A generalized Asymmetric Exclusion Process with $U_q(\mathfrak{sl}_2)$ stochastic duality

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Abstract

We study a new process, which we call ASEP(q, j), where particles move asymmetrically on a one-dimensional integer lattice with a bias determined by $q \in (0, 1)$ and where at most $2j \in \mathbb{N}$ particles per site are allowed. The process is constructed from a $(2j+1)$-dimensional representation of a quantum Hamiltonian with $U_q(\mathfrak{sl}_2)$ invariance by applying a suitable ground-state transformation. After showing basic properties of the process ASEP(q, j), we prove self-duality with several self-duality functions constructed from the symmetries of the quantum Hamiltonian. By making use of the self-duality property we compute the first q-exponential moment of the current for step initial conditions (both a shock or a rarefaction fan) as well as when the process is started from an homogeneous product measure.

1 Introduction

1.1 Motivation

The *Asymmetric Simple Exclusion Process* (ASEP) on $\mathbb{Z}$ is one of the most popular interacting particle system. For each $q \in (0, 1]$, the process is defined, up to an irrelevant time-scale factor, by the following two rules: i) each site is vacant or occupied; ii) particles sitting at occupied sites try to jump at rate q to the left and at rate q^{-1} to the right and they succeed if the arrival site is empty. The ASEP plays a crucial role in the development of the mathematical theory of non-equilibrium statistical mechanics, similar to the role of Ising model for equilibrium statistical mechanics. However, whereas the Ising model – defined for dichotomic spin variables – is easily generalizable to variables taking more than two values (Potts model), there are a-priori different possibilities to define the ASEP process with more than one particle per site and it is not clear what the best option is.

In the analysis of the standard (i.e., maximum one particle per site) Exclusion Process a very important property of the model is played by *self-duality*. First established in the context of the Symmetric Simple Exclusion Process (SSEP) [13], self-duality is a key tool that allows to study the process using only a finite number of dual particles. For instance, using self-duality and coupling techniques Spitzer and Liggett were able to show that the only extreme translation invariant measures for the SSEP on $\mathbb{Z}^d$ are the Bernoulli product measures and to identify the domain of attraction of them. The extension of duality to ASEP is due to Schütz [20] and has played an important role in showing that ASEP is included in the KPZ universality class, see e.g. [2, 7]. As a general rule, the extension of a duality relation from a symmetric to an asymmetric process is far from trivial.

It is the aim of this paper to provide a generalization of the ASEP with multiple occupation per site for which (self-)duality can be established. A guiding principle in the search of such process will be the connection between Exclusion Processes and Quantum Spin Chains. The duality property will then be used to study the statistics of the current of particles for the process on the infinite lattice.

1.2 Previous extensions of the ASEP

Several extensions of the ASEP model allowing multiple occupancy at each site have been provided and studied in the literature. Among them we mention the following.

a) It is well known that the XXX Heisenberg quantum spin chain with spin $j = 1/2$ is related (by a change of basis) to the SSEP. In this mapping the spins are represented by 2×2 matrices satisfying the $\mathfrak{sl}_2$ algebra. By considering higher values of the spins, represented by $(2j + 1)$-dimensional matrices with $j \in \mathbb{N}/2$, one obtains the *generalized Symmetric Simple Exclusion Process with up to* $2j$ *particles per site* (SSEP($2j$) for short), sometimes also called "partial exclusion" [4, 21, 9]. Namely, denoting by $\eta_i \in \{0, 1, \dots 2j\}$ the number of particles at site $i \in \mathbb{Z}$, the process that is obtained has rates $\eta_i(2j - \eta_{i+1})$ for a particle jump from site i to site $i + 1$ and rate $\eta_{i+1}(2j - \eta_i)$ for the reversed jump. For such extension of the SSEP, duality can be formulated and (extreme) translation invariant measures are provided by the Binomial product measures with parameters $2j$ (the number of trials) and ρ (the success probability in each trial).

   The naive asymmetric version of this process, i.e., considering a rate $q\, n_{i+1}(2j - n_i)$ for the jump of a particle from site $i + 1$ to site i and a rate $q^{-1}n_i(2j - n_{i+1})$ for the jump of a particle from site i to site $i + 1$, with $q \in (0, 1)$, loses the $\mathfrak{sl}_2$ symmetry and has no other symmetries from which duality functions can be obtained. In fact in this model, there is no self-duality property expect in the case $j = 1/2$ where it coincides with ASEP [20].

b) Another possibility is to consider the so-called *K-exclusion process* [17] that simply gives rates 1 to particle jumps from occupied sites together with the exclusion rule that prevents more than K particles to accumulate on each site ($K \in \mathbb{N}$). Namely, denoting by $\mathbf{1}_A$ the indicator function of the set A, the K-exclusion process on $\mathbb{Z}$ has rates $\mathbf{1}_{\{\eta_i>1,\, \eta_{i+1}<K\}}$ for the jump from site i to site $i + 1$ and $\mathbf{1}_{\{\eta_{i+1}>1,\, \eta_i<K\}}$ for the jump from site $i + 1$ to site i. For the symmetric version of this process it has been

shown in [17] that extremal translation invariant measures are product measures (with truncated-geometric marginals). The asymmetric version of this process obtained by giving rate q to (say) the left jumps and rate q^{-1} for the right jumps, has been studied by Seppäläinen (see [23] and references therein). For the asymmetric process, invariant measures are unknown, and non-product, nevertheless many properties of this process (e.g. hydrodynamic limit) could be established. For this process, both in the symmetric and asymmetric case, there is no duality.

1.3 Informal description of the results

The fact that self-duality is known for the Symmetric Exclusion Process for any $j \in \mathbb{N}/2$ [9] and it is unknown in all the other cases (except ASEP with $j = 1/2$) can be traced back to the link that it exists between self-duality and the algebraic structure of interacting particle systems. Such underlying structure is usually provided by a *Lie algebra* naturally associated to the generator of the process. The first result in this direction was given in [21] for the *symmetric* process, while a systematic and general approach has been described in [9], [5]. When passing from symmetric to asymmetric processes, one has to change from the original Lie algebra to the corresponding *deformed quantum Lie algebra*, where the deformation parameter is related to the asymmetry. This was noticed in [20] for the standard ASEP, which corresponds to a representation of the $U_q(\mathfrak{sl}_2)$ algebra with spin $j = 1/2$.

In this paper we fully unveil the relation between the deformed $U_q(\mathfrak{sl}_2)$ algebra and a suitable generalization of the Asymmetric Simple Exclusion Process. *For a given $q \in (0, 1)$ and $j \in \mathbb{N}/2$, we construct a new process, that we name ASEP(q, j)*, which provides an extension of the standard ASEP process to a situation where sites can accommodate more than one (namely $2j$) particles. The construction is based on a quantum Hamiltonian [3], which up to a constant can be obtained from the Casimir operator and a suitable co-product structure of the quantum Lie algebra $U_q(\mathfrak{sl}_2)$. For this Hamiltonian we construct a ground-state which is a tensor product over lattice sites. This ground-state is used to transform the Hamiltonian into the generator of the Markov process ASEP(q, j) via a ground-state transformation. As a result of the symmetries of the Hamiltonian, we obtain several self-duality functions of the associated ASEP(q, j). Those functions are then used in the study of the statistics of the current that flows through the system for different initial conditions.

For $j = 1/2$ the ASEP(q, j) reduces to the standard ASEP. For $j \to \infty$, after a proper time-rescaling, ASEP(q, j) converges to the so-called q-TAZRP (Totally Asymmetric Zero Range process), see Remark 3.3 below and [2] for more datails.

We mention also [16] and [15] for other processes with $U_q(\mathfrak{sl}_2)$ symmetry. In particular the process in [15] is a $(2j + 1)$ state partial exclusion process constructed using the Temperley-Lieb algebra, in which multiple jumps of particles between neighboring sites are allowed. We remark that for $j = 1$ the process depends on a parameter β and for the special value $\beta = 0$ it reduces to ASEP$(q, 1)$.

1.4 From quantum Lie algebras to self-dual Markov processes

By analyzing in full details the case of the $U_q(\mathfrak{sl}_2)$ we will elucidate a general scheme that can be applied to other algebras, thus providing asymmetric version of other interacting particle systems (e.g. independent random walkers, zero-range process, inclusion process).

We highlight below the main steps of the scheme (at the end of each step we point to the section where such step is made for $U_q(\mathfrak{sl}_2)$).

i) (*Quantum Lie Algebra*): Start from the quantization $U_q(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (Sect. 4.1).

ii) (*Co-product*): Consider a co-product $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ making the quantized enveloping algebra a bialgebra (Sect. 4.2).

iii) (*Quantum Hamiltonian*): For a given representation of the quantum Lie algebra $U_q(\mathfrak{g})$ compute the co-product $\Delta(C)$ of a Casimir element C (or an element in the centre of the algebra). For a one-dimensional chain of size L construct the quantum Hamiltonian $H_{(L)}$ by summing up copies of $\Delta(C)$ over nearest neighbor edges. (Sect. 4.3).

iv) (*Symmetries*): Basic symmetries (i.e. commuting operators) of the quantum Hamiltonian are constructed by applying the co-product to the generators of the quantum Lie algebra. (Sect. 4.4).

v) (*Ground state transformation*): Apply a ground state transformation to the quantum Hamiltonian $H_{(L)}$ to turn it into the generator $\mathscr{L}^{(L)}$ of a Markov stochastic process (Sect. 5).

vi) (*Self-duality*): Self-duality functions of the Markov process are obtained by acting with (a function) of the basic symmetries on the reversible measure of the process. (Sect. 6).

Whereas steps i)–iv) depend on the specific choice of the quantum Lie algebra, the last two steps are independent of the particular choice but require additional hypotheses. In particular whether step v) is possible depends on the specific properties of the Hamiltonian and its ground state. They are further discussed in Section 2.

1.5 Organization of the paper

The rest of the paper is organized as follows. In Section 2 we give the general strategy to construct a self-dual Markov process from a quantum Hamiltonian, a positive ground state and a symmetry. In the case where the quantum Hamiltonian is given by a finite dimensional matrix the strategy actually amounts to a similarity transformation with the diagonal matrix constructed from the ground state components.

In Section 3 we start by defining the ASEP(q, j) process. After proving some of its basic properties in theorem 3.1 (e.g. existence of non-homogenous product measure and absence of translation invariant product measure), we enunciate our main results. They include: the self-duality property of the (finite or infinite) ASEP(q, j) (theorem 3.2) and its use in the computation of some exponential moments of the total integrated current via a single dual asymmetric walker (lemma 3.1). The explicit computation are shown for the step initial conditions (theorem 3.3) and when the process is started from an homogenous product measure (theorem 3.4).

The remaining Sections contain the algebraic construction of the ASEP(q, j) process by the implementation of the steps described in the above scheme for the case of the quantum

Lie algebra $U_q(\mathfrak{g})$. In particular, in Section 4 we introduce the quantum Hamiltonian and its basic symmetries on which we base our construction of the ASEP(q, j). In Section 5 we exhibit a non trivial q-exponential symmetry and a positive ground state of the quantum Hamiltonian that allows to define a Markov process. In Section 6 we prove the main self-duality result for the ASEP(q, j). In Section 7 we explore other choices for the symmetries of the Hamiltonian and, as a consequence, prove the existence of an alternative duality function that reduces to the Schütz duality function for the case $j = 1/2$.

2 Ground state transformation and self-duality

In this section we describe a general strategy to construct a Markov process from a quantum Hamiltonian. Furthermore we illustrate how to derive self-duality functions for that Markov process from symmetries of the Hamiltonian. The construction of a Markov process from a Hamiltonian and a positive ground state has been used at several places, e.g. the Ornstein-Uhlenbeck process is constructed in this way from the harmonic oscillator Hamiltonian, see e.g. [22]. In lemma 2.1 below we recall the procedure, and how to recover symmetries of the Markov process from symmetries of the Hamiltonian. In general this procedure to be applied requires some condition on the Hamiltonian. In the discrete setting this condition boils down to non-negative out-of-diagonal elements and the existence of a positive ground state. In the more general setting the Hamiltonian has to be a Markov generator up to mass conservation (cfr. (1)).

2.1 Ground state transformation and symmetries

Lemma 2.1. *Let h be a bounded continuous function and let L be the generator of a Markov process on a metric space Ω. Let A be an operator of the form*

$$Af = Lf - hf \tag{1}$$

Suppose that there exists ψ such that e^{ψ} is in the domain of A, and

$$Ae^{\psi} = 0. \tag{2}$$

Then the following holds:

a) The operator defined by

$$L_{\psi} f = e^{-\psi} A(e^{\psi} f) \tag{3}$$

is a Markov generator.

b) There is a one-to-one correspondence between symmetries (commuting operators) of A and symmetries of L_ψ: $[S, A] = SA - AS = 0$ if and only if $[L_\psi, S_\psi] = 0$ with $S_\psi = e^{-\psi} S e^{\psi}$.

c) If A is self-adjoint on the space $L^2(\Omega, d\alpha)$ for some σ-finite measure α on Ω, then L_ψ is self-adjoint on $L^2(\Omega, d\mu)$ with $\mu(dx) = e^{2\psi(x)}\alpha(dx)$. In particular, if $\int e^{2\psi(x)}\alpha(dx) = 1$ then μ is a reversible probability measure for the Markov process with generator L_ψ.

PROOF. For item a): for every φ such that e^φ is in the domain of L, the operator

$$L_\varphi f = e^{-\varphi} L(e^\varphi f) - (e^{-\varphi} L(e^\varphi)) f \tag{4}$$

defines a Markov generator, see e.g. [8] section 1.2.2, and [18]. Now choosing $\varphi = \psi$, we obtain from the assumption (2) that

$$e^{-\psi} L e^\psi = h$$

Hence,

$$\begin{aligned} L_\psi f &= e^{-\psi} L(e^\psi f) - (e^{-\psi} L(e^\psi)) f \\ &= e^{-\psi} L(e^\psi f) - h f = e^{-\psi}(L-h)(e^\psi f) \\ &= e^{-\psi} A(e^\psi f) \end{aligned}$$

For item b) suppose that S commutes with A, then

$$\begin{aligned} L_\psi S_\psi &= e^{-\psi} A e^\psi e^{-\psi} S e^\psi \\ &= e^{-\psi} A S e^\psi = e^{-\psi} S A e^\psi \\ &= S_\psi L_\psi \end{aligned}$$

For item c), we compute

$$\begin{aligned} \int g L_\psi(f) d\mu &= \int g (e^{-\psi} A(e^\psi f)) e^{2\psi} d\alpha \\ &= \int e^\psi g A(e^\psi f) d\alpha \\ &= \int A(e^\psi g)(e^\psi f) d\alpha = \int (L_\psi g) f d\mu \end{aligned}$$

where in the third equality we used $A = A^*$ in $L^2(\Omega, d\alpha)$. $\square$

The following is a restatement of lemma 2.1 in the context of a finite state space Ω with cardinality $|\Omega| < \infty$. In this case the condition $A = L - h$ just means that A has non-negative off diagonal elements.

COROLLARY 2.1. *Let A be a $|\Omega| \times |\Omega|$ matrix with non-negative off diagonal elements. Suppose there exists a column vector $e^\psi := g \in \mathbb{R}^{|\Omega|}$ with strictly positive entries and such that $Ag = 0$. Let us denote by G the diagonal matrix with entries $G(x,x) = g(x)$ for $x \in \Omega$. Then we have the following*

a) The matrix

$$\mathscr{L} = G^{-1} A G$$

with entries

$$\mathscr{L}(x,y) = \frac{A(x,y) g(y)}{g(x)}, \qquad x, y \in \Omega \times \Omega \tag{5}$$

is the generator of a Markov process $\{X_t : t \geq 0\}$ taking values on Ω.

b) S commutes with A if and only if $G^{-1}SG$ commutes with $\mathscr{L}$.

c) If $A = A^*$, where * denotes transposition, then the probability measure μ on Ω

$$\mu(x) = \frac{(g(x))^2}{\sum_{x\in\Omega}(g(x))^2} \tag{6}$$

is reversible for the process with generator $\mathscr{L}$.

Proof. The proof of the corollary is obtained by specializing the statements of the lemma 2.1 to the finite dimensional setting. In particular for item a), the operator L_φ in (4) reads

$$(L_\varphi f)(x) = \sum_{y\in\Omega} L(x,y)e^{\varphi(y)-\varphi(x)}(f(y) - f(x)) \; .$$

Putting $\varphi(x) = \psi(x)$ and using the condition $\sum_{y\in\Omega} L(x,y)e^{\psi(y)} = h(x)e^{\psi(x)}$ one finds

$$(L_\psi f)(x) = \sum_{y\in\Omega} A(x,y)e^{\psi(y)-\psi(x)} f(y)$$

from which (5) follows. □

Remark 2.1. *Notice that for every column vector f we have that if $Af = 0$ then for any S commuting with A (symmetry of A) we have $ASf = SAf = 0$. This will be useful later on (see section 5.3) when starting from a vector f with some entries equal to zero, we can produce, by acting with a symmetry S, a vector $g = Sh$ which has all entries strictly positive.*

2.2 Self-duality and symmetries

For the discussion of self-duality, we restrict to the case of a finite state space Ω.

Definition 2.1 (Self-duality)**.** *We say that a Markov process $X := \{X_t : t \geq 0\}$ on Ω is self-dual with self-duality function $D : \Omega \times \Omega \to \mathbb{R}$ if for all $x, y \in \Omega$ and for all $t > 0$*

$$\mathbb{E}_x D(X_t, y) = \mathbb{E}_y D(x, Y_t) \; . \tag{7}$$

Here $\mathbb{E}_x(\cdot)$ denotes expectation with respect to the process X initialed at x at time $t = 0$ and Y denotes a copy of the process started at y.

This is equivalent to its infinitesimal reformulation, i.e., if the Markov process X has generator $\mathscr{L}$ then (7) holds if and only if

$$\mathscr{L}D = D\mathscr{L}^* \tag{8}$$

where D is the $|\Omega| \times |\Omega|$ matrix with entries $D(x, y)$ for $x, y \in \Omega$. We recall two general results on self-duality from [9].

a) *Trivial duality function from a reversible measure.*

   If the process $\{X_t : t \geq 0\}$ has a reversible measure $\mu(x) > 0$, then by the detailed balance condition, it is easy to check that the diagonal matrix

$$D(x, y) = \frac{1}{\mu(x)}\delta_{x,y} \tag{9}$$

   is a self-duality function.

b) *New duality functions via symmetries.*

If D is a self-duality function and S is a symmetry of $\mathscr{L}$, then SD is a self-duality function.

We can then combine corollary 2.1 with these results to obtain the following.

Proposition 2.1. *Let $A = A^*$ be a matrix with non-negative off-diagonal elements, and g an eigenvector of A with eigenvalue zero, with strictly positive entries. Let $\mathscr{L} = G^{-1}AG$ be the corresponding Markov generator. Let S be a symmetry of A, then $G^{-1}SG^{-1}$ is a self-duality function for the process with generator $\mathscr{L}$.*

Proof. By item c) of the corollary 2.1 combined with item a) of the general facts on self-duality we conclude that G^{-2} is a self-duality function. By item b) of corollary 2.1 we conclude that if S is a symmetry of A then $G^{-1}SG$ is a symmetry of $\mathscr{L}$. Then, using item b) of the general facts on self-duality we conclude that $G^{-1}SGG^{-2} = G^{-1}SG^{-1}$ is a self-duality function for the process with generator $\mathscr{L}$. □

3 The asymmetric exclusion process with parameters (q, j) (ASEP(q, j))

Notation. For $q \in (0, 1)$ and $n \in \mathbb{N}_0$ we introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{10}$$

satisfying the property $\lim_{q\to 1}[n]_q = n$. The first q-number's are thus given by

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \dots$$

We also introduce the q-factorial

$$[n]_q! := [n]_q \cdot [n-1]_q \cdot \dots \cdot [1]_q \ ,$$

and the q-binomial coefficient

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \ .$$

3.1 Process definition

We start with the definition of a novel interacting particle systems.

Definition 3.1 (ASEP(q, j) process)**.** *Let $q \in (0, 1)$ and $j \in \mathbb{N}/2$. For a given vertex set V, denote by $\eta = (\eta_i)_{i\in V}$ a particle configuration belonging to the state space $\{0, 1, \dots, 2j\}^V$ so that η_i is interpreted as the number of particles at site $i \in V$. Let $\eta^{i,k}$ denotes the particle configuration that is obtained from η by moving a particle from site i to site k.*

a) The Markov process ASEP(q,j) on $[1,L]\cap\mathbb{Z}$ with closed boundary conditions is defined by the generator

$$(\mathscr{L}^{(L)}f)(\eta)=\sum_{i=1}^{L-1}(\mathscr{L}_{i,i+1}f)(\eta) \qquad \textit{with}$$

$$\begin{aligned}(\mathscr{L}_{i,i+1}f)(\eta) &= q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ &+ q^{\eta_i-\eta_{i+1}+(2j+1)}[2j-\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta)) \qquad (11)\end{aligned}$$

b) We call the infinite-volume ASEP(q,j) on $\mathbb{Z}$ the process whose generator is given by

$$(\mathscr{L}^{(\mathbb{Z})}f)(\eta)=\sum_{i\in\mathbb{Z}}(\mathscr{L}_{i,i+1}f)(\eta) \qquad (12)$$

c) The ASEP(q,j) on the torus $\mathbf{T}_L:=\mathbb{Z}/L\mathbb{Z}$ with periodic boundary conditions is defined as the Markov process with generator

$$(\mathscr{L}^{(T_L)}f)(\eta)=\sum_{i\in\mathbf{T}_L}(\mathscr{L}_{i,i+1}f)(\eta) \qquad (13)$$

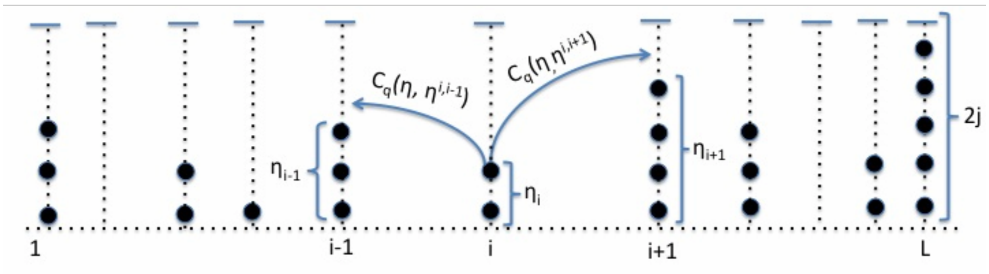

Figure 1: Schematic description of the ASEP$((q,j))$. The arrows represent the possible transitions and the corresponding rates $c_q(\eta,\xi)$ are given in (14) below. Each site can accomodate at most $2j$ particles.

$$c_q(\eta,\xi)=\begin{cases} q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q & \textit{if } \xi=\eta^{i,i+1} \\ q^{\eta_{i-1}-\eta_i+(2j+1)}[2j-\eta_{i-1}]_q[\eta_i]_q & \textit{if } \xi=\eta^{i,i-1} \\ 0 & \textit{otherwise}\end{cases} \qquad (14)$$

Remark 3.1 (The standard ASEP)**.** *In the case $j=1/2$ each site can accommodate at most one particle and the ASEP(q,j) reduces to the standard ASEP with jump rate to the left equal to q and jump rate to the right equal to q^{-1}.*

Remark 3.2 (The symmetric process)**.** *In the limit $q\to 1$ the ASEP(q,j) reduces to the SSEP$(2j)$, i.e. the generalized simple symmetric exclusion process with up to $2j$ particles per site (also called partial exclusion) (see [4, 21, 9, 10]). All the results of the present paper apply also to this symmetric case. In particular, for $q\to 1$, the duality functions that will be given in theorem 3.2 below reduce to the duality functions of the SSEP.*

Remark 3.3 (Connection with the q-TAZRP)**.** *Consider the process $y_t^{(j)} := \{y_i^{(j)}(t)\}_{i\in\mathbb{Z}}$ obtained from the ASEP(q,j) after the time scale transfomation $t \to (1-q^2)q^{4j-1}t$ (i.e. $y_i^{(j)}(t) := \eta_i((1-q^2)q^{4j-1}t)$) then, in the limit $j \to \infty$, $y_t^{(j)}$ converges to the q-TAZRP (Totally Asymmetric Zero Range process) in $\mathbb{Z}$ whose generator is given by:*

$$(\mathscr{L}^{(q-TAZRP)}f)(y) = \sum_{i\in\mathbb{Z}} \frac{1-q^{2y_i}}{1-q^2}\,[f(y^{i,i+1}) - f(y)], \qquad f : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R} \tag{15}$$

see e.g. [2] for more details on this process.

3.2 Basic properties of the ASEP(q,j)

We summarize basic properties of the ASEP(q,j) in the following theorem. We recall that a function f is said to be monotonous if $f(\eta) \le f(\eta')$ whenever $\eta \le \eta'$ (in the sense of partial order) and a Markov process with semigroup $S(t)$ is said to be monotonous if, for every time $t \ge 0$, $S(t)f$ is monotonous function if f is a monotonous function. In this paper we do not investigate the consequence of monotonicity which is for instance very useful for the hydrodynamic limit (see [1]).

Theorem 3.1 (Properties of ASEP(q,j) process)**.**

a) For all $L \in \mathbb{N}$, the ASEP(q,j) on $[1,L]\cap\mathbb{Z}$ with closed boundary conditions admits a family (labeled by $\alpha > 0$) of reversible product measures with marginals given by

$$\mathbb{P}^{(\alpha)}(\eta_i = n) = \frac{\alpha^n}{Z_i^{(\alpha)}} \binom{2j}{n}_q \cdot q^{2n(1+j-2ji)} \qquad n = 0,1,\dots,2j \tag{16}$$

for $i \in \{1,\dots,L\}$ and

$$Z_i^{(\alpha)} = \sum_{n=0}^{2j} \binom{2j}{n}_q \cdot \alpha^n q^{2n(1+j-2ji)} \tag{17}$$

b) The infinite volume ASEP(q,j) is well-defined and admits the reversible product measures with marginals given by (16)-(17).

c) Both the ASEP(q,j) on $[1,L]\cap\mathbb{Z}$ with closed boundary conditions and its infinite volume version are monotone processes.

d) For $L \ge 3$, the ASEP(q,j) on the Torus $\mathbf{T}_L$ with periodic boundary conditions does not have translation invariant stationary product measures for $j \ne 1/2$.

e) The infinite volume ASEP(q,j) does not have translation invariant stationary product measures for $j \ne 1/2$.

Remark 3.4. *Notice that of course we could have absorbed the factor $q^{2(1+j)}$ into α in (16). However in remark 5.2 below we will see that the case $\alpha = 1$ exactly corresponds to a natural ground state.*

Proof.

a) Let μ be a reversible measure, then, from detailed balance we have

$$\mu(\eta)c_q(\eta,\eta^{i,i+1}) = \mu(\eta^{i,i+1})c_q(\eta^{i,i+1},\eta) \tag{18}$$

where $c_q(\eta,\xi)$ are the hopping rates from η to ξ given in (14). Suppose now that μ is a product measure of the form $\mu = \otimes_{i=1}^{L}\mu_i$ then (18) holds if and only if

$$\mu_i(\eta_i-1)\mu_{i+1}(\eta_{i+1}+1)q^{2j}[2j-\eta_i+1]_q[\eta_{i+1}+1]_q = \mu_i(\eta_i)\mu_{i+1}(\eta_{i+1})q^{-2j}[\eta_i]_q[2j-\eta_{i+1}]_q \tag{19}$$

which implies that there exists $\beta \in \mathbb{R}$ so that for all $i = 1,\ldots,L$

$$\frac{\mu_i(n)}{\mu_i(n-1)} = \beta q^{-4ji}\frac{[2j-n+1]_q}{[n]_q} \tag{20}$$

then (16) follows from (20) after using an induction argument on n and choosing $\beta = \alpha q^{2(j+1)}$.

b) The fact that the process is well-defined follows from standard existence criteria of [13], chapter 1, while the proof of the statement on the reversible product measure is the same as in item a).

c) This follows from the fact that the rate to go from η to $\eta^{i,i+1}$ is of the form $b(\eta_i,\eta_{i+1})$ where $k,l \mapsto b(k,l)$ is increasing in k and decreasing in l, and the same holds for the rate to go from η to $\eta^{i,i-1}$, and the general results in [6].

d) We will prove the absence of homogeneous product measures for $j=1$, the proof for larger j is similar. Suppose that there exists an homogeneous stationary product measure $\bar{\mu}(\eta) = \prod_{i=1}^{L}\mu(\eta_i)$, then, for any function $f : \{0,\ldots,2j\}^{\mathbb{Z}} \to \mathbb{R}$

$$0 = \sum_{\eta}[\mathscr{L}^{(T_L)}f](\eta)\bar{\mu}(\eta) = \sum_{\eta} f(\eta)[\mathscr{L}^{(T_L)*}\bar{\mu}](\eta) \tag{21}$$

where

$$[\mathscr{L}^{(T_L)*}\bar{\mu}](\eta) = \sum_{i\in\mathbf{T}_L} F(\eta_i,\eta_{i+1})\bar{\mu}(\eta) \tag{22}$$

with

$$\begin{aligned} F(\xi_1,\xi_2) &= q^{\xi_1-\xi_2-2j+1}[\xi_1+1]_q[2j-\xi_2+1]_q\frac{\mu(\xi_1+1)\mu(\xi_2-1)}{\mu(\xi_1)\mu(\xi_2)} \\ &+ q^{\xi_1-\xi_2+2j-1}[\xi_2+1]_q[2j-\xi_1+1]_q\frac{\mu(\xi_2+1)\mu(\xi_1-1)}{\mu(\xi_1)\mu(\xi_2)} \\ &- q^{\xi_1-\xi_2}\left(q^{-(2j+1)}+q^{2j+1}\right)[\xi_1]_q[2j-\xi_2]_q \end{aligned} \tag{23}$$

Then, from (21) and (22) we have that $\bar{\mu}$ is an homogeneous product measure if and only if, for all f,

$$\sum_{\eta} f(\eta)\bar{\mu}(\eta)\left(\sum_{i\in\mathbf{T}_L} F(\eta_i,\eta_{i+1})\right) = 0 \tag{24}$$

which is true if and only if

$$G(\eta) := \sum_{i\in \mathbf{T}_L} F(\eta_i,\eta_{i+1}) \equiv 0 \tag{25}$$

Let Δ_i be the discrete derivative with respect to the i-th coordinate, i.e. let $f : \{0,\dots,2j\}^N \to \mathbb{R}$, for some $N\in\mathbb{N}$, then $\Delta_i f(n) := f(n+\delta_i) - f(n)$, $n=(n_1,\dots,n_N)$. From (25) it follows that, for any $i\in\{1,\dots,L\}$,

$$0 = \Delta_i G(\eta) = \Delta_2 F(\eta_{i-1},\eta_i) + \Delta_1 F(\eta_i,\eta_{i+1}) \qquad \text{for any} \quad \eta_{i-1},\eta_i,\eta_{i+1} \tag{26}$$

this implies in particular that $\Delta_2 F(\xi_1,\xi_2)$ does not depend on ξ_1 and that $\Delta_1 F(\xi_1,\xi_2)$ does not depend on ξ_2. Therefore, necessarily $F(\xi_1,\xi_2)$ is of the form

$$F(\xi_1,\xi_2) = g(\xi_1) + h(\xi_2) \tag{27}$$

for some functions $g,h : \{0,\dots,2j\} \to \mathbb{R}$. By using again (25) it follows in particular that $F(\xi_1,\xi_1)=0$, then, from this fact and (27) we deduce that $h(\xi_1) = -g(\xi_1)$. As a consequence (25) holds if and only if there exists a function g as above such that, for each $i\in\mathbf{T}_L$,

$$F(\eta_i,\eta_{i+1}) = g(\eta_i) - g(\eta_{i+1}) \tag{28}$$

(the opposite implication following from the fact that the sum $\left(\sum_{i\in\mathbf{T}_L} F(\eta_i,\eta_{i+1})\right)$ is now telescopic and hence zero because of periodicity).

We are going to prove now that (28) cannot hold for the function F given in (23). Denote by

$$\gamma := \frac{\mu(1)^2}{\mu(2)\mu(0)} \qquad \text{and} \qquad \alpha := q^3 + q - q^{-1} - q^{-3}\,, \tag{29}$$

fix i and define $\bar\eta := (\eta_i,\eta_{i+1})$; then, for $j=1$ the expression in (23) becomes

$$\begin{aligned} & \alpha(\mathbf{1}_{\bar\eta=(1,0)} - \mathbf{1}_{\bar\eta=(0,1)}) + \alpha(\mathbf{1}_{\bar\eta=(2,1)} - \mathbf{1}_{\bar\eta=(1,2)}) \\ + \quad & \left[\gamma q^3 - q - 2q^{-1} - q^{-3}\right] \mathbf{1}_{\bar\eta=(2,0)} - \left[q^3 + 2q + q^{-1} - \gamma q^{-3}\right] \mathbf{1}_{\bar\eta=(0,2)} \\ + \quad & \left[\gamma^{-1}(q^3+3q+3q^{-1}+q^{-3}) - q^3 - q^{-3}\right] \mathbf{1}_{\bar\eta=(1,1)} \\ = \quad & g(\eta_i) - g(\eta_{i+1}) \end{aligned} \tag{30}$$

The condition (30) for $\bar\eta=(1,1)$ yields that the coefficient in front of $\mathbf{1}_{\bar\eta=(1,1)}$ has to be zero, which gives

$$\gamma = \frac{q^3+3q+3q^{-1}+q^{-3}}{q^3+q^{-3}} \tag{31}$$

with this choice of γ (30) gives

$$\begin{aligned} & \alpha(\mathbf{1}_{\bar\eta=(1,0)} - \mathbf{1}_{\bar\eta=(0,1)}) + \alpha(\mathbf{1}_{\bar\eta=(2,1)} - \mathbf{1}_{\bar\eta=(1,2)}) + \delta(\mathbf{1}_{\bar\eta=(2,0)} - \mathbf{1}_{\bar\eta=(0,2)}) \\ = \quad & g(\eta_i) - g(\eta_{i+1}) \end{aligned} \tag{32}$$

with

$$\delta := \gamma q^3 - q - 2q^{-1} - q^{-3}. \tag{33}$$

This yields $g(1)-g(0) = g(2)-g(1) = \alpha, g(2)-g(0) = \delta$ from which we conclude $\delta = 2\alpha$ which is in contradiction with (29), (31) and (33).

e) The proof is analogous to the proof of item d), but it requires an extra limiting argument. Namely, we want to show that the assumption of the existence of a translation invariant product measure $\bar{\mu}$ implies that $\int \mathscr{L}^{(\mathbb{Z})} f d\bar{\mu} = 0$ for every local function f. This leads to

$$\sum_{i\in\mathbb{Z}} \int f(\eta) F(\eta_i, \eta_{i+1}) d\bar{\mu}(\eta) = 0$$

for every local function f and where $F(\eta_i, \eta_{i+1})$ is defined in (23). In the same spirit of point d), the proof in [19] implies that $F(\eta_i, \eta_{i+1})$ has to be of the form $g(\eta_i) - g(\eta_{i+1})$ which leads to the same contradiction as in item d).

□

3.3 Self-duality properties of the ASEP(q, j)

The following self-duality theorem, together with the subsequent corollary, is the main result of the paper.

Theorem 3.2 (Self-duality of the finite ASEP(q,j))**.** *The ASEP(q,j) on $[1, L] \cap \mathbb{Z}$ with closed boundary conditions is self-dual with the following self-duality functions*

$$D_{(L)}(\eta, \xi) = \prod_{i=1}^{L} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q} \cdot q^{(\eta_i - \xi_i)\left[2\sum_{k=1}^{i-1} \xi_k + \xi_i\right] + 4ji\xi_i} \cdot \mathbf{1}_{\xi_i \le \eta_i} \tag{34}$$

and

$$D'_{(L)}(\eta, \xi) = \prod_{i=1}^{L} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q} \cdot q^{(\eta_i - \xi_i)\left[2\sum_{k=1}^{i-1} \eta_k - \eta_i\right] + 4ji\xi_i} \cdot \mathbf{1}_{\xi_i \le \eta_i} \tag{35}$$

Corollary 3.1 (Self-duality of the infinite ASEP(q,j))**.** *The ASEP(q,j) on $\mathbb{Z}$ is self-dual with the following self-duality functions*

$$D(\eta, \xi) = \prod_{i\in\mathbb{Z}} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q} \cdot q^{(\eta_i - \xi_i)\left[2\sum_{k=1}^{i-1} \xi_k + \xi_i\right] + 4ji\xi_i} \cdot \mathbf{1}_{\xi_i \le \eta_i} \tag{36}$$

and

$$D'(\eta, \xi) = \prod_{i\in\mathbb{Z}} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q} \cdot q^{(\eta_i - \xi_i)\left[2\sum_{k=1}^{i-1} \eta_k - \eta_i\right] + 4ji\xi_i} \cdot \mathbf{1}_{\xi_i \le \eta_i} \tag{37}$$

where the configurations η and ξ are such that the exponents in (36) *and* (37) *are finite.*

The following rewriting of the duality function in (36) will be useful in the analysis of the current statistics.

Remark 3.5. *For $l \in \mathbb{N}$, let $\xi^{(i_1,\ldots,i_\ell)}$ be the configurations such that*

$$\xi_m^{(i_1,\ldots,i_\ell)} = \begin{cases} 1 & \text{if } m \in \{i_1, \ldots, i_\ell\} \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Define

$$N_i(\eta) := \sum_{k\geq i} \eta_k\,, \tag{39}$$

then

$$D(\eta,\xi^{(i)}) = \frac{q^{4ji-1}}{q^{2j}-q^{-2j}} \cdot (q^{2N_i(\eta)} - q^{2N_{i+1}(\eta)}) \tag{40}$$

and more generally

$$D(\eta,\xi^{(i_1,\dots,i_\ell)}) = \frac{q^{4j\sum_{k=1}^{\ell} i_k - \ell^2}}{(q^{2j}-q^{-2j})^\ell} \cdot \prod_{k=1}^{\ell} (q^{2N_{i_k}(\eta)} - q^{2N_{i_k+1}(\eta)})$$

3.4 Computation of the first q-exponential moment of the current for the infinite volume ASEP(q,j)

We start by defining the current for the ASEP(q,j) process on $\mathbb{Z}$.

Definition 3.2 (Current)**.** *The total integrated current $J_i(t)$ in the time interval $[0,t]$ is defined as the net number of particles crossing the bond $(i-1,i)$ in the right direction. Namely, let $(t_i)_{i\in\mathbb{N}}$ be sequence of the process jump times. Then*

$$J_i(t) = \sum_{k:t_k\in[0,t]} (\mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i-1,i}\}} - \mathbf{1}_{\{\eta(t_k)=\eta(t_k^-)^{i,i-1}\}}) \tag{41}$$

Lemma 3.1 (Current q-exponential moment via a dual walker)**.** *The total integrated current of a trajectory $(\eta(s))_{0\leq s\leq t}$ is given by*

$$J_i(t) := N_i(\eta(t)) - N_i(\eta(0)) \tag{42}$$

where $N_i(\eta)$ is defined in (39). The first q-exponential moment of the current when the process is started from a configuration η at time $t=0$ is given by

$$\mathbb{E}_\eta\left[q^{2J_i(t)}\right] = q^{2(N(\eta)-N_i(\eta))} - \sum_{k=-\infty}^{i-1} q^{-4jk}\, \mathbf{E}_k\left[q^{4jx(t)}\left(1-q^{-2\eta_{x(t)}}\right) q^{2(N_{x(t)}(\eta)-N_i(\eta))}\right] \tag{43}$$

where $N(\eta) := \sum_{i\in\mathbb{Z}} \eta_i$ denotes the total number of particle (that is conserved by the dynamics), $x(t)$ denotes a continuous time asymmetric random walker on $\mathbb{Z}$ jumping left at rate $q^{2j}[2j]_q$ and jumping right at rate $q^{-2j}[2j]_q$ and $\mathbf{E}_k$ denotes the expectation with respect to the law of $x(t)$ started at site $k\in\mathbb{Z}$ at time $t=0$. Furthermore $N(\eta)-N_i(\eta) = \sum_{k<i}\eta_k$ and the first term on the right hand side of (43) is zero when there are infinitely many particles to the left of $i\in\mathbb{Z}$ in the configuration η.

Proof. (42) immediately follows from the definition of $J_i(t)$. To prove (43) we start from the duality relation

$$\mathbb{E}_\eta\left[D(\eta(t),\xi^{(i)})\right] = \mathbb{E}_{\xi^{(i)}}\left[D(\eta,\xi^{(x(t))})\right] \tag{44}$$

where $\xi^{(i)}$ is the configuration with a single dual particle at site i (cfr. (38)). Since the ASEP(q,j) is self-dual the dynamics of the single dual particle is given an asymmetric random walk $x(t)$ whose rates are computed from the process definition and coincides with those in the statement of the lemma. By (40) the left-hand side of (44) is equal to

$$\mathbb{E}_\eta \left[D(\eta(t), \xi^{(i)}) \right] = \frac{q^{4ji-1}}{q^{2j} - q^{-2j}} \, \mathbb{E}_\eta \left[q^{2N_i(t)} - q^{2N_{i+1}(t)} \right]$$

whereas the right-hand side gives

$$\mathbb{E}_{\xi^{(i)}} \left[D(\eta, \xi^{(x(t))}) \right] = \frac{q^{-1}}{q^{2j} - q^{-2j}} \, \mathbf{E}_i \left[q^{4jx(t)} (q^{2N_{x(t)}(\eta)} - q^{2N_{x(t)+1}(\eta)}) \right]$$

As a consequence, for any $i \in \mathbb{Z}$

$$\mathbb{E}_\eta \left[q^{2N_i(\eta(t))} \right] = \mathbb{E}_\eta \left[q^{2N_{i+1}(\eta(t))} \right] + q^{-4ji} \, \mathbf{E}_i \left[q^{4jx(t)} (q^{2N_{x(t)}(\eta)} - q^{2N_{x(t)+1}(\eta)}) \right] \tag{45}$$

In the case of the infinite-volume ASEP(q, j) the duality relation (45) is significant only for configurations such that $N_i(\eta(t))$ is finite for all t. For this reason it is convenient to divide both sides of (45) by $q^{2N_i(\eta)}$ in order to obtain a recursive relation for the current. Then we get from (42)

$$\begin{aligned} \mathbb{E}_\eta \left[q^{2J_i(t)} \right] &= q^{-2\eta_i} \, \mathbb{E}_\eta \left[q^{2J_{i+1}(t)} \right] \\ &+ q^{-4ji} \, \mathbf{E}_i \left[q^{4jx(t)} (q^{2(N_{x(t)}(\eta) - N_i(\eta))} - q^{2(N_{x(t)+1}(\eta) - N_i(\eta))}) \right] \end{aligned} \tag{46}$$

Notice that both $J_i(t)$ and $N_{x(t)}(\eta) - N_i(\eta)$ are finite quantities, for all i and t. By iterating the relation in (46) and using the fact that $\lim_{i \to -\infty} N_i(\eta(t)) = N(\eta(t)) = N(\eta)$ we obtain (43). $\square$

Notice that all the quantities in (43) are finite for finite t, since $N(\eta) - N_i(\eta) > 0$ and $q \leq 1$.

3.5 Step initial condition

Theorem 3.3 (q-moment for step initial condition)**.** *Consider the step configurations $\eta^\pm \in \{0, \ldots, 2j\}^{\mathbb{Z}}$ defined as follows*

$$\eta_i^+ := \begin{cases} 0 & \textit{for } i < 0 \\ 2j & \textit{for } i \geq 0 \end{cases} \qquad \eta_i^- := \begin{cases} 2j & \textit{for } i < 0 \\ 0 & \textit{for } i \geq 0 \end{cases} \tag{47}$$

then, for the infinite volume ASEP(q, j) we have

$$\mathbb{E}_{\eta^+} \left[q^{2J_i(t)} \right] = q^{4j \max\{0, i\}} \left\{ 1 + q^{-4ji} \, \mathbf{E}_i \left[\left(1 - q^{4jx(t)} \right) \mathbf{1}_{x(t) \geq 1} \right] \right\} \tag{48}$$

and

$$\mathbb{E}_{\eta^-} \left[q^{2J_i(t)} \right] = q^{-4j \max\{0, i\}} \left\{ 1 - \mathbf{E}_i \left[\left(1 - q^{4jx(t)} \right) \mathbf{1}_{x(t) \geq 1} \right] \right\} \tag{49}$$

In the formulas above $x(t)$ denotes the random walk of Lemma 3.1 and

$$\mathbf{E}_i(f(x(t)) = \sum_{x \in \mathbb{Z}} f(x) \cdot \mathbf{P}_i(x(t) = x)$$

with

$$\begin{aligned}\mathbf{P}_i(x(t)=x) &= \mathbb{P}(x(t)=x \mid x(0)=i)\\ &= e^{-[4j]_q t} q^{-2j(x-i)} I_{x-i}(2[2j]_q t)\end{aligned} \tag{50}$$

and $I_n(t)$ denotes the modified Bessel function.

Proof. We prove only (48) since the proof of (49) is analogous. From the definition of η^+ and (43), we have

$$\mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{2(N(\eta^+)-N_i(\eta^+))} - (1-q^{-4j}) \sum_{k=-\infty}^{i-1} q^{-4jk} \sum_{x\geq 0} q^{4jx}\, q^{2(N_x(\eta^+)-N_i(\eta^+))}\, \mathbf{P}_k\left(x(t)=x\right)$$

where $N(\eta^+) - N^i(\eta^+) = 2j\max\{0,i\}$ and $N_x(\eta^+) - N_i(\eta^+) = 2j(\max\{0,i\}-x)$ for any $x \geq 0$. Then we have

$$\mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left\{1 + (q^{-4j}-1)F_i(t)\right\}$$

with

$$\begin{aligned}F_i(t) &:= \sum_{k=-\infty}^{i-1} q^{-4jk}\,\mathbf{P}_k\left(x(t)\geq 0\right) = \sum_{k=-\infty}^{i-1} q^{-4jk}\,\mathbf{P}_0\left(x(t)\geq -k\right)\\ &= \sum_{r=-i+1}^{+\infty}\sum_{\ell=r}^{+\infty} q^{4jr}\,\mathbf{P}_0\left(x(t)=-\ell\right) = \sum_{\ell=-i+1}^{+\infty}\sum_{r=-i+1}^{\ell} q^{4jr}\,\mathbf{P}_0\left(x(t)=-\ell\right)\\ &= \frac{q^{-4j(i-1)}}{1-q^{4j}} \sum_{\ell=-i+1}^{+\infty}\left(1-q^{4j(\ell+i)}\right)\mathbf{P}_0\left(x(t)=\ell\right)\\ &= \frac{q^{-4j(i-1)}}{1-q^{4j}}\,\mathbf{E}_i\left[\left(1-q^{4jx(t)}\right)\mathbf{1}_{x(t)\geq 1}\right].\end{aligned}$$

Thus (48) is proved. □

Remark 3.6. *Since for $q\in(0,1)$*

$$\lim_{t\to\infty}\mathbf{E}_i\left[\left(1-q^{4jx(t)}\right)\mathbf{1}_{x(t)\geq 1}\right] = 1 \tag{51}$$

from (48) and (49) we have that

$$\lim_{t\to\infty}\mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left(1+q^{-4ji}\right) \tag{52}$$

and

$$\lim_{t\to\infty}\mathbb{E}_{\eta^-}\left[q^{2J_i(t)}\right] = 0 \tag{53}$$

The limits in (52) and (53) are consistent with a scenario of a shock, respectively, rarefaction fan. Namely, in the case of shock for a fixed location i, the current $J_i(t)$ in (52) remains bounded as $t\to\infty$ because particles for large times can jump and produce a current only at the location of the moving shock. On the contrary, in (53) the current $J_i(t)$ goes to ∞ as $t\to\infty$, i.e. the average current $J_i(t)/t$ converges to its stationary value.

It is possible to rewrite (48), (49) as contour integral. We do this in the following corollary in order to recover in the case $j = 1/2$ the results of [2].

Corollary 3.2. *The explicit expression of the q-moment in terms of contour integral reads*

$$\mathbb{E}_{\eta^+}\left[q^{2J_k(t)}\right] = \frac{q^{4j\max\{0,k\}}}{2\pi i}\oint e^{-\frac{q^{2j}[2j]_q^3(q^{-1}-q)^2 z}{(1+q^{4j}z)(1+z)}t}\left(\frac{1+z}{1+q^{4j}z}\right)^k\frac{dz}{z} \tag{54}$$

where the integration contour includes 0 *and* $-q^{-4j}$ *but does not include* -1*, and*

$$\mathbb{E}_{\eta^-}\left[q^{2J_k(t)}\right] = \frac{q^{-4j\max\{0,k\}}}{2\pi i}\oint e^{-\frac{q^{-2j}[2j]_q^3(q^{-1}-q)^2 z}{(1+q^{-4j}z)(1+z)}t}\left(\frac{1+z}{1+q^{-4j}z}\right)^k\frac{dz}{z} \tag{55}$$

where the integration contour includes 0 *and* $-q^{4j}$ *but does not include* -1*.*

Proof. In order to get (54) and (55) it is sufficient to exploit the contour integral formulation of the modified Bessel function appearing in (50), i.e.

$$I_n(x) := \frac{1}{2\pi i}\oint e^{(\xi+\xi^{-1})\frac{x}{2}}\xi^{-n-1}\,d\xi \tag{56}$$

where the integration contour includes the origin. From (50) and (56) we have

$$\begin{aligned}\mathbf{E}_k\left[\left(1-q^{4jx(t)}\right)\mathbf{1}_{x(t)\geq 1}\right] &= \sum_{x\geq 1}(1-q^{4jx})e^{-[4j]_q t}q^{-2j(x-k)}\,I_{x-k}\left(2[2j]_q t\right)\\ &= \frac{q^{2jk}}{2\pi i}\,e^{-[4j]_q t}\oint e^{[2j]_q(\xi+\xi^{-1})t}\,\xi^{k-1}\sum_{x\geq 1}\left(1-q^{4jx}\right)\frac{1}{(\xi q^{2j})^x}\,d\xi\end{aligned} \tag{57}$$

In order to have the convergence of the series in (57) it is necessary to assume $|\xi| \geq q^{-2j}$. Under such assumption we have

$$\sum_{x\geq 1}\left(1-q^{4jx}\right)\frac{1}{(\xi q^{2j})^x} = \frac{\left(1-q^{4j}\right)\xi}{\left(q^{2j}\xi-1\right)\left(\xi-q^{2j}\right)} \tag{58}$$

and therefore

$$\mathbf{E}_k\left[\left(1-q^{4jx(t)}\right)\mathbf{1}_{x(t)\geq 1}\right] = \frac{q^{2jk}}{2\pi i}\oint_\gamma f_k(\xi)\,d\xi, \tag{59}$$

$$\text{with}\qquad f_k(\xi) := e^{\{[2j]_q(\xi+\xi^{-1})-[4j]_q\}t}\,\frac{\left(1-q^{4j}\right)\xi^k}{\left(q^{2j}\xi-1\right)\left(\xi-q^{2j}\right)} \tag{60}$$

where, from the assumption above, the integration contour γ includes 0, q^{2j} and q^{-2j}. From (48), (49) and (59) we have

$$\mathbb{E}_{\eta^\pm}\left[q^{2J_k(t)}\right] = q^{\pm 4j\max\{0,k\}}\left\{1 \pm \frac{q^{\mp 2jk}}{2\pi i}\oint_\gamma f_k(\xi)\,d\xi\right\} \tag{61}$$

It is easy to verify that $q^{\pm 2j}$ are two simple poles for $f_k(\xi)$ such that

$$\mathrm{Res}_{q^{\pm 2j}}(f_k) = \mp q^{\pm 2jk} \tag{62}$$

then

$$\mathbb{E}_{\eta_\pm}\left[q^{2J_k(t)}\right] = \pm q^{\pm 4j\max\{0,k\}}\frac{1}{2\pi i}\oint_{\gamma_\pm} q^{\mp 2jk} f_k(\xi)\, d\xi \tag{63}$$

where $\gamma_\pm$ are now two different contours which include 0 and $q^{\mp 2j}$ and do not include $q^{\pm 2j}$. In order to get the results in (54) it is sufficient to perform the change of variable

$$\xi := \frac{1+z}{1+q^{4j}z}\, q^{2j} \tag{64}$$

to get

$$\mathbb{E}_{\eta^+}\left[q^{2J_k(t)}\right] = -\frac{q^{4j\max\{0,k\}}}{2\pi i}\oint_{\tilde\gamma_+} e^{-\frac{q^{2j}[2j]_q^3(q^{-1}-q)^2 z}{(1+q^{4j}z)(1+z)}t}\left(\frac{1+z}{1+q^{4j}z}\right)^k \frac{dz}{z} \tag{65}$$

where now the integral is done clockwise over the contour $\tilde\gamma_+$ which includes 0 and q^{-4j} but does not include -1. This yields (54) after changing the integration sense. (55) is obtained similarly from (63) after performing tha change of variables $\xi := \frac{1+z}{1+q^{-4j}z}\, q^{-2j}$. □

Remark 3.7. *In the case $j = 1/2$ formula (54) coincides with the expression in Theorem 1.2 of Borodin, Corwin, Sasamoto [2] for $n = 1$. Indeed defineing*

$$J_k(t) = -N_{k-1}^{BCS}(\eta(t)) + N_{k-1}^{BCS}(\eta(0)), \qquad N_k^{BCS}(\eta) := \sum_{i\le k}\eta_i \tag{66}$$

then, if $\eta(0) = \eta_+$ it holds $J_k(t) = -N_{k-1}^{BCS}(\eta(t)) + 2j\max\{0,k\}$. As a consequence, from (54), for $j = 1/2$ we have

$$\mathbb{E}_{\eta^+}\left[q^{-2N_{k-1}^{BCS}(t)}\right] = \frac{1}{2\pi i}\oint e^{-\frac{(q^{-1}-q)^2 z}{(q^{-1}+qz)(1+z)}t}\left(\frac{1+z}{1+q^2 z}\right)^k \frac{dz}{z} \tag{67}$$

where the integration contour includes 0 and $-q^{-2}$ but does not include -1. Notice that (67) recovers the expression in Theorem 1.2 of [2] for $\tau = q^{-2}, p = q^{-1}$ (up to a shift $k \to k-1$ which comes from the fact that in η_+ the first occupied site is 0 in our case while is it choosen to be 1 in [2]).

3.6 Product initial condition

We start with a lemma that is useful in the following.

Lemma 3.2. *Let $x(t)$ be the random walk defined in Lemma 3.1, $a \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ then*

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbf{E}_0\left[a^{x(t)} \mid x(t)\in A\right] = \sup_{x\in A}\{x\log a - \mathscr{I}(x)\} - \inf_{x\in A}\mathscr{I}(x) \tag{68}$$

with

$$\mathscr{I}(x) = [4j]_q - x + x\log\left[q^{2j}\left(\frac{x}{2[2j]_q} + \sqrt{\left(\frac{x}{2[2j]_q}\right)^2 + 1}\right)\right] \tag{69}$$

PROOF. From large deviations theory [12] we know that $x(t)/t$, conditional on $x(t)/t \in A$, satisfies a large deviation principle with rate function $\mathscr{I}(x) - \inf_{x\in A}\mathscr{I}(x)$ where $\mathscr{I}(x)$ is given by

$$\mathscr{I}(x) := \sup_z \{zx - \Lambda(z)\} \tag{70}$$

with

$$\Lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{zx(t)}\right] = [2j]_q \left((e^z - 1)\, q^{-2j} + \left(e^{-z} - 1\right) q^{2j}\right) \tag{71}$$

from which it easily follows (69). The application of Varadhan's lemma yields (68). □

We denote by $\mathbb{E}^{\otimes\mu}$ the expectation of the ASEP(q,j) process on $\mathbb{Z}$ initialized with the omogeneous product measure on $\{0,1,\dots 2j\}^{\mathbb{Z}}$ with marginals μ at time 0, i.e. $\mathbb{E}^{\otimes\mu}[f(\eta(t))] = \sum_\eta \left(\otimes_{i\in\mathbb{Z}}\mu(\eta_i)\right) \mathbb{E}_\eta[f(\eta(t))]$.

THEOREM 3.4 (q-moment for product initial condition)**.** *Consider a probability measure μ on $\{0,1,\dots 2j\}$. Then, for the infinite volume ASEP(q,j), we have*

$$\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] = \mathbf{E}_0\left[\left(\frac{q^{4j}}{\lambda_q}\right)^{x(t)} \mathbf{1}_{x(t)\le 0}\right] + \mathbf{E}_0\left[q^{4jx(t)}\left(\lambda_{1/q}^{x(t)} - \lambda_{1/q} + \lambda_q^{-1}\right)\mathbf{1}_{x(t)\ge 1}\right] \tag{72}$$

where $\lambda_y := \sum_{n=0}^{2j} y^n \mu(n)$ and $x(t)$ is the random walk defined in Lemma 3.1. In particular we have

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}^{\otimes\mu}[q^{2J_i(t)}] = \sup_{x\ge 0}\{x\log M_q - \mathscr{I}(x)\} - \inf_{x\ge 0}\mathscr{I}(x) \tag{73}$$

with $M_q := \max\{\lambda_q, q^{4j}\lambda_{1/q}\}$ and $\mathscr{I}(x)$ given by (69).

PROOF. From (43) we have

$$\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] = \int \otimes\mu(d\eta)\, \mathbb{E}_\eta\left[q^{2J_i(t)}\right]$$
$$= \int \otimes\mu(d\eta) q^{2(N(\eta)-N_i(\eta))} + \sum_{k=-\infty}^{i-1} q^{-4jk} \int \otimes\mu(d\eta)\mathbf{E}_k\left[q^{4jx(t)}\left(q^{-2\eta_{x(t)}} - 1\right) q^{2(N_{x(t)}(\eta)-N_i(\eta))}\right].$$

Since

$$\int \otimes\mu(d\eta) q^{2(N_x(\eta)-N_i(\eta))} = \lambda_q^{i-x}\,\mathbf{1}_{\{x\le i\}} + \lambda_{1/q}^{x-i}\,\mathbf{1}_{\{x>i\}} \tag{74}$$

then, in particular, $\int \otimes\mu(d\eta) q^{2(N(\eta)-N_i(\eta))} = 0$ since $\lambda_q < 1$, where we recall the interpretation of $N(\eta) - N_i(\eta)$ from lemma 3.1. Hence

$$\begin{aligned}\mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \sum_{k=-\infty}^{i-1} q^{-4jk} \sum_{x\in\mathbb{Z}} \mathbf{P}_k\left(x(t) = x\right) q^{4jx} \int \otimes\mu(d\eta)\left[q^{2(N_{x+1}(\eta)-N_i(\eta))} - q^{2(N_x(\eta)-N_i(\eta))}\right] \\ &= \left(\lambda_q^{-1} - 1\right) A(t) + \left(\lambda_{1/q} - 1\right) B(t)\end{aligned} \tag{75}$$

with

$$A(t) := \sum_{k\le i-1} q^{-4jk} \sum_{x\le i} \mathbf{P}_k\left(x(t) = x\right) q^{4jx}\lambda_q^{i-x} \tag{76}$$

and

$$B(t) := \sum_{k\leq i-1} q^{-4jk} \sum_{x\geq i+1} \mathbf{P}_k\left(x(t)=x\right) q^{4jx}\lambda_{1/q}^{x-i} \tag{77}$$

Now, let $\alpha := q^{4j}\lambda_q^{-1}$, then

$$\begin{aligned} A(t) &= \sum_{k\leq i-1} q^{-4jk}\lambda_q^i \sum_{x\leq i} \mathbf{P}_k\left(x(t)=x\right)\alpha^x \\ &= \sum_{n\geq 1}\lambda_q^n \sum_{m\leq n} \mathbf{P}_0\left(x(t)=m\right)\alpha^m \\ &= \sum_{m\leq 0}\alpha^m \mathbf{P}_0\left(x(t)=m\right)\sum_{n\geq 1}\lambda_q^n + \sum_{m\geq 1}\alpha^m \mathbf{P}_0\left(x(t)=m\right)\sum_{n\geq m}\lambda_q^n \\ &= \frac{1}{1-\lambda_q}\left\{\lambda_q\,\mathbf{E}_0\left[\alpha^{x(t)}\,\mathbf{1}_{x(t)\leq 0}\right] + \mathbf{E}_0\left[q^{4jx(t)}\,\mathbf{1}_{x(t)\geq 1}\right]\right\} \end{aligned} \tag{78}$$

Analogously one can prove that

$$B(t) = \frac{1}{\lambda_{1/q}-1}\left\{\mathbf{E}_0\left[\beta^{x(t)}\,\mathbf{1}_{x(t)\geq 2}\right] - \lambda_{1/q}\mathbf{E}_0\left[q^{4jx(t)}\,\mathbf{1}_{x(t)\geq 2}\right]\right\} \tag{79}$$

with $\beta = q^{4j}\lambda_{1/q}$ then (72) follows by combining (75), (78) and (79).

In order to prove (73) we use the fact that $x(t)$ has a Skellam distribution with parameters $([2j]_q q^{-2j}t, [2j]_q q^{2j}t)$, i.e. $x(t)$ is the difference of two independent Poisson random variables with those parameters. This implies that

$$\mathbf{E}_0\left[\left(\frac{q^{4j}}{\lambda_q}\right)^{x(t)}\mathbf{1}_{x(t)\leq 0}\right] = \mathbf{E}_0\left[\lambda_q^{x(t)}\mathbf{1}_{x(t)\geq 0}\right] .$$

Then we can rewrite (72) as

$$\begin{aligned} \mathbb{E}^{\otimes\mu}\left[q^{2J_i(t)}\right] &= \mathbf{E}_0\left[\left(\lambda_q^{x(t)} + \left(q^{4j}\lambda_{1/q}\right)^{x(t)}\right)\mathbf{1}_{x(t)\geq 1}\right] + \mathbf{P}_0\left(x(t)=0\right) \\ &+ \left(\lambda_q^{-1}-\lambda_{1/q}\right)\mathbf{E}_0\left[q^{4jx(t)}\mathbf{1}_{x(t)\geq 1}\right] \\ &= \mathbf{E}_0\left[M_q^{x(t)}\mathbf{1}_{x(t)\geq 0}\right]\left(1+\mathscr{E}_1(t)+\mathscr{E}_2(t)+\mathscr{E}_3(t)\right) \end{aligned} \tag{80}$$

with

$$\mathscr{E}_1(t) := \frac{\mathbf{E}_0\left[\left(\lambda_q^{x(t)} + \left(q^{4j}\lambda_{1/q}\right)^{x(t)}\right)\mathbf{1}_{x(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{x(t)}\mathbf{1}_{x(t)\geq 0}\right]}, \qquad \mathscr{E}_2(t) := \frac{\mathbf{P}_0\left(x(t)=0\right)}{\mathbf{E}_0\left[M_q^{x(t)}\mathbf{1}_{x(t)\geq 0}\right]}$$

and

$$\mathscr{E}_3(t) := \frac{\left(\lambda_q^{-1}-\lambda_{1/q}\right)\mathbf{E}_0\left[q^{4jx(t)}\mathbf{1}_{x(t)\geq 1}\right]}{\mathbf{E}_0\left[M_q^{x(t)}\mathbf{1}_{x(t)\geq 0}\right]} . \tag{81}$$

To identify the leaden term in (80) it remains to prove that, for each $i = 1, 2, 3$ there exists $c_i > 0$ such that

$$\sup_{t \geq 0} |\mathscr{E}_i(t)| \leq c_i \tag{82}$$

This would imply, making use of Lemma 3.2, the result in (73). The bound in (82) is immediate for $i = 1, 2$. To prove it for $i = 3$ it is sufficient to show that there exists $c > 0$ such that

$$\lambda_q^{-1} \mathbf{E}_0 \left[q^{4jx(t)} \mathbf{1}_{x(t) \geq 1} \right] \leq c\, \mathbf{E}_0 \left[\left(q^{4j} \lambda_{1/q} \right)^{x(t)} \mathbf{1}_{x(t) \geq 1} \right] . \tag{83}$$

This follows since there exists $x_* \geq 1$ such that for any $x \geq x_*$ $\lambda_q^{-1} \leq \lambda_{1/q}^x$ and then

$$\begin{aligned} \lambda_q^{-1} \mathbf{E}_0 \left[q^{4jx(t)} \mathbf{1}_{x(t) \geq 1} \right] &\leq \lambda_q^{-1} \mathbf{E}_0 \left[q^{4jx(t)} \mathbf{1}_{1 \leq x(t) < x_*} \right] + \mathbf{E}_0 \left[q^{4jx(t)} \lambda_{1/q}^{x(t)} \mathbf{1}_{x(t) \geq x_*} \right] \\ &\leq \lambda_q^{-1} \mathbf{E}_0 \left[q^{4jx(t)} \mathbf{1}_{1 \leq x(t)} \right] + \mathbf{E}_0 \left[q^{4jx(t)} \lambda_{1/q}^{x(t)} \mathbf{1}_{x(t) \geq 1} \right] \\ &\leq \left(1 + \lambda_q^{-1} \right) \mathbf{E}_0 \left[\left(q^{4j} \lambda_{1/q} \right)^{x(t)} \mathbf{1}_{x(t) \geq 1} \right] . \end{aligned} \tag{84}$$

This concludes the proof. □

The rest of our paper is devoted to the construction of the process ASEP(q, j) from a quantum spin chain Hamiltonian with $U_q(\mathfrak{sl}_2)$ symmetry of which we show that it admits a positive ground state. The self-duality functions will then be constructed from application of suitable symmetries to this ground state and application of proposition 2.1.

4 Algebraic structure and symmetries

4.1 The quantum Lie algebra $U_q(\mathfrak{sl}_2)$

For $q \in (0, 1)$ we consider the algebra with generators J^+, J^-, J^0 satisfying the commutation relations

$$[J^+, J^-] = [2J^0]_q, \qquad [J^0, J^\pm] = \pm J^\pm , \tag{85}$$

where $[\cdot, \cdot]$ denotes the commutator, i.e. $[A, B] = AB - BA$, and

$$[2J^0]_q := \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}} . \tag{86}$$

This is the quantum Lie algebra $U_q(\mathfrak{sl}_2)$, that in the limit $q \to 1$ reduces to the Lie algebra $\mathfrak{sl}_2$. Its irreducible representations are $(2j + 1)-$dimensional, with $j \in \mathbb{N}/2$. They are labeled by the eigenvalues of the Casimir element

$$C = J^- J^+ + [J^0]_q [J^0 + 1]_q . \tag{87}$$

A standard representation [14] of the quantum Lie algebra $U_q(\mathfrak{sl}_2)$ is given by $(2j+1) \times (2j+1)$ dimensional matrices defined by

$$\begin{cases} J^+ |n\rangle &= \sqrt{[2j - \eta]_q [\eta + 1]_q}\ |n + 1\rangle \\ J^- |n\rangle &= \sqrt{[\eta]_q [2j - \eta + 1]_q}\ |n - 1\rangle \\ J^0 |n\rangle &= (\eta - j)\ |n\rangle . \end{cases} \tag{88}$$

Here the collection of column vectors $|n\rangle$, with $n \in \{0, \dots, 2j\}$, denote the standard orthonormal basis with respect to the Euclidean scalar product, i.e. $|n\rangle = (0, \dots, 0, 1, 0, \dots, 0)^T$ with the element 1 in the n^{th} position and with the sympol ^{T} denoting transposition. Here and in the following, with abuse of notation, we use the same symbol for a linear operator and the matrix associated to it in a given basis. In the representation (88) the ladder operators J^+ and J^- are one the adjoint of the other, namely

$$(J^+)^* = J^- \tag{89}$$

and the Casimir element is given by the diagonal matrix

$$C|n\rangle = [j]_q[j+1]_q|n\rangle \ .$$

Later on, in the construction of the q-deformed asymmetric simple exclusion process, we will consider other representations for which the ladder operators are not adjoint of each other. For later use, we also observe that the $U_q(\mathfrak{sl}_2)$ commutation relations in (85) can be rewritten as follows

$$\begin{aligned} q^{J_0} J^+ &= q\ J^+ q^{J_0} \\ q^{J_0} J^- &= q^{-1}\ J^- q^{J_0} \\ [J^+, J^-] &= [2J^0]_q \end{aligned} \tag{90}$$

4.2 Co-product structure

A co-product for the quantum Lie algebra $U_q(\mathfrak{sl}_2)$ is defined as the map $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$

$$\begin{aligned} \Delta(J^\pm) &= J^\pm \otimes q^{-J^0} + q^{J^0} \otimes J^\pm \ , \\ \Delta(J^0) &= J^0 \otimes 1 + 1 \otimes J^0 \ . \end{aligned} \tag{91}$$

The co-product is an isomorphism for the quantum Lie algebra $U_q(\mathfrak{sl}_2)$, i.e.

$$[\Delta(J^+), \Delta(J^-)] = [2\Delta(J^0)]_q, \qquad [\Delta(J^0), \Delta(J^\pm)] = \pm\Delta(J^\pm) \ . \tag{92}$$

Moreover it can be easily checked that the co-product satisfies the co-associativity property

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \ . \tag{93}$$

Since we are interested in extended systems we will work with the tensor product over copies of the $U_q(\mathfrak{sl}_2)$ quantum algebra. We denote by J_i^+, J_i^-, J_i^0, with $i \in \mathbb{Z}$, the generators of the i^{th} copy. Obviously algebra elements of different copies commute. As a consequence of (93), one can define iteratively $\Delta^n : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes(n+1)}$, i.e. higher power of Δ, as follows: for $n = 1$, from (91) we have

$$\begin{aligned} \Delta(J_i^\pm) &= J_i^\pm \otimes q^{-J_{i+1}^0} + q^{J_i^0} \otimes J_{i+1}^\pm \\ \Delta(J_i^0) &= J_i^0 \otimes 1 + 1 \otimes J_{i+1}^0 \ , \end{aligned} \tag{94}$$

for $n \geq 2$,

$$\begin{aligned} \Delta^n(J_i^\pm) &= \Delta^{n-1}(J_i^\pm) \otimes q^{-J_{n+i}^0} + q^{\Delta^{n-1}(J_i^0)} \otimes J_{n+i}^\pm \\ \Delta^n(J_i^0) &= \Delta^{n-1}(J_i^0) \otimes 1 + \underbrace{1 \otimes \ldots \otimes 1}_{n \text{ times}} \otimes J_{n+i}^0 \ . \end{aligned} \tag{95}$$

4.3 The quantum Hamiltonian

Starting from the quantum Lie algebra $U_q(\mathfrak{sl}_2)$ (Section 4.1) and the co-product structure (Section 4.2) we would like to construct a linear operator (called "the quantum Hamiltonian" in the following and denoted by $H_{(L)}$ for a system of length L) with the following properties:

1. it is $U_q(\mathfrak{sl}_2)$ symmetric, i.e. it admits non-trivial symmetries constructed from the generators of the quantum algebra; the non-trivial symmetries can then be used to construct self-duality functions;

2. it can be associated to a continuos time Markov jump process, i.e. there exists a representation given by a matrix with non-negative out-of-diagonal elements (which can therefore be interpreted as the rates of an interacting particle systems) and with zero sum on each column.

We will approach the first issue in this subsection, whereas the definition of the related stochastic process is presented in Section 5.

A natural candidate for the quantum Hamiltonian operator is obtained by applying the co-product to the Casimir operator C in (87). Using the co-product definition (91), simple algebraic manipulations (cfr. also [3]) yield the following definition.

Definition 4.1 (Quantum Hamiltonian). *For every $L \in \mathbb{N}$, $L \geq 2$, we consider the operator $H_{(L)}$ defined by*

$$H_{(L)} := \sum_{i=1}^{L-1} H_{(L)}^{i,i+1} = \sum_{i=1}^{L-1} \left(h_{(L)}^{i,i+1} + c_{(L)} \right) , \tag{96}$$

where the two-site Hamiltonian is the sum of

$$c_{(L)} = \frac{(q^{2j} - q^{-2j})(q^{2j+1} - q^{-(2j+1)})}{(q - q^{-1})^2} \underbrace{1 \otimes \cdots \otimes 1}_{L \text{ times}} \tag{97}$$

and

$$h_{(L)}^{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1) \text{ times}} \otimes \Delta(C_i) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i-1) \text{ times}} \tag{98}$$

and, from (87) *and* (91),

$$\Delta(C_i) = \Delta(J_i^-)\Delta(J_i^+) + \Delta([J_i^0]_q)\Delta([J_i^0 + 1]_q) . \tag{99}$$

Explicitely

$$\begin{aligned} \Delta(C_i) \;\; = \;\; -q^{J_i^0} \Bigg\{ & J_i^+ \otimes J_{i+1}^- + J_i^- \otimes J_{i+1}^+ + \frac{(q^j + q^{-j})(q^{j+1} + q^{-(j+1)})}{2} [J_i^0]_q \otimes [J_{i+1}^0]_q \\ & + \frac{[j]_q [j+1]_q}{2} \left(q^{J_i^0} + q^{-J_i^0} \right) \otimes \left(q^{J_{i+1}^0} + q^{-J_{i+1}^0} \right) \Bigg\} q^{-J_{i+1}^0} \end{aligned} \tag{100}$$

Remark 4.1. *The diagonal operator $c_{(L)}$ in* (97) *has been added so that the ground state $|0\rangle_{(L)} := \otimes_{i=1}^{L}|0\rangle_i$ is a right eigenvector with eigenvalue zero, i.e. $H_{(L)}|0\rangle_{(L)} = 0$ as it is immediately seen using* (88).

Proposition 4.1. *In the representation* (88) *the operator $H_{(L)}$ is self-adjoint.*

Proof. It is enough to consider the non-diagonal part of $H_{(L)}$. Using (89) we have

$$\begin{aligned} & \left(q^{J_i^0} J_i^+ \otimes J_{i+1}^- q^{-J_{i+1}^0} + q^{J_i^0} J_i^- \otimes J_{i+1}^+ q^{-J_{i+1}^0}\right)^* \\ = \; & J_i^- q^{J_i^0} \otimes q^{-J_{i+1}^0} J_{i+1}^+ + J_i^+ q^{J_i^0} \otimes q^{-J_{i+1}^0} J_{i+1}^- \\ = \; & q^{J_i^0+1} J_i^- \otimes J_{i+1}^+ q^{-J_{i+1}^0-1} + q^{J_i^0-1} J_i^+ \otimes J_{i+1}^- q^{-J_{i+1}^0+1} \end{aligned}$$

where the last identity follows by using the commutation relations (90). This concludes the proof. □

4.4 Basic symmetries

It is easy to construct symmetries for the operator $H_{(L)}$ by using the property that the co-product is an isomorphism for the $U_q(\mathfrak{sl}_2)$ algebra.

Theorem 4.1 (Symmetries of $H_{(L)}$)**.** *Recalling* (95)*, we define the operators*

$$\begin{aligned} J_{(L)}^{\pm} &:= \Delta^{L-1}(J_1^{\pm}) = \sum_{i=1}^{L} q^{J_1^0} \otimes \cdots \otimes q^{J_{i-1}^0} \otimes J_i^{\pm} \otimes q^{-J_{i+1}^0} \otimes \ldots \otimes q^{-J_L^0} \,, \\ J_{(L)}^{0} &:= \Delta^{L-1}(J_1^{0}) = \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\ times} \otimes J_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\ times} \,. \end{aligned} \tag{101}$$

They are symmetries of the Hamiltonian (96)*, i.e.*

$$[H_{(L)}, J_{(L)}^{\pm}] = [H_{(L)}, J_{(L)}^{0}] = 0 \,. \tag{102}$$

Proof. We proceed by induction and prove only the result for $J_{(L)}^{\pm}$ (the case $J_{(L)}^0$ is similar). By construction $J_{(2)}^{\pm} := \Delta(J^{\pm})$ are symmetries of the two-site Hamiltonian $H_{(2)}$. Indeed this is an immediate consequence of the fact that the co-product defined in (92) conserves the commutation relations and the Casimir operator (87) commutes with any other operator in the algebra :

$$[H_{(2)}, J_{(2)}^{\pm}] = [\Delta(C_1), \Delta(J_1^{\pm})] = \Delta([C_1, J_1^{\pm}]) = 0 \,.$$

For the induction step assume now that it holds $[H_{(L-1)}, J_{(L-1)}^{\pm}] = 0$. We have

$$[H_{(L)}, J_{(L)}^{\pm}] = [H_{(L-1)}, J_{(L)}^{\pm}] + [h_{(L)}^{L-1,L}, J_{(L)}^{\pm}] \tag{103}$$

The first term on the right hand side of (103) can be seen to be zero using (95) with $i = 1$ and $n = L - 1$:

$$[H_{(L-1)}, J_{(L)}^{\pm}] = [H_{(L-1)}, J_{(L-1)}^{\pm} q^{-J_L^0} + q^{J_{(L-1)}^0} J_L^{\pm}]$$

Distributing the commutator with the rule $[A, BC] = B[A, C] + [A, B]C$, the induction hypothesis and the fact that spins on different sites commute imply the claim. The second term on the right hand side of (103) is also seen to be zero by writing

$$[h_{(L)}^{L-1,L}, J_{(L)}^{\pm}] = [h_{(L)}^{L-1,L}, J_{(L-2)}^{\pm} q^{-\Delta(J_{L-1}^0)} + q^{J^0_{(L-2)}} \Delta(J_{L-1}^{\pm})] = 0 \ .$$

□

Remark 4.2. *In the case $q = 1$, the quantum Hamiltonian in Definition 4.1 reduces to the (negative of the) well-known Heisenberg ferromagnetic quantum spin chain with spins J_i satisfying the $\mathfrak{sl}_2$ Lie-algebra. With abuse of notation for the tensor product, the Heisenberg quantum spin chain reads*

$$H_{(L)}^{Heis} = \sum_{i=1}^{L-1} \left(J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^0 J_{i+1}^0 - 2j^2 \right) \ , \tag{104}$$

whose symmetries are given by

$$J_{(L)}^{\pm,Heis} = \sum_{i=1}^{L} J_i^{\pm} \qquad \text{and} \qquad J_{(L)}^{0,Heis} = \sum_{i=1}^{L} J_i^0 \ .$$

5 Construction of the ASEP(q, j)

In order to construct a Markov process from the quantum Hamiltonian $H_{(L)}$, we apply item a) of Corollary 2.1 with $A = H_{(L)}$. At this aim we need a non-trivial symmetry which yields a non-trivial ground state. Starting from the basic symmetries of $H_{(L)}$ described in Section 4.4, and inspired by the analysis of the symmetric case ($q = 1$), it will be convenient to consider the *exponential* of those symmetries.

5.1 The q-exponential and its pseudo-factorization

Definition 5.1 (q-exponential)**.** *We define the q-analog of the exponential function as*

$$\exp_q(x) := \sum_{n\geq 0} \frac{x^n}{\{n\}_q!} \tag{105}$$

where

$$\{n\}_q := \frac{1-q^n}{1-q} \tag{106}$$

Remark 5.1. *The q-numbers in* (106) *are related to the q-numbers in* (10) *by the relation $\{n\}_{q^2} = [n]_q q^{n-1}$. This implies $\{n\}_{q^2}! = [n]_q!\, q^{n(n-1)/2}$ and therefore*

$$\exp_{q^2}(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!} q^{-n(n-1)/2} \tag{107}$$

One could also have defined the q-exponential directly in terms of the q-numbers (10), *namely*

$$\widetilde{\exp}_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!} \tag{108}$$

The reason to prefer definition of the q-deformed exponential given in (105), rather than (108), is that with the first choice we have then a pseudo-factorization property as described in the following.

Proposition 5.1 (Pseudo-factorization). *Let $\{g_1, \dots, g_L\}$ and $\{k_1, \dots, k_L\}$ be operators such that for $L \in \mathbb{N}$ and $g \in \mathbb{R}$*

$$k_i g_i = r g_i k_i \qquad \textit{for} \quad i = 1, \dots, L\ . \tag{109}$$

Define

$$g^{(L)} := \sum_{i=1}^{L} k^{(i-1)} g_i, \quad \textit{with} \quad k^{(i)} := k_1 \cdot \dots \cdot k_i \quad \textit{for } i \geq 1 \textit{ and } \quad k^{(0)} = 1, \tag{110}$$

then

$$\exp_r(g^{(L)}) = \exp_r(g_1) \cdot \exp_r(k^{(1)} g_2) \cdot \dots \cdot \exp_r(k^{(L-1)} g_L) \tag{111}$$

Moreover let

$$\hat{g}^{(L)} := \sum_{i=1}^{L} g_i\, h^{(i+1)}, \quad \textit{with} \quad h^{(i)} := k_i^{-1} \cdot \dots \cdot k_L^{-1} \quad \textit{for } i \leq L \textit{ and } \quad h^{(L+1)} = 1, \tag{112}$$

then

$$\exp_r(\hat{g}^{(L)}) = \exp_r(g_1\, h^{(2)}) \cdot \dots \cdot \exp_r(g_{L-1}\, h^{(L)}) \cdot \exp_r(g_L) \tag{113}$$

In this section we prove only (111) since the proof of (113) is similar. We first give a series of Lemma that are useful in the proof.

Lemma 5.1. *Let*

$$Bin_r\{n, m\} := \frac{\{n\}_r!}{\{m\}_r!\{n-m\}_r!} \tag{114}$$

then

$$r^m \mathrm{Bin}_r\{n, m\} + \mathrm{Bin}_r\{n, m-1\} = \mathrm{Bin}_r\{n+1, m\} \tag{115}$$

Proof. It follows from an immediate computation □

Lemma 5.2. *For any $n, L \in \mathbb{N}$, $L \geq 2$*

$$\left(g^{(L)}\right)^n = \sum_{m=0}^{n} \mathrm{Bin}\{n, m\}_r \left(g^{(L-1)}\right)^{n-m} (k^{(L-1)} g_L)^m \tag{116}$$

Proof. We prove it by induction on n. For $n = 1$ it is true because for each $L \geq 2$

$$g^{(L)} = g^{(L-1)} + k^{(L-1)} g_L \tag{117}$$

By (109), for any $\ell \in \mathbb{N}$

$$\left(k^{(\ell)}\right)^m g^{(\ell)} = r^m g^{(\ell)} \left(k^{(\ell)}\right)^m \tag{118}$$

Suppose that (116) holds for n for any $L \geq 2$, then, using (115) and (118) we have

$$\begin{aligned}
\left(g^{(L)}\right)^{n+1} &= \left(g^{(L-1)} + k^{(L-1)} g_L\right)^{n+1} \\
&= \sum_{m=0}^{n} \mathrm{Bin}_r\{n,m\} \left(g^{(L-1)}\right)^{n-m} \left(k^{(L-1)} g_L\right)^m \cdot \left[g^{(L-1)} + k^{(L-1)} g_L\right] \\
&= \sum_{m=1}^{n} \left[r^m \mathrm{Bin}_r\{n,m\} + \mathrm{Bin}_r\{n,m-1\}\right] \left(g^{(L-1)}\right)^{n+1-m} \left(k^{(L-1)} g_L\right)^m \\
&\quad + \left(g^{(L-1)}\right)^{n+1} + \left(k^{(L-1)} g_L\right)^{n+1} \\
&= \sum_{m=0}^{n+1} \mathrm{Bin}_r\{n+1,m\} \left(g^{(L-1)}\right)^{n+1-m} \left(k^{(L-1)} g_L\right)^m
\end{aligned} \tag{119}$$

that proves the lemma. $\square$

Lemma 5.3. *For any $n, L \in \mathbb{N}$, $L \geq 2$ we have*

$$\left(g^{(L)}\right)^n = \{n\}_r! \sum_{m_L=0}^{n} \sum_{m_{L-1}=0}^{n-m_L} \cdots \sum_{m_2=0}^{n-\sum_{i=3}^{L} m_i} \frac{g_1^{n-\sum_{i=2}^{L} m_i}}{\{n - \sum_{i=2}^{L} m_i\}_r!} \cdot \prod_{i=2}^{L} \frac{(k^{(i-1)} g_i)^{m_i}}{\{m_i\}_r!} \tag{120}$$

Proof. We prove it by induction on L. From (116), for any $n \in \mathbb{N}$ we have

$$\left(g^{(2)}\right)^n = (g_1 + k_1 g_2)^n = \{n\}_r! \sum_{m=0}^{n} \frac{(g_1)^{n-m}}{\{n-m\}_r!} \frac{(k_1 g_2)^m}{\{m\}_r!} \tag{121}$$

thus (120) is true for $L = 2$, $n \in \mathbb{N}$. Suppose that it holds for L for any $n \in \mathbb{N}$ then, using

(116) we have

$$\begin{aligned}
\left(g^{(L+1)}\right)^n &= \left(g^{(L)} + k^{(L)} g_{L+1}\right)^n \\
&= \sum_{m_{L+1}=0}^{n} \mathrm{Bin}_r\{n, m_{L+1}\} \left(g^{(L)}\right)^{n-m_{L+1}} \left(k^{(L)} g_{L+1}\right)^{m_{L+1}} \\
&= \sum_{m_{L+1}=0}^{n} \mathrm{Bin}_r\{n, m_{L+1}\} \Bigg(\{n-m_{L+1}\}_r! \sum_{m_L=0}^{n-m_{L+1}} \cdots\cdots\cdots \\
&\cdots \sum_{m_2=0}^{n-m_{L+1}-\sum_{i=3}^{L} m_i} \frac{g_1^{n-m_{L+1}-\sum_{i=2}^{L} m_i}}{\{n-m_{L+1}-\sum_{i=2}^{L} m_i\}_r!} \cdot \prod_{i=2}^{L} \frac{(k^{(i-1)} g_i)^{m_i}}{\{m_i\}_r!} \Bigg) \cdot \left(k^{(L)} g_{L+1}\right)^{m_{L+1}} \\
&= \{n\}_r! \sum_{m_{L+1}=0}^{n} \sum_{m_L=0}^{n-m_{L+1}} \cdots \sum_{m_2=0}^{n-\sum_{i=3}^{L+1} m_i} \frac{g_1^{n-\sum_{i=2}^{L+1} m_i}}{\{n-\sum_{i=2}^{L+1} m_i\}_r!} \cdot \prod_{i=2}^{L+1} \frac{(k^{(i-1)} g_i)^{m_i}}{\{m_i\}_r!}
\end{aligned}$$

this proves the lemma. □

Lemma 5.4. *Let $L \in \mathbb{N}$, $L \geq 2$ and for any $i = 1, \ldots, L$ let $\mathbf{X}_i \in \mathbb{R}^{\mathbb{N}}$ a sequence of real numbers, $\mathbf{X}_i = \{X_i(m)\}_{m \in \mathbb{N}}$, then*

$$\sum_{n=0}^{\infty} \sum_{m_L=0}^{n} \sum_{m_{L-1}=0}^{n-m_L} \cdots \sum_{m_2=0}^{n-\sum_{i=3}^{L} m_i} X_1(n - \sum_{i=2}^{L} m_i) \cdot \prod_{i=2}^{L} X_i(m_i) = \prod_{i=1}^{L} \sum_{m_1=0}^{\infty} X_i(m_i) \tag{122}$$

Proof. It is sufficient to prove it for $L = 2$, the proof of (122) follows by an analogous argument. By performing the change of variable $n := m_1 + m_2$ we obtain

$$\begin{aligned}
&\sum_{m_i=0}^{\infty} \prod_{i=1}^{2} X_i(m_i) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} X_1(m_1) X_2(m_2) \\
&= \sum_{m_2=0}^{\infty} \sum_{n=m_2}^{\infty} X_1(n-m_2) X_2(m_2) = \sum_{n=0}^{\infty} \sum_{m_2=0}^{n} X_1(n-m_2) X_2(m_2)
\end{aligned}$$

that yields (122) for $L = 2$. □

Proof of proposition 5.1. From (120) we have

$$\exp_r(g^{(L)}) = \sum_{n=0}^{\infty} \frac{\left(g^{(L)}\right)^n}{\{n\}_r!} \tag{123}$$

$$= \sum_{n=0}^{\infty} \sum_{m_L=0}^{n} \sum_{m_{L-1}=0}^{n-m_L} \cdots \sum_{m_2=0}^{n-\sum_{i=3}^{L} m_i} \frac{g_1^{n-\sum_{i=2}^{L} m_i}}{\{n-\sum_{i=2}^{L} m_i\}_r!} \cdot \prod_{i=2}^{L} \frac{(k^{(i-1)} g_i)^{m_i}}{\{m_i\}_r!} \tag{124}$$

$$= \prod_{i=1}^{L} \sum_{m_i=0}^{\infty} \frac{(k^{(i-1)} g_i)^{m_i}}{\{m_i\}_r!} \tag{125}$$

$$= \prod_{i=1}^{L} \exp_r(k^{(i-1)} g_i) \tag{126}$$

where the passage from (124) to (125) follows from Lemma 5.4. □

5.2 The exponential symmetry $S^+_{(L)}$

In this Section we identify the symmetry that will be used in the construction of the process ASEP(q, j). To have a symmetry that has quasi-product form over the sites we preliminary define more convenient generators of the $U_q(\mathfrak{sl}_2)$ quantum Lie algebra. Let

$$E := q^{J^0} J^+, \qquad F := J^- q^{-J^0} \qquad \text{and} \qquad K := q^{2J^0} \tag{127}$$

From the commutation relations (85) we deduce that (E, F, K) verify the relations

$$KE = q^2 EK \qquad \text{and} \qquad KF = q^{-2} FK \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \;. \tag{128}$$

Moreover, from Theorem 4.1, the following co-products

$$\Delta(E_1) := \Delta(q^{J_1^0}) \cdot \Delta(J_1^+) = E_1 \otimes \mathbf{1} + K_1 \otimes E_2 \tag{129}$$

$$\Delta(F_1) := \Delta(J_1^-) \cdot \Delta(q^{-J_1^0}) = F_1 \otimes K_2^{-1} + \mathbf{1} \otimes F_2 \tag{130}$$

are still symmetries of $H_{(2)}$. In general we can extend (129) and (130) to L sites, then we have that

$$\begin{aligned} E^{(L)} &:= \Delta^{(L-1)}(E_1) \\ &= \Delta^{(L-1)}(q^{J_1^0}) \cdot \Delta^{(L-1)}(J_1^+) \\ &= q^{J_1^0} J_1^+ + q^{2J_1^0 + J_2^0} J_2^+ + ... + q^{2\sum_{i=1}^{L-1} J_i^0 + J_L^0} J_L^+ \\ &= E_1 + K_1 E_2 + K_1 K_2 E_3 + ... + K_1 \cdot ... \cdot K_{L-1} E_L \end{aligned} \tag{131}$$

$$\begin{aligned} F^{(L)} &:= \Delta^{(L-1)}(F_1) \\ &= \Delta^{(L-1)}(J_1^-) \cdot \Delta^{(L-1)}(q^{-J_1^0}) \\ &= J_1^- q^{-J_1^0 - 2\sum_{i=2}^{L} J_i^0} + \cdots + J_{L-1}^- q^{-J_{L-1}^0 - 2J_L^0} + J_L^- q^{-J_L^0} \\ &= F_1 \cdot K_2^{-1} \cdot ... \cdot K_L^{-1} + \cdots + F_{L-1} \cdot K_L^{-1} + F_L \end{aligned} \tag{132}$$

are symmetries of H. If we consider now the symmetry obtained by q-exponentiating $E^{(L)}$ then this operator will pseudo-factorize by Proposition 5.1.

Lemma 5.5. *The operator*

$$S^+_{(L)} := \exp_{q^2}(E^{(L)}) \tag{133}$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$\langle \eta_1, ..., \eta_L | S^+ | \xi_1, ..., \xi_L \rangle = \prod_{i=1}^{L} \sqrt{\binom{\eta_i}{\xi_i}_q \binom{2j-\xi_i}{2j-\eta_i}_q} \cdot \mathbf{1}_{\eta_i \geq \xi_i} q^{(\eta_i - \xi_i)[1-j+\xi_i+2\sum_{k=1}^{i-1}(\xi_k - j)]} \tag{134}$$

Proof. From (128) we know that the operators E_i, K_i, copies of the operators defined in (127), verify the conditions (109) with $r = q^2$. As a consequence, from (131), (133) and Proposition 5.1, we have

$$\begin{aligned}
S^+_{(L)} &= \exp_{q^2}(E^{(L)}) \\
&= \exp_{q^2}(E_1) \cdot \exp_{q^2}(K_1 E_2) \cdots \exp_{q^2}(K_1 \cdots K_{L-1} E_L) \\
&= \exp_{q^2}\left(q^{J_1^0} J_1^+\right) \cdot \exp_{q^2}\left(q^{2J_1^0} q^{J_2^0} J_2^+\right) \cdots \exp_{q^2}\left(q^{2\sum_{i=1}^{L-1} J_i^0 + J_L^0} J_L^+\right) \\
&= S_1^+ S_2^+ \cdots S_L^+
\end{aligned} \tag{135}$$

where $S_i^+ := \exp_{q^2}\left(q^{2\sum_{k=1}^{i-1} J_k^0 + J_i^0} J_i^+\right)$ has been defined. Using (107), we find

$$\begin{aligned}
S_i^+ |\xi_1, \dots, \xi_L\rangle &= \sum_{\ell_i \geq 0} \frac{1}{[\ell_i]_q!} \left(q^{2\sum_{k=1}^{i-1} J_k^0 + J_i^0} J_i^+\right)^{\ell_i} q^{-\frac{1}{2}\ell_i(\ell_i - 1)} |\xi_1, \dots, \xi_L\rangle \\
&= \sum_{\ell_i \geq 0} \sqrt{\binom{2j-\xi_i}{\ell_i}_q \cdot \binom{\xi_i + \ell_i}{\xi_i}_q} \cdot q^{\ell_i(1-j+\xi_i) + 2\ell_i \sum_{k=1}^{i-1}(\xi_k - j)} |\xi_1, \dots, \xi_i + \ell_i, \dots, \xi_L\rangle
\end{aligned} \tag{136}$$

where in the last equality we used (88). Thus we find

$$\begin{aligned}
S^+_{(L)} |\xi_1, \dots, \xi_L\rangle &= S_1^+ S_2^+ \dots S_L^+ |\xi_1, \dots, \xi_L\rangle \\
&= \sum_{\ell_1, \ell_2, \dots, \ell_L \geq 0} \prod_{i=1}^{L} \left(\sqrt{\binom{2j-\xi_i}{\ell_i}_q \cdot \binom{\xi_i + \ell_i}{\xi_i}_q} \right. \\
&\qquad \left. \cdot\, q^{\ell_i(1-j+\xi_i) + 2\ell_i \sum_{k=1}^{i-1}(\xi_k - j)} \right) |\xi_1 + \ell_1, \dots, \xi_L + \ell_L\rangle
\end{aligned} \tag{137}$$

form which the matrix elements in (134) are immediately found. □

5.3 Construction of a positive ground state and the associated Markov process ASEP(q, j)

By applying Corollary 2.1 we are now ready to identify the stochastic process related to the Hamiltonian $H_{(L)}$ in (96).

We start from the state $|\mathbf{0}\rangle = |0,\dots,0\rangle$ which is obviously a trivial ground state of $H_{(L)}$. We then produce a non-trivial ground state by acting with the symmetry $S^+_{(L)}$ in (133), as described in Remark 2.1. Using (137) we obtain

$$|g\rangle \;=\; S^+_{(L)}|0,\dots,0\rangle = \sum_{\ell_1,\ell_2,\dots,\ell_L\geq 0}\prod_{i=1}^{L}\sqrt{\binom{2j}{\ell_i}_q}\cdot\, q^{\ell_i(1+j-2ji)}\,|\ell_1,...,\ell_L\rangle$$

Therefore we arrived to a positive ground state (cfr. Remark 2.1). Following the scheme in Corollary 2.1 we construct the operator $G_{(L)}$ defined by

$$G_{(L)}|\eta_1,\dots,\eta_L\rangle = |\eta_1,\dots,\eta_L\rangle\langle\eta_1,\dots,\eta_L|S^+|0,\dots,0\rangle \tag{138}$$

In other words $G_{(L)}$ is represented by a diagonal matrix whose coefficients in the standard basis read

$$\langle\eta_1,\dots,\eta_L|G_{(L)}|\xi_1,\dots,\xi_L\rangle = \prod_{i=1}^{L}\sqrt{\binom{2j}{\eta_i}_q}\cdot q^{\eta_i(1+j-2ji)}\cdot\delta_{\eta_i=\xi_i} \tag{139}$$

Note that $G_{(L)}$ is factorized over the sites, i.e.

$$\langle\eta_1,\dots,\eta_L|G_{(L)}|\xi_1,\dots,\xi_L\rangle = \otimes_{i=1}^{L}\langle\eta_i|G_i|\xi_i\rangle \tag{140}$$

As a consequence of item a) of Corollary 2.1, the operator $\mathscr{L}^{(L)}$ conjugated to $H_{(L)}$ via $G^{-1}_{(L)}$, i.e.

$$\mathscr{L}^{(L)} = G^{-1}_{(L)}H_{(L)}G_{(L)} \tag{141}$$

is the generator of a Markov jump process $\eta(t) = (\eta_1(t),\dots,\eta_L(t))$ describing particles jumping on the line $\{1,\dots,L\}$. The state space of such a process is given by $\{0,\dots,2j\}^L$ and its elements are denoted by $\eta = (\eta_1,\dots,\eta_L)$, where η_i is interpreted as the number of particles at site i. The exclusion rule is due to the fact that on each site can sit no more than $2j$ particles. The asymmetry is controlled by the parameter $0 < q \leq 1$.

Proposition 5.2. *The action of the Markov generator $\mathscr{L}^{(L)} := G^{-1}_{(L)}H_{(L)}G_{(L)}$ is given by (11).*

Proof. From Proposition 4.1 we know that $H^*_{(L)} = H_{(L)}$, hence we have that the operator $\tilde{H}_{(L)} := G_{(L)}H_{(L)}G^{-1}_{(L)}$ is the transposed of the generator $\mathscr{L}^{(L)}$ defined by (141). Then we have to verify that the transition rates to move from η to ξ for the Markov process generated by (11) are equal to the elements $\langle\xi|\tilde{H}_{(L)}|\eta\rangle$.
Since we already know that $\mathscr{L}^{(L)}$ is a Markov generator, in order to prove the result it is sufficient to apply the similarity transformation given by the matrix $G_{(L)}$ defined in (139) to the non-diagonal terms of (100), i.e. $q^{J_i^0}J_i^{\pm}J_{i+1}^{\mp}q^{-J^0_{i+1}}$. We show here the computation only for the first term, being the computation for the other term similar.
We have

$$\begin{aligned}&\langle\xi_i,\xi_{i+1}|G_iG_{i+1}\cdot q^{J_i^0}J_i^+J_{i+1}^-q^{-J^0_{i+1}}\cdot G_i^{-1}G_{i+1}^{-1}|\eta_i,\eta_{i+1}\rangle\\ &= \langle\xi_i|G_iq^{J_i^0}J_i^+G_i^{-1}|\eta_i\rangle\otimes\langle\xi_{i+1}|G_{i+1}J_{i+1}^-q^{-J^0_{i+1}}G^{-1}_{i+1}|\eta_{i+1}\rangle\end{aligned} \tag{142}$$

Using (139) and (88) one has

$$\langle \xi_i | G_i q^{J_i^0} J_i^+ G_i^{-1} | \eta_i \rangle = q^{\eta_i + 2 - 2ji} \, [2j - \eta_i]_q \langle \xi_i | \eta_i + 1 \rangle \tag{143}$$

and

$$\langle \xi_{i+1} | G_{i+1} J_{i+1}^- q^{-J_{i+1}^0} G_{i+1}^{-1} | \eta_{i+1} \rangle = q^{-\eta_{i+1} + 2j - 1 + 2ji} \, [\eta_{i+1}]_q \langle \xi_{i+1} | \eta_{i+1} - 1 \rangle \tag{144}$$

Multiplying the last two expressions one has

$$\langle \eta^{i+1,i} | \tilde{H}_{(L)} | \eta \rangle = q^{\eta_i - \eta_{i+1} + (2j+1)} [2j - \eta_i]_q [\eta_{i+1}]_q \tag{145}$$

that corresponds indeed to the rate to move from η to $\eta^{i+1,i}$ in (11). This concludes the proof. □

Remark 5.2. *From item c) of Corollary 2.1, we have that the product measure $\mu_{(L)}$ defined by*

$$\mu_{(L)}(\eta) = \langle \eta | G^2_{(L)} | \eta \rangle \tag{146}$$

is a reversible measure of $\mathscr{L}^{(L)}$. Notice that it corresponds to the reversible measure $\mathbb{P}^{(\alpha)}$ defined in (16) *with the choice $\alpha = 1$.*

6 Self-Duality results for the ASEP(q,j)

We now use Proposition 2.1 and the exponential simmetry obtained in Section 5.2 to deduce a non-trivial duality function for the ASEP(q,j) process. We first have the following remark on trivial duality functions.

Remark 6.1. *From* (9) *and item a) of Theorem 3.1 it follows that all the functions*

$$d_\alpha(\eta, \xi) = \prod_{i=1}^{L} \left(\binom{2j}{\eta_i}_q \cdot \alpha^n q^{2\eta_i(1+j-2ji)} \right)^{-1} \cdot \delta_{\eta_i = \xi_i} \tag{147}$$

are diagonal duality functions for the Markov process with generator $\mathscr{L}^{(L)}$.

We then deduce the main result, i.e. a non-trivial duality function.

Proof of (34) in theorem 3.2. From Proposition 4.1 we know that $H_{(L)}$ is self-adjoint, then, using Proposition 2.1 with $A = H_{(L)}$, $G = G_{(L)}$ given by (139) and $S = S^+_{(L)}$ given by (134) it follows that

$$G^{-1}_{(L)} S^+_{(L)} G^{-1}_{(L)} \tag{148}$$

is a self-duality function for the process generated by $\mathscr{L}^{(L)}$. Its elements are computed as follows:

$$\langle\eta|G_{(L)}^{-1}S_{(L)}^{+}G_{(L)}^{-1}|\xi\rangle = \tag{149}$$

$$= \prod_{i=1}^{L}\left(\sqrt{\binom{2j}{\eta_i}_q}\cdot q^{\eta_i(1+j-2ji)}\right)^{-1}\langle\eta|S_i^{+}|\xi\rangle\left(\sqrt{\binom{2j}{\xi_i}_q}\cdot q^{\xi_i(1+j-2ji)}\right)^{-1} = \tag{150}$$

$$= \prod_{i=1}^{L}\sqrt{\binom{\eta_i}{\xi_i}_q\binom{2j-\xi_i}{2j-\eta_i}_q\Big/\binom{2j}{\eta_i}_q\binom{2j}{\xi_i}_q}\cdot q^{(\eta_i-\xi_i)\left[2\sum_{k=1}^{i-1}(\xi_k-j)+\xi_i\right]+(2ji-j-1)(\eta_i+\xi_i)}\cdot \mathbf{1}_{\xi_i\le\eta_i} =$$

$$= q^{\sum_{i=1}^{L}((j-1)\eta_i-(3j+1)\xi_i)}\prod_{i=1}^{L}\frac{[2j-\xi_i]_q![\eta_i]_q!}{[2j]_q![\eta_i-\xi_i]_q!}\cdot\ q^{(\eta_i-\xi_i)\left[2\sum_{k=1}^{i-1}\xi_k+\xi_i\right]+4ji\xi_i}\cdot\mathbf{1}_{\xi_i\le\eta_i}$$

Since both the original process and the dual process conserve the total number of particles it follows that $D_{(L)}$ in (34) is also a duality function. □

7 A second symmetry and associated self-duality

Up to now we worked with the symmetry $S_{(L)}^{+}$ defined in (133). In this Section we explore other choices for the symmetry and their consequences.

7.1 Construction of alternative symmetries

We already observed that the operator $F^{(L)}$ defined in (132) is a symmetry of $H_{(L)}$. The following Lemma gives the exponential symmetry that is further obtained.

Lemma 7.1. *The operator*

$$S_{(L)}^{-} := \exp_{q^{-2}}(F^{(L)}) \tag{151}$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$\langle\eta_1,...,\eta_L|S_{(L)}^{-}|\xi_1,...,\xi_L\rangle = \prod_{i=1}^{L}\sqrt{\binom{\xi_i}{\eta_i}_q\cdot\binom{2j-\eta_i}{2j-\xi_i}_q}\cdot\mathbf{1}_{\eta_i\le\xi_i}q^{-(\xi_i-\eta_i)\left[2\sum_{k=i+1}^{L}(\eta_k-j)+\eta_i-j+1\right]} \tag{152}$$

Proof. From (128) we know that the operators F_i, K_i, copies of the operator defined in (127), verify the conditions (109) with $r = q^{-2}$. Then, from (160) and Proposition 5.1

$$\begin{aligned} S_{(L)}^{-} &= \exp_{q^{-2}}(F^{(L)}) \\ &= \exp_{q^{-2}}(F_1K_2^{-1}\dots K_L^{-1})\cdot\dots\cdot\exp_{q^{-2}}(F_{L-1}K_L^{-1})\cdot\exp_{q^{-2}}(F_L) \\ &= \exp_{q^{-2}}\left(J_1^{-}q^{-J_1^0-2\sum_{i=2}^{L}J_i^0}\right)\cdot\dots\cdot\exp_{q^{-2}}\left(J_{L-1}^{-}q^{-J_{L-1}^0-2J_L^0}\right)\cdot\exp_{q^{-2}}\left(J_L^{-}q^{-J_L^0}\right) \\ &= S_1^{-}S_2^{-}\dots S_L^{-} \end{aligned} \tag{153}$$

where $S_i^- := \exp_{q^{-2}}\left(J_i^- q^{-J_i^0 - 2\sum_{k=i+1}^L J_k^0}\right)$. Using (107) and the fact that $[x]_{q^{-1}} = [x]_q$, we have

$$\begin{aligned} S_i^-|\xi_1,...,\xi_L\rangle &= \sum_{\ell_i\geq 0}\frac{1}{[\ell_i]_q!}\left(J_i^- q^{-J_i^0-2\sum_{k=i+1}^L J_k^0}\right)^{\ell_i} q^{\frac{1}{2}\ell_i(\ell_i-1)}|\xi_1,...,\xi_L\rangle \qquad (154)\\ &= \sum_{\ell_i\geq 0}\sqrt{\binom{\xi_i}{\ell_i}_q\cdot\binom{2j-\xi_i+\ell_i}{\ell_i}_q}\, q^{-2\ell_i\sum_{k=i+1}^L(\xi_k-j)}\, q^{\ell_i(\ell_i-\xi_i+j-1)}|\xi_1,...,\xi_i-\ell_i,...\xi_L\rangle \end{aligned}$$

then

$$\begin{aligned} S_{(L)}^-|\xi_1,...,\xi_L\rangle &= S_1^- S_2^- \dots S_L^-|\xi_1,...,\xi_L\rangle\\ &= \sum_{\ell_1,\ell_2,...,\ell_L\geq 0}\prod_{i=1}^L\Bigg(\sqrt{\binom{\xi_i}{\ell_i}_q\cdot\binom{2j-\xi_i+\ell_i}{\ell_i}_q}\\ &\qquad \cdot q^{-2\ell_i\sum_{k=i+1}^L(\xi_k-\ell_k-j)}\, q^{\ell_i(\ell_i-\xi_i+j-1)}\Bigg)|\xi_1-\ell_1,...,\xi_L-\ell_L\rangle \end{aligned}$$

From this the matrix elements in (152) immediately follows. □

Other symmetries can be obtained as follows. Similarly to Section 5.2, we consider

$$\tilde{E} := J^+ q^{-J^0}, \qquad \tilde{F} := q^{J^0} J^- \qquad \text{and} \qquad \tilde{K} := q^{2J^0} \qquad (155)$$

and notice that $(\tilde{E},\tilde{F},K)$ (as (E,F,K) in Section 5.2) verify the commutation relations

$$\tilde{K}\tilde{E} = q^2\tilde{E}\tilde{K} \qquad \text{and} \qquad \tilde{K}\tilde{F} = q^{-2}\tilde{F}\tilde{K} \qquad [\tilde{E},\tilde{F}] = \frac{\tilde{K}-\tilde{K}^{-1}}{q-q^{-1}}\,. \qquad (156)$$

Therefore the following co-products

$$\Delta(\tilde{E}_1) := \Delta(J_1^+)\cdot\Delta(q^{-J_1^0}) = \tilde{E}_1\otimes\tilde{K}_2^{-1} + \mathbf{1}\otimes\tilde{E}_2 \qquad (157)$$

$$\Delta(\tilde{F}_1) := \Delta(q^{J_1^0})\cdot\Delta(J_1^-) = \tilde{F}_1\otimes\mathbf{1} + \tilde{K}_1\otimes\tilde{F}_2 \qquad (158)$$

are symmetries of $H_{(2)}$. In general we can extend (157) and (158) to L sites, then we have that

$$\begin{aligned} \tilde{E}^{(L)} &:= \Delta^{(L-1)}\tilde{E}_1\\ &= \Delta^{(L-1)}(J_1^+)\cdot\Delta^{(L-1)}(q^{-J_1^0})\\ &= J_1^+ q^{-J_1^0-2\sum_{i=2}^L J_i^0} + \cdots + J_{L-1}^+ q^{-J_{L-1}^0-2J_L^0} + J_L^+ q^{-J_L^0}\\ &= \tilde{E}_1\cdot\tilde{K}_2^{-1}\cdot\ldots\cdot\tilde{K}_L^{-1} + \cdots + \tilde{E}_{L-1}\cdot\tilde{K}_L^{-1} + \tilde{E}_L \qquad (159) \end{aligned}$$

$$\begin{aligned} \tilde{F}^{(L)} &:= \Delta^{(L-1)}\tilde{F}_1\\ &= \Delta^{(L-1)}(q^{J_1^0})\cdot\Delta^{(L-1)}(J_1^-)\\ &= q^{J_1^0}J_1^- + q^{2J_1^0+J_2^0}J_2^- + \ldots + q^{2\sum_{i=1}^{L-1}J_i^0+J_L^0}J_L^-\\ &= \tilde{F}_1 + \tilde{K}_1\tilde{F}_2 + \tilde{K}_1\tilde{K}_2\tilde{F}_3 + \ldots + \tilde{K}_1\cdot\ldots\cdot\tilde{K}_{L-1}\tilde{F}_L \qquad (160) \end{aligned}$$

are symmetries of $H_{(L)}$.

Remark 7.1. *Notice that $\tilde{E}_{(L)}$ (respectively $\tilde{F}_{(L)}$) is related to $F_{(L)}$ (respectively $E_{(L)}$) by a transposition. More precisely, using (90), one has*

$$\begin{aligned}
(\tilde{E}^{(L)})^* &= q^{-J_1^0} J_1^- q^{-2\sum_{i=2}^L J_i^0} + \cdots + q^{-J_{L-1}^0} J_{L-1}^- q^{-2J_L^0} + q^{-J_L^0} J_L^- \\
&= q\left(J_1^- q^{-J_1^0} q^{-2\sum_{i=2}^L J_i^0} + \cdots + J_{L-1}^- q^{-J_{L-1}^0} q^{-2J_L^0} + J_L^- q^{-J_L^0}\right) \\
&= qF^{(L)} \qquad (161)
\end{aligned}$$

$$\begin{aligned}
(\tilde{F}^{(L)})^* &= J_1^+ q^{J_1^0} + q^{2J_1^0} J_2^+ q^{J_2^0} + \ldots + q^{2\sum_{i=1}^{L-1} J_i^0} J_L^+ q^{J_L^0} \\
&= q^{-1}\left(q^{J_1^0} J_1^+ + q^{2J_1^0 + J_2^0} J_2^+ + \ldots + q^{2\sum_{i=1}^{L-1} J_i^0 + J_L^0} J_L^+\right) \\
&= q^{-1} E^{(L)} \qquad (162)
\end{aligned}$$

By exponentiating $\tilde{E}_{(L)}$ and $\tilde{F}_{(L)}$ the following two symmetries $\tilde{S}^+_{(L)}$ and $\tilde{S}^-_{(L)}$ are obtained.

Lemma 7.2. *The operator*

$$\tilde{S}^+_{(L)} := \exp_{q^2}(\tilde{E}^{(L)}) \qquad (163)$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$\langle \eta_1, ..., \eta_L | \tilde{S}^+_{(L)} | \xi_1, ..., \xi_L \rangle = \prod_{i=1}^L \sqrt{\binom{2j-\xi_i}{2j-\eta_i}_q \cdot \binom{\eta_i}{\xi_i}_q}\, q^{-(\eta_i - \xi_i)\left[2\sum_{k=i+1}^L (\eta_k - j) + \eta_i - j - 1\right]} \cdot \mathbf{1}_{\xi_i \le \eta_i} \qquad (164)$$

Proof. From (156) we know that the operators $\tilde{E}_i, \tilde{K}_i$, copies of the operators defined in (155), verify the conditions (109) with $r = q^2$. Then, from (159) and Proposition 5.1

$$\begin{aligned}
\tilde{S}^+_{(L)} &= \exp_{q^2}(\tilde{E}^{(L)}) \\
&= \exp_{q^2}(\tilde{E}_1 \tilde{K}_2^{-1} \ldots \tilde{K}_L^{-1}) \cdot \cdots \cdot \exp_{q^2}(\tilde{E}_{L-1}\tilde{K}_L^{-1}) \cdot \exp_{q^2}(\tilde{E}_L) \\
&= \exp_{q^2}\left(J_1^+ q^{-J_1^0 - 2\sum_{i=2}^L J_i^0}\right) \cdot \cdots \cdot \exp_{q^2}\left(J_{L-1}^+ q^{-J_{L-1}^0 - 2J_L^0}\right) \cdot \exp_{q^2}\left(J_L^+ q^{-J_L^0}\right) \\
&= \tilde{S}_1^+ \tilde{S}_2^+ \ldots \tilde{S}_L^+ \qquad (165)
\end{aligned}$$

where $\tilde{S}_i^+ := \exp_{q^2}\left(J_i^+ q^{-J_i^0 - 2\sum_{k=i+1}^L J_k^0}\right)$. Using (107), we have

$$\tilde{S}_i^+ |\xi_1, ..., \xi_L\rangle = \sum_{\ell_i \ge 0} \frac{1}{[\ell_i]_q!}\left(J_i^+ q^{-J_i^0 - 2\sum_{k=i+1}^L J_k^0}\right)^{\ell_i} q^{-\frac{1}{2}\ell_i(\ell_i - 1)} |\xi_1, ..., \xi_L\rangle \qquad (166)$$

$$= \sum_{\ell_i \ge 0} \sqrt{\binom{2j-\xi_i}{\ell_i}_q \cdot \binom{\xi_i + \ell_i}{\xi_i}_q}\, q^{-2\ell_i \sum_{k=i+1}^L (\xi_k - j)}\, q^{-\ell_i(\xi_i + \ell_i - j - 1)} |\xi_1, ..., \xi_i + \ell_i, ...\xi_L\rangle$$

then

$$\begin{aligned}\tilde S^+_{(L)}|\xi_1,...,\xi_L\rangle &= \tilde S^+_1\tilde S^+_2\dots\tilde S^+_L|\xi_1,...,\xi_L\rangle\\ &= \sum_{\ell_1,\ell_2,...,\ell_L\geq 0}\prod_{i=1}^L\Bigg(\sqrt{\binom{2j-\xi_i}{\ell_i}_q\cdot\binom{\xi_i+\ell_i}{\xi_i}_q}\\ &\qquad\cdot q^{-2\ell_i\sum_{k=i+1}^L(\xi_k+\ell_k-j)}\,q^{-\ell_i(\xi_i+\ell_i-j-1)}\Bigg)|\xi_1+\ell_1,...,\xi_L+\ell_L\rangle\end{aligned}$$

Hence the matrix elements of $\tilde S^+_{(L)}$ are given by (164). □

Lemma 7.3. *The operator*

$$\tilde S^-_{(L)} := \exp_{q^{-2}}(\tilde F^{(L)}) \tag{167}$$

is a symmetry of $H_{(L)}$. Its matrix elements are given by

$$\langle\eta_1,...,\eta_L|\tilde S^-_{(L)}|\xi_1,...,\xi_L\rangle = \prod_{i=1}^L\sqrt{\binom{\xi_i}{\eta_i}_q\binom{2j-\eta_i}{2j-\xi_i}_q}\cdot q^{(\xi_i-\eta_i)\left[2\sum_{k=1}^{i-1}(\xi_k-j)-\xi_i+1+j\right]}\cdot\mathbf{1}_{\eta_i\leq\xi_i} \tag{168}$$

Proof. From (156) we know that the operators $\tilde F_i, \tilde K_i$, copies of the operators defined in (155), verify the conditions (109) with $r = q^{-2}$. Then, from (159) and Proposition 5.1

$$\begin{aligned}\tilde S^-_{(L)} &= \exp_{q^{-2}}(\tilde F^{(L)})\\ &= \exp_{q^{-2}}(\tilde F_1)\cdot\exp_{q^{-2}}(\tilde K_1\tilde F_2)\cdot\dots\cdot\exp_{q^{-2}}(\tilde K_1\cdot\dots\cdot\tilde K_{L-1}\tilde F_L)\\ &= \exp_{q^{-2}}\left(q^{J_1^0}J_1^-\right)\cdot\exp_{q^{-2}}\left(q^{2J_1^0}q^{J_2^0}J_2^-\right)\cdot\dots\cdot\exp_{q^{-2}}\left(q^{2\sum_{i=1}^{L-1}J_i^0+J_L^0}J_L^-\right)\\ &= \tilde S^-_1\tilde S^-_2\dots\tilde S^-_L\end{aligned} \tag{169}$$

where $\tilde S^-_i := \exp_{q^{-2}}\left(q^{2\sum_{k=1}^{i-1}J_k^0+J_i^0}J_i^-\right)$. Using (107) and the fact that $[x]_{q^{-1}} = [x]_q$, we have

$$\tilde S^-_i|\xi_1,...,\xi_L\rangle = \sum_{\ell_i\geq 0}\frac{1}{[\ell_i]_q!}\left(q^{2\sum_{k=1}^{i-1}J_k^0+J_i^0}J_i^-\right)^{\ell_i}q^{\frac12\ell_i(\ell_i-1)}|\xi_1,...,\xi_L\rangle \tag{170}$$

$$= \sum_{\ell_i\geq 0}\sqrt{\binom{2j-\xi_i+\ell_i}{\ell_i}_q\cdot\binom{\xi_i}{\ell_i}_q}\cdot q^{\ell_i(1+j-\xi_i)+2\ell_i\sum_{k=1}^{i-1}(\xi_k-j)}|\xi_1,...,\xi_i-\ell_i,...\xi_L\rangle$$

then

$$\begin{aligned}\tilde S^-_{(L)}|\xi_1,...,\xi_L\rangle &= \tilde S^-_1\tilde S^-_2\dots\tilde S^-_L|\xi_1,...,\xi_L\rangle\\ &= \sum_{\ell_1,\ell_2,...,\ell_L\geq 0}\prod_{i=1}^L\Bigg(\sqrt{\binom{2j-\xi_i+\ell_i}{\ell_i}_q\cdot\binom{\xi_i}{\ell_i}_q}\\ &\qquad\cdot q^{\ell_i(1+j-\xi_i)+2\ell_i\sum_{k=1}^{i-1}(\xi_k-j)}\Bigg)|\xi_1-\ell_1,...,\xi_L-\ell_L\rangle\end{aligned}$$

Hence the matrix elements of $\tilde{S}^-_{(L)}$ are given by (168). $\square$

As it was done with the ground state $S^+_{(L)}|0,\ldots,0\rangle$, one could wonder what Markov process is obtained if one uses the ground state $\tilde{S}^+_{(L)}|0,\ldots,0\rangle$. One can check by an explicit computation (not reported here) that if $H_{(L)}$ is transformed by a similarity transformation $\tilde{G}_{(L)}$ given by

$$\tilde{G}_{(L)}|\eta_1,\ldots,\eta_L\rangle = |\eta_1,\ldots,\eta_L\rangle\langle\eta_1,\ldots,\eta_L|\tilde{S}^+_{(L)}|0,\ldots,0\rangle \tag{171}$$

one recovers the ASEP(q,j) Markov jump process.

7.2 Construction of alternative self-duality functions

One can wonder what other dualities are found using the other symmetries of the previous Section. Using $S^-_{(L)}$ one finds a duality function which is the transpose of (34). In the same way $\tilde{S}^+_{(L)}$ and $\tilde{S}^-_{(L)}$ give duality functions that are related by a transposition. Such duality function is different from (34) and is given by (35) that we are going to prove below.

Proof of (35) **in Theorem 3.2**. From Proposition 4.1 we know that $H_{(L)}$ is self-adjoint, then, using Proposition 2.1 with $A = H_{(L)}$, $G = G_{(L)}$ given by (139) and $S = \tilde{S}^-_{(L)}$ given by (134) it follows that

$$G^{-1}_{(L)}\tilde{S}^-_{(L)}G^{-1}_{(L)} \tag{172}$$

is a self-duality function for the process generated by $\mathscr{L}^{(L)}$. Its elements are computed as follows:

$$\langle\eta|G^{-1}_{(L)}\tilde{S}^-_{(L)}G^{-1}_{(L)}|\xi\rangle = \tag{173}$$

$$= \prod_{i=1}^{L}\left(\sqrt{\binom{2j}{\eta_i}_q}\cdot q^{\eta_i(1+j-2ji)}\right)^{-1}\langle\eta|\tilde{S}^-_i|\xi\rangle\left(\sqrt{\binom{2j}{\xi_i}_q}\cdot q^{\xi_i(1+j-2ji)}\right)^{-1} = \tag{174}$$

$$= \prod_{i=1}^{L}\sqrt{\binom{\xi_i}{\eta_i}_q\binom{2j-\eta_i}{2j-\xi_i}_q\Big/\binom{2j}{\eta_i}_q\binom{2j}{\xi_i}_q}\cdot q^{(\xi_i-\eta_i)\left[2\sum_{k=1}^{i-1}(\xi_k-j)-\xi_i\right]+(2ji-j-1)(\eta_i+\xi_i)}\cdot\mathbf{1}_{\eta_i\le\xi_i} =$$

$$= q^{\sum_{i=1}^{L}((j-1)\xi_i-(3j+1)\eta_i)}\prod_{i=1}^{L}\frac{[2j-\eta_i]_q![\xi_i]_q!}{[2j]_q![\xi_i-\eta_i]_q!}\cdot q^{(\xi_i-\eta_i)\left[2\sum_{k=1}^{i-1}\xi_k-\xi_i\right]+4ji\eta_i}\cdot\mathbf{1}_{\eta_i\le\xi_i}$$

Since both the original process and the dual process conserve the total number of particles it follows that $D'_{(L)}$ in (35) is also a duality function. $\square$

7.3 Comparison with the Schütz duality in the case $j = 1/2$.

Consider the duality matrix D' computed in (35), then the associated duality function is

$$D'_{(L)}(\eta,\xi) = \prod_{i=1}^{L}\frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q}\cdot q^{(\eta_i-\xi_i)\left[2\sum_{k=1}^{i-1}\eta_k-\eta_i\right]+4ji\xi_i}\cdot\mathbf{1}_{\xi_i\le\eta_i}$$

For $j = 1/2$ both ξ_i and η_i take values in $\{0, 1\}$ then

$$\eta_i^2 \equiv \eta_i \qquad \text{and for } \xi_i \leq \eta_i, \qquad \xi_i \eta_i \equiv \xi_i \tag{175}$$

hence, assuming that $\xi_i \leq \eta_i$ for all i, we have

$$\sum_{i=1}^{L} (\eta_i - \xi_i)\eta_i = \sum_{i=1}^{L} \eta_i^2 - \sum_{i=1}^{L} \xi_i \eta_i = \sum_{i=1}^{L} \eta_i - \sum_{i=1}^{L} \xi_i = N - M$$

where N and M are the total numbers of particles respectively in the configurations η and ξ. Thus

$$\prod_{i=1}^{L} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{2j}{\xi_i}_q} \cdot q^{-(\eta_i - \xi_i)\eta_i} \cdot \mathbf{1}_{\xi_i \leq \eta_i} = q^{-\sum_{i=1}^{L} (\eta_i - \xi_i)\eta_i} \cdot \prod_{i=1}^{L} \mathbf{1}_{\xi_i \leq \eta_i} = c \cdot \mathbf{1}_{\{\xi_i \leq \eta_i, \forall i\}}$$

On the other hand, assuming that $\xi_i \leq \eta_i$, we have

$$\eta_i - \xi_i = \mathbf{1}_{\eta_i = 1, \xi_1 = 0}, \qquad \text{then} \qquad \prod_{i=1}^{L} q^{2(\eta_i - \xi_i) \sum_{k=1}^{i-1} \eta_k} = \prod_{i: \eta_i = 1, \xi_i = 0} q^{2 \sum_{k=1}^{i-1} \eta_k}$$

then, for $j = 1/2$

$$D'(\eta, \xi) = c \cdot \mathbf{1}_{\{\xi_i \leq \eta_i, \forall i\}} \cdot q^{2 \sum_{i=1}^{L} i \xi_i} \prod_{i: \eta_i = 1, \xi_i = 0} q^{2 \sum_{k=1}^{i-1} \eta_k}$$

Now, using the Schütz notation, one may represent a given M-particles configuaration by the set C of occupied sites. More precisely, let M be the total number of the configuartion ξ, we denote by $C := \{k_1, \ldots, k_M\}$ the set of occupies sites $k_i \in \{1, \ldots, L\}$ $k_i \leq k_{i+1}$. With this notation we have

$$\sum_{i=1}^{L} i \xi_i = \sum_{m=1}^{M} k_m$$

On the other hand, for the configuration η we denote by N_i, $i = 1, \ldots, L$ the number of particles at the left of i (with site i included):

$$N_i := \sum_{k=1}^{i} \eta_k$$

With this notation we have

$$D'_{(L)}(\eta, \xi) = c \cdot \mathbf{1}_{\{\xi_i \leq \eta_i, \forall i\}} \cdot q^{2 \sum_{m=1}^{M} k_m} \; q^{2 \sum_{i: \eta_i = 1, \xi_i = 0} N_{i-1}} \tag{176}$$

Now, assuming that $\xi_i \leq \eta_i$ for all i, we have

$$\sum_{i: \eta_i = 1, \xi_i = 0} N_{i-1} = \sum_{i: \eta_i = 1} N_{i-1} - \sum_{i: \eta_i = 1, \xi_i = 1} N_{i-1} \tag{177}$$

Let now N be the total number of particles in the configuration η, then we prove that

$$\sum_{i:\eta_i=1} N_{i-1} = \frac{N(N-1)}{2} \tag{178}$$

We have

$$\sum_{i:\eta_i=1} N_{i-1} = \sum_{i:\eta_i=1} \eta_i N_{i-1} = \sum_{i:\eta_i=1} \sum_{k=1}^{i-1} \eta_i \eta_k$$

On the other hand

$$\begin{aligned} N^2 &= \left(\sum_{i=1}^{L} \eta_i \right)^2 = \sum_{i=1}^{L} \sum_{k=1}^{L} \eta_i \eta_k \\ &= \sum_{i=1}^{L} \sum_{k=1}^{i-1} \eta_i \eta_k + \sum_{i=1}^{L} \eta_i^2 + \sum_{i=1}^{L} \sum_{k=i+1}^{L} \eta_i \eta_k \\ &= 2 \sum_{i=1}^{L} \sum_{k=1}^{i-1} \eta_i \eta_k + N \end{aligned}$$

where the last identity follows because

$$\sum_{i=1}^{L} \sum_{k=1}^{i-1} \eta_i \eta_k = \sum_{i=1}^{L} \sum_{k=i+1}^{L} \eta_i \eta_k$$

and since , from the left identity in (175),

$$\sum_{i=1}^{L} \eta_i^2 = \sum_{i=1}^{L} \eta_i = N$$

then (178) is proved. On the other hand, from the right identity in (175) we have

$$\begin{aligned} \sum_{i:\eta_i=1,\xi_i=1} N_{i-1} &= \sum_{i=1}^{L} \eta_i \xi_i \sum_{k=1}^{i-1} \eta_k \\ &= \sum_{i=1}^{L} \xi_i \sum_{k=1}^{i-1} \eta_k \\ &= \sum_{m=1}^{M} \sum_{k=1}^{k_m-1} \eta_k \\ &= \sum_{m=1}^{M} N_{k_m-1} \end{aligned} \tag{179}$$

Finally from (177), (178) and (179) we have

$$\sum_{i:\eta_i=1,\xi_i=0} N_{i-1} = \frac{N(N-1)}{2} - \sum_{m=1}^{M} N_{k_m-1} \tag{180}$$

Finally we have that $\xi_i \leq \eta_i$ for all i if and only if all the sites $\{k_1, \dots, k_M\}$ are occupied sites for the configuration η, then from (176) and (180) we have

$$\begin{aligned} D'({}_{(L)}\eta, \xi) &= c' \cdot \mathbf{1}_{\{\xi_i \leq \eta_i, \forall i\}} \cdot q^{2\sum_{m=1}^{M} k_m} \; q^{-2\sum_{m=1}^{M} N_{k_m - 1}} \\ &= c' \cdot \prod_{m=1}^{M} q^{2k_m} \; q^{-2N_{k_m-1}} \cdot \eta_{k_m} \end{aligned} \tag{181}$$

that is the Schütz self-duality function (up to a sign, i.e. q^{2k_m} instead of q^{-2k_m}).

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References

[1] C. Bahadoran, H. Guiol, K. Ravishankar, E. Saada, Strong hydrodynamic limit for attractive particle systems on $\mathbb{Z}$ Electronic Journal of Probability, **15**, 1–43 (2010)

[2] A. Borodin, I. Corwin, T. Sasamoto From duality to determinants for q-TASEP and ASEP, preprint arXiv:1207.5035 (2012).

[3] A. Bytsko, On integrable Hamiltonians for higher spin XXZ chain, Journal of Mathematical Physics 44, 3698 (2003).

[4] P. Caputo, Energy gap estimates in XXZ ferromagnets and stochastic particle systems Markov Process. Related Fields 11, 189–210 (2005).

[5] G. Carinci, C. Giardinà, C. Giberti, F. Redig, Dualities in population genetics: a fresh look with new dualities preprint arXiv:1302.3206 (2013).

[6] Cocozza-Thivent, Christiane Processus des misanthropes. (French) [Misanthropic processes] Z. Wahrsch. Verw. Gebiete **70** , 509–523 (1985).

[7] I. Corwin, Two ways to solve ASEP, preprint arXiv:1212.2267 (2012).

[8] J. Feng and T. G. Kurtz. Large Deviations for Stochastic Processes. American Mathematical Society (2006).

[9] C. Giardinà, J. Kurchan, F. Redig, K. Vafayi, Duality and hidden symmetries in interacting particle systems, Journal of Statistical Physics, **135**, 25–55 (2009).

[10] C. Giardinà, F. Redig, K. Vafayi, Correlation inequalities for interacting particle systems with duality, Journal of Statistical Physics, **141**, 242–263 (2010).

[11] B. C. Hall, Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Springer, Graduate Texts in Mathematics 222 (2003).

[12] F. den Hollander, *Large deviations* (Vol. 14), American Mathematical Society (2008).

[13] T.M. Liggett, *Interacting particle systems*, Springer (2005).

[14] G. Lusztig, Introduction to Quantum Groups. Cambridge MA, Birkhäuser (2010).

[15] C. Matsui, Multi-state asymmetric simple exclusion processes, preprint http://arxiv.org/pdf/1311.7473

[16] B. Nachtergaele, W. Spitzer, S. Starr Ferromagnetic Ordering of Energy Levels for $U_q(\mathfrak{sl}_2)$ Symmetric Spin Chains. Letters in Mathematical Physics, 100(3), 327–356 (2012).

[17] J.D. Keisling, An ergodic theorem for the symmetric generalized exclusion process, Markov Processes and Related Fields **4**, 351–379 (1998).

[18] Z. Palmowski, T. Rolski. A technique for exponential change of measure for Markov processes. Bernoulli, **8**, 767–785 (2002).

[19] L. Fajfrova, T. Gobron, E. Saada, Invariant measures for mass migration processes, *preprint* (2014).

[20] G.M. Schütz, Duality relations for asymmetric exclusion processes, Journal of statistical physics **86**, 1265–1287 (1997).

[21] G. Schütz, S. Sandow, Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems", Phys. Rev. E **49**, 2726 (1994).

[22] B. Simon, Functional integration and quantum mechanics, Academic Press, (1979).

[23] T. Seppäläinen, Existence of hydrodynamics for the totally asymmetric simple K-exclusion process, The Annals of Probability, **27**, 361–415 (1999).