{+\infty} w(x,y)\exp(\imath s x)\mathrm{d}x$$

which is analytic in the complex half-plane $\mathrm{Im}(s) > -\alpha_1$ provided that $\exists\, \alpha_1 > 0$ such that $w(x,y) < W_1(y)\exp(-\alpha_1 x)$ as $x \to +\infty$ (for a study of the regularity of this transform see [10]), and

$$\mathscr{F}^-[w](s,y) = \overline{w}^-(s,y) \doteq \int_{-\infty}^{0} w(x,y)\exp(\imath s x)\mathrm{d}x,$$

which is analytic in the complex half-plane $\mathrm{Im}(s) < \alpha_2$ provided that $\exists\, \alpha_2 > 0$ such that $w(x,y) > W_2(y)\exp(\alpha_2 x)$ as $x \to -\infty$. Consequently, the bilateral Fourier transform is related to the unilateral transforms through

$$\overline{w}(s,y) = \overline{w}^+(s,y) + \overline{w}^-(s,y) \tag{1.4}$$

and it is analytic in the infinite strip $\mathscr{S} = \{s \in \mathbb{C} : -\alpha_1 < \mathrm{Im}(s) < \alpha_2\}$ containing the real axis. Taking the Fourier transform of Eq.(1.3) in the x variable, a linear constant coefficient ODE is obtained whose general solution is

$$\begin{aligned}\overline{w}(s,y) = A_1\exp(-\sqrt{\Lambda_1}|y|) + B_1\exp(\sqrt{\Lambda_1}|y|)\\ + A_2\exp(-\sqrt{\Lambda_2}|y|) + B_2\exp(\sqrt{\Lambda_2}|y|),\end{aligned}$$

wherein

$$\Lambda_{1,2} = s^2 \mp \sqrt{\eta^2 s^2 - 1}, \tag{1.5}$$

such that $\Lambda_1\Lambda_2 = s^4 - \eta^2 s^2 + 1$. Hereinafter, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote the real and the imaginary part of $s \in \mathbb{C}$, respectively, and a superscript asterisk denotes complex conjugation, i.e. $s^* = \mathrm{Re}(s) - \imath\,\mathrm{Im}(s)$. Let $w(x,y^+)$ $(w(x,y^-))$ be the restriction of the displacement $w(x,y)$ in the upper (lower) half of the (x,y)-plane, respectively. It is understood that $y^+ \in (0,+\infty)$ and $y^- \in (-\infty,0)$. The general solution of the ODE (1.3), bounded at infinity, is

$$\overline{w}(s,y^\pm) = A_1^\pm\exp(-\sqrt{\Lambda_1}|y|) + A_2^\pm\exp(-\sqrt{\Lambda_2}|y|), \tag{1.6}$$

where $A_1^\pm, A_2^\pm$ are four *complex-valued* functions of s to be determined. The square root in (1.6) is made defined by choosing the Riemann sheet such that $\mathrm{Re}(\sqrt{\Lambda_{1,2}}) > 0$. Besides, we let the shorthand notation for the restriction of (1.6) to the x-axis

$$\overline{w}_{0^\pm}(s) = A_1^\pm + A_2^\pm \tag{1.7}$$

and let $A_i^\pm$ split into symmetric and skew-symmetric parts

$$A_i^\pm = \tfrac{1}{2}\left(\overline{A}_i \pm \Delta A_i\right), \quad i = 1,2. \tag{1.8}$$

1.2 Boundary conditions

Let the bending moment and equivalent shearing force (deprived of the factor D)

$$m = -(\partial_{yy} + \nu\partial_{xx})\,w, \quad v = -\partial_y\,[\partial_{yy} + (2-\nu)\partial_{xx}]\,w,$$

together with the slope

$$\phi = \partial_y w.$$

The boundary conditions (BCs) across the line $y=0$ ahead of the crack tip are of the kinematic type as they warrant continuity for displacement and slope

$$[\![w_0(x)]\!] = [\![\phi_0(x)]\!] = 0, \quad x > 0, \tag{1.9}$$

and of the static type, for they demand continuity for bending moment and equivalent shearing force

$$[\![m_0(x)]\!] = [\![v_0(x)]\!] = 0, \quad x > 0. \tag{1.10}$$

Here, $[\![f(0)]\!]$ denotes the jump of the function $f(y)$ across $y=0$, namely $f(0^+) - f(0^-)$, while a subscript zero stands for evaluation on the real axis. Conversely, it is assumed that the crack flanks are loaded in a continuous fashion by a harmonic loading. Then, the BCs at the crack line $y=0$ are

$$m(x,0^{\pm}) = M_0 \exp(\imath a x), \quad v(x,0^{\pm}) = V_0 \exp(\imath a x), \quad x < 0, \tag{1.11}$$

where $M_0 = M_0(a)$, $V_0 = V_0(a)$ and

$$\mathrm{Im}(a) < 0 \tag{1.12}$$

to ensure a decay condition as $x \to -\infty$. We observe that the system (1.11), just like the system (1.9,1.10), entails four conditions, for it applies at both flanks of the crack (denoted by $y = 0^{\pm}$). As a consequence of loading continuity, Eqs.(1.10) hold on the entire real line. Furthermore, we note that a similar problem arises in the realm of couple-stress theory, although with a different set of boundary conditions [9, 5, 8].

Eq.(1.9) may be written in terms of the generalized plus transform as

$$[\![\overline{w}_0^+]\!] = [\![\overline{\phi}_0^+]\!] = 0,$$

whence, using the general solution (1.6) and in view of Eqs.(1.4, 1.8), it is

$$\Delta A_1 + \Delta A_2 = [\![\overline{w}_0^-]\!], \tag{1.13a}$$

$$-\sqrt{\Lambda_1}\overline{A}_1 - \sqrt{\Lambda_2}\overline{A}_2 = [\![\overline{\phi}_0^-]\!]. \tag{1.13b}$$

Likewise, the Fourier unilateral transform of Eqs.(1.11) gives

$$\overline{m}_0^- = -\imath\frac{M_0}{s+a}, \quad \overline{v}_0^- = -\imath\frac{V_0}{s+a},$$

in the strip $\mathscr{S}_1 = \{s : \mathrm{Im}(s) < -\mathrm{Im}(a)\}$. In particular, inequality (1.12) guarantees the existence of the minus transform up to a little above the real axis in the lower complex half-plane. Thus

$$-\left[(\Lambda_1 - \nu s^2)A_1^{\pm} + (\Lambda_2 - \nu s^2)A_2^{\pm}\right] = \overline{m}_0^{+} - \imath\frac{M_0}{s+a}, \tag{1.14a}$$

$$\pm\left\{\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]A_1^{\pm} + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]A_2^{\pm}\right\} = \overline{v}_0^{+} - \imath\frac{V_0}{s+a}, \tag{1.14b}$$

in the strip $\mathscr{S}_0 = \mathscr{S} \cap \mathscr{S}_1$. Finally, the Fourier transforming Eqs.(1.10), which are holding on the entire real line, gives

$$(\Lambda_1 - \nu s^2)\Delta A_1 + (\Lambda_2 - \nu s^2)\Delta A_2 = 0, \tag{1.15a}$$

$$\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\overline{A}_1 + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\overline{A}_2 = 0, \tag{1.15b}$$

according to which the system (1.14) becomes

$$-\tfrac{1}{2}\left[(\Lambda_1 - \nu s^2)\overline{A}_1 + (\Lambda_2 - \nu s^2)\overline{A}_2\right] = \overline{m}_0^{+} - \imath\frac{M_0}{s+a}, \tag{1.16a}$$

$$\tfrac{1}{2}\left\{\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\Delta A_1 + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\Delta A_2\right\} = \overline{v}_{y0}^{+} - \imath\frac{V_0}{s+a}. \tag{1.16b}$$

Conditions (1.15) are immediately fulfilled through letting

$$\begin{aligned}
\Delta A_1 &= -(\Lambda_2 - \nu s^2)\Delta A,\\
\Delta A_2 &= (\Lambda_1 - \nu s^2)\Delta A,\\
\overline{A}_1 &= -\sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\overline{A},\\
\overline{A}_2 &= \sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\overline{A}.
\end{aligned}$$

Then, Eqs.(1.13) become

$$(\Lambda_1 - \Lambda_2)\,\Delta A = [\![\overline{w}_0^{-}]\!], \tag{1.18a}$$

$$-\sqrt{\Lambda_1\Lambda_2}\,(\Lambda_1 - \Lambda_2)\,\overline{A} = [\![\overline{\phi}_0^{-}]\!]. \tag{1.18b}$$

Likewise, the system (1.16) gives

$$(\Lambda_2 - \Lambda_1)\,K(s)\overline{A} = \overline{m}_0^{+} - \imath\frac{M_0}{s+a}, \tag{1.19a}$$

$$(\Lambda_2 - \Lambda_1)\,K(s)\Delta A = \overline{v}_0^{+} - \imath\frac{V_0}{s+a}, \tag{1.19b}$$

where the *kernel function* $K(s)$ is let as follows:

$$2\left(\Lambda_2-\Lambda_1\right)K(s)=-\sqrt{\Lambda_1}\left[\Lambda_1-(2-\nu)s^2\right]\left(\Lambda_2-\nu s^2\right)\\ +\sqrt{\Lambda_2}\left[\Lambda_2-(2-\nu)s^2\right]\left(\Lambda_1-\nu s^2\right). \quad (1.20)$$

With the help of Eq.(1.5), Eq.(1.20) may be rewritten as

$$4\sqrt{\eta^2s^2-1}K(s)=\sqrt{s^2-\sqrt{\eta^2s^2-1}}\left(\nu_0s^2+\sqrt{\eta^2s^2-1}\right)^2\\ -\sqrt{s^2+\sqrt{\eta^2s^2-1}}\left(\nu_0s^2-\sqrt{\eta^2s^2-1}\right)^2, \quad (1.21)$$

having let $\nu_0=1-\nu$. In particular, in the limit as $\eta\to0$ and with $\sqrt{\eta^2s^2-1}\to\imath$, the kernel $4\imath K(s)$ in Eq.(1.21) reduces to Eq.(24) of [1]. Solving the system (1.18) for the unknown functions $\overline{A},\Delta A$ and plugging the result in Eqs.(1.19) provides the following two uncoupled Wiener-Hopf (W-H) equations, namely

$$K(s)[\![\overline{w}_0^-]\!]+\overline{v}_0^+=\imath\frac{V_0}{s+a},$$
$$(\Lambda_1\Lambda_2)^{-1/2}K(s)[\![\overline{\phi}_0^-]\!]-\overline{m}_0^+=-\imath\frac{M_0}{s+a}.$$

Making use of Eq.(1.5), this system of functional equations may be rewritten as

$$K(s)[\![\overline{w}_0^-]\!]+\overline{v}_0^+=\imath\frac{V_0}{s+a}, \quad (1.22a)$$
$$\frac{K(s)}{\sqrt{s^2-\beta^2}\sqrt{s^2-\beta^{*2}}}[\![\overline{\phi}_0^-]\!]-\overline{m}_0^+=-\imath\frac{M_0}{s+a}, \quad (1.22b)$$

having taken the factor decomposition $\Lambda_1\Lambda_2=(s^2-\beta^2)(s^2-\beta^{*2})$, with

$$\beta=\sqrt{\tfrac{1}{2}\eta^2+\sqrt{\left(\tfrac{1}{2}\eta^2\right)^2-1}}.$$

It is observed that β is a complex number with unit modulus and it is located in the first quadrant of the complex plane inasmuch as $0\le\eta<\sqrt{2}$. Alternatively, when $\eta\ge\sqrt{2}$, we define the positive real numbers $\beta_1\ge\beta_2$,

$$\beta_{1,2}=\sqrt{\tfrac{1}{2}\eta^2\pm\sqrt{\left(\tfrac{1}{2}\eta^2\right)^2-1}}$$

and the following manipulations still hold formally, with the understanding that β stands for β_1 and β^* stands for β_2. Furthermore, we observe that

$$\beta\beta^*=\beta_1\beta_2=1.$$

1.3 Wiener-Hopf factorization

The kernel $K(s)$ is an even function and it possesses 6 roots

$$K(s) = 0 \text{ for } s = \pm s_1, \pm s_1^*, \text{ and } s = \pm \imath r_1, \tag{1.23}$$

all of which are of multiplicity 1, in the general case. Here, s_1 is taken to sit in the first quadrant of the complex plane,

$$s_1 = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}, \tag{1.24}$$

having let $\eta_R = \sqrt{2}\gamma_e$, with

$$\gamma_e = \sqrt[4]{(1-\nu)(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1})} \tag{1.25}$$

and $\gamma_e \in [\sqrt{2}(\sqrt{5}-2)^{1/4}, 1]$ is a well-known bending edge-wave constant [4, 3, 2]. The double roots in the kernel $K(s)$ bring in four non-straight branch cuts which extend from $s = \pm\zeta_1, \pm\zeta_1^*$ to $\pm\imath\infty$.

It is observed that for $\nu = 0$, we have $\gamma_e = 1$ and the roots $\pm s_1, \pm s_1^*$ coincide with the branch points for the double roots $\pm\beta, \pm\beta^*$, whence their multiplicity goes down to $1/2$. Besides, it is easy to see that s_1 is complex-valued for $\eta < \eta_R$ (subsonic regime) and it sits in the complex plane on the circle of radius γ_e^{-1}, i.e.

$$s_1 = \gamma_e^{-1} \exp(\imath\theta/2), \quad \tan\theta = \frac{1 - (\eta/\eta_R)^4}{(\eta/\eta_R)^2}.$$

In a similar fashion, letting $\eta_S = \sqrt{2}\gamma_m$, we find the location of the purely imaginary roots $\pm\imath r_1$

$$r_1 = \gamma_m^{-1} \sqrt{\left(\frac{\eta}{\eta_S}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_S}\right)^4 + 1}},$$

where we have defined the monotonic decreasing function of ν

$$\gamma_m = \sqrt[4]{(1-\nu)(-3\nu + 1 + 2\sqrt{2\nu^2 - 2\nu + 1})},$$

We note that $\gamma_m \in [\frac{1}{\sqrt{2}}(2\sqrt{2}-1)^{1/4}, \sqrt{2}(\sqrt{5}+2)^{1/4}]$ and r_1 is a real-valued monotonic decreasing (increasing) function of ν (of η), whose minimum $r_1 = \gamma_m^{-1}$ is attained in the static case $\eta = 0$, i.e. unlike s_1, this root never reaches the real axis. In the special case of $\eta = 0$ (stationary crack), we have

$$r_1 = \gamma_e^{-1} \quad \text{and} \quad s_1 = \frac{\sqrt{2}}{2\gamma_e}(1 + \imath),$$

whence s_1 sits on the bisector of the first-third quadrants of the complex plane. In contrast, for $\eta > \eta_R$ (hypersonic regime) s_1 turns real-valued and the root landscape (1.23) switches to

$$\pm s_-, \pm s_+, \pm \imath r_1, \tag{1.26}$$

where now $0 < s_- < s_+$ are real values

$$s_\mp = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 \mp \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}. \tag{1.27}$$

Let us define, for any value $b > \alpha_1 = \max\left(\mathrm{Im}(s_1), r_1\right)$,

$$F(s) = \frac{\sqrt{s - \imath b}\sqrt{s + \imath b}}{c(s - s_1)(s + s_1)(s - s_1^*)(s + s_1^*)} K(s), \quad \nu \neq 0, \tag{1.28}$$

along with the constant $c = \nu_0(3 + \nu)/4$. In the special case $\nu = 0$, we need set

$$F(s) = \frac{\sqrt{s - \imath b}\sqrt{s + \imath b}}{c\sqrt{s - s_1}\sqrt{s + s_1}\sqrt{s - s_1^*}\sqrt{s + s_1^*}(s - \imath r_1)(s + \imath r_1)} K(s), \quad \nu = 0.$$

The function $F(s)$ is deprived of zeros in an semi-infinite strip of analyticity about the real axis $\mathscr{S}$, extending along the imaginary axis up to $\alpha_1 = \alpha_2 = \mathrm{Im}(s_1)$, and it is such that $\lim_{|s| \to \infty} F(s) = 1$ in this strip. The Cauchy integral theorem gives

$$\ln F(s) = \frac{1}{2\pi\imath} \oint_C \frac{\ln F(z)}{z - s} \mathrm{d}z,$$

where C may be taken as the close path in the analyticity strip consisting of two parallel infinite lines a little above and a little below the real axis while s sits within this closed path. The former contribution brings along a minus function, $F^-(s)$, the latter a plus function, $F^+(s)$, for we may define

$$F^+(s) = \exp R(s) \quad \text{and} \quad F^-(s) = F^+(-s),$$

where

$$R(s) = \frac{1}{2\pi\imath} \int_{-\infty - \imath c}^{\infty - \imath c} \frac{\ln F(z)}{z - s} \mathrm{d}z, \quad 0 < c < \alpha_1. \tag{1.29}$$

Then, provided $|\mathrm{Im}(s)| < c$, we have

$$F(s) = F^+(s) F^-(s),$$

and the system (1.22) reads

$$K^-(s)[\![\overline{w}_0^-]\!] + \frac{\overline{v}_0^+}{K^+(s)} = \imath V_0 \frac{1}{(s + a)K^+(s)}, \tag{1.30a}$$

$$\frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}[\![\bar{\phi}_0^-]\!] - \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\bar{m}_0^+ = -\imath M_0 \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{(s+a)K^+(s)}, \quad (1.30\text{b})$$

where,

$$K^\pm(s) = \sqrt{c}\frac{(s\pm s_1)(s\mp s_1^*)}{\sqrt{s\pm \imath b}}F^\pm(s)\exp(\pm\imath\pi/4),$$

with the property that $K^-(s) = K^+(-s)$. Clearly, for large values of $|s|$, we get the asymptotic behavior

$$K^\pm(s) \sim \sqrt{c}\exp(\pm\imath\pi/4)|s|^{3/2}.$$

Finally, the RHS' are split in terms of a plus and a minus function, thus giving

$$\frac{\bar{v}_0^+}{K^+(s)} - \imath\frac{V_0}{s+a}\left(\frac{1}{K^+(s)} - \frac{1}{K^-(a)}\right) = \imath\frac{V_0}{(s+a)K^-(a)} - K^-(s)[\![\bar{w}_0^-]\!], \quad (1.31\text{a})$$

$$\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\bar{m}_0^+ - \imath\frac{M_0}{s+a}\left(\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)} - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\right)$$
$$= \imath\frac{M_0}{s+a}\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)} + \frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}[\![\bar{\phi}_0^-]\!], \quad (1.31\text{b})$$

Since the LHSs (RHSs) represent two analytic functions in the upper (lower) half complex plane with a common strip of regularity, they can be analitically continued to the whole complex plane giving two entire functions $E_1(s)$ and $E_2(s)$, i.e. they are holomorphic over the whole complex plane. It is observed that both hands of Eqs.(1.31) behave like s^{-1} as $s\to\infty$, whereupon $E_1(s)\equiv E_2(s)\equiv 0$, by Liouville's theorem. Indeed,

$$w(x)\sim x^{3/2} \quad \Rightarrow \bar{w}^-(s)\sim s^{-5/2}, \qquad \phi(x)\sim x^{1/2} \Rightarrow \quad \bar{\phi}^-(s)\sim s^{-3/2},$$
$$m(x)\sim x^{-1/2} \quad \Rightarrow \bar{m}^+(s)\sim s^{-1/2}, \qquad v(x)\sim x^{-3/2} \Rightarrow \quad \bar{v}^+(s)\sim s^{1/2},$$

the latter being meaningful in a distributional sense. Thus

$$[\![\bar{w}_0^-]\!] = \imath\frac{V_0}{K^-(a)}\frac{1}{(s+a)K^-(s)}, \quad (1.32)$$

and

$$[\![\bar{\phi}_0^-]\!] = \imath M_0\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\frac{\sqrt{s-\beta}\sqrt{s+\beta^*}}{(s+a)K^-(s)}. \quad (1.33)$$

Likewise, we obtain a direct expression for the unilateral Fourier transform of bending moment and shearing force along the co-ordinate axis $y=0$, namely

$$\bar{m}_0^+ = \imath\frac{M_0}{s+a}\left(1 - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{\sqrt{s+\beta}\sqrt{s-\beta^*}}\frac{K^+(s)}{K^-(a)}\right) \quad (1.34)$$

and

$$\bar{v}_0^+ = \imath \frac{V_0}{s+a}\left(1 - \frac{K^+(s)}{K^-(a)}\right). \tag{1.35}$$

It is observed that, according to Jordan's lemma [10], Eqs.(1.32) and (1.33) satisfy both BCs (1.9) and, by the same argument, Eqs.(1.35) and (1.34) convey the conditions (1.10).

References

1. D. D. Ang, E. S. Folias, and M. L. Williams. The bending stress in a cracked plate on an elastic foundation. *J. Appl. Mech.*, 30(2):245–251, 1963.
2. J Kaplunov and A Nobili. The edge waves on a Kirchhoff plate bilaterally supported by a two-parameter elastic foundation. *Journal of Vibration and Control*, page 1077546315606838, 2015.
3. J Kaplunov, DA Prikazchikov, GA Rogerson, and MI Lashab. The edge wave on an elastically supported Kirchhoff plate. *The Journal of the Acoustical Society of America*, 136(4):1487–1490, 2014.
4. YK Konenkov. A Rayleigh-type flexural wave. *Sov. Phys. Acoust*, 6:122–123, 1960.
5. G. Mishuris, A. Piccolroaz, and E. Radi. Steady-state propagation of a mode III crack in couple stress elastic materials. *International Journal of Engineering Science*, 61(0):112 – 128, 2012.
6. A. Nobili, E. Radi, and L. Lanzoni. A cracked infinite Kirchhoff plate supported by a two-parameter elastic foundation. *Journal of the European Ceramic Society*, 34(11):2737 – 2744, 2014. Modelling and Simulation meet Innovation in Ceramics Technology.
7. A. Nobili, E. Radi, and L. Lanzoni. On the effect of the backup plate stiffness on the brittle failure of a ceramic armor. *Acta Mechanica*, pages 1–14, 2015.
8. Andrea Piccolroaz, Gennady Mishuris, and Enrico Radi. Mode iii interfacial crack in the presence of couple-stress elastic materials. *Engineering Fracture Mechanics*, 80:60–71, 2012.
9. E. Radi. On the effects of characteristic lengths in bending and torsion on mode III crack in couple stress elasticity. *International Journal of Solids and Structures*, 45(10):3033 – 3058, 2008.
10. B. W. Roos. *Analytic functions and distributions in physics and engineering*. John Wiley & Sons, 1969.