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HARTMAN-TYPE CONDITIONS FOR MULTIVALUED DIRICHLET PROBLEM IN ABSTRACT SPACES

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Abstract. The classical Hartman's Theorem in [18] for the solvability of the vector Dirichlet problem will be generalized and extended in several directions. We will consider its multivalued versions for Marchaud and upper-Carathéodory right-hand sides with only certain amount of compactness in Banach spaces. Advanced topological methods are combined with a bound sets technique. Besides the existence, the localization of solutions can be obtained in this way.

1. **Introduction.** One of the most classical results in the theory of boundary value problems (b.v.p.) is Hartman's Theorem.

Theorem 1.1. ([18, Theorem 1], cf. also [19]) *Let us consider the vector Dirichlet problem*

$$\left.\begin{array}{ll} \ddot{x}(t) = f(t, x(t), \dot{x}(t)), & t \in [0, T], \\ x(T) = x(0) = 0, & \end{array}\right\} \tag{1}$$

where $f \colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Assume that there exists $R > 0$ such that

$$\langle f(t, x, y), x \rangle + ||y||^2 \geq 0, \tag{2}$$

for all $t \in [0, T]$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $||x|| = R$ and $\langle x, y \rangle = 0$.

Moreover, let

$$||f(t, x, y)|| \leq 2\alpha(\langle f(t, x, y), x \rangle + ||y||^2) + K, \text{ for all } t \in [0, T], \text{ with } ||x|| \leq R \tag{3}$$

and

$$||f(t, x, y)|| \leq \phi(||y||), \text{ for all } t \in [0, T] \text{ with } ||x|| \leq R \tag{4}$$

where α, K are nonnegative constants and $\phi(s)$ is a Nagumo function, i.e. $\phi \colon [0, \infty) \to (0, \infty)$ is a continuous function such that

$$\int^{\infty} \frac{s}{\phi(s)}\, ds = \infty. \tag{5}$$

Then the b.v.p. (1) has a solution.

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Although it was already published in 1960, as far as we know, it has not yet been improved, but only extended in many ways (see e.g. [1], [3], [5], [7], [9], [11]–[14], [17], [22], [24]–[28], [30]–[33]). In order to be more specific, better (more general) conditions were obtained e.g. in [23, 27], but they were not imposed strictly locally, but globally. Moreover, it is not clear whether or not, for instance, $\mathbb{R}^n$ can be replaced by a Hilbert space, when f is additionally completely continuous, or a continuous right-hand side (r.h.s.) can be replaced by a Carathéodory r.h.s., provided the same conditions (2)–(5) remain valid.

On the other hand, condition (2) can be assumed on the boundary of a convex subset of $\mathbb{R}^n$ or even on the boundary of its homeomorphic image or retract (cf. also [1]), which we regard rather as a generalization than an improvement of Theorem 1.1.

In the present paper, we will show that such extensions and generalizations are also possible for multivalued problems in abstract spaces. We will consider separately Dirichlet problems with not necessarily completely continuous Marchaud (i.e. globally upper semicontinuous) and upper-Carathéodory r.h.s. in Banach and, in particular, Hilbert spaces.

Remark 1.1. In fact, in [18] as well as in [19], the following problem with nonhomogeneous boundary conditions was considered

$$\left.\begin{aligned} &\ddot{x}(t) = f(t, x(t), \dot{x}(t)), \qquad t \in [0, T],\\ &x(0) = x_0,\ x(T) = x_T. \end{aligned}\right\} \tag{6}$$

On the other hand, problem (6) can be easily transformed via homogenization (i.e. by making the change of variables $y(t) = x(t) - u(t)$, where $u(t) = \frac{x_T - x_0}{T} t + x_0$; cf. e.g. [7, pp. 2–4]) into the form (1) in Theorem 1.1 which is more convenient for comparison with analogous results.

Remark 1.2. As pointed out in [18, 19], condition (3) can be omitted in the scalar case, i.e. for $n = 1$, or (for an arbitrary $n \in \mathbb{N}$) provided condition (4) is replaced by

$$||f(t, x, y)|| \le \gamma ||y||^2 + C, \text{ for all } t \in [0, T], x \in \mathbb{R}^n \text{ with } ||x|| \le R \text{ and } y \in \mathbb{R}^n, \tag{7}$$

where $\gamma,\ C$ are nonnegative constants such that $\gamma R < 1$.

Remark 1.3. Using the Nagumo-type result (cf. [32]), conditions (2)–(5) were reformulated in [15, Theorem V.24], [16] as follows:

- The non-strict inequality in (2) was replaced by the strict one, i.e.
$$\langle f(t, x, y), x\rangle + ||y||^2 > 0, \tag{8}$$
for all $t \in [0, T]$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $||x|| = R$ and $\langle x, y\rangle = 0$.
- Condition (3) was omitted.
- The function $\phi\colon [0, \infty) \to (0, \infty)$ appearing in the Nagumo-type conditions (4), (5) was this time a continuous, nondecreasing function such that (4) holds, jointly with
$$\lim_{s\to\infty} \frac{s^2}{\phi(s)} = \infty. \tag{9}$$

Remark 1.4. In [34], the Dirichlet problem (1) was studied for a completely continuous mapping $f\colon [0, T] \times E \times E \to E$, where E is a Banach space. If E is $\mathbb{R}^n$, or even a Hilbert space H, then the analogy of Theorem 1.1 holds, when

- The inequality in (2) is replaced by
$$\langle f(t, x, y), x\rangle \ge 0, \tag{10}$$
for all $t \in [0, T]$ and $(x, y) \in H \times H$ with $||x|| = R$ and $\langle x, y\rangle = 0$.
- Condition (3) is omitted.
- The function $\phi\colon [0, \infty) \to (0, \infty)$ appearing in the Nagumo-type condition (4) has the same properties as in Remark 1.3.

We would like to improve especially the statements in Remarks 1.3 and 1.4 and generalize them into the multivalued setting in abstract spaces.

The paper is organized as follows. After this introduction, we recall in the next section some elements of functional and multivalued analyses, jointly with the definition of a measure of non-compactness and condensing maps. Since our main theorems rely essentially on our earlier results in [3]–[6],[8], we formulate them in the form of propositions. The Nagumo-type auxiliary results are given separately in Section 3. The main results are formulated in the form of three theorems in Section 4. Section 5 contains an application to a control problem. Finally, in Section 6, the special finite-dimensional single-valued case of obtained results is compared in discussion with their classical analogous versions of the other authors.

2. **Preliminaries.** Let E be a Banach space having the Radon-Nikodym property (see e.g. [29, pp. 694–695]) and $[0,T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0,T],E)$, we shall mean the set of all Bochner integrable functions $x\colon [0,T] \to E$. For the definition and properties, see e.g. [29, pp. 693–701].

The symbol $AC^1([0,T],E)$ will denote the set of functions $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous. Then $\ddot{x} \in L^1([0,T],E)$ and the fundamental theorem of calculus (the Newton–Leibniz formula) holds (see e.g. [2, pp. 243–244], [29, pp. 695–696]).

Thus, the differentiability (derivatives) will be entirely understood in a strong sense of Fréchet. Similarly, solutions of differential inclusions will be considered in a (strong) Carathéodory sense, i.e. belonging to the AC^1-class. Of course, in the single-valued case, solutions of ordinary differential equations with continuous r.h.s. will automatically become C^2-solutions, like in Theorem 1.1.

We will denote by E' the Banach space dual to E and by $\langle \cdot, \cdot \rangle$ the pairing (the duality relation) between E and E', i.e., for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) := \langle \Phi, x \rangle$.

Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e. $B = \{x \in E \mid ||x|| < 1\}$.

We shall also need the following definitions and notions from the multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given.

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

We say that a multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is a *step multivalued mapping* if there exists a finite family of disjoint measurable subsets I_k, $k = 1,\dots,n$ such that $[0,T] = \cup I_k$ and F is constant, on every I_k. A multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is called *strongly measurable* if there exists a sequence of step multivalued mappings $\{F_n\}$ such that $d_H(F_n(t), F(t)) \to 0$ as $n \to \infty$, for a.a. $t \in [0,T]$, where d_H stands for the Hausdorff distance.

A multivalued mapping $F\colon [0,T] \times X \multimap Y$ is called an *upper-Carathéodory mapping* if the map $F(\cdot,x)\colon [0,T] \multimap Y$ is strongly measurable, for all $x \in X$, the map $F(t,\cdot)\colon X \multimap Y$ is u.s.c., for almost all $t \in [0,T]$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in [0,T] \times X$.

For more details concerning multivalued analysis, see e.g. [2, 10, 20, 21].

In the sequel, the measure of non-compactness will also be employed.

Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all nonempty subsets of E. A function $\beta\colon P(E) \to N$ is called a *measure of non-compactness* (m.n.c.) in E if $\beta(\overline{co\ \Omega}) = \beta(\Omega)$, for all $\Omega \in P(E)$, where $\overline{co\ \Omega}$ denotes the closed convex hull of Ω.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) := \inf\{\varepsilon > 0 \mid \exists x_1, \dots, x_n \in E \colon \Omega \subset \cup_{i=1}^{n} B(x_i, \varepsilon)\}.$$

Let E be a Banach space and $X \subset E$. A multivalued mapping $F \colon X \multimap E$ with compact values is called *condensing with respect to an m.n.c.* β (shortly, β-*condensing*) if, for every $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

For the properties and more details about an m.n.c. and condensing maps, see e.g. [2, 10, 20, 21].

We will also need the following slight modification of the result presented in [8].

Proposition 2.1. [8, Theorem 2.2] *Let E be a Banach space and $K \subset E$ a nonempty, open, convex, bounded set such that $0 \in K$. Assume that $V \colon E \to \mathbb{R}$ is a Fréchet differentiable function with $\dot{V}$ Lipschitzian in $\partial K + \varepsilon\overline{B}$, for some $\varepsilon > 0$, and that it satisfies*

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$,

$(H3)$ $||\dot{V}(x)|| \geq \delta$, for all $x \in \partial K$, where $\delta > 0$.

Then there exist $k \in (0, \varepsilon]$ and a bounded Lipschitzian function $\psi \colon \partial K + k\overline{B} \to E$ such that $\langle \dot{V}_x, \psi(x)\rangle = 1$, for every $x \in \partial K + k\overline{B}$.

Remark 2.1. In a Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle$, every derivative $\dot{V}_x$ can be identified with an element of H. Consequently, if $\dot{V}$ is Lipschitizian and if it satisfies condition $(H3)$, it is possible to find $k \in (0, \varepsilon]$ such that

$$x \longmapsto \psi(x) = \frac{\dot{V}_x}{\|\dot{V}_x\|}, \quad x \in \partial K + k\overline{B},$$

is well defined. Moreover, it holds that $\langle \dot{V}_x, \psi(x)\rangle = 1$, for all x in its domain.

We will also apply the following continuation principle developed in [4].

Proposition 2.2. *Let us consider the general multivalued b.v.p.*

$$\left.\begin{array}{c}\ddot{x}(t) \in \varphi(t, x(t), \dot{x}(t)), \text{ for a.a. } t \in [0,T],\\ x \in S,\end{array}\right\} \tag{11}$$

where $\varphi \colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory mapping and $S \subset AC^1([0,T], E)$. Let $H \colon [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset \varphi(t, c, d), \text{ for all } (t, c, d) \in [0,T] \times E \times E. \tag{12}$$

Moreover, assume that the following conditions hold:

(i) There exist a closed, bounded set $S_1 \subset S$ and a closed, convex set $Q \subset C^1([0,T], E)$ with a non-empty interior Int Q such that each associated problem

$$\left.\begin{array}{c}\ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \text{ for a.a. } t \in [0,T],\\ x \in S_1,\end{array}\right\} \tag*{$P(q,\lambda)$}$$

where $q \in Q$ and $\lambda \in [0,1]$, has a non-empty, convex set of solutions (denoted by $\mathfrak{T}(q, \lambda)$).

(ii) For every nonempty, bounded set $\Omega \subset E \times E$, there exists $\nu_\Omega \in L^1([0,T], [0,\infty))$ such that

$$||H(t, x, y, z, w, \lambda)|| \leq \nu_\Omega(t),$$

for a.a. $t \in [0,T]$ and all $(x, y, z, w) \in \Omega \times \Omega$ and $\lambda \in [0,1]$.

(iii) The solution mapping $\mathfrak{T}$ is quasi-compact and μ-condensing with respect to a monotone and nonsingular m.n.c. μ defined on $C^1([0,T], E)$.

(iv) For each $q \in Q$, the set of solutions of the problem $P(q, 0)$ is a subset of Int Q, i.e. $\mathfrak{T}(q, 0) \subset$ Int Q, for all $q \in Q$.

(v) For each $\lambda \in (0,1)$, the solution mapping $\mathfrak{T}(\cdot,\lambda)$ has no fixed points on the boundary ∂Q of Q.

Then the b.v.p. (11) has a solution in Q.

In [3] and [5], we proved in detail the following three results, stated here in the form of propositions, which deal with the existence and the localization of a solution of a multivalued Dirichlet problem. The first result concerns the case of globally upper semicontinuous r.h.s. and strictly localized (so called *bounding*) function V (cf. conditions (15) and (16) below). The second result concerns the case of upper-Carathéodory r.h.s. and non-strictly localized bounding function V (cf. conditions (20) and (21) below). The third result, obtained in [5] via the Scorza-Dragoni approximation technique, deals with the case of upper-Carathéodory r.h.s. and strictly localized bounding function V (cf. condition (23) below).

Proposition 2.3. (cf. [3, Theorem 5.2]) *Consider the Dirichlet b.v.p.*

$$\left.\begin{array}{r}\ddot{x}(t) \in F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [0,T],\\ x(T) = x(0) = 0,\end{array}\right\} \tag{13}$$

where E is a Banach space having the Radon-Nikodym property and $F\colon [0,T]\times E\times E \multimap E$ is an upper semicontinuous mapping with compact, convex values. Assume that $K \subset E$ is an open, bounded, convex set containing 0. Moreover, let the following conditions be satisfied:

(1_i) *$\gamma\left(F(t,\Omega_1\times\Omega_2)\right) \le g(t)\left(\gamma(\Omega_1)+\gamma(\Omega_2)\right)$, for a.a. $t\in[0,T]$ and each $\Omega_1 \subset \overline{K}$, and each bounded $\Omega_2 \subset E$, where $g \in L^1([0,T],[0,\infty))$ and γ is the Hausdorff m.n.c. in E.*

(1_{ii}) *For every non-empty, bounded set $\Omega \subset E$, there exists $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that*

$$\|F(t,x,y)\| \le \nu_\Omega(t), \tag{14}$$

for a.a. $t \in [0,T]$ and all $(x,y) \in \Omega \times E$.

(1_{iii})

$$(T+4)\|g\|_{L^1([0,T],[0,\infty))} < 2.$$

Furthermore, let there exist a function $V \in C^1(E,\mathbb{R})$ with a locally Lipschitzian Fréchet derivative $\dot{V}$ satisfying $(H1)$ and $(H2)$ of Proposition 2.1. Moreover, let, for all $x \in \partial K$, $t \in (0,T)$, $\lambda \in (0,1)$ and $y \in E$ satisfying

$$\langle \dot{V}_x, y\rangle = 0, \tag{15}$$

the following condition holds

$$\liminf_{h\to 0} \frac{\langle \dot{V}_{x+hy}, y\rangle}{h} + \langle \dot{V}_x,\, w\rangle > 0, \tag{16}$$

for all $w \in \lambda F(t,x,y)$. Then the Dirichlet b.v.p. (13) admits a solution whose values are located in $\overline{K}$. If, moreover, $0 \notin F(t,0,0)$, for a.a. $t \in [0,T]$, then the obtained solution is nontrivial.

Remark 2.2. The typical case occurs when $E = H$ is a Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the scalar product and

$$V(x) := \frac{1}{2}\left(||x||^2 - R^2\right) = \frac{1}{2}\left(\langle x,x\rangle - R^2\right), \tag{17}$$

for some $R > 0$. In this case, $V \in C^2(H,\mathbb{R})$ and it is not difficult to see that condition (16) becomes

$$\langle y,\, y\rangle + \lambda\langle x,\, w\rangle > 0 \tag{18}$$

with t,x,y as in Proposition 2.3 (where $K := \{x \in H |\ ||x|| < R\ \}$) and $w \in F(t,x,y)$.

Moreover, since the condition (18) is satisfied, when

$$\langle y,\, y\rangle + \langle x,\, w\rangle > 0 \tag{19}$$

holds, for t, x, y and w as above, we can assume, instead of (18), condition (19) which is more convenient for applications and especially more suitable for comparison with analogous results.

We will present the conditions under which the strict inequality in the condition (16) can be replaced by a non-strict one. In the last section, we will discuss the case when $E = \mathbb{R}^n$ and $f = F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Thus, we will obtain the generalizations of results described in Theorem 1.1, Remark 1.3 and Remark 1.4.

Proposition 2.4. (cf. [3, Theorem 5.1]) *Consider the Dirichlet b.v.p. (13), where $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping. Assume that $K \subset E$ is an open,bounded, convex set containing 0. Furthermore, let the conditions (1_i), (1_{ii}), (1_{iii}) from Proposition 2.3 be satisfied.*

Finally, let there exist a function $V \in C^1(E, \mathbb{R})$ with a locally Lipschitzian Fréchet derivative $\dot{V}$ satisfying conditions $(H1)$ and $(H2)$ of Proposition 2.1 and at least one of the conditions

$$\limsup_{h \to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y \rangle}{h} + \langle \dot{V}_x, w \rangle > 0, \tag{20}$$

$$\limsup_{h \to 0^+} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y \rangle}{h} + \langle \dot{V}_x, w \rangle > 0, \tag{21}$$

for a suitable $\varepsilon > 0$, all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,y)$. Then the Dirichlet b.v.p. (13) admits a solution whose values are located in $\overline{K}$. If, moreover, $0 \notin F(t,0,0)$, for a.a. $t \in [0,T]$, then the obtained solution is nontrivial.

Remark 2.3. Let us note that also in this case the analogy of previous remark holds, i.e. if $E = H$ is a Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the scalar product and V is defined by (17), then conditions (20) and (21) take the form (18). They can be satisfied e.g. by the condition (19) which is more convenient for comparisons with other results.

We will also present the conditions under which the strict inequality in the condition (18) can be replaced by a non-strict one.

Proposition 2.5. (cf. [5, Theorem 3.1]) *In a separable Banach space E, let us consider the Dirichlet b.v.p. (13) and suppose that $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory mapping which is either globally measurable or quasi-compact. Let $K \subset E$ be an open, bounded, convex set containing 0 and let the conditions (1_i), (1_{ii}) and the following condition (2_{iii}) be satisfied:*

(2_{iii})

$$(T+4)||g||_{L^1([0,T],[0,\infty))} < 4.$$

Furthermore, let there exist $\varepsilon > 0$ and a function $V \in C^2(E, \mathbb{R})$ with Fréchet derivative $\dot{V}$ Lipschitzian in $\overline{B(\partial K, \varepsilon)}$ satisfying $(H1)$, $(H2)$ and $(H3)$ of Proposition 2.1. Let there still exist $h > 0$ such that

$$\left\langle \ddot{V}_x(v), v \right\rangle \geq 0, \ \textit{for all } x \in B(\partial K, h),\ v \in E, \tag{22}$$

where $\ddot{V}_x(v)$ denotes the second Fréchet derivative of V at x in the direction $(v,v) \in E \times E$. Finally, let

$$\langle \dot{V}_x, w \rangle > 0, \tag{23}$$

for a.a. $t \in (0,T)$ and all $x \in \partial K$, $v \in E$, and $w \in F(t,x,v)$.

Then the Dirichlet b.v.p. (13) admits a solution whose values are located in $\overline{K}$. If, moreover, $0 \notin F(t,0,0)$, for a.a. $t \in [0,T]$, then the obtained solution is nontrivial.

3. **Nagumo-type auxiliary results.** Following the ideas in [34], for $x \in C^2([0,1],E)$ and F single-valued and completely continuous, we can state the following lemma which this time concerns $x \in AC^1([0,T],E)$ and a multivalued upper-Carathéodory r.h.s. F. For the sake of completeness, we shall prove it in detail.

Lemma 3.1. *Let $\phi\colon [0,\infty) \to (0,\infty)$ be a continuous, nondecreasing function such that*

$$\lim_{s\to\infty} \frac{s^2}{\phi(s)} = \infty. \tag{24}$$

Moreover, let E be a Banach space satisfying the Radon-Nikodym property and $F\colon [0,T]\times E\times E \multimap E$ be an upper-Carathéodory mapping satisfying (29). If $R > 0$ and $x \in AC^1([0,T],E)$ is a solution of the inclusion

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \qquad \textit{for a.a. } t \in [0,T], \tag{25}$$

satisfying $||x(t)|| \le R$, then there exists $M > 0$ (depending only on R and ϕ) such that $\|\dot{x}(t)\| \le M$, for all $t \in [0,T]$.

Proof. Let $x \in AC^1([0,T],E)$ be a solution of the inclusion (25) satisfying

$$||x|| := \left\{ \max_{t\in[0,T]} ||x(t)|| \right\} \le R.$$

Put

$$q := \max_{t\in[0,T]} ||\dot{x}(t)||$$

and let $t_0 \in [0,T]$ be such that $||\dot{x}(t_0)|| = q$.

If $a \in [-\frac{1}{2}, \frac{1}{2}]$ is arbitrary such that $t_0 + a \in [0,T]$, it holds that

$$x(t_0 + a) - x(t_0) = \int_0^1 \dot{x}(t_0 + sa)a\; ds = -\int_0^1 \dot{x}(t_0 + sa)a(1-s)'\; ds.$$

Using the per-partes formula and the integration in the sense of Lebesque, we then get that

$$\begin{aligned} x(t_0 + a) - x(t_0) &= -\left[\dot{x}(t_0 + sa)a(1-s)\right]_0^1 + \int_0^1 \ddot{x}(t_0 + sa)a^2(1-s)\; ds \\ &= a\dot{x}(t_0) + a^2 \int_0^1 \ddot{x}(t_0 + sa)(1-s)\; ds. \end{aligned}$$

Therefore,

$$q := ||\dot{x}(t_0)|| \le \frac{2R}{|a|} + |a| \int_0^1 ||\ddot{x}(t_0 + sa)||(1-s)\; ds \le \frac{2R}{|a|} + \frac{|a|\phi(q)}{2},$$

because the function ϕ is nondecreasing and satisfies condition (29).

Moreover, since

$$\lim_{s\to\infty} \frac{s^2}{\phi(s)} = \infty,$$

it is possible to find $Q(R) > 0$ such that if $s > Q(R)$, then $\frac{s^2}{\phi(s)} > 4R$.

We will now show that $q \le M$, where

$$M := \max\left\{Q(R), 8R\right\}. \tag{26}$$

For this purpose, let us assume that $q > Q(R)$. Then $\frac{q^2}{4R} > \phi(q)$, and so

$$q < \frac{2R}{|a|} + \frac{|a|q^2}{8R}. \tag{27}$$

If $\frac{4R}{q} \geq \frac{1}{2}$, then $q \leq 8R$, and if $\frac{4R}{q} < \frac{1}{2}$, then we obtain the contradiction to (27), when setting $|a| = \frac{4R}{q}$, because in such a case

$$q < \frac{2R}{\frac{4R}{q}} + \frac{\frac{4R}{q}q^2}{8R} = \frac{q}{2} + \frac{q}{2} = q.$$

Therefore, if $q > Q(R)$, then $q \leq 8R$, and so $q = ||\dot{x}|| \leq M$, where M is defined by (26). □

It was shown in [19] that if $E = \mathbb{R}^n$ and $F = f\colon [0,T] \times E \times E \to E$ is a continuous function, then condition (24) can be improved as follows.

Lemma 3.2. (cf. [19, Lemma XII.5.2]) *Let $f\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function satisfying condition (4), where $\phi\colon [0,\infty) \to (0,\infty)$ is a continuous function satisfying condition (5). Moreover, let there exist α, $\beta \geq 0$ such that, for a.a. $t \in [0,T]$ and all $x \in \mathbb{R}^n$ with $||x|| \leq R$ and $y \in \mathbb{R}^n$, condition (3) holds.*

If $R > 0$ and $x \in C^2([0,T], \mathbb{R}^n)$ is a solution of the equation

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)), \qquad \textit{for } t \in [0,T], \tag{28}$$

satisfying $||x(t)|| \leq R$, then there exists $M > 0$ (depending only on R, α, β and ϕ) such that $\|\dot{x}(t)\| \leq M$, for all $t \in [0,T]$.

Let us note that the standard usage of the Nagumo technique, i.e. Lemma 3.1 and Lemma 3.2 here, consists in its application to the inclusion $\ddot{x} \in F(t, x, r(\dot{x}))$, where $r\colon E \to \overline{B}_M$ is a retraction to the ball $\overline{B}_M := \{y \in E : \|y\| \leq M\}$ such that $r|_{\overline{B}_M} = \mathrm{id}|_{\overline{B}_M}$. Since $\|x\| \leq R$ implies $\|\dot{x}\| \leq M$, the obtained results can be then adopted to the original inclusion $\ddot{x} \in F(t, x, \dot{x})$.

4. **Main results.**

In this section, we will show conditions under which the strict inequalities (16), (20)–(21), (23) in Propositions 2.3–2.5 can be replaced by non-strict ones and we will also formulate the versions of the obtained results, when E is a Hilbert space. In the following, we always assume that F satisfies the Nagumo-type condition, i.e.

$$||F(t,x,y)|| \leq \phi(||y||), \text{ for a.a. } t \in [0,T], \text{ all } x \in E \text{ with } ||x|| \leq R \text{ and } y \in E, \tag{29}$$

with ϕ as in (24).

Theorem 4.1. *Let us consider the Dirichlet b.v.p. (13), where $F\colon [0,T] \times E \times E \multimap E$ is an upper semicontinuous mapping with compact, convex values satisfying conditions (1_i), (1_{iii}) from Proposition 2.3 and condition (2_{ii}), where:*

(2_{ii}) *For every non-empty, bounded set $\Omega \subset E \times E$, there exists $\nu_\Omega \in L^1([0,T], [0,\infty))$ such that*

$$\|F(t,x,y)\| \leq \nu_\Omega(t), \tag{30}$$

for a.a. $t \in [0,T]$ and all $(x,y) \in \Omega$.

Let $K \subset E$ be an open, bounded, convex set containing 0 and let there exist a function $V \in C^1(E, \mathbb{R})$ with a locally Lipschitzian Fréchet derivative $\dot{V}$ satisfying $(H1)$, $(H2)$ and $(H3)$ of Proposition 2.1. Furthermore, let, for all $x \in \partial K$, $t \in (0,T)$, $\lambda \in (0,1)$ and $y \in E$ such that

$$\langle \dot{V}_x, y \rangle = 0, \tag{31}$$

the following condition

$$\liminf_{h \to 0} \frac{\langle \dot{V}_{x+hy}, y \rangle}{h} + \langle \dot{V}_x, w \rangle \geq 0 \tag{32}$$

holds, for all $w \in \lambda F(t,x,y)$. Finally, let (29) be valid with ϕ as in (24) and

$$R = |K| := \sup_{k \in K} \|k\|.$$

Then the Dirichlet b.v.p. (13) admits a solution whose values are located in $\overline{K}$. If, moreover, $0 \notin F(t,0,0)$, for a.a. $t \in [0,T]$, then the obtained solution is nontrivial.

Proof. Consider a continuous function $\tau\colon E \to [0,1]$ such that $\tau(x) = 0$, for $x \notin \partial K + k\overline{B}$, while $\tau(x) = 1$, for $x \in \partial K + \frac{k}{2}\overline{B}$, where the constant k was introduced in Proposition 2.1. Let ψ be as in Proposition 2.1 and let us denote by L_ψ its Lipschitz constant. It is easy to see that the function $\overline{\psi}\colon E \to E$ given by

$$\overline{\psi}(x) = \begin{cases} \tau(x)\psi(x), \text{ for } x \in \partial K + k\overline{B}, \\ \qquad 0, \text{ otherwise}, \end{cases}$$

is well-defined, continuous and bounded.

Let us put $m := \sup_{x \in E} \|\overline{\psi}(x)\|$ and consider the sequence of Dirichlet problems

$$\left.\begin{aligned} &\ddot{x}(t) \in F_n(t, x(t), \dot{x}(t)), \qquad \text{for a.a. } t \in [0,T], \\ &x(T) = x(0) = 0, \end{aligned}\right\} \; (P_n)$$

with $F_n\colon [0,T] \times E \times E \multimap E$ such that $(t,x,y) \longmapsto F(t,x,y) + \dfrac{\overline{\psi}(x)}{n}$.

Since $\overline{\psi}$ is continuous, the mapping F_n is u.s.c., for all $n \in \mathbb{N}$. We will show that F_n satisfies condition (1_i) with $g_n = g + \frac{L_\psi}{n}$. In such a case, for a sufficiently large n, we have that

$$(T+4)\|g_n\|_{L^1([0,T],[0,\infty))} = (T+4)\|g\|_{L^1([0,T],[0,\infty))} + \frac{TL_\psi(T+4)}{n} < 2,$$

and so (1_{iii}) holds. In order to prove condition (1_i), for F_n, let $\Omega_1 \subset \overline{K}$ and $\Omega_2 \subset E$ be bounded. Firstly, notice that

$$\begin{aligned} \gamma\left(\left\{\tfrac{\overline{\psi}(\Omega_1)}{n}\right\}_n\right) &= \tfrac{1}{n}\gamma\left(\{\tau(x)\psi(x) \,:\, x \in \Omega_1 \cap \left(\partial K + k\overline{B}\right)\}\right) \\ &\le \tfrac{1}{n}\gamma\left(\cup_{\lambda\in[0,1]}\{\lambda\psi(x) \,:\, x \in \Omega_1 \cap \left(\partial K + k\overline{B}\right)\}\right). \end{aligned}$$

It is well known that $\gamma\left(\cup_{\lambda\in[0,1]}\Omega\right) = \gamma(\Omega)$, for every bounded $\Omega \subset E$ (see e.g. [6]). Therefore, according to the properties of the Hausdorff m.n.c.,

$$\gamma\left(\left\{\frac{\overline{\psi}(\Omega_1)}{n}\right\}_n\right) \le \frac{1}{n}\gamma\left(\psi(\Omega_1 \cap \left(\partial K + k\overline{B}\right))\right) \le \frac{L_\psi}{n}\gamma(\Omega_1 \cap \left(\partial K + k\overline{B}\right)).$$

Moreover, according to (1_i) and the additivity of the Hausdorff m.n.c., we obtain that

$$\begin{aligned} \gamma\left(F_n(t, \Omega_1 \times \Omega_2)\right) &\le g(t)\left(\gamma(\Omega_1) + \gamma(\Omega_2)\right) + \tfrac{L_\psi}{n}\gamma(\Omega_1 \cap \left(\partial K + k\overline{B}\right)) \\ &\le \left(g(t) + \tfrac{L_\psi}{n}\right)\left(\gamma(\Omega_1) + \gamma(\Omega_2)\right), \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, F_n satisfies (1_i), for all $n \in \mathbb{N}$, with g_n as announced. From now on, we assume that condition (1_{iii}) is satisfied, for all $n \in \mathbb{N}$.

Let $\overline{\phi}\colon E \to \mathbb{R}$ be such that $x \longmapsto \phi(x) + m$. Observe that $\overline{\phi}$ is nondecreasing, satisfies (24) and, for all $n \in \mathbb{N}$,

$$\|F_n(t,x,y)\| \le \overline{\phi}(\|y\|), \text{ for a.a. } t \in [0,T], \text{ all } x \in E \text{ with } ||x|| \le R \text{ and } y \in E. \tag{33}$$

Consider $x \in \partial K$, $t \in (0,1)$ and $y \in E$ satisfying $\langle \dot{V}_x, \, y\rangle = 0$. If $w_n \in \lambda F_n(t,x,y)$, there exists $w \in \lambda F(t,x,y)$ satisfying $w_n = w + \frac{\lambda\psi(x)}{n}$, and so

$$\langle \dot{V}_x, \, w_n\rangle = \langle \dot{V}_x, \, w\rangle + \langle \dot{V}_x, \, \frac{\lambda\psi(x)}{n}\rangle = \langle \dot{V}_x, \, w\rangle + \frac{\lambda}{n}.$$

Therefore,

$$\liminf_{h\to 0}\frac{\langle \dot V_{x+hy}, y\rangle}{h}+\langle \dot V_x,\, w_n\rangle=\frac{\lambda}{n}+\liminf_{h\to 0}\frac{\langle \dot V_{x+hy}, y\rangle}{h}+\langle \dot V_x,\, w\rangle>0.$$

Thanks to condition (29), for a sufficiently large n, the Dirichlet problem (P_n) satisfies all the assumptions of Proposition 2.3, and subsequently it admits a solution $x_n(t)$ such that $x_n(t)\in\overline{K}$, for all $t\in[0,T]$. Moreover, according to (33) (cf. Lemma 3.1), there exists $M>0$ such that $\|\dot x_n(t)\|\le M$, for a.a. $t\in[0,T]$.

If we denote by $G(t,s)$ the Green function associated to the homogeneous problem (see e.g. [3]), it is known that

$$x_n(t)=\int_0^T G(t,s)\ddot x_n(s)\,ds \quad\text{and}\quad \dot x_n(t)=\int_0^T\frac{\partial}{\partial t}G(t,s)\ddot x_n(s)\,ds, \tag{34}$$

for all $t\in[0,T]$ and $n\in\mathbb{N}$. Moreover, $|G(t,s)|\le\frac{T}{4}$ and $|\frac{\partial}{\partial t}G(t,s)|\le 1$, for all $(t,s)\in[0,T]\times[0,T]$. Hence, according to (1_{ii}) and Lemma 3.1, there exists $M>0$ such that

$$\|G(t,s)\ddot x_n(s)\|\le\frac{T}{4}\left(\nu_{\overline{K}\times\overline{B}(0,M)}(s)+\frac{m}{n}\right),$$

and

$$\left\|\frac{\partial}{\partial t}G(t,s)\ddot x_n(s)\right\|\le\nu_{\overline{K}\times\overline{B}(0,M)}(s)+\frac{m}{n},$$

for all $(t,s)\in[0,T]\times[0,T]$ and $n\in\mathbb{N}$. In addition, condition (1_i) and the semi-homogeneity of the m.n.c. γ imply that

$$\gamma(\{G(t,s)\ddot x_n(s)\}_n)\le\frac{T}{4}\mathcal{S}g_n(s)\quad\text{and}\quad\gamma\left(\left\{\frac{\partial}{\partial t}G(t,s)\ddot x_n(s)\right\}_n\right)\le\mathcal{S}g_n(s),$$

for a.a. $t\in[0,T]$ and $s\in[0,t]$, where

$$\mathcal{S}:=\sup_{t\in[0,T]}\gamma(\{x_n(t)\}_n)+\gamma(\{\dot x_n(t)\}_n).$$

Notice that, in this case, it is possible to exchange the m.n.c. γ with the integral (see e.g. [6]). Hence, in view of (34), we obtain, for all $t\in[0,T]$,

$$\gamma\left(\{x_n(t)\}_n\right)=\gamma\left(\left\{\int_0^T G(t,s)\ddot x_n(s)\,ds\right\}_n\right)\le\frac{T}{2}\mathcal{S}\|g_n\|_{L^1([0,T],[0,\infty))}, \tag{35}$$

and

$$\gamma\left(\{\dot x_n(t)\}_n\right)=\gamma\left(\left\{\int_0^T\frac{\partial}{\partial t}G(t,s)\ddot x_n(s)\,ds\right\}_n\right)\le 2\mathcal{S}\|g_n\|_{L^1([0,T],[0,\infty))}. \tag{36}$$

Combining the last two estimates, we obtain $\mathcal{S}\le\frac{T+4}{2}\mathcal{S}\|g_n\|_{L^1([0,T],[0,\infty))}$ and, according to (1_{iii}), we get that $\mathcal{S}=0$. Hence, $\gamma(\{x_n(t)\}_n)=\gamma(\{\dot x_n(t)\}_n)=0$, for all $t\in[0,T]$. Thus, according to (1_i) and (29), $\gamma(\{\ddot x_n(t)\}_n)=0$, and $\|\ddot x_n(t)\|\le\nu_{\overline{K}\times\overline{B}(0,M)}(t)$, for a.a. $t\in[0,T]$. Therefore, a classical convergence result (see e.g. [2, Lemma III.1.30]) can be used which guarantees the existence of a subsequence, denoted as the sequence, and of a function $x\in AC^1([0,T],E)$ such that $x_n\to x$, $\dot x_n\to\dot x$, in $C([0,T],E)$, and $\ddot x_n\rightharpoonup\ddot x$ in $L^1([0,T],E)$, when $n\to\infty$. Consequently, $x(t)\in\overline{K}$, for a.a. $t\in[0,T]$. Finally, by means of a classical closure result (see e.g. [21, Lemma 5.1.1.]), we can show that x is a solution of (13) which completes the proof. □

Remark 4.1. Theorem 4.1 is a significant generalization in several directions of [34, Theorem 4.1]. Since in finite-dimensional spaces K in Theorem 4.1 need not be convex, but only an absolute retract space (i.e. in particular contractible), Theorem 4.1 can be also regarded as a generalization of [1, Theorem 4.5], where only strict sign conditions were applied.

As a straightforward consequence of Theorem 4.1, we can obtain the following result, when E is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$, $K := B(0,R)$ and $V(x) := \frac{1}{2}\left(\|x\|^2 - R^2\right)$. In this case, condition (32) becomes

$$\langle x, w\rangle + \langle y, y\rangle \geq 0.$$

Corollary 4.1. *Let E be a Hilbert space and let us consider the Dirichlet b.v.p. (13), where $F\colon [0,T]\times E\times E \multimap E$ is an upper semicontinuous mapping with compact, convex values. Moreover, let conditions $(1_i),(1_{iii})$ from Proposition 2.3, jointly with condition (2_{ii}) from Theorem 4.1, be satisfied and let there exist $R>0$ such that*

$$\langle x,\, w\rangle + \langle y,\, y\rangle \geq 0, \tag{37}$$

holds, for all $t\in(0,T)$, $(x,y)\in E\times E$ with $||x|| = R$ and $\langle x,y\rangle = 0$ and all $w\in F(t,x,y)$. Furthermore, let (29) be valid, with ϕ as in (24).

Then the Dirichlet b.v.p. (13) admits a solution $x(\cdot)$ such that $||x(t)||\leq R$, for all $t\in[0,T]$. If, moreover, $0\notin F(t,0,0)$, for a.a. $t\in[0,T]$, then the obtained solution is nontrivial.

It will be shown in the following theorem that if F is an upper-Carathéodory multivalued mapping and the transversality condition is not strictly localized, i.e. not required exclusively for $x\in\partial K$, but assumed for all $x\in\overline{K}\cap B(\partial K,\varepsilon)$, it is possible to replace the strict inequalities in (20) and (21) by non-strict ones.

Theorem 4.2. *Let us consider the Dirichlet b.v.p. (13), where $F\colon[0,T]\times E\times E\multimap E$ is an upper-Carathéodory multivalued mapping. Assume that $K\subset E$ is an open, bounded, convex set containing 0. Moreover, let conditions $(1_i),(1_{iii})$ from Proposition 2.3, jointly with condition (2_{ii}) from Theorem 4.1, be satisfied. Furthermore, let there exist a function $V\in C^1(E,\mathbb{R})$ with a locally Lipschitzian Fréchet derivative $\dot V$ satisfying conditions $(H1)$ and $(H2)$ of Proposition 2.1 and at least one of the conditions*

$$\limsup_{h\to 0^-}\frac{\langle \dot V_{x+hy}-\dot V_x, y\rangle}{h} + \langle \dot V_x,\, w\rangle \geq 0, \tag{38}$$

$$\limsup_{h\to 0^+}\frac{\langle \dot V_{x+hy}-\dot V_x, y\rangle}{h} + \langle \dot V_x,\, w\rangle \geq 0, \tag{39}$$

for a suitable $\varepsilon>0$, all $x\in\overline{K}\cap B(\partial K,\varepsilon)$, $t\in(0,T)$, $y\in E$, $\lambda\in(0,1)$ and $w\in\lambda F(t,x,y)$. Finally, let (29) be valid with ϕ as in (24) and

$$R = |K| := \sup_{k\in K}\|k\|.$$

Then the Dirichlet b.v.p. (13) admits a solution whose values are located in $\overline{K}$. If, moreover, $0\notin F(t,0,0)$, for a.a. $t\in[0,T]$, then the obtained solution is nontrivial.

Proof. Consider the function $\overline{\psi}$ introduced in the proof of Theorem 4.1 and put $\hat\psi := j\overline{\psi}$ with $j>0$. Let us note that $\hat\psi$ is continuous and bounded with $\sup_{x\in E}\|\hat\psi(x)\| = jm$.

We will now apply the continuation principle of Proposition 2.2 in the case when $\varphi(t,x,y) := F(t,x,y)$, $S := \{x\in AC^1([0,T],E) : x(0)=x(T)=0\}$ and

$$\begin{aligned} H\colon\quad & [0,T]\times E\times E\times E\times E\times[0,1]\multimap E,\\ & H(t,x,y,z,w,\lambda) = (1-\lambda)\hat\psi(z) + \lambda F(t,z,w). \end{aligned}$$

At first, let us observe that both φ and H are upper-Carathéodory mappings and that condition (12) is obviously satisfied. In what follows, we will check, step by step, that all the assumptions of Proposition 2.2 are satisfied.

ad (i) Let $S_1 = S$ and put $Q = C^1([0,T], \overline{K})$. It is clear that S_1 is closed and, according to the properties of K, the set Q is closed, convex and has a nonempty interior. Moreover, given $q \in Q$, $\lambda \in [0,1]$ and $f \in L^1([0,T], E)$ with $f(t) \in F(t, q(t), \dot{q}(t))$, for a.a. $t \in [0,T]$, consider the Dirichlet b.v.p.

$$\begin{cases} \ddot{x}(t) &= (1-\lambda)\hat{\psi}(q(t)) + \lambda f(t), \qquad \text{for a.a. } t \in [0,T], \\ x(T) &= x(0) = 0. \end{cases}$$

This problem is uniquely solvable (see e.g. the proof of [3, Theorem 5.1]) which implies that the set $\mathfrak{T}(q,\lambda)$ of its solutions is nonempty. Since F is convex-valued, $\mathfrak{T}(q,\lambda)$ is also convex.

ad (ii) Condition (ii) is a direct consequence of (2_{ii}), when applied Lemma 3.1, and the boundedness of $\hat{\psi}$. More concretely, if $\Omega \subset E \times E$ is bounded, then there exists $\nu_\Omega \in L^1([0,T],[0,\infty))$ (cf. condition (2_{ii})) such that

$$\|H(t,x,y,z,w,\lambda)\| = \|(1-\lambda)\hat{\psi}(z) + \lambda F(t,z,w)\| \le jm + \nu_\Omega(t), \qquad \text{for a.a. } t \in [0,T].$$

Since $jm + \nu_\Omega \in L^1([0,T],[0,\infty))$, condition (ii) from Proposition 2.2 is satisfied.

ad (iii) Let $\{q_n\}_n \subset Q$ and let $\{\lambda_n\}_n \subset [0,1]$. Then

$$x_n(t) = \int_0^T G(t,s)\left[(1-\lambda_n)\hat{\psi}(q_n(s)) + \lambda_n f_n(s)\right] ds$$

and

$$\dot{x}_n(t) = \lambda_n \int_0^T \frac{\partial}{\partial t} G(t,s)\left[(1-\lambda_n)\hat{\psi}(q_n(s)) + \lambda_n f_n(s)\right] ds,$$

with $G(t,s)$ as in the proof of [3, Theorem 5.1] and $f_n(t) \in F(t, q_n(t), \dot{q}_n(t))$, for a.a. $t \in [0,T]$. Notice, in particular, that if $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$, then $(1-\lambda_n)\hat{\psi}(q_n(t)) \to (1-\lambda)\hat{\psi}(q(t))$, implying that $\{(1-\lambda_n)\hat{\psi}(q_n(t))\}_n$ is relatively compact, for all $t \in [0,T]$. Therefore, a very similar reasoning as in the proof of [3, Theorem 5.1] can be used in order to show that the solution mapping $\mathfrak{T}$ is quasi-compact and μ-condensing, where μ is the monotone non-singular m.n.c. defined in [3, p. 308].

ad (iv) For every $q \in Q$, $\mathfrak{T}(q,0)$ is the unique solution of the problem

$$\begin{cases} \ddot{x}(t) &= \hat{\psi}(q(t)), \qquad \text{for a.a. } t \in [0,T], \\ x(T) &= x(0) = 0. \end{cases}$$

Since

$$x(t) = \int_0^T G(t,s)\hat{\psi}(q(s))\, ds,$$

we have that

$$\|x(t)\| \le \frac{T}{4}\int_0^T \|\hat{\psi}(q(s))\|\, ds \le \frac{jT^2 m}{4}.$$

So, if j is small enough, then the ball $\frac{jT^2m}{4}B \subset K$, and so condition (iv) is also satisfied.

ad (v) Assume that condition (38) is satisfied.[1] If x is a fixed point of $\mathfrak{T}(\cdot,\lambda)$, for some $\lambda \in (0,1)$, then x is a solution of

$$\begin{cases} \ddot{x}(t) &\in (1-\lambda)\hat{\psi}(x(t)) + F(t,x(t),\dot{x}(t)), \qquad \text{for a.a. } t \in [0,T], \\ x(T) &= x(0) = 0. \end{cases} \tag{40}$$

We can then apply the bound set theory (cf. [3, Propositon 4.1]) to the problem (40) in order to show that x is necessarily in the interior of Q.

[1] Let us note that a similar reasoning can be developed under condition (39).

In fact, let $p = \min\left\{\frac{k}{2}, \varepsilon\right\}$, $x \in \overline{K} \cap B(\partial K, \frac{p}{2})$, $t \in (0,T)$ and $y \in E$ and let $w \in (1-\lambda)\hat{\psi}(x) + F(t,x,y)$. According to the definition of $\hat{\psi}$, we have that $\hat{\psi}(x) = j\psi(x)$, and, moreover, there exists $w_0 \in \lambda F(t,x,y)$ such that $w = (1-\lambda)j\psi(x) + w_0$. Hence,

$$\begin{aligned} \tfrac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x,\, w\rangle &= \tfrac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x,\, j(1-\lambda)\psi(x) + w_0\rangle \\ &= \tfrac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + (1-\lambda)j\langle \dot{V}_x,\, \psi(x)\rangle + \langle \dot{V}_x,\, w_0\rangle \\ &= \tfrac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x,\, w_0\rangle + (1-\lambda)j. \end{aligned}$$

Therefore, according to (38), we obtain that

$$\limsup_{h\to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x,\, w\rangle$$

$$= (1-\lambda)j + \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x,\, w_0\rangle > 0.$$

Now, we can apply [3, Propositon 4.1] in order to prove that $x(t) \in K$, for all $t \in [0,T]$, and so condition (v) from Proposition 2.2 is verified.

All the assumptions of the continuation principle from Proposition 2.2 are so satisfied, and the proof is complete. □

As a straightforward consequence of Theorem 4.2, we can obtain the following result, when E is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$, $K := B(0,R)$ and $V(x) := \frac{1}{2}\left(\|x\|^2 - R^2\right)$. In this case, both conditions (38) and (39) become

$$\langle x, w\rangle + \langle y, y\rangle \geq 0.$$

Corollary 4.2. *Let E be a Hilbert space and let us consider the Dirichlet b.v.p. (13), where $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping. Let the conditions $(1_i), (1_{iii})$ from Proposition 2.3, jointly with condition (2_{ii}) from Theorem 4.1, be satisfied Furthermore, let there exist $R > 0$ such that*

$$\langle x,\, w\rangle + \langle y,\, y\rangle \geq 0 \tag{41}$$

holds, for a suitable $\varepsilon > 0$, a.a. $t \in (0,T)$ and all $x \in E$ with $R - \varepsilon < ||x|| \leq R$, $y \in E$ and $w \in F(t,x,y)$. Finally, let (29) be valid with ϕ as in (24) and

$$R = |K| := \sup_{k\in K} \|k\|.$$

Then the Dirichlet b.v.p. (13) admits a solution $x(\cdot)$ such that $||x(t)|| \leq R$, for all $t \in [0,T]$. If, moreover, $0 \notin F(t,0,0)$, for a.a. $t \in [0,T]$, then the obtained solution is nontrivial.

By the similar arguments as in the proof of Theorem 4.1, the following result dealing with an upper-Carathéodory mapping F and a strictly localized bounding function V can be proven. In this case, it is possible to construct a sequence of approximating problems which satisfy assumptions of Proposition 2.5. Moreover, since the Nagumo-type condition (29) is used, it is guaranteed that the desired bounded solutions have bounded derivatives, which enables us to replace condition (1_{ii}) in Proposition 2.5 by the less restrictive condition (2_{ii}).

Theorem 4.3. *Let E be a separable Banach space and let us consider the Dirichlet b.v.p. (13), where $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory mapping which is either globally measurable or quasi-compact satisfying conditions (1_i) from Proposition 2.3, (2_{ii}) from Theorem 4.1 and (2_{iii}) from Proposition 2.5.*

Assume that $K \subset E$ is an open, bounded, convex set containing 0, *let*

$$R = |K| := \sup_{k \in K} \|k\|$$

and let (29) *be valid, with ϕ as in* (24). *Moreover, let there exist $\varepsilon > 0$ and a function $V \in C^2(E, \mathbb{R})$ with Fréchet derivative $\dot{V}$ Lipschitzian in $\overline{B(\partial K, \varepsilon)}$ satisfying* $(H1)$, $(H2)$ *and* $(H3)$ *of Proposition 2.1. Moreover, let condition* (22) *hold, jointly with*

$$\langle \dot{V}_x, w \rangle \geq 0, \tag{42}$$

for a.a. $t \in (0, T)$ and all $x \in \partial K$, $v \in E$, and $w \in F(t, x, v)$.

Then the Dirichlet b.v.p. (13) *admits a solution whose values are located in $\overline{K}$. If, moreover, $0 \notin F(t, 0, 0)$, for a.a. $t \in [0, T]$, then the obtained solution is nontrivial.*

Proof. Let us define the function $\overline{\psi}$ and the sequence of problems (P_n), where $F_n \colon [0, T] \times E \times E \multimap E$, in the same way as in the proof of Theorem 4.1.

Since $\overline{\psi}$ is continuous, the mapping F_n is upper-Carathéodory, for all $n \in \mathbb{N}$. By the analogous reasonings as in the proof of Theorem 4.1, we can obtain that F_n satisfies condition (1_i) with $g_n = g + \frac{L_\psi}{n}$, and so

$$(T+4)\|g_n\|_{L^1([0,T],[0,\infty])} = (T+4)\|g\|_{L^1([0,T],[0,\infty])} + \frac{T L_\psi (T+4)}{n} < 4,$$

for n sufficiently large, and that also condition (33) is valid, with $\overline{\phi}$ defined as in the proof of Theorem 4.1.

Therefore, we can consider, instead of condition (1_{ii}), the less restrictive condition (2_{ii}), which is ensured directly by the definition of mapping F_n.

For $x \in \partial K$, $y \in E$ and $w_n \in F_n(t, x, y)$, there exists $w \in F(t, x, y)$ satisfying $w_n = w + \frac{\psi(x)}{n}$, and so

$$\langle \dot{V}_x,\, w_n \rangle = \langle \dot{V}_x,\, w \rangle + \langle \dot{V}_x,\, \frac{\psi(x)}{n} \rangle = \langle \dot{V}_x,\, w \rangle + \frac{1}{n} > 0,$$

for a.a. $t \in (0, T)$, which ensures the validity of condition (23).

Hence, thanks to condition (29), for a sufficiently large n the Dirichlet problem (P_n) satisfies all the assumptions of Proposition 2.5, and subsequently it admits a solution $x_n(t)$ such that $x_n(t) \in \overline{K}$, for all $t \in [0, T]$. Moreover, according to (33) (cf. Lemma 3.1), there exists $M > 0$ such that $\|\dot{x}_n(t)\| \leq M$, for a.a. $t \in [0, T]$. Since E is separable, less restrictive estimates than (35) and (36) are valid implying $\mathcal{S} \leq \frac{T+4}{4}\mathcal{S}\|g_n\|_{L^1([0,T],[0,\infty])}$ with $\mathcal{S}$ defined as in the proof of Theorem 4.1. The convergence of obtained sequence of solutions to the solution of the original Dirichlet problem (13) can then be shown by the same arguments as in the proof of Theorem 4.1. □

In the case when E is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$, $K := B(0, R)$ and $V(x) := \frac{1}{2}\left(\|x\|^2 - R^2\right)$, condition (42) becomes

$$\langle x, w \rangle \geq 0,$$

and we can immediately obtain the following consequence of Theorem 4.3.

Corollary 4.3. *Let E be a separable Hilbert space and let us consider the Dirichlet b.v.p.* (13), *where $F \colon [0, T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping which is either globally measurable or quasi-compact. Let the conditions (1_i) from Proposition 2.3, (2_{ii}) from Theorem 4.1 and (2_{iii}) from Proposition 2.5 be satisfied. Moreover, let there exist $R > 0$ such that*

$$\langle x, w \rangle \geq 0 \tag{43}$$

holds, for a.a. $t \in (0, T)$ and all $x \in E$ satisfying $||x|| = R$, $v \in E$, and $w \in F(t, x, v)$. Finally, let (29) *be valid, with ϕ as in* (24).

Then the Dirichlet b.v.p. (13) *admits a solution* $x(\cdot)$ *such that* $||x(t)|| \le R$*, for all* $t \in [0,T]$*. If, moreover,* $0 \notin F(t,0,0)$*, for a.a.* $t \in [0,T]$*, then the obtained solution is nontrivial.*

5. **Applications.** In [5, Example 4.1 and Example 4.2], we proposed some applications of our techniques to integro-differential equations respectively with discontinuous r.h.s. and non-homogeneous conditions. In the following, we briefly show their usage in the study of control problems. Consider the integro-differential equation

$$u_{tt}(t,x) + u_t(t,x) = bu(t,x) + \int_{\mathbb{R}} k(x,y)u(t,y)\ dy + f(u(t,x)) + \omega(x) \tag{44}$$

with $t \in [0,T]$, $x \in \mathbb{R}$, $k\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$. The function $\omega\colon \mathbb{R} \to \mathbb{R}$ accounts the external forces implemented into the model. We assume that

(a) $k \in L^2(\mathbb{R}\times\mathbb{R})$ with $||k||_{L^2(\mathbb{R}\times\mathbb{R})} = 1$;
(b) f is L-Lipschitzian with $f(0) = 0$ and $xf(x) > 0$, for all $x \neq 0$;
(c) there exist $\tilde{\omega}_0 \in AC([\alpha,\beta])$ and $\ell > 0$ satisfying $|\tilde{\omega}_0'(x)| \le \ell$ a.e. in $[\alpha,\beta] \subset \mathbb{R}$, $0 \le \tilde{\omega}_0(x) \le \omega(x) \le |u(t,x)| + \tilde{\omega}_0(x)$ for a.a. $x \in [\alpha,\beta]$ and $t \in [0,T]$, $\omega \in AC([\alpha,\beta])$ with $|\omega'(x)| \le \ell$ a.e. in $[\alpha,\beta]$, $\omega(\alpha) = \tilde{\omega}_0(\alpha)$ and $\omega \equiv 0$ outside $[\alpha,\beta]$;
(d) $b > 1$ and $(T+4)(b+L+2)T < 2$.

We prove in the following, by means of Corollary 4.1, the existence of $u(t,x)$ with $u \in C^1([0,T], L^2(\mathbb{R}))$ and $\omega \in AC([\alpha,\beta])$ as in *(c)* satisfying (44) as well as the Dirichlet condition

$$u(0,x) = u(T,x) = 0, \quad \forall x \in \mathbb{R}. \tag{45}$$

To this aim, we rewrite (44)–(45) into the abstract setting

$$\begin{cases} \ddot{y}(t) \in F(y(t), \dot{y}(t)), & t \in [0,T], \\ y(T) = y(0) = 0, \end{cases} \tag{46}$$

where $y(t) := u(t,\cdot) \in L^2(\mathbb{R})$, and $F\colon L^2(\mathbb{R}) \times L^2(\mathbb{R}) \multimap L^2(\mathbb{R})$ is defined by

$$F(y,v) := -v + by + K(y) + \hat{f}(y) + \mathcal{D}(y)$$

where $K\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $z \to (x \to \int_{\mathbb{R}} k(x,y)z(y)\ dy)$, $\hat{f}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $y \to (x \to f(y(x)))$ and the multivalued map $\mathcal{D}\colon L^2(\mathbb{R}) \multimap L^2(\mathbb{R})$ is given by

$$\mathcal{D}(y) = \left\{ z \in L^2(\mathbb{R})\colon \begin{array}{l} z \in AC([\alpha,\beta]), \\ \tilde{\omega}_0(x) \le z(x) \le |y(x)| + \tilde{\omega}_0(x) \text{ and } |z'(x)| \le \ell \text{ a.e. in } [\alpha,\beta], \\ z(\alpha) = \tilde{\omega}_0(\alpha) \text{ and } z \equiv 0 \text{ otherwise.} \end{array} \right\}$$

By similar arguments as in [5, Example 4.1] it is possible to show that K and $\hat{f}$ are well-defined with K 1-Lipschitzian and $\hat{f}$ L-Lipschitzian. Concerning $\mathcal{D}$, it is clear that it is convex valued and

$$0 \le z(x) \le \omega_0(0) + \ell(x-\alpha),\ x \in [\alpha,\beta]\ z \in \mathcal{D}(y) \text{ and } y \in L^2(\mathbb{R}). \tag{47}$$

We prove now that $\mathcal{D}$ is also compact. Indeed, let $\{z_n\} \subset \mathcal{D}(L^2(\mathbb{R}))$. According to (47) and the definition of $\mathcal{D}$, we can apply the Ascoli–Arzelá Theorem in $[\alpha,\beta]$. Passing to a subsequence, denoted as the sequence, it implies that $z_n \to z$ in $C[\alpha,\beta]$. Assuming $z \equiv 0$ outside $[\alpha,\beta]$ we have that $z_n \to z$ also in $L^2(\mathbb{R})$. By similar arguments we can also show that $\mathcal{D}$ has a closed graph and so, since it is compact, it is compact valued and u.s.c. In conclusion F is u.s.c. with convex, compact values. Moreover condition (1_i) in Proposition 2.3 is satisfied and $g(t) \equiv b + L + 2$. According to *(d)* also condition (1_{iii}) in Proposition 2.3 is satisfied. As a consequence of (47), we have that $\mathcal{D}$ is bounded and let $r > 0$ be such that $\|z\|_{L^2(\mathbb{R})} \le r$ for all $z \in \mathcal{D}(L^2(\mathbb{R}))$. Let $y, v \in L^2(\mathbb{R})$ with $\|y\| \le R$ for some $R > 0$; notice that $\|F(y,v)\| \le (b+L+1)R + r + \|v\|$ and so (24) and (29) are satisfied, with $\phi(s) = (b+L+1)R + r + s$. Since F is autonomous, also condition (2_{ii}) in Theorem 4.1 is true. Now we prove condition (37). So, let $y, v \in L^2(\mathbb{R})$ with $\langle y, v\rangle = 0$ and take

$w \in F(x,y)$, implying $w = -v + by + K(y) + \hat{f}(y) + \omega_1$ with $\omega_1 \in \mathcal{D}(y)$. Condition (a) implies that $\langle y, K(y)\rangle \geq -\|y\|$ while (b) yields that $\langle y, \hat{f}(y)\rangle \geq 0$. Consequently, since $b > 1$ (cfr. (d)), $\langle y, w\rangle + \langle v, v\rangle \geq bR^2 - R^2 - Rr > 0$ when R is sufficiently large. All the assumptions of Corollary 4.1 are then satisfied and so the problem (44)–(45) is solvable.

6. **Comparison with classical results.** In the last section, we will discuss the relations between classical theorems (for $E = \mathbb{R}^n$ and a continuous function f) on one side and the main results introduced in the previous parts of the paper (formulated for a Banach space E and multivalued u.s.c. or upper-Carathéodory r.h.s. F) on the other side.

At first, let us note that if $E = \mathbb{R}^n$ and $f = F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, (24) can be replaced by (3) and (5) (cf. Lemma 3.2).

Moreover, in the case when $E = \mathbb{R}^n$ and $f = F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, the function g appearing in (1_i) and (1_{iii}) is equal to 0, and since f is continuous, it is guaranteed by the second Weierstrass theorem that there exists, for every nonempty, bounded set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$, a constant $M_{\overline{\Omega}}$ such that

$$||F(t,x,y)|| \leq M_{\overline{\Omega}}, \tag{48}$$

for all $(t,x,y) \in [0,T] \times \overline{\Omega}$. Thus, condition (2_{ii}) is in the finite-dimensional, continuous case automatically satisfied.

Therefore, we are able to obtain as a direct consequence of Theorem 4.1 and Lemma 3.2 exactly Hartman's Theorem 1.1.

Moreover, we could also immediately obtain (applying Proposition 2.3, Theorem 4.1 or Theorem 4.2) the following improvements of results from Remark 1.3:

- The strict inequality in (8) can be replaced by a non-strict one (2).

The result from Remark 1.4 can be also improved using Proposition 2.3, Theorem 4.1 or Theorem 4.2 as follows:

- The inequality (10) can be replaced by the more general (2).

As a very particular case of Theorem 4.1, we are able to obtain the following generalization of results from Remarks 1.3 and 1.4.

Corollary 6.1. *Let us consider the vector Dirichlet problem* (1), *where* $f\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function and assume that there exists* $R > 0$ *such that the non-strict inequality*

$$\langle f(t,x,y), x\rangle + ||y||^2 \geq 0$$

holds, for all $t \in [0,T]$ *and* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ *with* $||x|| = R$ *and* $\langle x, y\rangle = 0$.

Moreover, let $\phi\colon [0,\infty) \to (0,\infty)$ *be a continuous, nondecreasing function such that*

$$\lim_{s\to\infty} \frac{s^2}{\phi(s)} = \infty$$

and that

$$||f(t,x,y)|| \leq \phi(||y||), \text{ for all } t \in [0,T], \text{ and all } x \in \mathbb{R}^n \text{ with } ||x|| \leq R \text{ and } y \in \mathbb{R}^n.$$

Then the b.v.p. (1) *has a solution* $x(\cdot)$ *such that* $||x(t)|| \leq R$, *for all* $t \in [0,T]$.

As already mentioned before, as a direct consequence of Theorem 4.1 and Lemma 3.2, exactly Hartman's Theorem 1.1 can be obtained. On the other hand, a variant of Theorem 4.1 in $\mathbb{R}^n$, with (29), (24) replaced (in view of Lemma 3.2) by (3), (5), is a significant generalization of Hartman's Theorem 1.1 (see also Remark 4.1). This, besides other things, demonstrates, jointly with Corollary 6.1, the power of our main theorems formulated in abstract spaces.

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