Also available on SCIENCE DIRECT www.sciencedirect.com

European Journal of Combinatorics 24 (2003) 797–807

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Primitive collineation groups of ovals with a fixed point

Arrigo Bonisoli[a], Gloria Rinaldi[b]

[a]*Dipartimento di Scienze Sociali, Cognitive e Quantitative, Università di Modena e Reggio Emilia, via Giglioli Valle 9, 42100 Reggio Emilia, Italy*

[b]*Dipartimento di Scienze Agrarie, della Formazione Primaria e della Programmazione Sanitaria, Università di Modena e Reggio Emilia, via Kennedy 17, 42100 Reggio Emilia, Italy*

Received 28 October 2002; received in revised form 19 June 2003; accepted 3 July 2003

Dedicato a Pasquale Quattrocchi

Abstract

We investigate collineation groups of a finite projective plane of odd order n fixing an oval and having two orbits on it, one of which is assumed to be primitive. The situation in which the group fixes a point off the oval is considered. We prove that it occurs in a Desarguesian plane if and only if $(n + 1)/2$ is an odd prime, the group lying in the normalizer of a Singer cycle of $PGL(2, n)$ in this case. For an arbitrary plane we show that the group cannot contain Baer involutions and derive a number of structural and numerical properties in the case where the group has even order. The existence question for a non-Desarguesian example is addressed but remains unanswered, although such an example cannot have order $n \leq 23$ as computer searches carried out with GAP show.

1. Introduction

Let π be a finite projective plane of odd order n with an oval Ω and let G be a collineation group of π fixing Ω. The additional assumptions in this situation may concern the action of G on Ω, for instance primitivity [4] and transitivity [5], they may concern the action of G on π, most typically irreducibility [5, 6], or they may concern the algebraic nature of the group G, like for example if G is assumed to be simple [5].

Situations in which the given collineation group is neither transitive on the oval nor irreducible on the plane have been considered and satisfactory classifications have been achieved, at least partially, from time to time, see [1, 12, 13, 15].

E-mail addresses: bonisoli.arrigo@unimore.it (A. Bonisoli), rinaldi.gloria@unimore.it (G. Rinaldi).

0195-6698/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0195-6698(03)00090-8

The paper [2] considers the case in which G fixes a triangle and has two orbits on Ω, with a primitive action on at least one of them. While maintaining the latter assumption, in this paper we treat the "reducible" situation in which the group G fixes a point off the oval and we prove that it occurs in a Desarguesian plane if and only if $(n + 1)/2$ is a prime, the group lying in the normalizer of a Singer cycle of $PGL(2, n)$ in this case.

For an arbitrary plane we show that such a group of even order cannot contain Baer involutions: this piece of information is sufficient for deriving many of the properties holding in [5, Proposition 4.3(1)] where the group is assumed to act transitively on Ω. In particular, the Sylow 2-subgroups are either cyclic or generalized quaternion groups, with additional numerical constraints on n.

It is remarked in [21, p. 60] that a group fixing an oval in a projective plane of odd order and acting transitively on its points "may be soluble as well as fixing a non-incident point–line pair"; despite that, the author feels that "it seems probable" that a transitive oval "can occur only in the classical case". We are aware of no example in a non-Desarguesian plane for the situation which is being considered in this paper.

Unfortunately, no additional information seems to arise from such group-theoretic tools as the O'Nan–Scott theorem, which have found successful application elsewhere. In the final Section 5 we give a brief report on the computer searches that we carried out with GAP [16]: a non-Desarguesian plane of order $n \leq 23$ cannot yield an example for the situation under consideration.

2. Preliminaries

We recall that a group action on a given set is said to be *faithful* if the kernel of the action is trivial. We adopt the exponential notation for collineations acting on points, lines and, possibly, other configurations of a projective plane. For a finite group U, we denote by $O(U)$ the largest normal subgroup of odd order in U, a characteristic subgroup of U.

Let π be a finite projective plane of odd order n with an oval Ω and let G be a collineation group of π fixing Ω. We observe first of all that the action of G on Ω is faithful. Throughout the rest of the paper, the group G is assumed to satisfy the following properties:

(a) the action of G on Ω yields precisely two orbits Ω_1 and Ω_2 with $|\Omega_1| \geq 3$, $|\Omega_2| \geq 3$;
(b) G fixes a point O which is not on the oval;
(c) G has a primitive action on Ω_2.

Proposition 1. *The fixed point O is internal. Each secant through O contains precisely one point of Ω_1 and precisely one point of Ω_2. The action of G on Ω_2 is faithful and is equivalent to the action of G on the set of secants through O and to the action of G on Ω_1. In particular we have $|\Omega_1| = |\Omega_2| = (n + 1)/2$, the group G is primitive on Ω_1 and every collineation in G induces an even permutation on Ω.*

Proof. If O is an external point and X, Y denote the points at which the tangents through O meet Ω, then G either fixes or interchanges X and Y. If one of these points is in Ω_1 and the other one is in Ω_2, then both X and Y are fixed points of G and that contradicts the transitivity of G on Ω_2, since $|\Omega_2| \geq 3$.

If both X and Y are in Ω_i, where $i = 1$ or 2, then $\{X, Y\}$ is (at least setwise) fixed by G and that contradicts the transitivity of G on Ω_i since $|\Omega_i| \geq 3$. We conclude that O is an internal point.

Take a point $P \in \Omega_2$ and let Q be the further point of intersection of the line OP with Ω. Assume $Q \in \Omega_2$. The pairs $\{P^g, Q^g\}$, as g varies in G, form then a system of non-trivial blocks of imprimitivity for G on Ω_2, a contradiction to the primitivity of G on Ω_2, unless $|\Omega_2| = 2$ which is excluded. We conclude that $Q \in \Omega_1$. The transitivity of G on the two orbits shows that the action of G on each orbit is equivalent to the action of G on the secants through the point O and that each secant through O meets Ω in points of distinct orbits.

Assume $g \in G$ fixes Ω_2 pointwise; then since g fixes O it will also fix each secant through O and so g will fix Ω_1 pointwise; consequently g fixes each point of Ω and so g is the identity on the whole plane π. □

It follows from the faithfulness of G on Ω_2, that a non-trivial normal subgroup N of G cannot fix Ω_2 pointwise; since the orbits of N on Ω_2 form a system of blocks of imprimitivity for G on Ω_2, the primitivity of G on Ω_2 yields that N must be transitive on Ω_2.

Proposition 2. *There cannot exist a Baer involution in G. Consequently G contains involutory homologies if and only if it has even order.*

Proof. Assume G contains a Baer involution. We have $n \equiv 1 \bmod 4$ in this case and consequently $|\Omega_1| = |\Omega_2| = (n + 1)/2$ is odd. An involution in G must have at least one fixed point in each orbit. If g is a Baer involution in G, then its fixed Baer subplane has a non-empty intersection with the oval; consequently case (II) of [4, Proposition 2.2] occurs and g fixes no internal point, a contradiction.

A non-trivial perspectivity fixing an oval in a projective plane of odd order must be an involutory homology by [4, Proposition 2.1]. Since a collineation of order two of a finite projective plane is either a perspectivity or a Baer involution, the assertion follows. □

We know from [4, Proposition 2.1] that for an involutory homology fixing Ω either the center is an external point and the axis is a secant line or the center is an internal point and the axis is an external line; furthermore, the involutory homology is uniquely determined by its center or by its axis respectively.

Proposition 3. *If h is an involutory homology in G, then the center C of h is different from O and so the axis is incident with O; if C is external then $n \equiv 1 \bmod 4$, while if C is internal then $n \equiv -1 \bmod 4$; the line OC is external.*

Proof. Assume $C = O$. Each secant through O is fixed by h and meets Ω in points from distinct orbits Ω_1 and Ω_2. Consequently h fixes Ω pointwise, yielding the contradiction $h = \mathrm{id}$.

Let ℓ be the axis of h.

If C is external then ℓ is a secant through O meeting Ω at points $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$ which are fixed by h. Each secant through C meets Ω at two points which are exchanged by h, hence they lie in the same orbit. We conclude that $(n - 1)/2 = |\Omega_1 \setminus \{P_1\}| = |\Omega_2 \setminus \{P_2\}|$ is even in this case.

If C is internal then ℓ is an external line through O. Each secant through C meets Ω at two points which are exchanged by h, hence they lie in the same orbit. We conclude that $(n+1)/2 = |\Omega_1| = |\Omega_2|$ is even in this case.

Since O is an internal point by Proposition 1, the line OC cannot be a tangent. If OC is a secant, then Proposition 1 shows that the two points which OC shares with Ω lie in distinct orbits Ω_1 and Ω_2, but we have just observed that h exchanges these two points, a contradiction. We conclude that OC is an external line. □

3. The situation in Desarguesian planes

We want to prove that examples for the situation described in the previous section can be found in a Desarguesian plane. More precisely, we prove the following.

Proposition 4. *If π is Desarguesian and G is a collineation group of π satisfying* (a)–(c), *then $(n+1)/2$ is an odd prime and G is a subgroup of the normalizer of a Singer cycle of $PGL(2,n)$ acting on a conic.*

Proof. Let q be an odd prime and let r be a positive integer with $n = q^r > 3$. Every oval of $\pi = PG(2,n)$ is a conic by Segre's theorem [24]. Let therefore Ω be a conic of $PG(2,n)$ and let E denote the setwise stabilizer of Ω in the full collineation group of $PG(2,n)$. The group E acts on Ω as $P\Gamma L(2,n)$ in its natural 3-transitive permutation representation and the linear collineations in E form a subgroup L isomorphic to $PGL(2,n)$, [19, Theorem 2.37].

Let O be a point of $PG(2,n)$ which is internal with respect to Ω. The stabilizer of O in L is a dihedral subgroup D of order $2(n+1)$, hence D is the normalizer in L of a Singer cycle S [20, II Section 8.27]. The group L is thus transitive on internal points and consequently so is E. If we denote by V the stabilizer of the internal point O in E, then we have $|V| = 2r(n+1)$ and it follows from $D = V \cap L$ and the fact that L is a normal subgroup of E that D is a normal subgroup of V. Furthermore, the relation $n+1 > 2$ shows that S is the unique cyclic subgroup of order $n+1$ in D, consequently S is a normal subgroup of V. Singer cycles form a single conjugacy class in $PGL(2,n)$, whence also in $P\Gamma L(2,n)$, and so the normalizer of a Singer cycle in $P\Gamma L(2,n)$ has order $2r(n+1)$, showing that V is precisely the normalizer of S in E.

Let G be an arbitrary subgroup of V satisfying the requested conditions (a)–(c), and let Ω_1 and Ω_2 be the orbits of G on Ω. We know from Proposition 1 that $|\Omega_1| = |\Omega_2| = (n+1)/2$.

Setting $M = G \cap L$ we have that M is normal in G, hence by the primitivity of G, it is transitive on Ω_1 (resp. Ω_2) and $(n+1)/2$ divides the order of M. Therefore it is either $|M| = (n+1)/2$ or $|M| = n+1$.

In the latter case the group M is cyclic or dihedral; if M is cyclic it coincides with S, so it is transitive on Ω, which is false. Therefore M is dihedral of order $n+1$ and contains W, the subgroup of order $(n+1)/2$ of S. In particular, Ω_1 and Ω_2 are precisely the two orbits of W on Ω. The setwise stabilizer Γ of Ω_1 in V coincides with the setwise stabilizer of Ω_2 in V and we have $|V : \Gamma| = 2$.

A Singer cycle has the property that its normalizer in $P\Gamma L(2, n)$ coincides with the normalizer of any one of its non-trivial subgroups [20, II Section 7.3]. In particular, if $|\Omega_1| = |\Omega_2| = |W| = (n+1)/2$ is not a prime, then W admits at least one non-trivial proper subgroup. Since such a subgroup is normal in Γ hence also in G, its orbits will form a system of non-trivial blocks of imprimitivity for G on Ω_2. A transitive permutation group of prime degree is primitive [20, II Section 1.3] and so we conclude that Γ is primitive on Ω_2 if and only if $(n+1)/2$ is a prime. If that is the case, we can take for G any subgroup of Γ containing W and such that its linear part is dihedral of order $n+1$.

Assume now $|M| = (n+1)/2$. Then M is either the subgroup of order $(n+1)/2$ of S or M is dihedral intersecting S in a subgroup U of order $(n+1)/4 \geq 2$. In the former case we have $M = W$ and we know the situation from the previous discussion. In the latter case we know that U is normal in V, [20, II Section 7.3] and so U is normal in G as well; in particular since $(n+1)/4$ is a proper divisor of $(n+1)/2$ in this case, it turns out that the orbits of U on Ω_2 form a system of non-trivial blocks of imprimitivity for the action of G on Ω_2 and so this case is excluded.

We conclude that $(n+1)/2$ is an odd prime, yielding in particular $n \equiv 1 \bmod 4$. Furthermore the group G is a subgroup of Γ containing W and such that its linear part is either W or is dihedral of order $(n+1)$ (if this is the case the linear part of G coincides with the linear part of Γ). □

The homologies of $PG(2, n)$ are linear collineations and so the involutory homologies in Γ lie in the dihedral subgroup D. The group D has $n+2$ involutions, all of which are involutory homologies. One of these lies in S and has center O, while the centers of the involutory homologies in $D \backslash S$ are precisely the points of the polar line of O with respect to Ω, an external line [18, Theorem 8.16]. Half of these centers are external points, these are namely the centers of the involutory homologies in Γ (see Proposition 3).

If G has even order, it contains involutory homologies (see Proposition 2) and these are precisely the involutory homologies of Γ. We conclude that the linear part of G coincides with W if and only if G has odd order. On the other hand G has even order if and only if its linear part coincides with that of Γ.

Since Γ fixes a line, the previous situation admits an "affine" representation, which can be entirely stated in terms of finite fields. Since $GF(n^2)$ is a two-dimensional vector space over $GF(n)$ we may consider its elements as the points of the affine plane $AG(2, n)$. The subset $\Omega = \{z \in GF(n^2) : z^{n+1} = 1\}$ is an ellipse in $AG(2, n)$.

The group $A\Gamma L(2, n)$ of plane affine collineations contains the subgroup of all transformations

$$GF(n^2) \to GF(n^2), \qquad z \mapsto az^{q^i} + b$$

where $a, b \in GF(n^2)$, $a \neq 0$ and $0 \leq i < 2r$. Let ζ be a primitive element of $GF(n^2)$ and take ω to be $\zeta^{2(n-1)}$ or any other element of order $(n+1)/2$ in $GF(n^2)$. Consider the subgroup of $A\Gamma L(2, n)$ generated by the transformations

$$f : z \mapsto \omega z, \qquad g : z \mapsto z^q.$$

The subgroup $\langle f \rangle$ fixes Ω and yields two orbits of length $(n+1)/2$ on Ω, namely $\Omega_1 = \{1, \omega, \ldots, \omega^{(n-1)/2}\}$ and $\Omega_2 = \{\xi, \xi\omega, \ldots, \xi\omega^{(n-1)/2}\}$ where $\xi \in \Omega \backslash \Omega_1$. Since g is a field automorphism of $GF(n^2)$, it fixes each multiplicative subgroup of $GF(n^2)^*$, in particular g fixes Ω. The group $\langle f, g\rangle$ has order $r \cdot (n+1)$, fixes the internal point given by the zero element of $GF(n^2)$ and has two orbits on Ω, namely Ω_1, Ω_2: we conclude that $\langle f, g\rangle$ is precisely the group Γ of the previous "projective" description and the group $\langle f\rangle$ is precisely the group W.

If $(n+1)/2$ is an odd prime then the order of a Sylow 2-subgroup of Γ is 2 times the largest power of 2 dividing r; since the order of the transformation g is $2r$, we can conclude that if Γ is primitive on Ω_2, then a Sylow 2-subgroup of Γ is cyclic (the case $n = 3$ being the unique exception which is excluded by the assumption $|\Omega_i| \geq 3$).

The stabilizer in Γ of the field element 1 is precisely $\langle g\rangle$; a necessary condition for Γ to act 2-transitively on Ω_1 and Ω_2 is that $(n-1)/2 = (q^r - 1)/2$ divides $2r$, yielding $q = 3$ and $r = 2$: the action of Γ on Ω_1 and Ω_2 is sharply 2-transitive in this case.

4. The situation for groups of even order in arbitrary planes

We shall assume in this section that the group G has even order and show that it retains many of the properties proved in [5, Proposition 4.3]: the plane there was assumed of order $n \equiv 1 \bmod 4$ and G was assumed to act transitively on Ω with four dividing $|G|$.

Observe that our group G contains involutory homologies by Proposition 2. If we assume that G contains just one such involutory homology h, then the subgroup $\langle h\rangle$ is normal in G. Since the action of G on Ω_2 is faithful and primitive, the subgroup $\langle h\rangle$ is transitive on Ω_2, contradicting $|\Omega_2| \geq 3$. We conclude that G contains at least two distinct involutory homologies.

In what follows denote by $\mathcal{Z}$ and $\mathcal{A}$ the sets consisting of the centers and axes of the involutory homologies in G respectively.

Proposition 5. *Let h_1, h_2 be distinct involutory homologies in G with centers C_1, C_2 and axes a_1, a_2 respectively. We have $h_1h_2 \neq h_2h_1$, $C_1 \notin a_2$, $C_2 \notin a_1$. The product h_1h_2 has odd order and acts semiregularly on the set of points not on C_1C_2 and distinct from O. The line C_1C_2 is an external line not through O.*

Proof. If h_1, h_2 commute then h_1h_2 is an involution, whence an involutory homology by Proposition 2; since O lies on the axes a_1, a_2 it follows from [11, 3.1.7] that O is the center of the involutory homology h_1h_2, contradicting Proposition 3.

The involutory homology $h_2^{-1}h_1h_2$ has center $(C_1)^{h_2}$; if $C_1 \in a_2$ then we have $(C_1)^{h_2} = C_1$, consequently $h_2^{-1}h_1h_2$ has center C_1 and must therefore coincide with h_1, which means h_1, h_2 commute, a contradiction. Similarly, we cannot have $C_2 \in a_1$.

The subgroup $\langle h_1, h_2\rangle$ is a dihedral group of order $2d$, where d is the order of the collineation h_1h_2. If d is even then $\langle h_1, h_2\rangle$ contains a central involution, whence an involutory homology commuting in particular with, say, h_1, a contradiction. Consequently d is odd.

Let t be a positive integer with $t < d$ and let P be a fixed point of $(h_1h_2)^t$ off C_1C_2 and distinct from O. Consider the collineation $g = (h_1h_2)^t h_1$. As $gh_1 = (h_1h_2)^t$ has odd order, g is conjugate to h_1, whence g and h_1 are distinct involutory homologies.

However $P^g = P^{(h_1h_2)^t h_1} = P^{h_1}$, but that is only possible if g and h_1 have the same axis or the same center, a contradiction.

We prove that the line C_1C_2 is external. Assume the line C_1C_2 is a tangent to Ω at, say, P. Let R_1 and R_2 denote the (distinct) points at which the tangents through C_1 and C_2 other than C_1C_2 touch Ω respectively. Then P turns out to be the common point of the lines PR_1, PR_2, which are the axes of h_1 and h_2 respectively, contradicting the fact that the common point of these axes is O.

Assume C_1C_2 is a secant meeting Ω at, say, R, S. Certainly both R and S lie in Ω_1 or in Ω_2. Assume the former and observe that the collineation h_1h_2 acts semiregularly on the $(n+1)/2-2$ points of $\Omega_1\backslash\{R,S\}$ as well as on the $(n+1)/2$ points of Ω_2, yielding that d should divide $\gcd((n+1)/2-2,(n+1)/2)=1$, a contradiction.

Finally, assume O lies on C_1C_2 and observe that h_1h_2 acts semiregularly on the $(n+1)(n-2)+1$ points off $\Omega \cup C_1C_2$ as well as on the $(n+1)/2$ points of Ω_1. Hence d should divide $\gcd((n+1)(n-2)+1,(n+1)/2)=1$, a contradiction. □

Proposition 6. *The group G contains an elementary Abelian normal subgroup M of odd order acting regularly on Ω_2, as well as on the sets $\mathcal{Z}$ and $\mathcal{A}$, respectively. In particular, $(n+1)/2 = p^e$ holds for an odd prime p and a positive integer e, whence $n \equiv 1 \bmod 4$.*

The set $\mathcal{A}$ coincides with the set of secants through O and O is the unique fixed point of G.

The subgroup H generated by the involutory homologies in G has order $2p^e$ and it is the semidirect product of M by a group of order two; H is primitive on Ω_1 respectively, Ω_2 if and only if $(n+1)/2$ is a prime.

Proof. Let H be the subgroup of G generated by the involutory homologies. Any two involutory homologies in G generate a dihedral group of order twice an odd number, hence they are conjugate in this dihedral group and so they are also conjugate in H. By Corollary 3 in [17] the group H possesses a subgroup N of odd order which is normal in G with $|H : N| = 2$.

Let h be an involutory homology in G. As $H = N\langle h\rangle$, every involutory homology in G is conjugate to h under a collineation in N. The group N is non-trivial, as G does not have a normal subgroup of order 2. By the Feit–Thompson theorem [14] the group N is solvable and consequently a minimal normal subgroup M of G contained in N must be an elementary Abelian group acting regularly on Ω_2 [20, II Section 3.2]. In particular, $(n+1)/2$ is a power p^e of an odd prime p and consequently the relation $n \equiv 1 \bmod 4$ holds. Proposition 3 shows now that the center of each involutory homology in G is an external point, and so the axis is a secant line through O. In particular, h fixes a point in Ω_2, say P.

Since M is transitive on Ω_2, we have $N = MN_P$. For every $g \in N_P$ the product hh^g acts semiregularly on Ω_2 by Proposition 5. However hh^g fixes P as h does. Hence $hh^g = \mathrm{id}$ for all $g \in N_P$. In other words h centralizes N_P, whence the conjugates of h under $N = MN_P$ coincide with those under M. As the latter ones generate $M\langle h\rangle$, we have $N\langle h\rangle = H = M\langle h\rangle$, whence $N = M$ and $\langle h\rangle$ is the stabilizer of P in H.

In particular, H is primitive on Ω_2 (which is equivalent to the maximality of $\langle h\rangle$ in H) if and only if $p^e = |M|$ is a prime.

For each secant line s through O we have $|s \cap \Omega_1| = |s \cap \Omega_2| = 1$ and since $(n+1)/2$ is the total number of secants through O we have the regularity of M on $\mathcal{A}$ and $\mathcal{Z}$.

Finally, assume G fixes another point O'. For an involutory homology in G we have that either the point O' is the center or the line OO' is the axis. Since the number $(n+1)/2$ of involutory homologies in G is at least three, we see that at least two distinct involutory homologies in G have either the same center or the same axis, which is not the case. □

Proposition 7. *The group G fixes an external line ℓ not through O containing $\mathcal{Z}$.*

Proof. We have seen that H contains an elementary Abelian normal subgroup M acting regularly on $\mathcal{Z}$ and Ω_2. Consequently M acts regularly on the set of external lines through O as well as on the set of secants through O.

We observe first of all that O is the unique fixed point of M, otherwise the line joining O to a further fixed point of M would also be fixed by M, contradicting the previous observation. The number of external lines is $(2p^e - 1)(p^e - 1) \equiv 1 \bmod p$, therefore M fixes at least one external line and such a line cannot contain O as we have already observed. Should M fix another external line, then the common point of these two fixed external lines would be a fixed point of M other than O, a contradiction. We conclude that M fixes precisely one external line.

The collineation group G leaves invariant the set of external lines; furthermore M is normal in G; consequently the unique external line which is fixed by M must also be fixed by G. Such an external line is therefore fixed by each involutory homology in G and, being different from its axis (which is a secant), it must contain its center. □

A finite group is said to have 2-rank k if 2^k is the largest order of an elementary Abelian 2-subgroup. The 2-rank of the given group is clearly an upper bound for the 2-rank of its 2-subgroups, in particular of its Sylow 2-subgroups. For a finite group of even order having 2-rank 1 is obviously equivalent to having a unique involution in each Sylow 2-subgroup. A finite 2-group with a unique involution is known to be either a cyclic group or a generalized quaternion group [8, p. 132]; see also [11, 4.2.2].

A remarkable property of a collineation group fixing an oval in a finite projective plane of odd order is given by [4, Proposition 2.4], stating that the 2-rank of such a group is at most 3. Our situation is even sharper, as the next property shows.

Proposition 8. *The group G has 2-rank 1. If $(n+1)/2$ is a prime then G is the semidirect product of $O(G)$, the largest normal subgroup of odd order of G, by a cyclic 2-group.*

Proof. If the 2-rank of G is strictly greater than 1, then G possesses a Klein group K; the three involutions in K are involutory homologies by Proposition 2 and so the center of one of them must be O by [11, 3.1.7], which is impossible by Proposition 3.

If $(n+1)/2 = p$, then M is cyclic and $\mathrm{Aut}(M)$ is cyclic of order $p-1$. For $X \in \Omega$, we have that $|G : G_X|$ is odd and so G_X contains a Sylow 2-subgroup of G, which must be cyclic because $G_X \leq \mathrm{Aut}(M)$, as $C_G(M) = M$. Now G splits over $O(G)$ by [20, IV Section 2.8]. □

It is proved in [3, Satz 2] that a transitive finite permutation group in which every involution fixes precisely one point either (i) possesses a transitive normal subgroup of odd order and Sylow 2-subgroups which are cyclic or generalized quaternion groups, or (ii) is 2-transitive.

Proposition 8 tells us that our collineation group G, viewed as a primitive permutation group on Ω_2 (resp. Ω_1) is of *affine* type in the terminology of the O'Nan–Scott theorem, see [23], all other classes having namely 2-rank at least 2.

The next proposition adds some numerical constraints in case our collineation group G contains 2-elements other than involutions. Let namely T be a non-trivial 2-subgroup of G and let h be the unique involutory homology in T; denote by ℓ and C the axis and the center of h, respectively. Set $\Omega \cap \ell = \{P_1, P_2\}$ with $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$.

Proposition 9. *The group T acts semiregularly on $\Omega_1 \setminus \{P_1\}$ and on $\Omega_2 \setminus \{P_2\}$, respectively. We have $n \equiv 1 \bmod 2|T|$. If g is a collineation in T of order $2^t > 2$, then g partitions the $(n-1)/2$ external points of ℓ into cycles of length 2^{t-1}. If 2^m is the largest order of a cyclic 2-subgroup of G containing g, then g fixes at least 2^{m-t+1} internal points of ℓ; moreover, each line through C and one such point is external.*

Proof. For each point $Q \in \Omega_2 \setminus \{P_2\}$ the homology h does not lie in the stabilizer T_Q. Consequently $|T_Q| = 1$ and each T-orbit on $\Omega_2 \setminus \{P_2\}$ has length $|T|$. Hence $|\Omega_2 \setminus \{P_2\}| = (n+1)/2 - 1$ is a multiple of $|T|$, yielding $n \equiv 1 \bmod 2|T|$.

If S is an external point of ℓ, then S is fixed by $\langle g^{2^{t-1}} \rangle$ and by no larger subgroup of $\langle g \rangle$, that is a subgroup $\langle g^{2^i} \rangle$ with $i < t-1$. In fact, if Q and R are the points of $\Omega \setminus \{P_1, P_2\}$ lying on the tangents through S, then g^{2^i} should fix $\{Q, R\}$ at least setwise, contradicting the fact that the $\langle g \rangle$-orbit of Q on Ω has length 2^t. Each $\langle g \rangle$-orbit on the external points of ℓ has thus length 2^{t-1}.

Let now Z be a cyclic subgroup of G with $|Z| = 2^m$ and $g \in Z$. Let f be a collineation in Z of order 2^m such that $g = f^{2^{m-t}}$ holds. We have $n \equiv 1 \bmod 2^{m+1}$ by what we have seen above. Since the involutory homology $h = f^{2^{m-1}}$ fixes each point of ℓ, the action of f on ℓ yields orbits of length at most 2^{m-1}. The fixed points of g on ℓ appear in $\langle f \rangle$-orbits of length less than 2^{m-t+1}. If x is the number of these points, the remaining $n+1-x$ points of ℓ appear in $\langle f \rangle$-orbits of length at least 2^{m-t+1}. For $i = 1, \ldots, t-1$ let c_i be the number of $\langle f \rangle$-orbits of length 2^{m-t+i}. We count $n+1-x = c_1 \cdot 2^{m-t+1} + \cdots + c_{t-1} \cdot 2^{m-1}$ and obtain that 2^{m-t+1} divides $n+1-x$. As $n \equiv 1 \bmod 2^{m+1}$, we conclude that 2^{m-t+1} divides $x-2$. Furthermore, since P_1, P_2 and O are fixed points of g on ℓ, we have $x \geq 3$ and so x is at least $2^{m-t+1}+2$. Since P_1, P_2 are the two fixed points of g on Ω, we conclude that there are at least 2^{m-t+1} fixed internal points of g on ℓ.

Let I be a fixed (internal) point of g on ℓ. The line CI is external, because if CI intersects Ω in, say, X and Y, then g fixes $\{X, Y\}$ at least setwise, contradicting the fact that each $\langle g \rangle$-orbit on $\Omega \setminus \{P_1, P_2\}$ has length 2^t. □

5. Some computational data

Let π be a finite projective plane of odd order n with an oval Ω and let G be a collineation group of π fixing Ω. Is it possible to find an example in which G has even order and satisfies properties (a)–(c) of Section 2 while π is non-Desarguesian?

The information gathered in the previous sections does not seem sufficient for attempting to provide an answer. As a matter of fact, even in the Desarguesian case the group G turns out to be fairly small when compared with the full collineation group fixing the oval.

It sounds likely that the kind of "global" properties which are generally involved in the proof that a given plane is Desarguesian cannot be derived from the "local" information arising from G.

The unique projective plane of order $n = 5$ is Desarguesian; the integer $3 = (n + 1)/2$ is a prime and so we have an example for our situation in this plane.

There are four planes of order $n = 9$ up to isomorphism [22]. The Desarguesian plane does yield an example because $5 = (n+1)/2$ is a prime. The order of the setwise stabilizer of an oval in the Hall plane, in its dual plane or in the Hughes plane is one of the integers 16, 32 or 48, [10]: such a group contains no subgroup of order 10 and so none of these planes yields an example for our situation according to Proposition 6.

We decided to test the next values of n for which $(n + 1)/2$ is a prime power, using the GAP library of primitive permutation groups of small degree. Our search was exhaustive for $n = 13, 17$. In the former case, since $7 = (n + 1)/2$ is a prime, the Desarguesian plane does furnish an example for the given situation and we verified it is the only one for $n = 13$. In the latter case, since $9 = (n + 1)/2$ is not a prime, the Desarguesian plane does not yield an example for the given situation and we verified the non-existence of such an example for $n = 17$ altogether.

We have chosen the approach of Buekenhout ovals [7]. These are namely suitable subsets of involutory permutations on Ω and allow an implementation inside $\mathrm{Sym}(\Omega)$, which is thus suitable for GAP. The idea was to take the primitive groups of even order and given degree $p^e = (n + 1)/2$ from the GAP library `PrimitveGroups`.

The first step was to check whether such a group fulfilled the structural constraints like, for instance those on the 2-subgroups. Then the given permutation representation had to be doubled on twice as many elements, since the action of G on Ω_1 and Ω_2 is the same. The main part of the computation was the procedure reconstructing the Buekenhout oval arising from our putative *projective* oval Ω in the plane π; see [9] for the terminology. The reconstruction process stops before completion in all cases under consideration, except for the dihedral group of order 14 in its representation of degree 7, and that yields precisely the case of a conic in $PG(2, 13)$.

GAP source code for the two cases that were considered is available from the authors.

References

[1] V. Abatangelo, M.R. Enea, G. Korchmáros, B. Larato, Ovals and unitals in commutative twisted field planes, Discrete Math. 208–209 (1999) 3–8.

[2] A. Aguglia, A. Bonisoli, Intransitive collineation groups of ovals fixing a triangle, J. Combin. Theory Ser. A 102 (2003) 273–282.

[3] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, J. Algebra 17 (1971) 527–554.

[4] M. Biliotti, G. Korchmàros, Collineation groups which are primitive on an oval of a projective plane of odd order, J. London Math. Soc. 33 (1986) 525–534.

[5] M. Biliotti, G. Korchmáros, Collineation groups preserving an oval in a projective plane of odd order, J. Aust. Math. Soc. 48 (1990) 156–170.

[6] A. Bonisoli, M.R. Enea, G. Korchmàros, Irreducible collineation groups fixing an oval, Abh. Math. Sem. Univ. Hamburg 69 (1999) 259–264.
[7] F. Buekenhout, Etude intrinsèque des ovales, Rend. Mat. Appl. (5) 125 (1996) 87–94.
[8] W. Burnside, Theory of Groups of Finite Order, second ed., Cambridge, 1911. Dover, New York, 1955 (reprint).
[9] W. Cherowitzo, On the projectivity of B-ovals, J. Geometry 27 (1986) 119–139.
[10] W.E. Cherowitzo, D.I. Kiel, R. Killgrove, Ovals and other configurations in the known planes of order nine, Congr. Numer. 55 (1986) 167–179.
[11] P. Dembowski, Finite Geometries, Springer, Berlin, 1968.
[12] M.R. Enea, G. Korchmáros, Ovals in commutative semifield planes, Arch. Math. (Basel) 69 (1997) 259–264.
[13] M.R. Enea, G. Korchmáros, $\mathcal{I}$-transitive ovals in projective planes of odd order, J. Algebra 208 (1998) 604–618.
[14] W. Feit, J.G. Thompson, Solvability of groups of odd order, Pacific J. Math. 13 (1963) 775–1029.
[15] E. Francot, Ovals with 2-transitive groups fixing a point, Note Mat. 19 (1999) 19–24.
[16] The GAP Group, GAP—Groups, Algorithms, and Programming, version 4.2, Aachen, St. Andrews, 2000. Available from http://www-gap.dcs.st-and.ac.uk/~gap.
[17] G. Glauberman, Central elements in core-free groups, J. Algebra 4 (1966) 403–420.
[18] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, second ed., Oxford University Press, 1998.
[19] D.R. Hughes, F.C. Piper, Projective Planes, Springer, Berlin, 1970.
[20] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.
[21] G. Korchmáros, Old and new results on ovals in finite projective planes, in: A.D. Keedwell (Ed.), Surveys in Combinatorics, 1991, London Math. Soc. Lecture Note Ser., vol. 166. Cambridge University Press, Cambridge, 1991, pp. 41–72.
[22] C.W.H. Lam, G. Kolesova, L. Thiel, A computer search for finite projective planes of order 9, Discrete Math. 92 (1991) 187–195.
[23] M. Liebeck, C. Praeger, J. Saxl, On the O'Nan–Scott theorem for finite primitive permutation groups, J. Aust. Math. Soc. 44 (1988) 389–396.
[24] B. Segre, Ovals in a finite projective plane, Canad. J. Math. 7 (1955) 414–416.