Chapter 1
A Wiener-Hopf system of equations in the steady-state propagation of a rectilinear crack in an infinite elastic plate

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1.1 Introduction

Let us consider a Kirchhoff-Love infinite plate with thickness h, supported by a Winkler elastic foundation (Fig.1.1). The plate has a linear crack and a Cartesian reference frame is introduced such that the crack corresponds to the negative part of the x_1-axis. Besides, the crack is linearly propagating at a constant speed v (steady-state propagation) and the origin of the Cartesian reference frame is attached to and moves along with the crack tip. The governing equation for the transverse displacement of the plate w reads

$$D\triangle\triangle w + kw = -\rho h \partial_{tt} w + q, \tag{1.1}$$

being $\triangle = \partial_{x_1 x_1} + \partial_{x_2 x_2}$ the Laplace operator in two dimensions, q the transverse distributed load per unit area, D the plate bending stiffness, k the Winkler modulus and ρ the mass density per unit volume [6]. We let $w = w(x_1 - vt, x_2)$ and Eq.(1.1) may be rewritten as

$$\triangle\triangle w + \kappa^{-2} \partial_{x_1 x_1} w + \lambda^{-4} w = \frac{q}{D}, \tag{1.2}$$

having let the *characteristic lengths*

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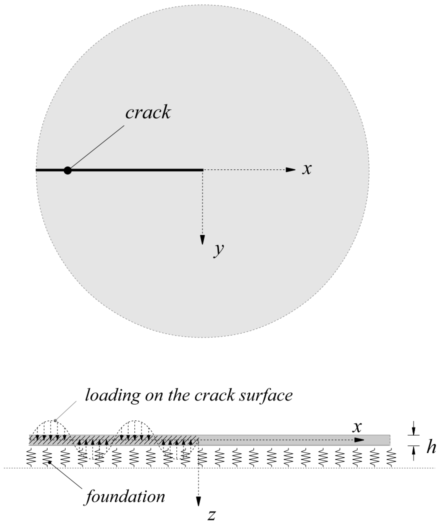

Fig. 1.1 Cracked Kirchhoff plate resting on a Winkler elastic foundation. The crack is propagating in the x direction with speed v

$$\lambda = \sqrt[4]{\frac{D}{k}}, \quad \text{and} \quad \kappa = \sqrt{\frac{D}{\rho h v^2}},$$

together with the positive dimensionless ratio

$$\eta = \lambda / \kappa.$$

We rescale the co-ordinate axes $(x,y) = \lambda^{-1}(x_1, x_2)$ and take $q \equiv 0$, with no loss of generality. Then, Eq.(1.2) becomes

$$\hat{\triangle}\hat{\triangle} w + \eta^2 \partial_{xx} w + w = 0, \tag{1.3}$$

where $\hat{\triangle} = \partial_{xx} + \partial_{yy}$ is the Laplacian operator in the dimensionless co-ordinates x, y. The special case $\eta = 0$ corresponds to the static problem, whose solution is given in [1] and extended in [7]. The Fourier transform of $w(x,y)$ along x is defined on the real axis in the usual way

$$\mathscr{F}[w](s,y) = \overline{w}(s,y) \doteq \int_{-\infty}^{+\infty} w(x,y) \exp(isx) \mathrm{d}x$$

along with the inverse transform

$$\mathscr{F}^{-1}[\overline{w}](x,y) = w(x,y) \doteq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{w}(s,y) \exp(-isx) \mathrm{d}s.$$

In the same fashion, the unilateral (or generalized) transforms are introduced:

$$\mathscr{F}^{+}[w](s,y) = \overline{w}^{+}(s,y) \doteq \int_{0}^{+\infty} w(x,y)\exp(\imath s x)\mathrm{d}x$$

which is analytic in the complex half-plane $\mathrm{Im}(s) > -\alpha_1$ provided that $\exists\,\alpha_1 > 0$ such that $w(x,y) < W_1(y)\exp(-\alpha_1 x)$ as $x \to +\infty$ (for a study of the regularity of this transform see [10]), and

$$\mathscr{F}^{-}[w](s,y) = \overline{w}^{-}(s,y) \doteq \int_{-\infty}^{0} w(x,y)\exp(\imath s x)\mathrm{d}x,$$

which is analytic in the complex half-plane $\mathrm{Im}(s) < \alpha_2$ provided that $\exists\,\alpha_2 > 0$ such that $w(x,y) > W_2(y)\exp(\alpha_2 x)$ as $x \to -\infty$. Consequently, the bilateral Fourier transform is related to the unilateral transforms through

$$\overline{w}(s,y) = \overline{w}^{+}(s,y) + \overline{w}^{-}(s,y) \tag{1.4}$$

and it is analytic in the infinite strip $\mathscr{S} = \{s \in \mathbb{C} : -\alpha_1 < \mathrm{Im}(s) < \alpha_2\}$ containing the real axis. Taking the Fourier transform of Eq.(1.3) in the x variable, a linear constant coefficient ODE is obtained whose general solution is

$$\begin{aligned}\overline{w}(s,y) = A_1\exp(-\sqrt{\Lambda_1}|y|) + B_1\exp(\sqrt{\Lambda_1}|y|)& \\ + A_2\exp(-\sqrt{\Lambda_2}|y|) + B_2\exp(\sqrt{\Lambda_2}|y|)&,\end{aligned}$$

wherein

$$\Lambda_{1,2} = s^2 \mp \sqrt{\eta^2 s^2 - 1}, \tag{1.5}$$

such that $\Lambda_1\Lambda_2 = s^4 - \eta^2 s^2 + 1$. Hereinafter, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote the real and the imaginary part of $s \in \mathbb{C}$, respectively, and a superscript asterisk denotes complex conjugation, i.e. $s^* = \mathrm{Re}(s) - \imath\,\mathrm{Im}(s)$. Let $w(x,y^+)$ ($w(x,y^-)$) be the restriction of the displacement $w(x,y)$ in the upper (lower) half of the (x,y)-plane, respectively. It is understood that $y^+ \in (0,+\infty)$ and $y^- \in (-\infty,0)$. The general solution of the ODE (1.3), bounded at infinity, is

$$\overline{w}(s,y^{\pm}) = A_1^{\pm}\exp(-\sqrt{\Lambda_1}|y|) + A_2^{\pm}\exp(-\sqrt{\Lambda_2}|y|), \tag{1.6}$$

where $A_1^{\pm}, A_2^{\pm}$ are four *complex-valued* functions of s to be determined. The square root in (1.6) is made defined by choosing the Riemann sheet such that $\mathrm{Re}(\sqrt{\Lambda_{1,2}}) > 0$. Besides, we let the shorthand notation for the restriction of (1.6) to the x-axis

$$\overline{w}_{0^{\pm}}(s) = A_1^{\pm} + A_2^{\pm} \tag{1.7}$$

and let $A_i^{\pm}$ split into symmetric and skew-symmetric parts

$$A_i^{\pm} = \tfrac{1}{2}\left(\overline{A}_i \pm \Delta A_i\right), \quad i = 1,2. \tag{1.8}$$

1.2 Boundary conditions

Let the bending moment and equivalent shearing force (deprived of the factor D)

$$m = -\left(\partial_{yy} + \nu\partial_{xx}\right) w, \quad v = -\partial_y \left[\partial_{yy} + (2-\nu)\partial_{xx}\right] w,$$

together with the slope

$$\phi = \partial_y w.$$

The boundary conditions (BCs) across the line $y = 0$ ahead of the crack tip are of the kinematic type as they warrant continuity for displacement and slope

$$[\![w_0(x)]\!] = [\![\phi_0(x)]\!] = 0, \quad x > 0, \tag{1.9}$$

and of the static type, for they demand continuity for bending moment and equivalent shearing force

$$[\![m_0(x)]\!] = [\![v_0(x)]\!] = 0, \quad x > 0. \tag{1.10}$$

Here, $[\![f(0)]\!]$ denotes the jump of the function $f(y)$ across $y = 0$, namely $f(0^+) - f(0^-)$, while a subscript zero stands for evaluation on the real axis. Conversely, it is assumed that the crack flanks are loaded in a continuous fashion by a harmonic loading. Then, the BCs at the crack line $y = 0$ are

$$m(x, 0^\pm) = M_0 \exp(\imath a x), \quad v(x, 0^\pm) = V_0 \exp(\imath a x), \quad x < 0, \tag{1.11}$$

where $M_0 = M_0(a)$, $V_0 = V_0(a)$ and

$$\mathrm{Im}(a) < 0 \tag{1.12}$$

to ensure a decay condition as $x \to -\infty$. We observe that the system (1.11), just like the system (1.9,1.10), entails four conditions, for it applies at both flanks of the crack (denoted by $y = 0^\pm$). As a consequence of loading continuity, Eqs.(1.10) hold on the entire real line. Furthermore, we note that a similar problem arises in the realm of couple-stress theory, although with a different set of boundary conditions [9, 5, 8].

Eq.(1.9) may be written in terms of the generalized plus transform as

$$[\![\overline{w}_0^+]\!] = [\![\overline{\phi}_0^+]\!] = 0,$$

whence, using the general solution (1.6) and in view of Eqs.(1.4, 1.8), it is

$$\Delta A_1 + \Delta A_2 = [\![\overline{w}_0^-]\!], \tag{1.13a}$$

$$-\sqrt{\Lambda_1}\overline{A}_1 - \sqrt{\Lambda_2}\overline{A}_2 = [\![\overline{\phi}_0^-]\!]. \tag{1.13b}$$

Likewise, the Fourier unilateral transform of Eqs.(1.11) gives

$$\overline{m}_0^- = -\imath\frac{M_0}{s+a}, \quad \overline{v}_0^- = -\imath\frac{V_0}{s+a},$$

in the strip $\mathscr{S}_1 = \{s : \mathrm{Im}(s) < -\mathrm{Im}(a)\}$. In particular, inequality (1.12) guarantees the existence of the minus transform up to a little above the real axis in the lower complex half-plane. Thus

$$-\left[(\Lambda_1 - \nu s^2)A_1^{\pm} + (\Lambda_2 - \nu s^2)A_2^{\pm}\right] = \overline{m}_0^{+} - \imath \frac{M_0}{s+a}, \tag{1.14a}$$

$$\pm\left\{\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]A_1^{\pm} + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]A_2^{\pm}\right\} = \overline{v}_0^{+} - \imath \frac{V_0}{s+a}, \tag{1.14b}$$

in the strip $\mathscr{S}_0 = \mathscr{S} \cap \mathscr{S}_1$. Finally, the Fourier transforming Eqs.(1.10), which are holding on the entire real line, gives

$$(\Lambda_1 - \nu s^2)\Delta A_1 + (\Lambda_2 - \nu s^2)\Delta A_2 = 0, \tag{1.15a}$$

$$\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\overline{A}_1 + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\overline{A}_2 = 0, \tag{1.15b}$$

according to which the system (1.14) becomes

$$-\tfrac{1}{2}\left[(\Lambda_1 - \nu s^2)\overline{A}_1 + (\Lambda_2 - \nu s^2)\overline{A}_2\right] = \overline{m}_0^{+} - \imath \frac{M_0}{s+a}, \tag{1.16a}$$

$$\tfrac{1}{2}\left\{\sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\Delta A_1 + \sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\Delta A_2\right\} = \overline{v}_{y0}^{+} - \imath \frac{V_0}{s+a}. \tag{1.16b}$$

Conditions (1.15) are immediately fulfilled through letting

$$\begin{aligned}
\Delta A_1 &= -(\Lambda_2 - \nu s^2)\Delta A, \\
\Delta A_2 &= (\Lambda_1 - \nu s^2)\Delta A, \\
\overline{A}_1 &= -\sqrt{\Lambda_2}\left[\Lambda_2 - (2-\nu)s^2\right]\overline{A}, \\
\overline{A}_2 &= \sqrt{\Lambda_1}\left[\Lambda_1 - (2-\nu)s^2\right]\overline{A}.
\end{aligned}$$

Then, Eqs.(1.13) become

$$(\Lambda_1 - \Lambda_2)\,\Delta A = [\![\overline{w}_0^{-}]\!], \tag{1.18a}$$

$$-\sqrt{\Lambda_1 \Lambda_2}\,(\Lambda_1 - \Lambda_2)\,\overline{A} = [\![\overline{\phi}_0^{-}]\!]. \tag{1.18b}$$

Likewise, the system (1.16) gives

$$(\Lambda_2 - \Lambda_1)\,K(s)\overline{A} = \overline{m}_0^{+} - \imath \frac{M_0}{s+a}, \tag{1.19a}$$

$$(\Lambda_2 - \Lambda_1)\,K(s)\Delta A = \overline{v}_0^{+} - \imath \frac{V_0}{s+a}, \tag{1.19b}$$

where the *kernel function* $K(s)$ is let as follows:

$$2(\Lambda_2-\Lambda_1)K(s) = -\sqrt{\Lambda_1}\left[\Lambda_1-(2-\nu)s^2\right](\Lambda_2-\nu s^2) \\ +\sqrt{\Lambda_2}\left[\Lambda_2-(2-\nu)s^2\right](\Lambda_1-\nu s^2). \quad (1.20)$$

With the help of Eq.(1.5), Eq.(1.20) may be rewritten as

$$4\sqrt{\eta^2 s^2-1}K(s) = \sqrt{s^2-\sqrt{\eta^2 s^2-1}}\left(\nu_0 s^2+\sqrt{\eta^2 s^2-1}\right)^2 \\ -\sqrt{s^2+\sqrt{\eta^2 s^2-1}}\left(\nu_0 s^2-\sqrt{\eta^2 s^2-1}\right)^2, \quad (1.21)$$

having let $\nu_0 = 1-\nu$. In particular, in the limit as $\eta \to 0$ and with $\sqrt{\eta^2 s^2-1} \to \imath$, the kernel $4\imath K(s)$ in Eq.(1.21) reduces to Eq.(24) of [1]. Solving the system (1.18) for the unknown functions $\overline{A}, \Delta A$ and plugging the result in Eqs.(1.19) provides the following two uncoupled Wiener-Hopf (W-H) equations, namely

$$K(s)[\![\overline{w}_0^-]\!]+\overline{v}_0^+ = \imath\frac{V_0}{s+a},$$
$$(\Lambda_1\Lambda_2)^{-1/2}K(s)[\![\overline{\phi}_0^-]\!]-\overline{m}_0^+ = -\imath\frac{M_0}{s+a}.$$

Making use of Eq.(1.5), this system of functional equations may be rewritten as

$$K(s)[\![\overline{w}_0^-]\!]+\overline{v}_0^+ = \imath\frac{V_0}{s+a}, \quad (1.22a)$$
$$\frac{K(s)}{\sqrt{s^2-\beta^2}\sqrt{s^2-\beta^{*2}}}[\![\overline{\phi}_0^-]\!]-\overline{m}_0^+ = -\imath\frac{M_0}{s+a}, \quad (1.22b)$$

having taken the factor decomposition $\Lambda_1\Lambda_2 = (s^2-\beta^2)(s^2-\beta^{*2})$, with

$$\beta = \sqrt{\tfrac{1}{2}\eta^2+\sqrt{\left(\tfrac{1}{2}\eta^2\right)^2-1}}.$$

It is observed that β is a complex number with unit modulus and it is located in the first quadrant of the complex plane inasmuch as $0 \le \eta < \sqrt{2}$. Alternatively, when $\eta \ge \sqrt{2}$, we define the positive real numbers $\beta_1 \ge \beta_2$,

$$\beta_{1,2} = \sqrt{\tfrac{1}{2}\eta^2 \pm \sqrt{\left(\tfrac{1}{2}\eta^2\right)^2-1}}$$

and the following manipulations still hold formally, with the understanding that β stands for β_1 and β^* stands for β_2. Furthermore, we observe that

$$\beta\beta^* = \beta_1\beta_2 = 1.$$

1.3 Wiener-Hopf factorization

The kernel $K(s)$ is an even function and it possesses 6 roots

$$K(s) = 0 \text{ for } s = \pm s_1, \pm s_1^*, \text{ and } s = \pm \imath r_1, \tag{1.23}$$

all of which are of multiplicity 1, in the general case. Here, s_1 is taken to sit in the first quadrant of the complex plane,

$$s_1 = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}, \tag{1.24}$$

having let $\eta_R = \sqrt{2}\gamma_e$, with

$$\gamma_e = \sqrt[4]{(1-\nu)(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1})} \tag{1.25}$$

and $\gamma_e \in [\sqrt{2}(\sqrt{5}-2)^{1/4}, 1]$ is a well-known bending edge-wave constant [4, 3, 2]. The double roots in the kernel $K(s)$ bring in four non-straight branch cuts which extend from $s = \pm\zeta_1, \pm\zeta_1^*$ to $\pm\imath\infty$.

It is observed that for $\nu = 0$, we have $\gamma_e = 1$ and the roots $\pm s_1, \pm s_1^*$ coincide with the branch points for the double roots $\pm\beta, \pm\beta^*$, whence their multiplicity goes down to $1/2$. Besides, it is easy to see that s_1 is complex-valued for $\eta < \eta_R$ (subsonic regime) and it sits in the complex plane on the circle of radius γ_e^{-1}, i.e.

$$s_1 = \gamma_e^{-1} \exp(\imath\theta/2), \quad \tan\theta = \frac{1 - (\eta/\eta_R)^4}{(\eta/\eta_R)^2}.$$

In a similar fashion, letting $\eta_S = \sqrt{2}\gamma_m$, we find the location of the purely imaginary roots $\pm\imath r_1$

$$r_1 = \gamma_m^{-1} \sqrt{\left(\frac{\eta}{\eta_S}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_S}\right)^4 + 1}},$$

where we have defined the monotonic decreasing function of ν

$$\gamma_m = \sqrt[4]{(1-\nu)(-3\nu + 1 + 2\sqrt{2\nu^2 - 2\nu + 1})},$$

We note that $\gamma_m \in [\frac{1}{\sqrt{2}}(2\sqrt{2}-1)^{1/4}, \sqrt{2}(\sqrt{5}+2)^{1/4}]$ and r_1 is a real-valued monotonic decreasing (increasing) function of ν (of η), whose minimum $r_1 = \gamma_m^{-1}$ is attained in the static case $\eta = 0$, i.e. unlike s_1, this root never reaches the real axis. In the special case of $\eta = 0$ (stationary crack), we have

$$r_1 = \gamma_e^{-1} \quad \text{and} \quad s_1 = \frac{\sqrt{2}}{2\gamma_e}(1 + \imath),$$

whence s_1 sits on the bisector of the first-third quadrants of the complex plane. In contrast, for $\eta > \eta_R$ (hypersonic regime) s_1 turns real-valued and the root landscape (1.23) switches to

$$\pm s_-, \pm s_+, \pm \imath r_1, \tag{1.26}$$

where now $0 < s_- < s_+$ are real values

$$s_\mp = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 \mp \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}. \tag{1.27}$$

Let us define, for any value $b > \alpha_1 = \max(\mathrm{Im}(s_1), r_1)$,

$$F(s) = \frac{\sqrt{s - \imath b}\sqrt{s + \imath b}}{c(s - s_1)(s + s_1)(s - s_1^*)(s + s_1^*)} K(s), \quad \nu \neq 0, \tag{1.28}$$

along with the constant $c = \nu_0(3 + \nu)/4$. In the special case $\nu = 0$, we need set

$$F(s) = \frac{\sqrt{s - \imath b}\sqrt{s + \imath b}}{c\sqrt{s - s_1}\sqrt{s + s_1}\sqrt{s - s_1^*}\sqrt{s + s_1^*}(s - \imath r_1)(s + \imath r_1)} K(s), \quad \nu = 0.$$

The function $F(s)$ is deprived of zeros in an semi-infinite strip of analyticity about the real axis $\mathscr{S}$, extending along the imaginary axis up to $\alpha_1 = \alpha_2 = \mathrm{Im}(s_1)$, and it is such that $\lim_{|s| \to \infty} F(s) = 1$ in this strip. The Cauchy integral theorem gives

$$\ln F(s) = \frac{1}{2\pi\imath} \oint_C \frac{\ln F(z)}{z - s} \mathrm{d}z,$$

where C may be taken as the close path in the analyticity strip consisting of two parallel infinite lines a little above and a little below the real axis while s sits within this closed path. The former contribution brings along a minus function, $F^-(s)$, the latter a plus function, $F^+(s)$, for we may define

$$F^+(s) = \exp R(s) \quad \text{and} \quad F^-(s) = F^+(-s),$$

where

$$R(s) = \frac{1}{2\pi\imath} \int_{-\infty - \imath c}^{\infty - \imath c} \frac{\ln F(z)}{z - s} \mathrm{d}z, \quad 0 < c < \alpha_1. \tag{1.29}$$

Then, provided $|\mathrm{Im}(s)| < c$, we have

$$F(s) = F^+(s) F^-(s),$$

and the system (1.22) reads

$$K^-(s)[\![\overline{w}_0^-]\!] + \frac{\overline{v}_0^+}{K^+(s)} = \imath V_0 \frac{1}{(s + a)K^+(s)}, \tag{1.30a}$$

$$\frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}[\![\overline{\phi}_0^-]\!] - \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\overline{m}_0^+ = -\imath M_0 \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{(s+a)K^+(s)}, \quad (1.30b)$$

where,

$$K^\pm(s) = \sqrt{c}\frac{(s\pm s_1)(s\mp s_1^*)}{\sqrt{s\pm \imath b}}F^\pm(s)\exp(\pm\imath\pi/4),$$

with the property that $K^-(s) = K^+(-s)$. Clearly, for large values of $|s|$, we get the asymptotic behavior

$$K^\pm(s) \sim \sqrt{c}\exp(\pm\imath\pi/4)|s|^{3/2}.$$

Finally, the RHS' are split in terms of a plus and a minus function, thus giving

$$\frac{\overline{v}_0^+}{K^+(s)} - \imath\frac{V_0}{s+a}\left(\frac{1}{K^+(s)} - \frac{1}{K^-(a)}\right) = \imath\frac{V_0}{(s+a)K^-(a)} - K^-(s)[\![\overline{w}_0^-]\!], \quad (1.31a)$$

$$\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\overline{m}_0^+ - \imath\frac{M_0}{s+a}\left(\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)} - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\right)$$
$$= \imath\frac{M_0}{s+a}\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)} + \frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}[\![\overline{\phi}_0^-]\!], \quad (1.31b)$$

Since the LHSs (RHSs) represent two analytic functions in the upper (lower) half complex plane with a common strip of regularity, they can be analitically continued to the whole complex plane giving two entire functions $E_1(s)$ and $E_2(s)$, i.e. they are holomorphic over the whole complex plane. It is observed that both hands of Eqs.(1.31) behave like s^{-1} as $s\to\infty$, whereupon $E_1(s)\equiv E_2(s)\equiv 0$, by Liouville's theorem. Indeed,

$$\begin{aligned} w(x) &\sim x^{3/2} &&\Rightarrow \overline{w}^-(s)\sim s^{-5/2}, & \phi(x) &\sim x^{1/2} \Rightarrow & \overline{\phi}^-(s) &\sim s^{-3/2},\\ m(x) &\sim x^{-1/2} &&\Rightarrow \overline{m}^+(s)\sim s^{-1/2}, & v(x) &\sim x^{-3/2} \Rightarrow & \overline{v}^+(s) &\sim s^{1/2}, \end{aligned}$$

the latter being meaningful in a distributional sense. Thus

$$[\![\overline{w}_0^-]\!] = \imath\frac{V_0}{K^-(a)}\frac{1}{(s+a)K^-(s)}, \quad (1.32)$$

and

$$[\![\overline{\phi}_0^-]\!] = \imath M_0\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\frac{\sqrt{s-\beta}\sqrt{s+\beta^*}}{(s+a)K^-(s)}. \quad (1.33)$$

Likewise, we obtain a direct expression for the unilateral Fourier transform of bending moment and shearing force along the co-ordinate axis $y=0$, namely

$$\overline{m}_0^+ = \imath\frac{M_0}{s+a}\left(1 - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{\sqrt{s+\beta}\sqrt{s-\beta^*}}\frac{K^+(s)}{K^-(a)}\right) \quad (1.34)$$

and

$$\overline{v}_0^+ = \imath \frac{V_0}{s+a}\left(1 - \frac{K^+(s)}{K^-(a)}\right). \tag{1.35}$$

It is observed that, according to Jordan's lemma [10], Eqs.(1.32) and (1.33) satisfy both BCs (1.9) and, by the same argument, Eqs.(1.35) and (1.34) convey the conditions (1.10).

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