A non-existence result on cyclic cycle-decompositions of the cocktail party graph[☆]

Marco Buratti [a], Gloria Rinaldi [b,*]

[a] Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli 1, I-06123 Perugia, Italy

[b] Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia, Viale Amendola 2, I-42100 Reggio Emilia, Italy

Dedicated to Anthony Hilton

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ABSTRACT

We prove that in every cyclic cycle-decomposition of $K_{2n} - I$ (the cocktail party graph of order $2n$) the number of cycle-orbits of odd length must have the same parity of $n(n-1)/2$. This gives, as corollaries, some useful non-existence results one of which allows to determine when the two table Oberwolfach Problem $OP(3, 2\ell)$ admits a 1-rotational solution.

1. Introduction

Throughout the paper K_v and C_ℓ will denote, as usual, the *complete graph of order* v and the ℓ*-cycle*, respectively. Also, the ℓ-cycle whose edges are $[a_0, a_1], [a_1, a_2], \dots, [a_{\ell-1}, a_0]$ will be denoted by $(a_0, a_1, \dots, a_{\ell-1})$.

We recall that the *circulant graph* of order v and connection set Ω is the *Cayley graph* Cay$[Z_v : \Omega]$, namely the simple graph with vertex-set Z_v and edge-set E defined by $[x, y] \in E$ if and only if $x - y \in \Omega$. Of course Ω must be a subset of $Z_v - \{0\}$ with the property that $-\omega \in \Omega$ for every $\omega \in \Omega$.

A *cycle-decomposition* of a graph K is a set $\mathcal{D}$ of subcycles of K whose edges partition $E(K)$. If all cycles of $\mathcal{D}$ have the same length ℓ one also says that $\mathcal{D}$ is a C_ℓ*-decomposition of* K or a (K, C_ℓ)*-design* or an ℓ*-cycle system of* K.

An r*-factorization* of a graph K is a set of r-factors of K (namely, r-regular spanning subgraphs of K) whose edges partition $E(K)$. So, in particular, a 2-factorization of K is a cycle-decomposition of K whose cycles have been arranged into 2-factors. Obviously, different 2-factorizations of K could have the same underlying cycle-decomposition.

A solution for the Oberwolfach problem $OP(\ell_1, \ell_2, \dots, \ell_r)$ is a 2-factorization of the complete graph $K_{\ell_1+\ell_2+\dots+\ell_r}$ whose 2-factors are all isomorphic to the graph $C_{\ell_1} \cup C_{\ell_2} \cup \dots \cup C_{\ell_r}$.

For cycle-decompositions and factorizations of graphs in general, we refer to [9,1], respectively. Here we are interested in *cyclic* cycle-decompositions and in 1-*rotational* 2-factorizations.

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* Corresponding author.
E-mail addresses: buratti@mat.uniroma1.it (M. Buratti), gloria.rinaldi@unimore.it (G. Rinaldi).

A cycle-decomposition $\mathcal{D}$ of a circulant graph $K = \mathrm{Cay}[Z_v : \Omega]$ is said to be *cyclic* if for every $A \in \mathcal{D}$ we also have $A + 1 \in \mathcal{D}$ where $A + 1$ denotes the cycle obtainable from A by replacing each vertex a of it with the vertex $a + 1 \pmod v$.

Similarly, a 2-factorization $\mathcal{F}$ of the graph $\{\infty\} + \mathrm{Cay}[Z_v : \Omega]$ (where ∞ is an element not belonging to Z_v) is *1-rotational* if $F + 1 \in \mathcal{F}$ for every $F \in \mathcal{F}$. Of course $F + 1$ is the r-factor obtainable from F by replacing each vertex $x \neq \infty$ with the vertex $x + 1 \pmod v$.

We are still quite far from the complete solution of the existence problem for cyclic ℓ-cycle decompositions of the complete graph. Partial answers have been given by several authors [3,4,6,7,11,12,15,18–20,22] and the few known non-existence results can be summarized as follows:

- [10] There is no cyclic (K_9, C_3)-design.
- [7] There is no cyclic (K_ℓ, C_ℓ)-design if $\ell = 15$ or $\ell = p^\alpha$ with p an odd prime and $\alpha > 1$.
- [5,12] There is no cyclic (K_v, C_ℓ)-design with $\gcd(v, \ell)$ a prime power and $\ell < v < 2\ell$.

Indeed it is conjectured that for all other admissible pairs (v, ℓ), a cyclic (K_v, C_ℓ)-design exists.

In this paper we present a necessary condition for the existence of cyclic cycle-decompositions of the *cocktail party graph* $K_{2n} - I$, that is the complete graph of order $2n$ with one 1-factor I removed. Using this condition we show, in particular, that there are infinitely many classes of admissible pairs $(2n, \ell)$ for which a cyclic $(K_{2n} - I, C_\ell)$-design does not exist. The same condition also allows us to determine the spectrum of values of ℓ for which there exists a 1-rotational solution for $OP(3, 2\ell)$.

2. The main result

From now on K_{2n} will be always seen as the circulant graph $\mathrm{Cay}[Z_{2n} : Z_{2n} - \{0\}]$. Observe that $\mathrm{Cay}[Z_{2n} : \{n\}]$ is a 1-factor of K_{2n}. Thus we will always represent the cocktail party graph $K_{2n} - I$ of order $2n$ as the circulant graph $\mathrm{Cay}[Z_{2n} : Z_{2n} - \{0, n\}]$. Note that the orbit of every edge $[x, y]$ of $K_{2n} - I$ under the natural action of Z_{2n}, denoted by $\mathrm{Orb}[x, y]$, has full length $2n$ and that it is the edge-set of the circulant graph $\mathrm{Cay}[Z_{2n} : \{x - y, y - x\}]$. For a given subcycle A of K_{2n} we denote by $\mathrm{Stab}(A)$ and $\mathrm{Orb}(A)$ the stabilizer and the orbit of A under Z_{2n}, respectively.

Although the following lemma can be easily deduced from [5], we will prove it in all details for convenience of the reader.

Lemma 2.1. *Let $A = (a_0, a_1, \ldots, a_{\ell-1})$ be a cycle belonging to a cyclic cycle-decomposition of $K_{2n} - I$ and let t be the order of* $\mathrm{Stab}(A)$. *Then* $\mathrm{Orb}(A)$ *is an ℓ-cycle decomposition of* $\mathrm{Cay}[Z_{2n} : \{\pm(a_{i-1} - a_i) \mid 1 \leq i \leq \frac{\ell}{t}\}]$.

Proof. Observe that $\mathrm{Stab}(A)$ can be viewed as a group of automorphisms of A. Hence, since the full automorphism group of an ℓ-cycle is $D_{2\ell}$, the dihedral group of order 2ℓ, we deduce that $\mathrm{Stab}(A)$ is isomorphic to a subgroup of $D_{2\ell}$. Assume that there exists an element of $\mathrm{Stab}(A)$ acting on A as a *reflection*. Then this element must be the only involution of Z_{2n}, moreover, it acts fixed-point free on A, so A has even length, say $\ell = 2k$, and we have

$$A = (a_0, a_1, \ldots, a_{k-1}, a_{k-1} + n, \ldots, a_1 + n, a_0 + n).$$

This is absurd since A would contain the edge $[a_0, a_0 + n]$ belonging to the 1-factor $I = \mathrm{Cay}[Z_{2n} : \{n\}]$.

Hence, $\mathrm{Stab}(A)$ is a group of rotations of order t. Thus, if ρ is a generator of $\mathrm{Stab}(A)$, we have that the map $\hat{\rho} : a_i \longrightarrow a_i + \rho$ is the map sending each vertex a_i into the vertex $a_{i+\ell/t \pmod \ell}$:

$$a_{i+\ell/t \pmod \ell} = a_i + \rho \quad \text{for } i = 0, 1, \ldots, \ell - 1. \tag{1}$$

This means, explicitly, that A has the following form:

$$(a_0, a_1, \ldots, a_{\ell/t-1}, a_0 + \rho, a_1 + \rho, \ldots, a_{\ell/t-1} + \rho, a_0 + 2\rho, a_1 + 2\rho, \ldots, a_{\ell/t-1} + 2\rho, \ldots,$$
$$a_0 + (t - 1)\rho, a_1 + (t - 1)\rho, \ldots, a_{\ell/t-1} + (t - 1)\rho).$$

Let $\overline{A}$ be the graph whose edges are precisely those covered by the cycles of $\mathrm{Orb}(A)$ and set

$$\partial(A) = \left\{\pm(a_{i-1} - a_i) \mid 1 \leq i \leq \frac{\ell}{t}\right\}.$$

We have to prove that $\overline{A} = \mathrm{Cay}[Z_{2n} : \partial(A)]$.

For $|\mathrm{Stab}(A)| = t$ we have $|\mathrm{Orb}(A)| = 2n/t$ and hence, since the cycles of $\mathrm{Orb}(A)$ belong to the cyclic cycle-decomposition, they are edge-disjoint and we have $|E(\overline{A})| = 2n\ell/t$. Also, for the same reason, the set $E(\overline{A})$ is a disjoint union of orbits of edges of A. Their number is then given by $\frac{2n\ell}{t} \cdot \frac{1}{2n} = \frac{\ell}{t}$ since, as previously observed, any edge-orbit of $K_{2n} - I$ has full length $2n$. Let us prove that such a disjoint union can be written as follows:

$$E(\overline{A}) = \mathrm{Orb}[a_0, a_1] \cup \mathrm{Orb}[a_1, a_2] \cup \cdots \cup \mathrm{Orb}[a_{\ell/t-1}, a_{\ell/t}].$$

By the "pigeon-hole principle" it is enough to show that every edge of A belongs to the orbit of an edge of the path $P = (a_0, a_1, \ldots, a_{\ell/t})$. In fact, given $[a_{i-1}, a_i] \in E(A)$, let

$$i = q \cdot \frac{\ell}{t} + r \quad 0 \leq r < \frac{\ell}{t}$$

be the Euclidean division of i by $\frac{\ell}{t}$. By (1), we have $a_i = a_r + q\rho$ and $a_{i-1} = a_{r-1} + q\rho$ so that $[a_{i-1}, a_i] = [a_{r-1}, a_r] + q\rho$ is in the same orbit of $[a_{r-1}, a_r] \in \vec{E}(P)$.

Then, having just observed that $\mathrm{Orb}[a_{i-1}, a_i]$ is the edge-set of $\mathrm{Cay}[Z_{2n} : \{a_{i-1} - a_i, a_i - a_{i-1}\}]$, we can write:

$$\bar{A} = \bigcup_{i=1}^{\ell/t} \mathrm{Cay}[Z_{2n} : \{a_{i-1} - a_i, a_i - a_{i-1}\}] = \mathrm{Cay}[Z_{2n} : \{\pm(a_i - a_{i-1}) \mid 1 \le i \le \ell/t\}] = \mathrm{Cay}[Z_{2n} : \partial(A)]$$

and the assertion follows. □

We are now ready for proving our main result.

Theorem 2.2. *The number of cycle-orbits of odd length in a cyclic cycle-decomposition of $K_{2n} - I$ has the same parity as $n(n-1)/2$.*

Proof. Let $\mathcal{D}$ be a cyclic cycle-decomposition of $K_{2n} - I$. For every cycle $A = (a_0, a_1, \ldots, a_{\ell-1})$ of $\mathcal{D}$ set

$$\sigma(A) = \sum_{i=1}^{\ell/t} (a_{i-1} - a_i) = (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{\ell/t-1} - a_{\ell/t})$$

where t is the order of $\mathrm{Stab}(A)$.

Obviously, we have $\sigma(A) = a_0 - a_{\ell/t}$. On the other hand, by (1), we have $a_{\ell/t} = a_0 + \rho$ where ρ is an element of Z_{2n} of order t so that we have

$$\sigma(A) = \frac{2nx}{t} \quad \text{with } \gcd(x, t) = 1.$$

This implies that $\sigma(A)$ is even if and only if t is a divisor of n.

Now note that the length of $\mathrm{Orb}(A)$ is $2n/t$ so that, also here, $|\mathrm{Orb}(A)|$ is even if and only if t is a divisor of n. We conclude that $\sigma(A)$ has the same parity as $|\mathrm{Orb}(A)|$:

$$\sigma(A) \equiv |\mathrm{Orb}(A)| \pmod 2 \ \forall A \in \mathcal{D}. \tag{2}$$

Let $\mathcal{S} = \{A_1, \ldots, A_s\}$ be a set of *base-cycles* of $\mathcal{D}$, namely a complete system of representatives for the orbits of the cycles of $\mathcal{D}$ so that we have

$$\mathcal{D} = \mathrm{Orb}(A_1) \cup \mathrm{Orb}(A_2) \cup \cdots \cup \mathrm{Orb}(A_s).$$

By Lemma 2.1, the cycles of $\mathrm{Orb}(A_i)$ form a decomposition of $\mathrm{Cay}[Z_{2n} : \partial(A_i)]$ so that we have

$$\mathrm{Cay}[Z_{2n} : Z_{2n} - \{0, n\}] = \bigcup_{i=1}^{s} \mathrm{Cay}[Z_{2n} : \partial(A_i)] = \mathrm{Cay}\left[Z_{2n} : \bigcup_{i=1}^{s} \partial(A_i)\right]$$

which implies that

$$\bigcup_{i=1}^{s} \partial(A_i) = Z_{2n} - \{0, n\}. \tag{3}$$

Now note that $\partial(A_i)$ is a disjoint union of the set of addends of $\sigma(A_i)$ and the set of their opposites. It follows, by (3), that $Z_{2n} - \{0, n\}$ is a disjoint union of the set of all addends of the sum $\sum_{i=1}^{s} \sigma(A_i)$ and the set of all their opposites. Thus, having $Z_{2n} - \{0, n\} = \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$, we can write:

$$\sum_{i=1}^{s} \sigma(A_i) = s_1 + s_2 + \cdots + s_{n-1}$$

where $s_i = i$ or $-i$ for each i. So, since i and $-i$ have the same parity, we have:

$$\sum_{i=1}^{s} \sigma(A_i) \equiv 1 + 2 + \cdots + (n-1) \pmod 2.$$

Using (2) the above congruence can be rewritten as

$$\sum_{i=1}^{s} |\mathrm{Orb}(A_i)| \equiv \frac{n(n-1)}{2} \pmod 2.$$

This means that the number of cycles A_i of $\mathcal{S}$ whose orbit has odd length has the same parity as $n(n-1)/2$ and the assertion follows. □

3. Some non-existence conditions

In this section the complete graph K_{2n+1} will be seen as the graph $\{\infty\} + \mathrm{Cay}[Z_{2n} : Z_{2n} - \{0\}]$.

We need this special case of a more general theorem given in [8]:

Theorem 3.1. *Let $\ell_1, \ldots, \ell_t$ be integers greater than 2 with $\sum_{i=1}^{t} \ell_i = 2n + 1$. Then $\mathcal{F}$ is a 1-rotational solution for $OP(\ell_1, \ell_2, \ldots, \ell_t)$ if and only if we have $\mathcal{F} = \mathrm{Orb}(F)$ where F is a 2-factor of K_{2n+1} having the following properties:*

- $F \simeq C_{\ell_1} \cup C_{\ell_2} \cup \cdots \cup C_{\ell_t}$;
- $\mathrm{Stab}(F) = \{0, n\}$;
- *every non-zero element of Z_{2n} can be expressed as a difference of two adjacent vertices of F.*

Now, applying Theorem 2.2 together with the above theorem we get a non-existence result on 1-rotational solutions for Oberwolfach problems in which one parameter is 3 and all the remaining ones are even.

Theorem 3.2. *Let $n = k_1 + k_2 + \cdots + k_r + 1$ with $k_i \geq 2$ for each i. Then a 1-rotational solution for $OP(3, 2k_1, 2k_2, \ldots, 2k_r)$ cannot exist in each of the following cases:*

- $n \equiv 2 \pmod 4$;
- $\frac{n-1}{2} + r$ *is an odd integer.*

Proof. Assume that $\mathcal{F}$ is a 1-rotational solution for $OP(3, 2k_1, 2k_2, \ldots, 2k_r)$. By Theorem 3.1 we have $\mathcal{F} = \mathrm{Orb}(F)$ where F is a 2-factor of K_{2n+1} isomorphic to $C_3 \cup C_{2k_1} \cup \cdots \cup C_{2k_r}$ that is fixed by n. For $F + n = F$ we have that the stabilizer of every cycle of F is either trivial or $\{0, n\}$. This easily implies that the 3-cycle T of F is of the form $(t, \infty, t + n)$ for a suitable $t \in Z_{2n}$ while the remaining cycles of F can be split into two sets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ is the set of cycles of F on which n acts as a rotation and where $\mathcal{B}$ is the set of cycles of F with trivial stabilizer. It is then obvious that $\mathcal{B}$ can be written as a disjoint union $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$ with $\mathcal{B}'' = \{B + n \mid B \in \mathcal{B}'\}$.

Note that the edges of K_{2n+1} that are covered by the orbit of T are precisely those through ∞ plus those of the circulant graph $\mathrm{Cay}[Z_{2n} : \{n\}]$. It follows that the cycles of $\mathcal{F}$ not belonging to $\mathrm{Orb}(T)$ form a cyclic cycle-decomposition $\mathcal{D}$ of $\mathrm{Cay}[Z_{2n} : Z_{2n} - \{0, n\}]$, namely a cyclic cycle-decomposition of $K_{2n} - I$. Also note that $\mathcal{A} \cup \mathcal{B}'$ is a set of base-cycles for $\mathcal{D}$. Then, considering that the orbits of the cycles of $\mathcal{A}$ have length n while the orbits of the cycles of $\mathcal{B}'$ have length $2n$, the number of cycle-orbits of $\mathcal{D}$ having odd length is 0 or $|\mathcal{A}|$ according to whether n is even or odd, respectively. On the other hand we have $|\mathcal{A}| + |\mathcal{B}'| + |\mathcal{B}''| = r$ and $|\mathcal{B}'| = |\mathcal{B}''|$, so that $|\mathcal{A}|$ has the same parity as r. Thus, by Theorem 2.2, we can write:

$$\frac{n(n-1)}{2} \underset{(\mathrm{mod}\ 2)}{\equiv} \begin{cases} 0 & \text{if } n \text{ is even;} \\ r & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

If n is even, (4) immediately gives $n \equiv 0 \pmod 4$. If n is odd, then $\frac{n(n-1)}{2}$ has the same parity as $\frac{n-1}{2}$ and hence (4) gives $\frac{n-1}{2} \equiv r \pmod 2$, i.e., $\frac{n-1}{2} + r$ is even. The assertion follows. □

It is known that the obvious necessary condition for the existence of an ℓ-cycle decomposition of $K_{2n} - I$, namely $2n(n-1) \equiv 0 \pmod{\ell}$, is also sufficient [2,21]. Now we see, as another consequence of Theorem 2.2, that this is not always true if we ask the decomposition to be cyclic.

Theorem 3.3. *A cyclic ℓ-cycle decomposition of $K_{2n} - I$ cannot exist in each of the following cases:*

- $n \equiv 2$ *or* $3 \pmod 4$ *and* $\ell \not\equiv 0 \pmod 4$;
- $n \equiv 0$ *or* $1 \pmod 4$ *and* ℓ *does not divide* $n(n-1)$.

Proof. The number of cycles of an ℓ-cycle decomposition of $K_{2n} - I$ is obviously given by $|E(K_{2n} - I)|/\ell = 2n(n-1)/\ell$. It follows that the number of cycle-orbits of odd length of a cyclic ℓ-cycle decomposition of $K_{2n} - I$ has the same parity as $2n(n-1)/\ell$. So, by Theorem 2.2, we have $2n(n-1)/\ell \equiv n(n-1)/2 \pmod 2$. The assertion easily follows. □

4. On the existence of a 1-rotational solution for $OP(3, 2\ell)$

Every *two table Oberwolfach problem*, i.e., every $OP(2h+1, 2\ell)$, has been solved as a consequence of more general results given in [14]. Nevertheless, as far as we are aware, the spectrum of values $2h + 1$ and ℓ for which a 1-rotational solution for $OP(2h + 1, 2\ell)$ exists has not been established yet. In this section, using *graceful labelings* of a cycle we are able to say when $OP(3, 2\ell)$ admits a 1-rotational solution.

We recall that a *graceful labeling* of C_ℓ is an ℓ-cycle $\Gamma = (a_0, a_1, \ldots, a_{\ell-1})$ with vertices in the set of integers $\{0, 1, 2, \ldots, \ell\}$ and the property that

$$\{|a_i - a_{i-1}| \mid 1 \leq i \leq \ell\} = \{1, 2, \ldots, \ell\}$$

where $a_\ell = a_0$ is understood. In [17] Rosa proved

Theorem 4.1. *There exists a graceful labeling of C_ℓ if and only if $\ell \equiv 0$ or $3 \pmod 4$.*

For graceful labelings of arbitrary graphs we refer to the rich survey of Gallian [13]. Now we see how the above theorem of Rosa and our Theorem 3.2 allow us to completely determine the spectrum of values of ℓ for which there exists a 1-rotational solution for $OP(3, 2\ell)$.

Theorem 4.2. *There exists a 1-rotational solution for $OP(3, 2\ell)$ if and only if $\ell \equiv 2$ or $3 \pmod 4$.*

Proof. Applying Theorem 3.2 we can see that a 1-rotational solution for $OP(3, 2\ell)$ cannot exist for $\ell \equiv 0$ or $1 \pmod 4$.

Now assume $\ell \equiv 2$ or $3 \pmod 4$. In this case we have $\ell + 1 \equiv 3$ or $0 (\mathrm{mod}\ 4)$ and hence, by Theorem 4.1, there exists a graceful labeling $\Gamma = (a_0, a_1, \ldots, a_\ell)$ of $C_{\ell+1}$. By the definition of a graceful labeling, it is obvious that $[0, \ell + 1]$ is an edge of Γ so that we can assume, without loss of generality, that $a_0 = 0$ and $a_\ell = \ell + 1$. It is also obvious that $\{1, 2, \ldots, \ell\} - \{a_1, \ldots, a_{\ell-1}\}$ is a singleton $\{t\}$. Consider the triangle $T = (t, \infty, t + \ell + 1)$, the 2ℓ-cycle $A = (a_0, a_1, \ldots, a_\ell, a_1 + \ell + 1, a_2 + \ell + 1, \ldots, a_{\ell-1} + \ell + 1)$, and set $F = T \cup A$. Thinking of the vertices of $F - \{\infty\}$ as elements of $Z_{2+2\ell}$ rather than integers, it is easy to see that F is a 2-factor of $K_{3+2\ell}$ and that it satisfies the hypothesis of Theorem 3.1 so that Orb(F) is a 1-rotational solution for $OP(3, 2\ell)$. □

Example 4.3. If we apply Theorem 4.2 using the graceful labeling (0, 1, 3) of C_3, we get the following 1-rotational solution for $OP(3, 4)$:

$$\begin{array}{ll}(2,\infty,5) & (0,1,3,4)\\ (3,\infty,0) & (1,2,4,5)\\ (4,\infty,1) & (2,3,5,0).\end{array}$$

Example 4.4. If we apply Theorem 4.2 using the graceful labeling (0, 3, 2, 4) of C_4, we get the following 1-rotational solution for $OP(3, 6)$:

$$\begin{array}{ll}(1,\infty,5) & (0,3,2,4,7,6)\\ (2,\infty,6) & (1,4,3,5,0,7)\\ (3,\infty,7) & (2,5,4,6,1,0)\\ (4,\infty,0) & (3,6,5,7,2,1).\end{array}$$

For its connection with this subject, we recall that Ollis [16] has found 1-rotational solutions to the three table Oberwolfach Problem $OP(r, r, 2s + 1)$ for several values of r and s. We point out, however, that he calls a *cyclic solution* what we call a 1-rotational solution.

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