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AN APPROXIMATION SOLVABILITY METHOD FOR NONLOCAL SEMILINEAR DIFFERENTIAL PROBLEMS IN BANACH SPACES

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Abstract. A new approximation solvability method is developed for the study of semilinear differential equations with nonlocal conditions without the compactness of the semigroup and of the nonlinearity. The method is based on the Yosida approximations of the generator of C_0-semigroup, the continuation principle, and the weak topology. It is shown how the abstract result can be applied to study the reaction-diffusion models.

1. **Introduction.** The paper deals with the nonlocal problem for semilinear differential equations of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \text{ for a.e. } t \in [0, T], \\ u(0) = Mu, \end{cases} \tag{1}$$

in a reflexive Banach space E having a Schauder basis satisfying the property (π_1) and a strictly convex dual space E^*, where $A\colon D(A) \subset E \to E$ is the generator of a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$; $f\colon [0,T] \times E \to E$ and $M\colon C([0,T]; E) \to E$.

The nonlocal problem for a semilinear differential equation with a C_0-semigroup generator was first studied by L. Byszewski [15]. The technique used in [15] is based on the Banach fixed point theorem for contraction mappings. Then others fixed point theorems (for example, Leray-Schauder fixed point theorem for compact mappings, fixed point theorem for condensing mappings, Kakutani fixed point theorem for multivalued mappings...) are used for the study of semilinear differential equations and inclusions in Banach spaces with various boundary conditions (see, e.g. [5], [18] and [26] and the references therein).

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In the present paper, we introduce a new method for the study of problem (1) which is the combination of the Yosida approximations of the generator of C_0-semigroups and an *approximation solvability method*. This method is a generalization of the well known *strong approximation* used by F.E. Browder and W.V. Petryshyn in [14] and by W.V. Petryshyn in [23]. It was developed in [20] for the study of periodic oscillations of differential inclusions and then it was combined with the bounding function method to investigate differential equations with nonlocal conditions in Hilbert spaces (see [6] and [7]). In this paper we extend it to Banach spaces.

The proofs of our main results are based on a continuation principle in Banach spaces due to Andres-Górniewicz [3]. The employment of fixed point theorems or continuation principles requires strong compactness conditions, which are usually not satisfied in an infinite dimensional framework, if the evolution operator associated to A fails to be compact. In this paper making use of weak topologies we avoid any compactness assumptions on both the nonlinear term and on the C_0-semigroup. Weak topology was first exploited to prove existence results of problems of type (1) in [9] and in [10]. There, the proofs are based on a continuation principle in Fréchet spaces. It requires to prove the *transversality condition*, known as *pushing condition* (introduced in [17]), which is more difficult than proving the corresponding condition in a Banach space. Unlike these results, due to the approximation scheme which enables to reduce to a finite dimensional setting, in this paper we are able to consider the usual transversality condition (see (A2) in Section 3). Thus, we can handle nonlinearities with superlinear growth and we obtain, as a by-result, the existence of a solution in a prescribed bounded set. Furthermore, we prove the existence of a bounded solution of the equation in (1) on the whole half-line $[0, +\infty)$.

Finally, we show how the abstract results can be applied to study various reaction-diffusion models. More precisely, with our techniques we can handle diffusion problems of the form

$$\frac{\partial u(t,\xi)}{\partial t} = \Delta u(t,\xi) + f\left(t,\xi,u,\int_\Omega k(\xi,\eta)u(t,\eta)\,d\eta\right)$$

in an open bounded domain with sufficiently regular boundary, with Neumann or Dirichlet conditions on the boundary and various boundary conditions. We consider three possible examples of map $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ arising from various mathematical models in applied sciences. In particular, in the case of $f\left(t,\xi,u,\int_\Omega k(\xi,\eta)\,d\eta\right) = h\left(t,\xi,\int_\Omega k(\xi,\eta)u(t,\eta)\,d\eta\right)$ we obtain a generalized version of the nonlocal FKPP equation. The same problem was studied also in [8] under different assumptions and techniques.

The paper is organized as follows. In Section 2 we recall some notions and notation from the theory of functional analysis. The main results are presented in Section 3, in which first we prove the theorems of existence and uniqueness of mild solutions (see Theorem 3.1 and Theorem 3.2) to problem (1) on compact intervals, then the existence and uniqueness of bounded mild solutions to problem (1) on the half line for the case when $f : [0, +\infty) \times E \to E$ (see Theorem 3.3 and Theorem 3.4). In Section 4, applications to various reaction-diffusion models are considered.

2. **Preliminaries.**

Throughout this paper, I represents the real interval $[0, T]$. By E^ω and $\langle \cdot, \cdot \rangle$ we denote respectively the space E endowed with the weak topology and the dual product between E and its dual.

We recall that a sequence $\{e_n\} \subset E$ is a Schauder basis for E if for every $x \in E$ there exists a unique sequence of scalars $\{\alpha_n(x)\} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} \alpha_i(x) e_i \right\| = 0.$$

Given $\{e_n\}_{n=1}^{\infty}$ a Schauder basis of E, for every $n \in \mathbb{N}$ let E_n be the $n-$dimensional subspace of E generated by the basis $\{e_k\}_{k=1}^{n}$ and $\mathbb{P}_n : E \to E_n$ be the projection of E onto E_n. We recall in particular that $\mathbb{P}_n x = \sum_{k=1}^{n} \alpha_k(x) e_k$ for every $x \in E$ with the coefficients $\{\alpha_k(x)\}$ linear and continuous, i.e. $\alpha_k \in E^*$, $\forall\, k \in \mathbb{N}$ (see [25, pp 18-20]), thus $\mathbb{P}_n x_j \to \mathbb{P}_n x_0$ for $x_j \rightharpoonup x_0$, i.e. $\mathbb{P}_n : E^\omega \to E_n$ is continuous. It is well known that the sequence $\{\|\mathbb{P}_n\|\}$ is bounded. The Schauder basis $\{e_n\}$ is said to satisfy property (π_1) if $\|\mathbb{P}_n\| = 1$ for every $n \in \mathbb{N}$.

Remark 1. For every $1 < p < \infty, L^p(\Omega, \mathbb{R})$ is a reflexive Banach space with topological dual the strictly convex space $L^{p'}(\Omega, \mathbb{R})$, where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover for each bounded subset $\Omega \subset \mathbb{R}^n, L^p(\Omega, \mathbb{R})$ has a Schauder basis satisfying property (π_1) (see, e.g. [13]).

Some of the main properties of the projection $\mathbb{P}_n$ are contained in the following. They were proved in [7, Lemma 6] for Hilbert spaces, but are valid also in Banach spaces.

Lemma 2.1. *The projection $\mathbb{P}_n : E \to E_n$ satisfies the following properties:*
(a) $\mathbb{P}_n : E^\omega \to E_n$ is continuous;
(b) if $x_n \rightharpoonup x$ in E then $\mathbb{P}_n x_n \rightharpoonup x$ in E.

Lemma 2.2. *If $f_n \rightharpoonup f$ in $L^1(I, E)$ then $\mathbb{P}_n f_n \rightharpoonup f$ in $L^1(I, E)$.*

Proof. Let $\Phi\colon L^1(I, E) \to \mathbb{R}$ be a linear and bounded functional. Hence, there is $\varphi \in L^\infty(I, E)$ such that

$$\Phi(g) = \int_0^T \langle g(t),\, \varphi(t) \rangle\, dt \quad \text{for all} \quad g \in L^1(I, E).$$

We have

$$\begin{aligned}
\Phi(\mathbb{P}_n f_n - f) &= \textstyle\int_0^T \langle \mathbb{P}_n f_n(t) - f(t),\, \varphi(t) \rangle\, dt \\
&= \textstyle\int_0^T \langle \mathbb{P}_n f_n(t) - \mathbb{P}_n f(t), \varphi(t) \rangle\, dt + \int_0^T \langle \mathbb{P}_n f(t) - f(t), \varphi(t) \rangle\, dt \\
&= \textstyle\int_0^T \langle f_n(t) - f(t),\, \mathbb{P}_n \varphi(t) \rangle\, dt + \int_0^T \langle \mathbb{P}_n f(t) - f(t),\, \varphi(t) \rangle\, dt \\
&= \textstyle\int_0^T \langle f_n(t) - f(t),\, \varphi(t) \rangle\, dt + \int_0^T \langle f_n(t) - f(t),\, \mathbb{P}_n \varphi(t) - \varphi(t) \rangle\, dt \\
&\quad + \textstyle\int_0^T \langle \mathbb{P}_n f(t) - f(t),\, \varphi(t) \rangle\, dt \\
&= \Phi(f_n - f) + \textstyle\int_0^T \langle f_n(t) - f(t),\, \mathbb{P}_n \varphi(t) - \varphi(t) \rangle\, dt + \Phi(\mathbb{P}_n f - f).
\end{aligned}$$

Recall that $f_n \overset{L^1(I,E)}{\rightharpoonup} f$ and trivially $\mathbb{P}_n g \overset{L^1(I,E)}{\longrightarrow} g$ for every $g \in L^1(I, E)$, therefore, $\Phi(\mathbb{P}_n f_n - f) \to 0$ as $n \to \infty$, i.e., $\mathbb{P}_n f_n \overset{L^1(I,E)}{\rightharpoonup} f$. □

We can introduce the adjoint projections as $\mathbb{P}_n^* : E^* \to E_n^*$, $\langle \mathbb{P}_n^* g, x \rangle = \langle g, \mathbb{P}_n x \rangle$, $n \in \mathbb{N}$. For every $x = \sum_{k=1}^{\infty} \alpha_k(x) e_k$, the range of $\mathbb{P}_n^*$, $\mathcal{R}(\mathbb{P}_n^*) = \text{span}\{\alpha_k\}_{k=1}^{n}$, for every $n \in \mathbb{N}$, see [25, Theorem 12.1]; moreover, being E a reflexive space it follows that $\mathbb{P}_n^* g \to g$, for every $n \in \mathbb{N}$, see [25, Corollary 12.2].

Furthermore, we denote with $J : E \to E^*$ the duality map

$$J(x) = \{ g \in E^* \ :\ \|g\| = \|x\| \text{ and } \langle x, g \rangle = \|x\|^2 \}.$$

Since E^* is a strictly convex Banach space, the duality map is single valued; moreover, since E has a basis satisfying property (π_1), it follows

$$\mathbb{P}_n^* J(x) = J(x), \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad x \in E_n$$

(see [13]).

Given $A \subset E$, let $\overline{A}$ be the closure of A, while $B_E(0,R)$ denotes the closed ball

$$B_E(0,R) = \{w \in E \colon \|w\| \leq R\}.$$

By $C(I,E)$ and $L^1(I,E)$ we denote respectively the Banach space of all continuous functions $x \colon I \to E$ with norm

$$\|x\|_C = \max_{t \in I} \|x(t)\|,$$

and the Banach space of summable functions with norm

$$\|x\|_1 = \int_0^T \|x(t)\| \, dt.$$

A ball of radius R centered at 0 in the space $C(I,E)$ is denoted by $B_C(0,R)$. Finally, with $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ we denote the Banach space of linear and bounded operators in E.

We recall the following characterization of weak convergence in the space of continuous functions.

Theorem 2.3. *(see [12]) A sequence of continuous functions $\{x_n\}_n \rightharpoonup x \in C(I;E)$ if and only if*

(i) there exists $N > 0$ such that, for every $n \in \mathbb{N}$ and $t \in I, \|x_n(t)\| \leq N$;
(ii) for every $t \in I, x_n(t) \rightharpoonup x(t)$.

It follows that $\{x_n\}_n \rightharpoonup x \in C(I;E)$ implies that $\{x_n\}_n \rightharpoonup x \in L^1(I;E)$.

Let $S \subseteq \mathbb{R}$ be a measurable subset. A subset $A \subset L^1(S,E)$ is called uniformly integrable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\Omega \subset S$ and $\mu(\Omega) < \delta$ implies

$$\left\| \int_\Omega f \, d\mu \right\| < \varepsilon \quad \text{for all } f \in A,$$

where μ is the Lebesgue measure on $\mathbb{R}$.

We propose now the continuation principle that we use to prove the main result.

Theorem 2.4. *(see, e.g. [3]) Let Q be a closed, convex subset of a Banach space F with nonempty interior and $T \colon Q \times [0,1] \to F$ be a compact map having a closed graph such that $\mathcal{T}(Q,0) \subset intQ$ and $\mathcal{T}(\cdot,\lambda)$ is fixed points free on the boundary of Q for all $\lambda \in [0,1)$. Then there exists $y \in F$ such that $y = \mathcal{T}(y,1)$.*

3. Main results.

3.1. Existence results on compact intervals.

We will study the existence of mild solutions to problem (1) under the following assumptions:

$(A1)$ M is a linear and bounded operator with $\|M\| \leq 1$;
$(A2)$ there exist $R_0 > r_0 > 0$ such that

$$\langle J(z), f(t,z) \rangle \leq 0,$$

for a.e. $\in I$ and for $z \in E \colon r_0 < \|z\| < R_0$;
$(A3)$ for every $z \in E$ the function $f(\cdot,z) \colon I \to E$ is measurable;

$(A4)$ for every bounded subset $\Omega \subset E$ there exists a function $v_\Omega \in L^1(I, \mathbb{R})$ such that

$$\|f(t,z)\| \leq v_\Omega(t),$$

for a.e. $t \in I$ and all $z \in \Omega$;

$(A5)$ for a.e. $t \in I$ the function $f(t, \cdot)\colon E^\omega \to E^\omega$ is continuous;

$(A5)'$ there exist $R \in (r_0, R_0)$ and a function $L(\cdot) \in L^1(I, E)$ such that

$$\|f(t,z) - f(t,w)\| \leq L(t)\|z - w\|,$$

for a.e. $t \in I$ and all $z, w \in B_E(0, R)$, where R_0, r_0 are the constants from $(A2)$.

For $R \in (r_0, R_0)$ denote $K = B_E(0, R)$ and $Q = C(I, K)$. For each $m \in \mathbb{N}$, set $Q^{(m)} = Q \cap C(I, E_m)$.

Remark 2. Let us note that:

(a) The transversality condition $(A2)$ assures that every solution of the equation in (1) located in a suitable bounded region of E is strictly contained in such a region for $t \in (0, T]$, i.e. it isn't tangent to its boundary. This guarantees the absence of fixed points of the solution operator in the boundary of a given candidate set of solutions, which is a key ingredient when attaching solvability by means of fixed point theorems. Notice that, in the special case when E is a Hilbert space, condition $(A2)$ reads as

$$\langle z, f(t,z) \rangle \leq 0,$$

for a.e. $\in I$ and for $z \in E\colon r_0 < \|z\| < R_0$, which is an usual assumption in this setting;

(b) It is clear that under conditions $(A3) - (A5)$ [or, $(A3) - (A4)$ and $(A5)'$] for every $q \in Q$ the function $f_*(t) := f(t, q(t))$ is in the space $L^1(I, E)$;

(c) The class of boundary value problems with the operator M satisfying condition $(A1)$ is sufficiently large. In particular, it includes the following well-known problems:

- (i) $Mx = 0$ (the general Cauchy condition $x(0) = x_0$ can be replaced by condition $z(0) = 0$ by a transformation $z = x - x_0$);
- (ii) $Mx = \pm\, x(T)$ (periodic and anti-periodic problems);
- (iii) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $0 \leq t_1 < \cdots < t_{k_0} \leq T$ (multi-point problem);
- (iv) $Mx = \frac{1}{T}\int_0^T x(t)dt$ (mean value problem);
- (v) $Mx = \int_0^T w(t)x(t)dt$ with $\|w\|_1 \leq 1$ (weighted condition).

Define the continuous operator $T\colon Q \to C(I, E)$,

$$T(x)(t) = S(t)Mx + \int_0^t S(t-s)f(s, x(s))ds.$$

By *mild solutions on* Q of problem (1), we mean continuous functions $x \in Q$ such that

$$x = T(x).$$

Theorem 3.1. *Let conditions $(A1) - (A5)$ hold. Then the set of mild solutions on Q of problem (1) is nonempty and weakly compact in $C(I, E)$. If, in addition, the C_0-semigroup generated by A is compact, then the set of mild solutions on Q of problem (1) is also strongly compact in $C([a, T]; E)$ for each $a \in (0, T)$.*

Proof. For each $n \in \mathbb{N}$, let $A_n := n^2 R(n, A) - nI$ be the *Yosida approximation* of A, where $R(n, A) = (nI - A)^{-1}$ and I is the identity operator. It is well known (see, e.g. [26, Lemma 3.2.1 and 3.2.2]) that $A_n \in \mathcal{L}(E)$ and $\{e^{tA_n}\}$ is a semigroup of contractions such that

$$\lim_{n\to\infty} e^{tA_n} x = S(t)x, \tag{2}$$

for every $t \in I$ and $x \in E$. Now, for every $n \in \mathbb{N}$ consider the problem

$$\begin{cases} u'(t) = A_n u(t) + f(t, u(t)), \quad \text{for a.e. } \ t \in I, \\ u(0) = Mu. \end{cases} \tag{3}$$

To prove the existence of solutions $u \in Q$ of the problem (3) we will use the bounding function technique and the approximation solvability method. In fact, consider the auxiliary problem in $C(I, E_m)$

$$\begin{cases} u'(t) = \mathbb{P}_m A_n u(t) + \mathbb{P}_m f(t, u(t)), \\ u(0) = \mathbb{P}_m Mu. \end{cases} \tag{4}$$

Step 1. Let us show that problem (4) has a (strong) solution $u_m \in Q^{(m)}$ for each $m \in \mathbb{N}$. Toward this goal, fix m and $q \in Q^{(m)}$. Consider the linear Cauchy problem

$$\begin{cases} u'(t) = \lambda \mathbb{P}_m A_n q(t) + \lambda \mathbb{P}_m f(t, q(t)), \\ u(0) = \lambda \mathbb{P}_m Mq, \end{cases} \tag{5}$$

where $\lambda \in [0, 1]$.

It is clear that problem (5) has a unique solution, therefore we can define the solution operator $\Sigma\colon Q^{(m)} \times [0, 1] \to C(I, E_m)$,

$$\Sigma(q, \lambda)(t) = \lambda \mathbb{P}_m Mq + \lambda \int_0^t \big(\mathbb{P}_m A_n q(s) + \mathbb{P}_m f(s, q(s))\big) ds.$$

Let us show that the operator Σ has closed graph and it is compact. Indeed, let $\{q_k\} \subset Q^{(m)}$ and $\{\lambda_k\} \subset [0, 1]$ be two strongly convergent sequences to $q_0 \in Q^{(m)}$ and $\lambda_0 \in [0, 1]$ respectively and assume that the sequence $\{\Sigma(q_k, \lambda_k)\}$ strongly converges to $x_0 \in C(I, E_m)$. We shall prove that $x_0 = \Sigma(q_0, \lambda_0)$. By the linearity and continuity of the operators M, $\mathbb{P}_m$ and A_n it follows that $\lambda_k \mathbb{P}_m Mq_k \overset{E}{\to} \lambda_0 \mathbb{P}_m Mq_0$ and $\mathbb{P}_m A_n q_k(t) \overset{E}{\to} \mathbb{P}_m A_n q_0(t)$ for every $t \in I$. By virtue of the continuity of f and observing that E_m is a finite dimensional space we have that $\mathbb{P}_m f(t, q_k(t)) \to \mathbb{P}_m f(t, q_0(t))$ for a.e. $t \in I$. Moreover, notice that $\{q_k\} \subset Q^{(m)}$, then by the boundedness of the operators $\mathbb{P}_m$, M and A_n and by $(A4)$ we have

$$\|\mathbb{P}_m A_n q_k(t) + \mathbb{P}_m f(t, q_k(t))\| \le R \|A_n\|_{\mathcal{L}} + v_K(t)$$

for a.e. $t \in I$. By the Lebesgue Dominated Convergence Theorem we have that

$$\lambda_k \int_0^t \big(\mathbb{P}_m A_n q_k(s) + \mathbb{P}_m f(s, q_k(s))\big)\, ds \overset{E}{\to} \lambda_0 \int_0^t \big(\mathbb{P}_m A_n q_0(s) + \mathbb{P}_m f(s, q_0(s))\big)\, ds,$$

for every $t \in I$. By the uniqueness of the limit we obtain that

$$x_0(t) = \lambda_0 \mathbb{P}_m Mq_0 + \lambda_0 \int_0^t \big(\mathbb{P}_m A_n q_0(s) + \mathbb{P}_m f(s, q_0(s))\big)\, ds,$$

for every $t \in I$.

Now, let $\{q_k\} \subset Q^{(m)}$ and $\{\lambda_k\} \subset [0,1]$ and denote $x_k = \Sigma(q_k, \lambda_k)$. Recalling that $\{x_k\}$ is a sequence of strong solutions of (5), we have that

$$\|x_k'(t)\| \leq R\|A_n\|_{\mathcal{L}} + v_K(t),$$

for a.e. $t \in I$, and hence $\{x_k'\}$ is uniformly integrable. It follows the equicontinuity of the sequence $\{x_k\}$. Moreover, according to $(A1)$, it also follows that

$$\|x_k(t)\| \leq R + \int_0^t \|x_k'(s)\| ds,$$

i.e. the equiboundedness of $\{x_k\}$. Then, applying the Ascoli-Arzelà Theorem we obtain the relative compactness of $\{x_k\}$ and the claimed result.

Observe that $\Sigma(\cdot, 0) = \{0\} \subset \operatorname{int} Q^{(m)}$ and assume that there exists $(q, \lambda) \in \partial Q^{(m)} \times (0,1)$ such that

$$q = \Sigma(q, \lambda),$$

or equivalently,

$$\begin{cases} q'(t) = \lambda \mathbb{P}_m A_n q(t) + \lambda \mathbb{P}_m f(t, q(t)), \\ q(0) = \lambda \mathbb{P}_m M q. \end{cases}$$

Since $q \in \partial Q^{(m)}$ it follows that there exists $t_0 \in [0,T]$ such that $\|q(t_0)\| = R$. If $t_0 = 0$, then

$$R = \|q(0)\| = \|\lambda \mathbb{P}_m M q\| \leq \lambda \|q\|_C < R,$$

giving a contradiction. So, $t_0 \in (0,T]$. Hence, according to $\|q(0)\| < R = \|q(t_0)\|$ we can take a sufficiently small $\varepsilon > 0$ such that $r_0 < \|q(t)\| \leq R$ for all $t \in (t_0 - \varepsilon, t_0)$ and $\|q(t_0 - \epsilon)\| < R$. Condition $(A2)$ implies that

$$\langle J(q(t)), f(t, q(t)) \rangle \leq 0 \quad \text{for a.e.} \quad t \in (t_0 - \varepsilon, t_0).$$

Since $q(t) \in E_m$ we have

$$\langle J(q(t)), \mathbb{P}_m f(t, q(t)) \rangle = \langle \mathbb{P}_m^* J(q(t)), f(t, q(t)) \rangle = \langle J(q(t)), f(t, q(t)) \rangle \leq 0,$$

for a.e. $t \in (t_0 - \varepsilon, t_0)$.

On the other hand, since $\|R(n, A)\|_{\mathcal{L}} \leq n^{-1}$ (see [26, Theorem 3.1.1]), we have

$$\begin{aligned} \langle J(q(t)), \mathbb{P}_m A_n q(t) \rangle &= \langle J(q(t)), n^2 R(n, A) q(t) \rangle - n \langle J(q(t)), q(t) \rangle \\ &\leq n^2 \|R(n, A)\|_{\mathcal{L}} \|q(t)\|^2 - n \|q(t)\|^2 \leq 0, \end{aligned}$$

for all $t \in [0,T]$.

Consequently,

$$\int_{t_0 - \varepsilon}^{t_0} \langle J(q(t)), \mathbb{P}_m A_n q(t) + \mathbb{P}_m f(t, q(t)) \rangle dt \leq 0.$$

However,

$$\begin{aligned} \int_{t_0-\varepsilon}^{t_0} \langle J(q(t)), \mathbb{P}_m A_n q(t) + \mathbb{P}_m f(t, q(t)) \rangle dt &= \int_{t_0-\varepsilon}^{t_0} \langle J(q(t)), q'(t) \rangle dt \\ &= \int_{t_0-\varepsilon}^{t_0} \tfrac{d}{dt} (\|q(t)\|^2) \\ &= \|q(t_0)\|^2 - \|q(t_0 - \varepsilon)\|^2 > 0, \end{aligned}$$

giving a contradiction.

Applying Theorem 2.4 we obtain that for each $m \in \mathbb{N}$ there exists $u_m \in Q^{(m)}$ which is a solution to (4).

Step 2. In this step, we prove that for every $n \in \mathbb{N}$ problem (3) has a (strong) solution. Denote $f_m(t) = A_n u_m(t) + f(t, u_m(t))$. From the condition $\{u_m\} \subset Q$ and according to $(A4)$ it follows that there exists $\nu_* \in L^1(I, E)$ such that $\|f_m(t)\| \leq \nu_*(t)$

for a.e. $t \in I$ and every m. Therefore, the sequence $\{f_m\}$ is bounded and uniformly integrable in $L^1(I,E)$. So, it is relatively weakly compact in $L^1(I,E)$. W.l.o.g. assume that

$$f_m \overset{L^1(I,E)}{\rightharpoonup} f_0.$$

From Lemma 2.2 we get that $u_m^{'} = \mathbb{P}_m f_m \overset{L^1(I,E)}{\rightharpoonup} f_0$. Moreover, the set $\{u_m(0) : m \in \mathbb{N}\}$ is bounded in the reflexive Banach space E. So, w.l.o.g. we can assume that

$$u_m(0) \overset{E}{\rightharpoonup} \gamma_0. \tag{6}$$

Consider the absolute continuous function

$$u_0(t) := \gamma_0 + \int_0^t f_0(s)\, ds, \qquad t \in I.$$

Clearly u_0 is absolutely continuous and $u_0'(t) = f_0(t)$ for a.e. t. Moreover, it is easy to see that

$$u_m(t) = u_m(0) + \int_0^t u_m^{'}(s)\, ds \overset{E}{\rightharpoonup} u_0(t)$$

for all $t \in I$. By (A5), Lemma 2.1 and the linearity and continuity of the operator A_n we get that for a.e. $t \in I$ and for every weak neighborhood V of $A_n u_0(t) + f(t, u_0(t))$ there exists $m_0 = m_0(t, V)$ such that

$$f_m(t) \in V \text{ for all } m \geq m_0.$$

Now, by the Mazur Lemma (see, e.g. [16], p. 16) there is a sequence of convex combinations $\{\overline{f}^{(m)}\}$,

$$\overline{f}^{(m)} = \sum_{k=m}^{\infty} \sigma_{mk} f_k,\ \sigma_{mk} \geq 0, \text{ and } \sum_{k=m}^{\infty} \sigma_{mk} = 1,$$

which converges to $u_0^{'}$ in $L^1(I,E)$. Applying e.g. [24, Theorem 38] we assume w.l.o.g that

$$\overline{f}^{(m)}(t) \overset{E}{\to} u_0^{'}(t) \ \text{ for a.e. } t \in I.$$

So, by the convexity of the set V we have

$$\overline{f}^{(m)}(t) \in V \text{ for all } m \geq m_0.$$

Therefore, by the uniqueness of the weak limit we get $u_0^{'}(t) = A_n u_0(t) + f\big(t, u_0(t)\big)$ for a.e. $t \in I$. Hence, Theorem 2.3 implies

$$u_m \overset{C(I,E)}{\rightharpoonup} u_0,$$

and thus, $Mu_m \overset{E}{\rightharpoonup} Mu_0$. Consequently, again according to Lemma 2.1, $u_m(0) = \mathbb{P}_m M u_m \overset{E}{\rightharpoonup} Mu_0$. From (6) we have $Mu_0 = \gamma_0 = u_0(0)$, obtaining that u_0 is a solution to problem (3).

Step 3. We will show now that problem (1) has a mild solution in Q. For each $n \in \mathbb{N}$ let $u_n \in Q$ be a solution to the problem (3), i.e.,

$$u_n(t) = e^{tA_n} M u_n + \int_0^t e^{(t-s)A_n} f(s, u_n(s)) ds.$$

From $\{u_n\} \subset Q$, $(A1)$ and the reflexivity of the space E we get that there exists $\overline{u} \in E$ such that, up to subsequence, $Mu_n \overset{E}{\rightharpoonup} \overline{u}$. Moreover, denoting $f_n(t) = f(t, u_n(t))$, from $u_n \in Q$ and $(A4)$ it follows that there exists $v_* \in L^1(I, \mathbb{R})$ such that

$$\|f_n(t)\| \leq v_*(t) \quad \text{for a.e.} \quad t \in I.$$

Hence, we get that $\{f_n\}$ is bounded and uniformly integrable in $L^1(I, E)$. W.l.o.g. assume that

$$f_n \overset{L^1(I,E)}{\rightharpoonup} f.$$

For every $t \in I$, we have

$$\int_0^t \|e^{(t-s)A_n} f_n(s)\| \, ds \leq \int_0^t \|f_n(s)\| \, ds \leq \|v_*\|_1$$

and similarly

$$\int_0^t \|S(t-s)f(s)\| \, ds \leq \|v_*\|_1.$$

Consequently, the maps $e^{(t-\cdot)A_n} f_n$ and $S(t-\cdot)f$ belong to the space $L^1([0,t]; E)$ for every $t \in I$. Now let us show that

$$e^{(t-\cdot)A_n} f_n \overset{L^1([0,t];E)}{\rightharpoonup} S(t-\cdot)f \quad \text{for each} \quad t \in I.$$

To this aim, let $\Phi \colon L^1([0,t]; E) \to \mathbb{R}$ be a linear and bounded functional. Hence, there is $\varphi \in L^\infty([0,t]; E^*)$ such that

$$\Phi(g) = \int_0^t \big\langle g(s),\, \varphi(s) \big\rangle \, ds \quad \text{for all} \quad g \in L^1([0,t]; E).$$

We have

$$\begin{aligned} \Phi(e^{(t-\cdot)A_n} f_n - S(t-\cdot)f) = & \int_0^t \big\langle e^{(t-s)A_n} f_n(s) - S(t-s)f(s),\, \varphi(s) \big\rangle \, ds \\ = & \int_0^t \big\langle e^{(t-s)A_n} f_n(s) - e^{(t-s)A_n} f(s), \varphi(s) \big\rangle \, ds + \\ & \int_0^t \big\langle e^{(t-s)A_n} f(s) - S(t-s)f(s), \varphi(s) \big\rangle \, ds. \end{aligned}$$

By (2) and the Lebesgue dominated convergence theorem, we have that

$$\int_0^t \big\langle e^{(t-s)A_n} f(s) - S(t-s)f(s), \varphi(s) \big\rangle \, ds \to 0,$$

for every $t \in I$. Further, let $A^* \colon D(A^*) \subset E^* \to E^*$ be the adjoint operator of A. It is well known (see, e.g. [26, Theorem 3.7.1]) that A^* is the generator of the C_0-semigroup of contractions $\{S^*(t)\}_{t \geq 0}$, where $S^*(t)$ is the adjoint of the operator $S(t)$. Moreover,

$$\lim_{n \to \infty} e^{tA_n^*} z^* = S^*(t) z^*, \tag{7}$$

for $t \in I$ and $z^* \in E^*$, where $A_n^* = n^2 R(n, A^*) - nI$.

Consequently,

$$\begin{aligned} & \textstyle\int_0^t \big\langle e^{(t-s)A_n} f_n(s) - e^{(t-s)A_n} f(s), \varphi(s) \big\rangle \, ds = \\ = & \textstyle\int_0^t \big\langle f_n(s) - f(s),\, e^{(t-s)A_n^*} \varphi(s) \big\rangle \, ds \\ = & \textstyle\int_0^t \big\langle f_n(s) - f(s),\, e^{(t-s)A_n^*} \varphi(s) - S^*(t-s)\varphi(s) \big\rangle \, ds \\ & + \textstyle\int_0^t \big\langle f_n(s) - f(s),\, S^*(t-s)\varphi(s) \big\rangle \, ds. \end{aligned}$$

Clearly $S^*(t-\cdot)\varphi(\cdot) \in L^\infty([0,t];E)$ thus

$$\int_0^t \big\langle f_n(s) - f(s),\, S^*(t-s)\varphi(s)\big\rangle\, ds \to 0 \;\; \text{as} \;\; n \to \infty.$$

Notice that, from (7) we get that $e^{(t-s)A_n^*}\varphi(s) \to S^*(t-s)\varphi(s)$ for a.e. $s \in [0,t]$. Moreover the convergence is dominated, thus

$$e^{(t-\cdot)A_n^*}\varphi(\cdot) \overset{L^\infty(I,E^*)}{\rightharpoonup} S^*(t-\cdot)\varphi(\cdot)$$

and we get that

$$\int_0^t \big\langle f_n(s) - f(s),\, e^{(t-s)A_n^*}\varphi(s) - S^*(t-s)\varphi(s)\big\rangle\, ds \to 0 \;\; \text{as} \;\; n \to \infty.$$

So $\Phi(e^{(t-\cdot)A_n^*}\varphi(\cdot)) \to \Phi(S(t-\cdot)\varphi(\cdot))$, i.e.

$$u_n(t) \rightharpoonup S(t)\overline{u} + \int_0^t S(t-s)f(s)\, ds := u(t)$$

for every $t \in I$. Thus, $u_n \overset{C(I,E)}{\rightharpoonup} u \in Q$.

Now let us show that if $\{z_n\} \subset E$, $z_n \overset{E}{\rightharpoonup} z$, then $e^{tA_n}z_n \overset{E}{\rightharpoonup} S(t)z$ for each $t \in I$. In fact, for every $g \in E^*$ we have

$$\big\langle g, e^{tA_n}z_n - S(t)z\big\rangle = \big\langle g, e^{tA_n}z - S(t)z\big\rangle + \big\langle g, e^{tA_n}(z_n - z)\big\rangle.$$

By virtue of (2) it follows that

$$\big\langle g, e^{tA_n}z - S(t)z\big\rangle \to 0 \;\; \text{as} \;\; n \to \infty.$$

Further,

$$\begin{aligned} \big\langle g, e^{tA_n}(z_n - z)\big\rangle = &\;\; \big\langle (e^{tA_n})^* g, z_n - z\big\rangle = \big\langle e^{tA_n^*}g, z_n - z\big\rangle \\ = &\;\; \big\langle e^{tA_n^*}g - S(t)^*g, z_n - z\big\rangle + \big\langle S(t)^*g, z_n - n\big\rangle \to 0 \;\; \text{as} \;\; n \to \infty. \end{aligned}$$

Thus, according to (A1) and (A5), we get

$$e^{tA_n}Mu_n \overset{E}{\rightharpoonup} S(t)Mu,$$

and

$$\int_0^t e^{(t-s)A_n}f(s,u_n(s))ds \overset{E}{\rightharpoonup} \int_0^t S(t-s)f(s,u(s))ds.$$

Therefore, by the uniqueness of the weak limit we have

$$u(t) = S(t)Mu + \int_0^t S(t-s)f(s,u(s))ds, \;\; \text{for every} \;\; t \in I,$$

obtaining the claimed result.

Step 4. Now we will show that the set of solutions to problem (1) is weakly compact in $C(I,E)$. Let $\{u_k\} \subset Q$ be a sequence of solutions to the problem (1), i.e.,

$$u_k(t) = S(t)Mu_k + \int_0^t S(t-s)f(s,u_k(s))ds.$$

Denoting with $f_k(t) = f(t,u_k(t)), k \in \mathbb{N}$, from $u_k \in Q$ and $(A4)$ it follows that there exists $v_* \in L^1(I,\mathbb{R})$ such that

$$\|f_k(t)\| \le v_*(t) \;\; \text{for a.e.} \;\; t \in I.$$

Hence, we get that $\{f_k\}$ is bounded and uniformly integrable in $L^1(I, E)$. W.l.o.g. assume that

$$f_k \overset{L^1(I,E)}{\rightharpoonup} f. \tag{8}$$

For every $t \in I$, it easily follows that

$$\int_a^t S(t-s)f_k(s)\,ds \overset{E}{\rightharpoonup} \int_a^t S(t-s)f(s)\,ds.$$

Moreover, from $\{u_k\} \subset Q$, $(A1)$ and the reflexivity of the space E we get that there exists $\overline{u} \in E$ such that, up to subsequence, $Mu_k \overset{E}{\rightharpoonup} \overline{u}$. Hence

$$u_k(t) \rightharpoonup S(t)\overline{u} + \int_0^t S(t-s)f(s)\,ds := u(t)$$

for every $t \in I$. Thus, $u_k \overset{C(I,E)}{\rightharpoonup} u \in Q$.

Thus, by the linearity and continuity of the semigroup $S(t)$ according to (A1) and (A5), we get

$$S(t)Mu_n \overset{E}{\rightharpoonup} S(t)Mu,$$

and

$$\int_0^t S(t-s)f(s,u_n(s))ds \overset{E}{\rightharpoonup} \int_0^t S(t-s)f(s,u(s))ds.$$

Therefore, by the uniqueness of the weak limit we have

$$u(t) = S(t)Mu + \int_0^t S(t-s)f(s,u(s))ds, \quad \text{for every} \quad t \in I.$$

Step 5. Finally, let the C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by A be compact and $a \in (0,T)$ be an arbitrary number. We will show that the set of solutions on Q to problem (1) is compact in $C([a,T],E)$.

Let $\{u_k\} \subset Q$ be a sequence of solutions to the problem (1), i.e.,

$$u_k(t) = S(t)Mu_k + \int_0^t S(t-s)f(s,u_k(s))ds.$$

Fix $t \in [0,T]$. Reasoning as in Step 4, we get that the sequence $h_{k,t}$ defined as $h_{k,t}(s) = S(t-s)f(s,u_k(s))$ is bounded and uniformly integrable in $L^1([0,t],E)$. Moreover, from the compactness of $S(t-s)$ for every $s < t$, it follows that the set $\{h_{k,t}(s)\}$ is relatively compact for a.a. $s < t$. Since the operator $\mathcal{P}\colon L^1([0,t],E) \to C([0,t],E)$ defined by $\mathcal{P}g(r) = \int_0^r g(s)\,ds$ is the Cauchy operator in the special case when the semigroup is identically equal to I, it satisfies conditions *(i)* and *(ii)* in [18, Theorem 5.1.1], thus according to the same theorem we obtain that the sequence

$$\left\{\int_0^r h_{k,t}(s)\,ds\right\}$$

is relatively compact in $C([0,t],E)$, hence, in particular, the sequence $\{v_k\} \subset C(I,E)$ defined as

$$v_k(t) = \left\{\int_0^t S(t-s)f(s,u_k(s))\,ds\right\}$$

is such that $\{v_k(t)\}$ is relatively compact in E for every $t \geq a$.

Now we can use Theorem 8.4.1 in [26] to get the conclusion. This ends our proof. □

Remark 3. Let us note that

(i) Theorem 3.1 can be proved also replacing $(A1)$ with
$(A1')$ M is a weakly continuous map and $\|Mx\| \le \|x\|_0$ for every $x \in C(I, E)$.

(ii) Whenever M is as in one of the specific cases (i), (ii) or (iii) with $t_1 > 0$ in item (c) of Remark 6 and the C_0-semigroup generated by A is compact, then the set of mild solutions on Q of problem (1) is strongly compact in $C([0,T], E)$. In fact, given a sequence $\{u_k\} \subset Q$ of mild solutions, we already proved that it is equicontinuous in $C([0,T], E)$ as well as pointwise relatively compact for every $t \in (0,T]$. Since $u_k(0) = \sum_{i=1}^{k_0} \alpha_i u_k(t_i)$ for some $\alpha_i \in \mathbb{R}$ and $0 < t_1 < \cdots < t_{k_0} \le T$, the relative compactness of $\{u_k(t_i)\}$ for every $i = 1, ...k_0$ yields also the relative compactness of $\{u_k(0)\}$, thus the claimed result from Ascoli-Arzelá theorem.

We recall that given two Banach spaces E_1 and E_2 endowed with two measures of non compactness β_1 and β_2, it is possible to define the (β_2, β_1)-norm of a bounded linear operator $L : E_1 \to E_2$ as

$$\|L\|^{(\beta_2,\beta_1)} = \inf\{C > 0 \;:\; \beta_2(L(\Omega)) \le C\beta_1(\Omega),\ \Omega \text{ bounded in } E_1\}.$$

Notice that since L is a bounded and linear operator its (β_1, β_2)-norm is finite (see e.g. [2, Theorem 2.4.1]). If we consider the modulus of fiber of noncompactness as measure of non compactness in the space $C([0,T], E)$, i.e.

$$\gamma(\Omega) = \sup_{t\in[0,T]} \chi(\Omega(t))$$

with $\Omega \subset C([0,T]; E)$ a bounded set and χ the Hausdorff measure of noncompactness in E (see, e.g. [18]), under the assumption that

$$\max\left\{\|M\|^{(\chi,\gamma)}, \|M\|\right\} + \|L\|_1 < 1, \tag{9}$$

we can use the condition $(A5)'$ to obtain the uniqueness of the mild solution to problem (1).

Theorem 3.2. *Let conditions $(A1)-(A3)$, $(A5)'$ and (9) hold. Then problem (1) has a unique mild solution in Q.*

Proof. For fixed $n \in \mathbb{N}$, reasoning as in Theorem 3.1 we can prove that for every $m \in \mathbb{N}$ problem (4) has a strong solution $u_m \in Q^{(m)}$ with

$$u_m(t) = e^{t\mathbb{P}_m A_n}\mathbb{P}_m M u_m + \int_0^t e^{(t-s)\mathbb{P}_m A_n}\mathbb{P}_m f(s, u_m(s))ds.$$

For any $t \in [0,T]$ we can compute the Hausdorff measure of noncompactness of $\{u_m(t)\}$. Precisely, since $\{u_m\}$ is a bounded sequence and M is a bounded linear operator we have

$$\begin{aligned}
&\chi\big(\{u_m(t)\}\big)\\
&\le \chi\left(\{e^{t\mathbb{P}_m A_n}\mathbb{P}_m M u_m\}\right) + \chi\left(\left\{\int_0^t e^{(t-s)\mathbb{P}_m A_n}\mathbb{P}_m f(s, u_m(s))ds\right\}\right)\\
&\le \|e^{t\mathbb{P}_m A_n}\|\chi\big(M\{u_m\}\big) + \chi\left(\left\{\int_0^t e^{(t-s)\mathbb{P}_m A_n}\mathbb{P}_m f(s, u_m(s))ds\right\}\right)\\
&\le \|e^{t\mathbb{P}_m A_n}\|\|M\|^{(\chi,\gamma)}\gamma\big(\{u_m\}\big) + \chi\left(\left\{\int_0^t e^{(t-s)\mathbb{P}_m A_n}\mathbb{P}_m f(s, u_m(s))ds\right\}\right).
\end{aligned}$$

Since A_n generates a semigroup of contraction, it is easy to prove that $\|e^{t\mathbb{P}_m A_n}\| \leq 1$ for every $t \in I, m \in \mathbb{N}$, hence $(A5)'$ implies that

$$\begin{aligned}\chi\Big(\big\{e^{(t-s)\mathbb{P}_m A_n}\mathbb{P}_m f(s,u_m(s))\big\}\Big) &\leq \|e^{(t-s)\mathbb{P}_m A_n}\|\,\chi\big(\{\mathbb{P}_m f(s,u_m(s))\}\big)\\ &\leq L(s)\,\chi\big(\{u_m(s)\}\big) \leq L(s)\,\gamma\big(\{u_m\}\big),\end{aligned}$$

for a.e. $s \in I$, and hence,

$$\chi(\{u_m(t)\}) \leq \|M\|^{(\chi,\gamma)}\,\gamma(\{u_m\}) + \gamma(\{u_m\}) \int_0^t L(s)\,ds.$$

Thus,

$$\gamma(\{u_m\}) \leq \gamma(\{u_m\})\left(\|M\|^{(\chi,\gamma)} + \|L\|_1\right)$$

From $\|M\|^{(\chi,\gamma)} + \|L\|_1 < 1$ we obtain $\gamma(\{u_m\}) = 0$, and so $\chi(\{u_m(t)\}) = 0$ for every $t \in I$. Hence, the set $\{u_m(t)\}$ is relatively compact for every $t \in I$. From $u_m \in Q^{(m)}$ and $(A5)'$ we get that $\{u_m\}$ is relative compact in $C(I, E_m)$ and $\{u'_m\}$ is bounded and uniformly integrable in $L^1(I, E)$, thus $u_m \to u$ in Q and $u'_m \rightharpoonup u'$ in $L^1(I, E)$ (see, e.g. [3, Lemma 1.30]). Condition $(A1)$ implies that

$$u(0) = \lim_{m\to\infty} u_m(0) = \lim_{m\to\infty} \mathbb{P}_m M u_m = Mu.$$

Moreover, from $(A5)'$ and the continuity of A_n we obtain that $f(t, u_m(t)) \to f(t, u(t))$ and $A_n u_m(t) \to A_n u(t)$ for a.e. $t \in I$. Consequently,

$$u'_m(t) = \mathbb{P}_m A_n u_m(t) + \mathbb{P}_m f(t, u_m(t)) \to A_n u(t) + f(t, u(t))$$

for every $t \in I$ and the convergence is dominated, implying

$$u'(t) = A_n u(t) + f(t, u(t))$$

by the uniqueness of the weak limit.

So, we proved that for every $n \in \mathbb{N}$ problem (3) has a strong solution $u_n \in Q$, i.e.

$$u_n(t) = e^{tA_n} M u_n + \int_0^t e^{(t-s)A_n} f(s, u_n(s))ds. \tag{10}$$

Reasoning as above we can prove that $u_n \to u_0 \in Q$. From $(A1)$ and $(A5)'$ it follows that $Mu_n \to Mu_0$ and $f(s, u_n(s)) \overset{E}{\to} f(s, u_0(s))$ thus $e^{(t-s)A_n} f(s, u_n(s)) \to S(t-s) f(s, u_0(s))$ for every $t \in I$ and a.e. $s \leq t$ and the convergence is dominated. Passing (10) to the limit we again obtain that

$$u_0(t) = S(t)Mu_0 + \int_0^t S(t-s) f(s, u_0(s))\,ds.$$

Now, assume that there exist two mild solutions u and v of problem (1) in Q, i.e.

$$u(t) = S(t)Mu + \int_0^t S(t-s) f(s, u(s))ds,$$

and

$$v(t) = S(t)Mv + \int_0^t S(t-s) f(s, v(s))ds.$$

Therefore, for every $t \in I$:

$$\|u(t) - v(t)\| \leq (\|M\| + \|L\|_1)\|u - v\|_C < \|u - v\|_C.$$

So, $u \equiv v$, i.e. problem (1) has a unique mild solution in Q. $\square$

Remark 4. We point out that in all classical conditions listed in Remark 2 it holds
$$\|M\|^{(\chi,\gamma)} \leq \|M\|,$$
thus (9) reads as
$$\|M\| + \|L\|_1 < 1.$$
Indeed, consider the multipoint condition $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$, which includes the Cauchy condition (with $\alpha_i = 0$ for every i) and the periodic and antiperiodic conditions (with $k_0 = 1, t_1 = T$ and $\alpha_1 = \pm 1$) and let $\Omega \subset C(I, E)$ be bounded. From $M(\Omega) \subset \sum_{i=1}^{k_0} \alpha_i \Omega(t_i)$ and the monotonicity, algebraically semiadditivity and semihomogeneity properties of the Hausdorff measure of noncompactness, it holds
$$\begin{aligned}\chi\big(M(\Omega)\big) &\leq \chi\left(\sum_{i=1}^{k_0} \alpha_i \Omega(t_i)\right) \leq \sum_{i=1}^{k_0} \chi(\alpha_i \Omega(t_i)) \\ &\leq \sum_{i=1}^{k_0} |\alpha_i| \chi(\Omega(t_i)) \leq \sum_{i=1}^{k_0} |\alpha_i| \gamma(\Omega) = \|M\| \gamma(\Omega).\end{aligned}$$
Take now the weighted condition $Mx = \int_0^T w(t)x(t)dt$, including the mean value condition (with $w(s) = \frac{1}{T}$ for every s) and let $\Omega \subset C(I, E)$ be bounded. Then, for every $s \in I$, since
$$\chi(\{\omega(s)x(s)\}) \leq |w(s)|\chi(\{\Omega(s)\}) \leq |w(s)|\gamma(\Omega),$$
it follows that
$$\chi\left(\int_0^T w(s)x(s)ds\right) \leq \int_0^T |w(s)|\gamma(\Omega)ds = \gamma(\Omega)\|w\|_1 = \gamma(\Omega)\|M\|.$$
In both cases we get $\|M\|^{(\chi,\gamma)} \leq \|M\|$.

3.2. **Existence results on noncompact intervals.** Given a function $f : [0, +\infty) \times E \to E$, as consequence of our method, we discuss the existence of entirely bounded solution of the equation
$$u'(t) = Au(t) + f(t, u(t)), \text{ for a.e. } t \in [0, +\infty). \tag{11}$$

Theorem 3.3. *Assume that $f : [0, +\infty) \times E \to E$ satisfies (A2)-(A5) on $[0, \infty)$. Then equation (11) admits at least one bounded mild solution on $[0, +\infty)$.*

Proof. According to Theorem 3.1, for every $k \in \mathbb{N}_+$ there exists a solution u_k to the problem
$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \text{ for a.e. } t \in [0, k], \\ u(0) = 0, \end{cases} \tag{12}$$
with $\|u_k(t)\| \leq R$ for every $k \in \mathbb{N}$ and $t \in [0, k]$. Consider the restriction of the sequence of mild solutions $\{u_k\}$ to the interval $[0, 1]$, namely $\{u_k/_{[0,1]}\}$. Since the set of mild solutions of problem (1) is weakly compact in $C([0, 1]; K)$, there exists a subsequence still denoted as the sequence converging to a function $\psi^1 \in C([0, 1]; K)$ mild solution of the problem (12) in $[0, 1]$. Now, let us consider the sequence $\{u_k/_{[0,2]}\}_{k\geq 2}$. Again, we get that there exists a subsequence $\{u_{k_n}/_{[0,2]}\}_{k\geq 2}$ weakly converging to a function $\psi^2 \in C([0, 2]; K)$ mild solution for the problem (12) in $[0, 2]$. By construction it follows that
$$\psi^2/_{[0,1]} = \psi^1 \ .$$
By iterating this process we obtain, for every $k \in \mathbb{N}_+$, a mild solution $\psi^k : [0, k] \to E$ for problem (12) such that for every integer $k \geq 2$, we have
$$\psi^k_{[0,k-1]} = \psi^{k-1} \ . \tag{13}$$

Hence by the continuity of the maps ψ^k for any $k \in \mathbb{N}_+$ and by (13) we have that the map $\psi : [0, +\infty) \to E$ defined as

$$\psi(t) = \begin{cases} \psi^1(t) & t \in [0,1] \\ \psi^2(t) & t \in]1,2] \\ \dots & \dots \\ \psi^k(t) & t \in]k-1,k] \\ \dots & \dots \end{cases}$$

is a mild solution to equation (11) on $[0, +\infty)$. Moreover, $\|\psi(t)\| \le R$ for every $t \in [0, +\infty)$, i.e. ψ is bounded on the half-line. □

Easily we obtain an analogous result under the hypothesis $(A5)'$.

Theorem 3.4. *Assume $f : [0, +\infty) \times E \to E$ satisfies (A2)-(A3)-(A5)$'$ on $[0, \infty)$ with $\int_0^{+\infty} L(t)dt < 1$, then equation (11) has a unique mild solution u with $\|u(t)\| \le R$ on $[0, \infty)$.*

Proof. According to Theorem 3.2 for every $k \in \mathbb{N}$ there exists a unique mild solution u_k to the problem (12) such that $\|u_k(t)\| \le R$ for all $k \in \mathbb{N}$ and $t \in [0, k]$. Hence the map defined as

$$u(t) = \begin{cases} u_1(t) & t \in [0,1] \\ u_2(t) & t \in]1,2] \\ \dots & \dots \\ u_k(t) & t \in]k-1,k] \\ \dots & \dots \end{cases}$$

is the unique mild solution of (11) on $[0, +\infty)$. □

4. Applications.

In this section we will apply our abstract result to study population diffusion models. Both our models develop from very simple and ancient population growth equations, respectively the confined exponential distribution $u' = a(N-u)$ and the logistic distribution $u' = au(1-u)$. These ordinary differential equations were then generalized to partial differential equations to take into account the fact that sometimes growth, transfer and diffusion all occur simultaneously in a phenomenon. For example, when an epidemic spreads through a geographical region, the number of infected people grows as the disease is transferred from those infected to susceptible.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1-boundary. Let us first consider a simple linear diffusion model of the form:

$$\begin{cases} \frac{\partial u(t,\xi)}{\partial t} = \Delta u(t,\xi) + a(t,\xi)\big(K(t,\xi) - u(t,\xi)\big), & (t,\xi) \in (0,T] \times \Omega, \\ \frac{\partial u(t,\xi)}{\partial n} = 0, \quad (t,\xi) \in [0,T] \times \partial\Omega \\ u(0,\xi) = \sum_{i=1}^k \alpha_i u(t_i,\xi), \quad \xi \in \Omega. \end{cases} \tag{14}$$

This model is used to describe various mathematical models arising, for example, in technology transfer, social sciences, agriculture (see [4] and [21]).

Here, $u(t,\xi)$, $K(t,\xi)$ and $a(t,\xi)$ represent the population size, the carrying capacity, which is the largest population that the resources in the environment can sustain, and the growth (or transfer) coefficient at time t and location ξ, respectively. The temporal change of the population size at location ξ is given by the diffusion term $\Delta u(t,\xi)$ and the growth component is

$$a(t,\xi)\big(K(t,\xi) - u(t,\xi)\big).$$

Let us mention that problem of reaction-diffusion with time and location dependent of carrying capacity and of growth coefficient and with initial condition

$$u(0,\xi)=u_0(\xi), \quad \text{for all} \quad \xi\in\Omega$$

is studied intensively by researchers (see, e.g. [19]). Here we will consider problem (14) with a multi-point condition.

Let $E=L^2(\Omega,\mathbb{R})$. We denote with $\|\cdot\|$ the norm in E. For each $t\in[0,T]$ set $x(t)=u(t,\cdot)$. Assume that:
(i) $a\colon[0,T]\times\overline{\Omega}\to[c,+\infty)$ is a continuous function, with $c>0$;
(ii) for every $t\in[0,T]$ the function $k(t,\cdot)\in L^2(\Omega,\mathbb{R})$ and

$$\sup_{t\in[0,T]}||k(t,\cdot)||_2=K_*<\infty$$

(iii) $0<t_1<\cdots<t_k\le T$ and $\alpha_i\in\mathbb{R}$ are such that $\sum_{i=1}^k|\alpha_i|\le 1$.
Then problem (14) can be substituted by the following semilinear differential equation

$$\begin{cases} x'(t)=Ax(t)+f(t,x(t)), & \text{for} \quad t\in(0,T],\\ x(0)=Mx=\sum_{i=1}^k\alpha_i x(t_i), \end{cases} \tag{15}$$

where the operator $A\colon D(A)\subseteq E\to E$ is defined by

$$\begin{cases} D(A)=\left\{u\in W^{2,2}(\Omega)\colon \frac{\partial u}{\partial n}=0 \text{ on } [0,T]\times\partial\Omega\right\},\\ Au=\Delta u, \quad \text{for} \quad u\in D(A) \end{cases}$$

and $f\colon[0,T]\times E\to E$,

$$f(t,z)(\xi)=a(t,\xi)\big(K(t,\xi)-z(\xi)\big).$$

It is well known (see, e.g. [26, Theorem 4.2.2]) that A is the generator of a C_0-semigroup of contractions on E.

By *a mild solution* of problem (14) we mean a continuous function $x\in C([0,T];E)$ that is a mild solution of (15).

Theorem 4.1. *If*

$$\sum_{i=1}^k|\alpha_i|+T||a||_0<1,$$

then in $B_C(0,\frac{a_*K_*}{c})$ *the problem* (14) *has a unique mild solution.*

Remark 5. (a) The existence of mild solutions to problem (14) is a quite general result, since we do not need any other constrained condition on the functions $a(t,\xi)$ and $K(t,\xi)$;
(b) the multi-point condition can be replaced by other conditions such that $\|M\|\le 1$ (see Remark 2).

Proof of Theorem 4.1. It is easy to see that the map f satisfies conditions $(A3)$ and $(A5)'$ with $L(t)=a_*$ for every t. Let us verify condition $(A2)$. In fact, since E is a separable Hilbert space, so we have $J(z)=z$ for every $z\in E$. For every $(t,z)\in[0,T]\times E$ we have

$$\begin{aligned}\big\langle z,f(t,z)\big\rangle &=\textstyle\int_\Omega z(\xi)a(t,\xi)\big(K(t,\xi)-z(\xi)\big)d\xi\\ &\le a_*K_*\|z\|-c\|z\|^2<0,\end{aligned}$$

provided

$$\|z\| > \frac{a_* K_*}{c}.$$

Therefore, from Theorem 3.2 and Remarks 2 and 4 we obtain the existence of a unique mild solution of problem (14). □

Now, let us consider again the reaction-diffusion equation in (14) for the case $a(t,\xi) = a > 0$ for all $(t,\xi) \in [0,T] \times \Omega$, but with the growth component depending also on the total population size over the considered area. More precisely, the population dynamic is described by the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u(t,\xi) + a\left(K(t,\xi) - \dfrac{bu(t,\xi)}{1+\int_\Omega |u(t,\xi)|d\xi}\right), \quad (t,\xi) \in (0,T] \times \Omega, \\ \frac{\partial u(t,\xi)}{\partial n} = 0, \quad (t,\xi) \in [0,T] \times \partial\Omega \\ u(0,\xi) = \sum_{i=1}^k \alpha_i u(t_i,\xi), \quad \xi \in \Omega. \end{cases} \tag{16}$$

Assume (ii) and (iii). Let $f\colon [0,T] \times E \to E$ be a map defined by

$$f(t,z)(\xi) = a\left(K(t,\xi) - \frac{bz(\xi)}{1+\int_\Omega |z(\xi)|d\xi}\right),$$

then problem (16) can be written as problem (15).

Theorem 4.2. *If $b > K_*|\Omega|^{\frac{1}{2}}$, and*

$$a < \frac{(1-\sum_{i=1}^k |\alpha_i|)(b - K_*|\Omega|^{\frac{1}{2}})}{b^2 T}.$$

then in $B_C(0, \frac{K_*}{b-K_*|\Omega|^{\frac{1}{2}}})$ the problem (16) has a unique mild solution.

Proof. It is clear that conditions $(A1)$ and $(A3)$ hold true. Let us verify condition $(A2)$ and $(A5)'$. Toward this goal, for every $(t,z) \in [0,T] \times E$ we have

$$\begin{aligned} \langle z, f(t,z)\rangle &= a\int_\Omega z(\xi)\left(K(t,\xi) - \tfrac{bz(\xi)}{1+\int_\Omega |z(\xi)|d\xi}\right)d\xi \\ &\le aK_*\|z\| - \tfrac{ab\|z\|^2}{1+|\Omega|^{\frac{1}{2}}\|z\|} \\ &= \tfrac{a\|z\|}{1+\|z\|}\left[(K_*|\Omega|^{\frac{1}{2}} - b)\|z\| + K_*\right] < 0, \end{aligned}$$

provided

$$\|z\| > \frac{K_*}{b - K_*|\Omega|^{\frac{1}{2}}}.$$

Hence, condition $(A2)$ holds true. For $t \in [0,T]$ and $z, w \in B(0, \frac{K_*}{b-K_*|\Omega|^{\frac{1}{2}}})$, denoting $\|z\|_1 = \int_\Omega |z(\xi)|d\xi$, we have

$$\begin{aligned} &\|f(t,z) - f(t,w)\| = \\ = \ &ab\left\|\tfrac{z}{1+\|z\|_1} - \tfrac{w}{1+\|w\|_1}\right\| \\ \le \ &\tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}\left\|z - w + z\|w\|_1 - w\|z\|_1\right\| \end{aligned}$$

$$\begin{aligned}
&\le \tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}\left(\|z-w\| + \left\| z\|w\|_1 - w\|z\|_1 \right\|\right)\\
&\le \tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}\left(\|z-w\| + \left\| z\|w\|_1 - w\|w\|_1 + w\|w\|_1 - w\|z\|_1 \right\|\right)\\
&\le \tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}\left(\|z-w\| + \|w\|_1\|z-w\| + \|w\|\left|\|w\|_1 - \|z\|_1\right|\right)\\
&\le \tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}\left(\|z-w\| + \|w\|_1\|z-w\| + \|w\|\|w-z\|_1\right)\\
&\le \tfrac{ab}{(1+\|z\|_1)(1+\|w\|_1)}(1+\|w\|_1+\|w\||\Omega|^{\frac{1}{2}})\|w-z\|\\
&\le ab\left(\tfrac{1}{1+\|z\|_1} + \tfrac{\|w\||\Omega|^{\frac{1}{2}}}{(1+\|z\|_1)(1+\|w\|_1)}\right)\|w-z\|\\
&\le ab\left(1+\tfrac{K_*|\Omega|^{\frac{1}{2}}}{b-K_*|\Omega|^{\frac{1}{2}}}\right)\|w-z\|.
\end{aligned}$$

Since $(A5)'$ is satisfied with $L = \frac{ab^2}{b-K_*|\Omega|^{\frac{1}{2}}}$, from Theorem 3.2 and Remarks 2 and 4 we obtain that problem (16) has a unique mild solution. □

Let now $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2-boundary. Consider the periodic problem

$$\begin{cases} u_t = \Delta u - bu(t,\xi) + u\, h\Big(t,\xi,\displaystyle\int_\Omega k(\xi,\eta)u(t,\eta)d\eta\Big) + l(t,\xi), & (t,\xi)\in(0,1]\times\Omega,\\ u(t,\xi) = 0, \quad (t,\xi)\in[0,1]\times\partial\Omega, \\ u(0,\xi) = u(1,\xi), \quad \xi\in\Omega. \end{cases} \tag{17}$$

Problem (17) is a generalized version of the nonlocal Fisher-Kolmogorov-Petrovskii-Piscounov equation (FKPP), which describes a model of spatial diffusion and evolution of a population with non local consumptions of resources (see [1, 11]).

In (17) $u(t,\xi)$ represents the density of the population at time t and position ξ. The coefficient $b>0$ is the death rate. The proliferation rate h depends on the time, the position and the total size of the population weighted by the kernel k corresponding to the probability of an individual to move from one position to an other. It contains also the forcing term l. The periodic condition aims to take under control the diffusion of the population.

Let $E = L^p(\Omega,\mathbb{R}),\ 1<p<\infty$. We denote with $\|\cdot\|_p$ the norm in E. By means of a reformulation of this problem we will prove the existence of a mild solution to (17).

To this aim, for each $t\in[0,1]$, set $x(t) = u(t,\cdot)$; let $W^{m,p}(\Omega,\mathbb{R})$ be the Sobolev space and $W_0^{m,p}(\Omega,\mathbb{R})$ the subspace containing all functions of $W^{m,p}(\Omega,\mathbb{R})$ vanishing at the boundary $\partial\Omega$.

Assume that:

(i) the function $h:[0,1]\times\Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(1) $h(\cdot,\cdot,c)$ is measurable for every $c\in\mathbb{R}$;
(2) $h(t,\xi,\cdot)$ is continuous for a.e. $t\in[0,1]$ and $\xi\in\Omega$;
(3) there exists a monotone increasing function $\beta:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$|h(t,\xi,c)| \le \beta(|c|),$$

for every $(t, \xi, c) \in [0,1] \times \Omega \times \mathbb{R}$.

(ii) for a.e. $\xi \in \Omega$ the function $k(\xi, \cdot) \in L^{p'}(\Omega, \mathbb{R})$ and

$$\sup_{\xi \in \Omega} ||k(\xi, \cdot)||_{p'} = \overline{k} < \infty.$$

(iii) for a.e. $t \in [0,1]$ the function $l(t, \cdot) \in L^p(\Omega, \mathbb{R})$ and

$$\sup_{t \in [0,1]} ||l(t, \cdot)||_p = \bar{l} < \infty.$$

We can substitute (17) by the following problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \text{ for a.e. } t \in [0,1], \\ x(0) = x(1), \end{cases} \tag{18}$$

where

$$\begin{cases} D(A) = W^{2,p}(\Omega, \mathbb{R}) \cap W_0^{1,p}(\Omega, \mathbb{R}), \\ A\colon D(A) \subset E \to E, \quad Aw = \Delta w, \end{cases}$$

and $f\colon [0,1] \times E \to E,\ f(t,w) = -bw + g(t,w) + l(t, \cdot)$ with

$$g\colon [0,1] \times E \to E,\ g(t,w)(\xi) = w(\xi) h\left(t, \xi, \int_\Omega k(\xi, \eta) w(\eta)\, d\eta\right).$$

It is known that A generates a compact C_0-semigroup of contractions $\{U(t)\}_{t \geq 0}$ on E (see, e.g. [26, Theorem 4.1.3]). Moreover we remark that g is well-defined. Indeed, from Hölder inequality, (3) and (ii), for $w \in E$ it follows

$$\left| h\Big(t, \xi, \int_\Omega k(\xi, \eta) w(\eta)\, d\eta\Big) \right| \leq \beta\left(\Big| \int_\Omega k(\xi, \eta) w(\eta) d\eta \Big| \right) \leq \beta\big(\overline{k} \|w\|_p\big), \tag{19}$$

for a.e. $t \in [0,1] \times \Omega$.

Again, by *a mild solution* of problem (17) we mean a continuous function $x \in C([0,T]; E)$ that is a mild solution of (18).

Theorem 4.3. *If $b > \frac{\bar{l}}{R} + \beta(\overline{k}R)$ for some $R > 0$, then problem (17) has a mild solution in $B_C(0,R)$. Moreover the set of mild solution of (17) is strongly compact in $C([0,1], L^p(\Omega, \mathbb{R}))$.*

Proof. We show that all assumptions of Theorem 3.1 hold true. First of all, according to (19), for every $w \in E$,

$$\|f(t,w)\|_p \leq [b + \beta(\overline{k}\|w\|_p)]\|w\|_p + \bar{l}.$$

The measurability of $f(\cdot, w)$ is trivially satisfied. So, conditions $(A3)$ and $(A4)$ hold true. We now prove that $f(t, \cdot)$ is weakly sequentially continuous for a.e. t. Indeed, it is sufficient to prove that $g(t, \cdot)$ is weakly sequentially continuous for a.e. t. Let $w_n \overset{E}{\rightharpoonup} w_0$, trivially we have

$$\int_\Omega k(\xi, \eta) w_n(\eta)\, d\eta \to \int_\Omega k(\xi, \eta) w_0(\eta)\, d\eta \ \text{ for a.e. } \ \xi \in \Omega.$$

Therefore, (2) implies that

$$h\Big(t, \xi, \int_\Omega k(\xi, \eta) w_n(\eta) d\eta\Big) \to h\Big(t, \xi, \int_\Omega k(\xi, \eta) w_0(\eta) d\eta\Big)$$

for a.e. $\xi \in \Omega$. Moreover from (3) we obtain that the convergence is dominated, hence, from the boundedness of $\{\|w_n\|_p\}$ we get that for every given $\varphi \in L^{p'}(\Omega, \mathbb{R})$ it holds that

$$\left| \int_\Omega \varphi(\xi) w_n(\xi) \Big[h\Big(t, \xi, \int_\Omega k(\xi,\eta) w_n(\eta) d\eta\Big) - h\Big(t, \xi, \int_\Omega k(\xi,\eta) w_0(\eta) d\eta\Big)\Big] d\xi \right| \leq$$
$$\|w_n\|_p \left[\int_\Omega |\varphi(\xi)|^{p'} \left| h\Big(t, \xi, \int_\Omega k(\xi,\eta) w_n(\eta) d\eta\Big) - h\Big(t, \xi, \int_\Omega k(\xi,\eta) w_0(\eta) d\eta\Big)\right|^{p'} d\xi \right]^{\frac{1}{p'}}$$
$$\to 0$$

i.e.

$$w_n \Big[h\Big(t, \cdot, \int_\Omega k(\cdot,\eta) w_n(\eta) d\eta\Big) - h\Big(t, \cdot, \int_\Omega k(\cdot,\eta) w_0(\eta) d\eta\Big)\Big] \rightharpoonup 0$$

in $L^p(\Omega, \mathbb{R})$. From (3) we also get that

$$h\Big(t, \cdot, \int_\Omega k(\cdot,\eta) w_0(\eta) d\eta\Big)(w_n - w_0) \rightharpoonup 0$$

in $L^p(\Omega, \mathbb{R})$, obtaining that

$$w_n h\Big(t, \cdot, \int_\Omega k(\cdot,\eta) w_n(\eta) d\eta\Big) - w_0 h\Big(t, \cdot, \int_\Omega k(\cdot,\eta) w_0(\eta) d\eta\Big) \rightharpoonup 0$$

i.e. the claimed result. Hence, condition $(A5)$ holds true.
Now, let us check the condition $(A2)$. To this aim, we recall (see, e.g. [13]) that for $w \in E$, $\|w\|_p > 0$, we have

$$\langle J(w), v\rangle = \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-2}\, w(\xi)\, v(\xi)\, d\xi.$$

Therefore, given $R > 0$, for a.e. $t \in [0,1]$ and $w \in E$, $0 < \|w\|_p < R$:

$$\begin{aligned} \big\langle J(w), g(t,w)\big\rangle &= \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^p\, h\Big(t, \xi, \int_\Omega k(\xi,\eta) w(\eta)\, d\eta\Big)\, d\xi \\ &\leq \tfrac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^p\, \beta\big(\overline{k}\, \|w\|_p\big) d\xi \\ &\leq \beta(\overline{k}R) \|w\|_p^2. \end{aligned}$$

Moreover

$$\begin{aligned} \big\langle J(w), l(t,\cdot)\big\rangle &= \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-2}\, w(\xi)\, l(t,\xi) d\xi \\ &\leq \tfrac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-1}\, |l(t,\xi)| d\xi \\ &\leq \tfrac{1}{\|w\|_p^{p-2}} \Big(\int_\Omega \big(|w(\xi)|^{p-1}\big)^{p'} d\xi \Big)^{\frac{1}{p'}} \Big(\int_\Omega |l(t,\xi)|^p d\xi \Big)^{\frac{1}{p}}. \end{aligned}$$

Since

$$\Big(\int_\Omega \big(|w(\xi)|^{p-1}\big)^{p'} d\xi \Big)^{\frac{1}{p'}} = \Big(\int_\Omega |w(\xi)|^p d\xi \Big)^{\frac{p-1}{p}} = \|w\|_p^{p-1},$$

we obtain

$$\big\langle J(w), l(t,\cdot)\big\rangle \leq \frac{\|l(t,\cdot)\|_p}{\|w\|_p^{p-2}}\, \|w\|_p^{p-1} \leq \bar{l} \|w\|_p.$$

Consequently,

$$\begin{aligned} \big\langle J(w), f(t,w)\big\rangle &= \big\langle J(w), -bw\big\rangle + \big\langle J(w), g(t,w)\big\rangle + \big\langle J(w), l(t,\cdot)\big\rangle \\ &\leq -\|w\|_p \big[\big(b - \beta(\overline{k}R)\big) \|w\|_p - \overline{l} \big] < 0. \end{aligned}$$

provided

$$\frac{\bar{l}}{b-\beta(\bar{k}R)} < \|w\|_p < R.$$

So, condition $(A2)$ holds true. Applying Theorem 3.1 we obtain the claimed result. □

Remark 6. In the case of Ω non-empty, open and unbounded subset of $\mathbb{R}^n$ the operator $A : D(A) \subset L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$ defined as

$$\begin{cases} D(A) = \{w \in W_0^{1,2}(\Omega, \mathbb{R}), \, \Delta w \in L^2(\Omega, \mathbb{R})\} \\ Aw = \Delta w, \end{cases}$$

is the generator of a non compact semigroup of contraction (see Theorem 4.1.2 in [26]). Hence, if $b > \frac{\bar{l}}{R} + \beta(\bar{k}R)$ for some $R > 0$, by Theorem 3.1 we obtain a non-empty solution set of (17) that is weakly compact in $C([0,1], L^2(\Omega, \mathbb{R}))$.

Remark 7. Let us note that

(a) In the classical case $\beta(c) = a(1+c)$ the condition in Theorem 4.3 reads as $b > 2\sqrt{a\bar{l}\,\bar{k}} + a$.

(b) In problem (17) we can consider an equation of this kind:

$$u_t = \Delta u - bu + h\left(t, \xi, \int_\Omega k(\xi, \eta) u(t, \eta)\, d\eta\right)$$

with h satisfying hypotheses (i)-(iii) above except for the growth condition (3) that is replaced by

(3)' there exists a function $\gamma \in L^p(\Omega, \mathbb{R}^+)$ such that

$$|h(t, \xi, c)| \le \gamma(\xi)\beta(|c|),$$

for every $(t, \xi, c) \in [0,1] \times \Omega \times \mathbb{R}$ with β as above.

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