AMERICAN OPTION PRICING WITH IMPRECISE RISK-NEUTRAL PROBABILITIES:
FROM PLAIN INTERVALS TO FUZZY SETS

S. Muzzioli, H. Reynaerts

1 INTRODUCTION

The aim of this paper is to price an American style option when there is uncertainty on the volatility of the underlying asset. An option contract can be either European or American style depending on whether the exercise is possible only at or also before the expiry date. A European option gives the holder the right to buy or sell the underlying asset only at the expiry date of the option. On the other hand, an American option gives the holder the right to buy or sell the underlying asset at any time up to the expiry date. Therefore, in American option pricing, the likelihood of the early exercise should be carefully taken into account. American option valuation is usually performed, under the risk-neutral valuation paradigm, by using numerical procedures such as the binomial option pricing model of Cox, Ross, Rubinstein (1979). A key input of the multiperiod binomial model is the volatility of the underlying asset, that is an unobservable parameter.
The volatility parameter can be estimated either from historical data (historical volatility) or implied from the price of European options (implied volatility). In the first case, the lenght of the time series, the frequency and the estimation methodology may lead to different estimates. In the second case, as options differ in strike price, time to expiration and option type (call or put), which option class yields implied volatilities that are most representative of the markets' volatility expectations, is still an open debate. Various papers have examined the predictive power of implied volatility extracted from different option classes. Christensen and Prabhala (1998) examine the relation between implied and realized volatility on SP&100 options. They found that at the money calls are good predictors of future realized volatility. Christensen and Strunk (2002) consider the relation between implied and realized volatility on the S&P100 options. They suggest to compute implied volatility as a weighted average of implied volatilities from both in the money and out of the money options and both puts and calls. Ederington and Guan (2005) examine how the information in implied volatility differs by strike price for options on S&P500 futures. They suggest to use implied volatilities obtained from high strike options (out of the money calls and in the money puts) since the information content in implied volatilities varies roughly in a mirror image of the implied volatility smile.

Several ways have been proposed to introduce non-constant volatility in an option pricing model. One way is to make the volatility deterministically dependent on strike price and time (see e.g. Dupire (1994), Derman and Kani (1994), Rubinstein (1994)). An other way is to consider stochastic volatility (see e.g. Hull and White (1987)).
In this paper we follow a different approach. As it is hard to precisely estimate the volatility parameter, we let it take interval values. When using plain intervals, cautiousness may lead to a severe overestimation of the interval width. Therefore, if some expert judgment is available about the actual value of the parameters, it is possible to assign a greater degree of membership to some values within the interval. A more optimistic expert can choose a smaller interval, while a more cautious expert will prefer a larger one. Also the same expert can give different possibility degrees to some values within the interval. If the knowledge is more precise, a nested set of subintervals, with increasing degree of membership, within which one or more values have membership equal to one, i.e. a fuzzy number can be found (for a detailed discussion on the relation between nested intervals and fuzzy sets and a deep discussion on the pooling of experts' opinions see e.g. Nguyen and Kreinovich (1996)).
Fuzzy numbers combine qualitative and quantitative assesments in a single tool that is able to handle uncertainty. They provide us with a simple framework that is intuitively appealing and computationally simple. Fuzzy numbers and possibility distributions can be considered as two faces of the same coin since they have a common mathematical expression and possibility distributions can be manipulated by the combination rules of fuzzy numbers (for more details see Dubois and Prade (1980, 1988)). Therefore in the following we will use the two terms as synonims, keeping in mind that, even if they have a common mathematical expression, the underlying concepts are different: while a fuzzy number can be seen as a fuzzy value that we assign to a variable, viewed as a possibility distribution, the fuzzy number is the set of non fuzzy values that can possibly be assigned to a variable.
Recent literature on option pricing in the presence of uncertainty has mixed probability with fuzziness. Probability is used to model the uncertainty of an event that can occur or not, while fuzziness is used to model the imprecision on a value. Fuzzy European option pricing has been examined in continuous time by Yoshida (2003a) and Wu (2004) and in discrete time by Muzzioli and Torricelli (2004). Fuzzy American option pricing has been examined both in discrete and continuous time by Yoshida (2003b). Yoshida (2003b) has addressed the issue by using fuzzy random variables and fuzzy expectation based on the decision maker‘s subjective judgement. The approach hinges on a symplifying assumption on the evolution of the fuzzy stochastic

process. In particular it assumes that the amount of fuzziness is constant through time and symmetrical w.r.t. the crisp stochastic process. By contrast, in this paper, we drop this assumption. By following the approach of Muzzioli and Torricelli (2004), we let the fuzziness amount decrease as time goes by and allow it to be non symmetrical w.r.t. the crisp stochastic process. Starting from the Cox Ross Rubinstein (1979) binomial model in which the option has a well known valuation formula, we investigate which is the effect on the option price of assuming the volatility as an uncertain parameter. In the binomial model of Cox Ross and Rubinstein (1979) the volatility is modelled by two jump factors, up and down, that describe the possible moves of the underlying asset in the next time period. In this paper we use fuzzy numbers in order to model the two jump factors. Three cases are examined. Since a fuzzy number can be considered as a nested set of subintervals, with increasing degree of membership, as a first step we study the case in which the up and down jump factors are modelled by plain intervals. Then we tackle the case in which the up and down jump factors are modelled by triangular or trapezoidal fuzzy numbers. In order to compute the option price we first show how to derive the risk-neutral probabilities, i.e. the probabilities of an up and a down move of the underlying asset in the next time period in a risk-neutral world. The existence of the risk-neutral probabilities is guaranteed by the no arbitrage condition. The risk-neutral probabilities derivation is a fundamental problem in finance since they are necessary for the pricing of any derivative security. The problem boils down to the solution of a linear system of equations with interval or fuzzy coefficients, depending on the case analyzed. Once the risk-neutral probabilities are derived, they are used in the option valuation.
The plan of the paper is the following: In section 2 we present the Cox Ross Rubinstein binary tree model for the pricing of American put options. Section 3 illustrates the case in which plain intervals are used in order to model the up and down jump factors. Sections 4 and 5 illustrate the case in which triangular and trapezoidal fuzzy numbers respectively are used. In particular, in section 4 a comparison between the Yoshida (2003b) approach is provided. The last section concludes.

2 THE BINARY TREE MODEL FOR THE PRICING OF AN AMERICAN PUT OPTION

The binary tree model of Cox et al. (1979) is used to price options and other derivative securities. As the price of an American call option written on a non dividend paying stock is the same of that of a European call option, in this paper we analyse the only interesting case of a put option. An American put option is a financial security that provides its holder, in exchange for the payment of a premium, the right but not the obligation to sell a certain underlying asset before or at the expiration date for a specified price K. In the binary tree model of Cox et al. (1979) the following assumptions are made: (A1) the markets have no transaction costs, no taxes, no restrictions on short sales, and assets are infinitely divisible; (A2) the lifetime T of the option is divided into N time steps of length T/N; (A3) the market is complete; (A4) no arbitrage opportunities are allowed, which implies for the risk-free interest factor, $1+r$, over one step of length T/N, that $d < 1+r < u$, where u is the up and d the down factor. In order to price the American put option, the American algorithm is applied (see for example S. Shreve (2004)):
Define the functions $v_n(s), n = N, N-1, \ldots, 0$ as follows

$$\begin{aligned} v_N(s) &= (K-s)_+, \qquad s = S_0 u^i d^{N-i}, \qquad i = 0, 1, \ldots N, \\ v_n(s) &= max\{K-s, \frac{1}{1+r}(p_u v_{n+1}(us) + p_d v_{n+1}(ds))\}, \\ n &= N-1, N-2, \ldots, 0 \qquad s = S_0 u^i d^{n-i}, \qquad i = 0, 1, \ldots n, \end{aligned}$$

where K is the exercise price, S_0 is the price of the underlying asset at time the contract begins, p_u and p_d are the resp. up and down risk-neutral transition probabilities. Fundamental for the option valuation is the derivation of the risk-neutral probabilities, which are obtained from the following system:

$$\begin{cases} p_u + p_d = 1 \\ up_u + dp_d = 1 + r. \end{cases} \tag{1}$$

The solution is given by:

$$p_u = \frac{(1+r) - d}{u - d} \qquad p_d = \frac{u - (1+r)}{u - d}.$$

In order to estimate the up and down jump factors from market data, the standard methodology (see Cox et al. (1979)) leads to set:

$$u = e^{\sigma\sqrt{T/N}}, d = e^{-\sigma\sqrt{T/N}},$$

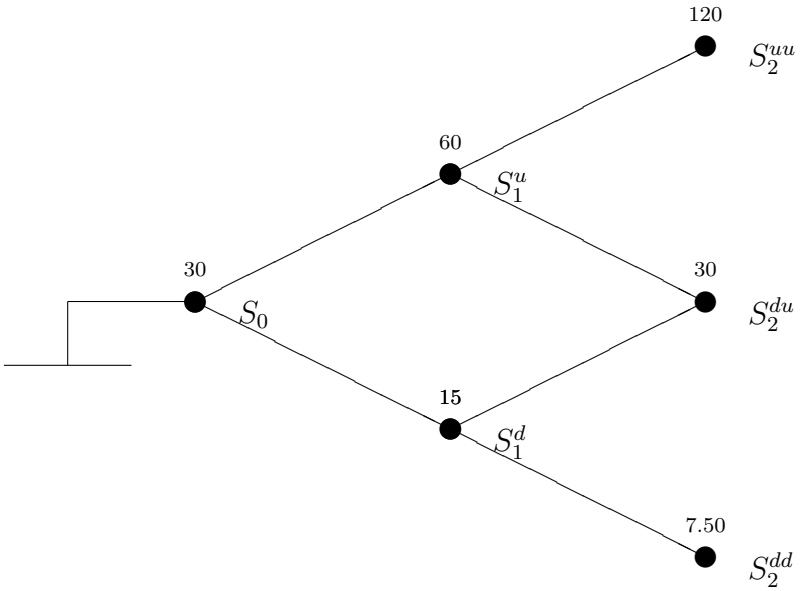

Figure 1: *Price for the underlying asset.*

where σ is the volatility of the underlying asset.
Numerical example
Consider the two-period model where $S_0 = 30$, $K = 35$, $u = 2$, $d = .50$ and $r = .02$.
The up and down probabilities are: $p_u = .3467$ and $p_d = .6633$.
The binomial tree for the price of the underlying asset is illustrated in Figure 1. By applying the American algorithm one obtains the American put option prices reported in Figure 2, as follows:

$$v_2(120) = (35 - 120)_+ = 0$$
$$v_2(30) = (35 - 30)_+ = 5$$
$$v_2(7.50) = (35 - 7.50)_+ = 27.50$$
$$v_1(60) =$$
$$max\{35 - 60, \frac{1}{1.02}[0.3467 v_2(60 * 2) + 0.6533 v_2(60 * .50)]\} = 3.2025$$
$$v_1(15) =$$
$$max\{35 - 15, \frac{1}{1.02}[0.3467 v_2(15 * 2) + 0.6533 v_2(15 * .50)]\} = 20$$
$$v_0(30) =$$
$$max\{35 - 30, \frac{1}{1.02}[0.3467 v_1(30 * 2) + 0.6533 v_1(30 * .50)]\} = 13.8989.$$

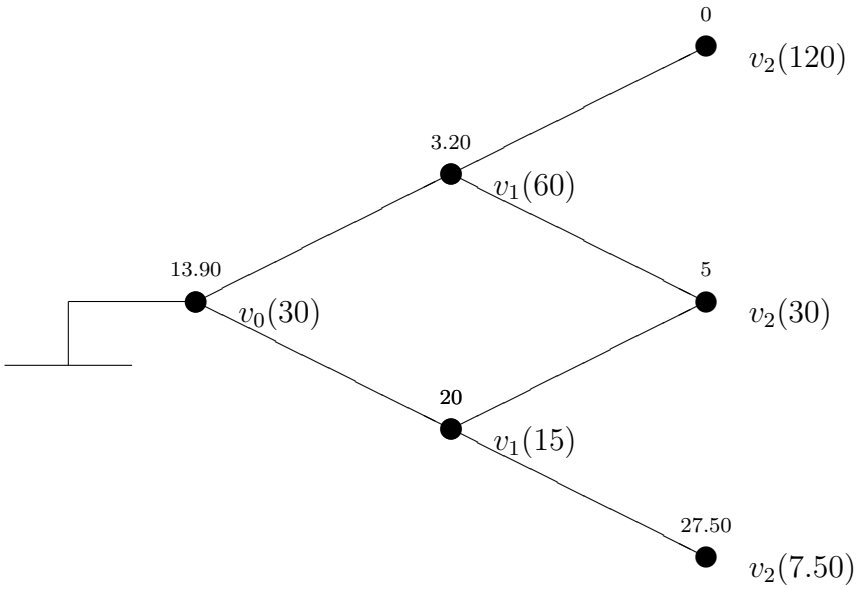

Figure 2: *American put option prices.*

3 THE CASE OF PLAIN INTERVALS

If there is some uncertainty about the value of the volatility, then it is also impossible to precisely estimate the up and down factors. In this section we examine the case in which the information about the up and down jump factors is so vague that we can only fix a lower and an upper bound for the possible values, i.e. u and d are represented by the real intervals $u = [u_1, u_2]$ and $d = [d_1, d_2]$. A real interval x is defined as the set of real numbers such that $x = [\underline{x}, \overline{x}] = \{\acute{x} \in \Re^n : \underline{x} \leq \acute{x} \leq \overline{x}\}$, where $\underline{x}$ and $\overline{x}$ are respectively the lower and the upper bound of the interval.
Basic operations on intervals are defined in Moore (1966). Assumptions (A1), (A2) and (A3) are still valid, while assumption (A4) changes as follows: $d_2 < 1 + r < u_1$.
In this setting system (1) is an interval linear system of the form:

$$Ax = b \tag{1}$$

where the elements, a_{ij}, $i = 1, 2$ $j = 1, 2$ of the matrix A are intervals and the elements, b_i, of the vector b are crisp. An interval matrix $A_{n,m}$ is defined as: $A = [\underline{A}, \overline{A}] = \{\acute{A} \in \Re^{n \times m} : \underline{A} \leq \acute{A} \leq \overline{A}\}$. Note that condition (A4)

guarantees that the resulting interval matrix:

$$\begin{bmatrix} 1 & 1 \\ [d_1, d_2] & [u_1, u_2] \end{bmatrix}$$

has always full rank for all $d \in [d_1, d_2]$ and for all $u \in [u_1, u_2]$.
There is no uncertainty in the risk-free rate of interest, since it is given at time zero.
The solution of the interval system is found by looking to the united solution set (USS)(see e.g. Kearfott (1996), Muzzioli and Reynaerts (2006a)). The USS looks for the set of all real vectors x that fulfil the set of equations $\acute{A}x = \acute{b}$, where $\acute{A} \in A$, is a real matrix contained in the interval matrix A and vector $\acute{b} \in b$, is a real vector contained in the right-hand side vector b:

$$X_{\exists\exists} = \{\acute{x} \in \Re^n : (\exists \acute{A} \in A)(\exists \acute{b} \in b)\acute{A}\acute{x} = \acute{b}\} = \{\acute{x} \in \Re^n : A\acute{x} \cap b \neq \emptyset\}$$

The USS looks at all the possible combinations between a real matrix $\acute{A} \in A$ and a real right-hand side vector $\acute{b} \in b$.
Usually the interval vector formed by the bounds on the coordinates of the USS is considered, this is called the solution hull:

$$hull(X_{\exists\exists}) = [infX_{\exists\exists}, supX_{\exists\exists}]$$

In order to compute the solution hull, it is useful to define the vertex solution set.
The vertex solution set (VSS) is the set of solutions $\acute{x} \in \Re^n$ of all real systems of equations, whose coefficients are all the possible combinations of the endpoints of the coefficients of the interval matrix A and of the interval vector b.

$$X_V = \{\acute{x} \in \Re^n \mid \exists \acute{E} \in vert(A), \exists \acute{e} \in vert(b) \mid s.t. \acute{E}\acute{x} = \acute{e}\}$$

where $vert(A) = \{\acute{A} \in A \mid \acute{a}_{i,j} \in \{\underline{a}_{i,j}, \overline{a}_{i,j}\}$, $vert(b) = \{\acute{b} \in b \mid \acute{b}_i \in \{\underline{b}_i, \overline{b}_i\}$
The VSS is a discrete and finite set, whose number of elements $t \leq 2^{n^2+n}$ depends on how many parameters in the matrix A and in the vector b are intervals.
As $hull(X_{\exists\exists}) = hull(X_V)$, we can compute the hull of the USS by solving a set of $t \leq 2^{n^2+n}$ real systems $Ax = b$, where the coefficients are all the possible combinations of the endpoints of the coefficients of the interval matrix A and of the interval vector b.
By following this procedure, we compute the solution of the interval linear system, as follows:

$$p_u = [\frac{(1+r) - d_2}{u_2 - d_2}, \frac{(1+r) - d_1}{u_1 - d_1}], p_d = [\frac{u_1 - (1+r)}{u_1 - d_1}, \frac{u_2 - (1+r)}{u_2 - d_2}],$$

In order to get the price of the American put option, the American algorithm should now be applied. Define the functions $v_n(s), n = N, N-1, \ldots, 0$ as follows

$$\begin{aligned} v_N(s) &= (K-s)_+, s = [S_0 u_1^i d_1^{N-i}, S_0 u_2^i d_2^{N-i}], i = 0, 1, \ldots N, \\ v_n(s) &= max\{K-s, \frac{1}{1+r}(p_u v_{n+1}(us) + p_d v_{n+1}(ds))\}, \\ & \quad n = N-1, N-2, \ldots, 0 \end{aligned}$$

where

$$p_u = [\frac{(1+r)-d_2}{u_2-d_2}, \frac{(1+r)-d_1}{u_1-d_1}], p_d = [\frac{u_1-(1+r)}{u_1-d_1}, \frac{u_2-(1+r)}{u_2-d_2}],$$

and the maximum between two intervals $f = [f_1, f_2]$ and $g = [g_1, g_2]$ is defined as:

$$max(f, g) = [max(f_1, g_1), max(f_2, g_2)],$$

In order to get the put price, we use the standard rules of addition and multiplication between intervals, as defined in Moore (1966).
Numerical example
Consider the two-period model where $S_0 = 30$, $K = 35$, $u = [1.50; 3]$, $d = [.35; .60]$ and $r = .02$.
The the up and down probabilities are: $p_u = [.1750; .5826]$ and $p_d = [.4174; .8250]$.
The binomial tree for the price of the underlying asset is illustrated in Figure 3. By applying the American algorithm one obtains the American put option prices reported in Figure 4, as follows:

$$\begin{aligned} & v_2(S_2^{uu}) = max\{(35 - [67.5; 270]), 0\} = 0 \\ & v_2(S_2^{ud}) = max\{(35 - [15.75; 54]), 0\} = [0; 19.25] \\ & v_2(S_2^{dd}) = max\{(35 - [3.675; 10.8]), 0\} = [24.2; 31.33] \\ & v_1(S_1^{u}) = max\{35 - [45; 90], \\ & \quad \frac{1}{1.02}([.1750; .5826]v_2(S_2^{uu}) + [.4174; .8250]v_2(S_2^{ud}))\} \\ & \quad = [0; 15.57] \end{aligned}$$

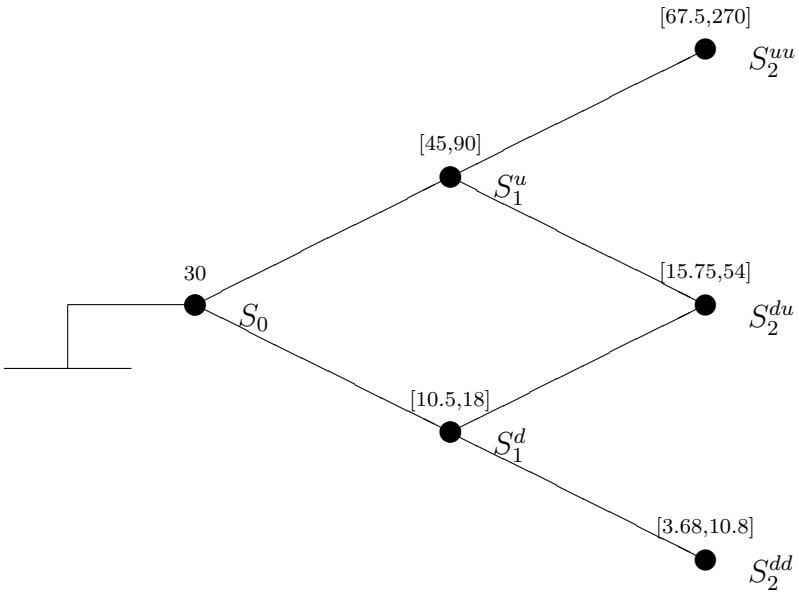

Figure 3: *Price for the underlying asset.*

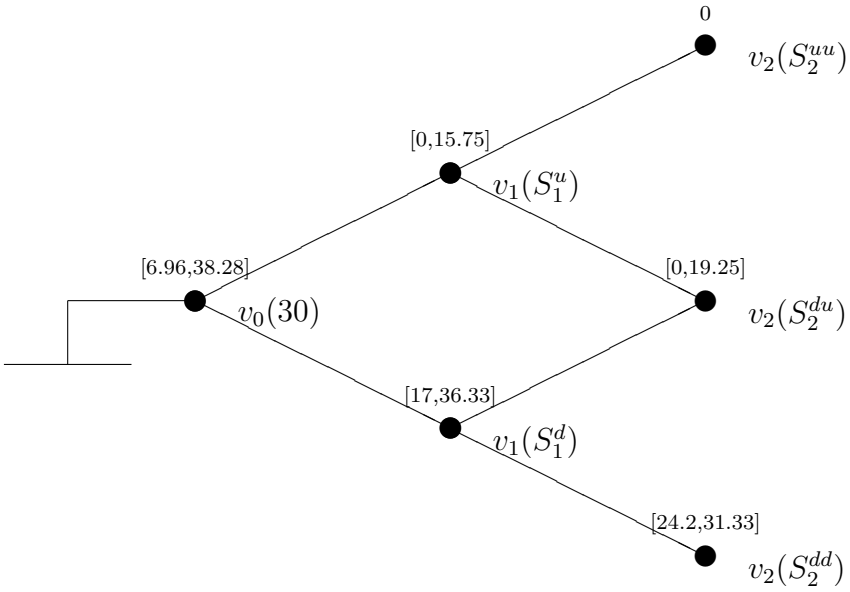

Figure 4: *American put option prices.*

$$
\begin{aligned}
v_1(S_1^d) &= max\{35 - [10.5; 18], \\
&\quad \frac{1}{1.02}([.1750; .5826]v_2(S_2^{ud}) + [.4174; .8250]v_2(S_2^{dd}))\} \\
&\quad = [17; 36.33] \\
v_0(30) &= max\{35 - 30, \\
&\quad \frac{1}{1.02}([.1750; .5826]v_1(S_1^u) + [.4174; .8250]v_1(S_1^d))\}, \\
&\quad = [6.96; 38.28]
\end{aligned}
$$

4 THE CASE OF TRIANGULAR FUZZY NUMBERS

In this section we assume that the information about the possible values of the jump factors can be described by means of a nested set of intervals within which a most possible value can be found. In order to introduce the notion of a triangular fuzzy number, some basic concepts about fuzzy sets have to be recalled. A fuzzy set F of $\mathbb{R}$ is a subset of $\mathbb{R}$, where the membership function of each element $x \in \mathbb{R}$, denoted by $\mu_F(x)$, is allowed to take any value in the closed interval [0,1]. $\mu_F(x) = 0$ indicates no membership, $\mu_F(x) = 1$ indicates full membership: the closer the value of the membership function is to 1, the more x belongs to F.
A fuzzy number N is a normal (i.e. at least one value x has full membership) and convex (the membership function should not have distinct local maximal points) fuzzy set of $\mathbb{R}$. Fuzzy numbers can be considered as possibility distributions (see e.g. Dubois and Prade, 1988): let a fuzzy number $A \in N$ and a real number $x \in \Re$, then $\mu_A(x)$ can be interpreted as the degree of possibility of the statement "x is A".
A triangular fuzzy number f is uniquely defined by the triplet (f_1, f_2, f_3) where f_1 and f_3 are the lower and the upper bounds of the interval of possible values and f_2 is the most possible. The membership function $\mu_{(f)}(x) = 0$ outside (f_1, f_3), and $\mu_{(f)}(x) = 1$ at $x = f_2$, the graph of the membership function is a straight line from $(f_1, 0)$ to $(f_2, 1)$ and from $((f_2, 1)$ to $(f_3, 0))$.
Alternatively, one can write a triangular fuzzy number in terms of its α-cuts, $f(\alpha)$, α in $[0, 1]$:

$$
\begin{aligned}
f(\alpha) &= [\underline{f}(\alpha), \overline{f}(\alpha)] \\
&= [f_1 + \alpha(f_2 - f_1), f_3 - \alpha(f_3 - f_2)].
\end{aligned}
$$

For simplicity of the notations the α-cuts will also be noted by $[\underline{f}, \overline{f}]$. Since the α-cuts of a triangular fuzzy number are compact intervals of the set of real numbers, the interval calculus of Moore (1966) can be applied on them. The up and down factors are represented by the triangular fuzzy numbers: $u = (u_1, u_2, u_3)$ and $d = (d_1, d_2, d_3)$. Assumptions (A1), (A2) and (A3) are still valid, while assumption (A4) changes as follows: $d_1 \leq d_2 \leq d_3 < 1 + r < u_1 \leq u_2 \leq u_3$.
There is no fuzziness in the risk-free rate of interest, since it is given at time zero.
In this setting system (1) is a fuzzy linear system of the form:

$$Ax = b \tag{-24}$$

where some of the elements, a_{ij}, $i = 1, 2$ $j = 1, 2$ of the matrix A are triangular fuzzy numbers and the elements, b_i, of the vector b are crisp. Note that the no arbitrage condition guarantees that the resulting fuzzy matrix:

$$\begin{bmatrix} 1 & 1 \\ (d_1, d_2, d_3) & (u_1, u_2, u_3) \end{bmatrix}$$

has always full rank for all $d \in [d_1, d_3]$ and for all $u \in [u_1, u_3]$.
As a fuzzy number can be considered as a set of nested intervals with membership greater than an increasing threshold, interval linear systems can be considered as a special case of fuzzy linear systems.
In order to investigate the solution of such a system, we use the concept of vector solution given in Buckley and Qu (1991), that is obtained by looking at the USS for each alpha-cut. In Muzzioli and Reynaerts (2006b) we proposed the following non linear programming problem in order to compute the vector solution of system (2):

$$\begin{aligned} & max_{u,d} \quad (\text{resp.}min_{u,d}) \quad \frac{1 + r - d}{u - d} \\ & \text{where} \quad (1 + r <) \underline{u} \leq u \leq \overline{u} \\ & \qquad \text{and} \quad \underline{d} \leq d \leq \overline{d} (< 1 + r) \end{aligned}$$

$$\begin{aligned} & max_{u,d} (\text{resp.}min_{u,d}) \frac{u - (1 + r)}{u - d} \\ & \text{where} \quad (1 + r <) \underline{u} \leq u \leq \overline{u} \\ & \qquad \text{and} \quad \underline{d} \leq d \leq \overline{d} (< 1 + r) \end{aligned}$$

Since $\frac{\partial p_u}{\partial u} = \frac{d - (1+r)}{(u-d)^2} < 0$ the maximum of p_u is obtained for $u^{max} = \underline{u}$ and the minimum for $u^{min} = \overline{u}$.

Since $\frac{\partial p_u}{\partial d} = \frac{(1+r)-u}{(u-d)^2} < 0$ the maximum of p_u is obtained for $d^{max} = \underline{d}$ and the minimum for $d^{min} = \overline{d}$.
Since $\frac{\partial p_d}{\partial u} = \frac{(1+r)-d}{(u-d)^2} > 0$ the maximum of p_d is obtained for $u^{max} = \overline{u}$ and the minimum for $u^{min} = \underline{u}$.
Since $\frac{\partial p_d}{\partial d} = \frac{u-(1+r)}{(u-d)^2} > 0$ the maximum of p_d is obtained for $d^{max} = \overline{d}$ and the minimum for $d^{min} = \underline{d}$
The solution to the system is:

$$p_u = [\frac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \frac{(1+r)-\underline{d}}{\underline{u}-\underline{d}}], p_d = [\frac{(\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \frac{(\overline{u}-(1+r)}{\overline{u}-\overline{d}}]$$

In order to get the price of the American put option, the American algorithm should now be applied. The functions $v_n(s), n = N, N-1, \ldots, 0$ are defined as:

$$\begin{aligned} v_N(s) &= (K-s)_+, s = S_0[\underline{u}^i, \overline{u}^i][\underline{d}^{N-i}, \overline{d}^{N-i}], i = 0, 1, \ldots N, \\ v_n(s) &= max\{K-s, \frac{1}{1+r}(p_u v_{n+1}(us) + p_d v_{n+1}(ds))\}, \\ & \quad n = N-1, N-2, \ldots, 0 \end{aligned}$$

with

$$\begin{aligned} p_d &= [\frac{(\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \frac{(\overline{u}-(1+r)}{\overline{u}-\overline{d}}] \\ p_u &= [\frac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \frac{(1+r)-\underline{d}}{\underline{u}-\underline{d}}] \end{aligned}$$

$$max(f,g)(\alpha) = [max(\underline{f}(\alpha), \underline{g}(\alpha), max(\overline{f}(\alpha), \overline{g}(\alpha)], \alpha \in [0,1].$$

For simple and fast computation between fuzzy numbers a restriction to triangular shaped fuzzy numbers is often preferable. Therefore we use the following approximations, let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers and $c \in \Re$ a crisp number:

$$\begin{aligned} A \circ B &= (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3), \\ max(A,B) &= (max(a_1, b_1), max(a_2, b_2), max(a_3, b_3)) \\ max(A,c) &= (max(a_1, c), max(a_2, c), max(a_3, c)) \end{aligned}$$

The risk-neutral probabilities are approximated by the following triangular numbers:

$$\begin{aligned} p_u &= (\frac{1+r-d_3}{u_3-d_3}, \frac{1+r-d_2}{u_2-d_2}, \frac{1+r-d_1}{u_1-d_1}) \\ p_d &= (\frac{u_1-(1+r)}{u_1-d_1}, \frac{u_2-(1+r)}{u_2-d_2}, \frac{u_3-(1+r)}{u_3-d_3},) \end{aligned}$$

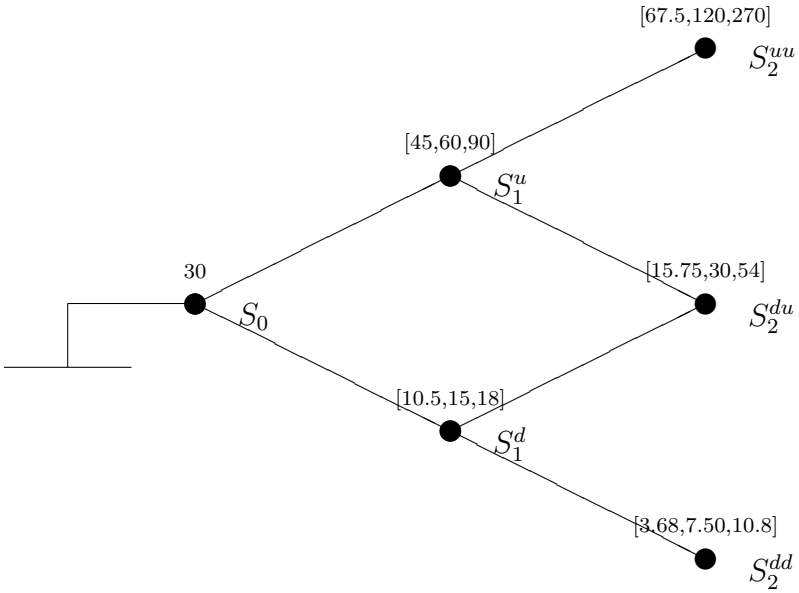

Figure 5: *Price for the underlying asset.*

Numerical example

Consider the two-period model where $S_0 = 30$, $K = 35$, $u = [1.50; 2; 3]$, $d = [.35; .50; .60]$ and $r = .02$.

The up and down probabilities are: $p_u = [.1750; .3467; .5826]$ and $p_d = [.4174; .6633; .8250]$.

The binomial tree for the price of the underlying asset is illustrated in Figure 5. By applying the American algorithm one obtains the American put option prices reported in Figure 6, as follows:

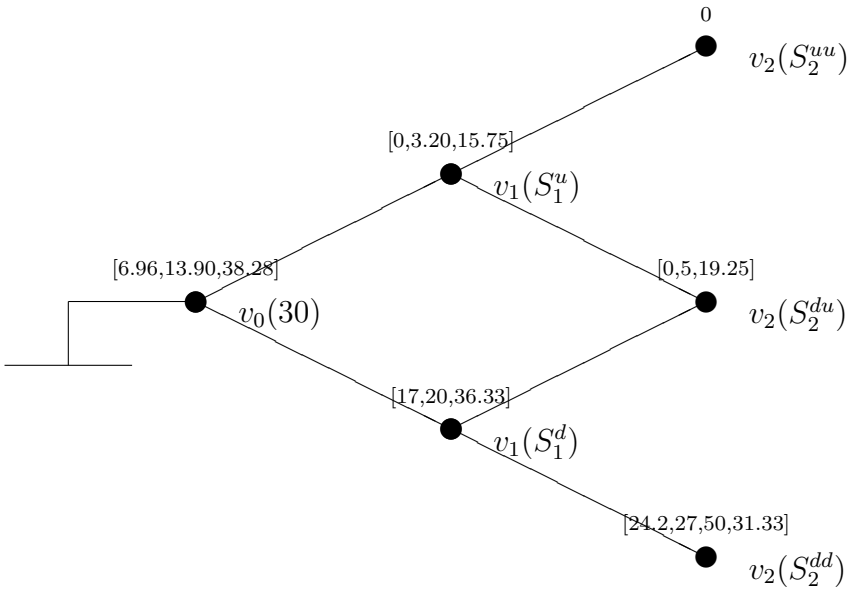

Figure 6: *American put option prices.*

$$
\begin{aligned}
&v_2(S_2^{uu}) = max\{(35 - [67.5; 120; 270]), 0\} = 0\\
&v_2(S_2^{ud}) = max\{(35 - [15.75; 30; 54]), 0\} = [0; 5; 19.25]\\
&v_2(S_2^{dd}) = max\{(35 - [3.675; 7.50; 10.8]), 0\} = [24.2; 27.50; 31.325]\}\\
&v_1(S_1^u) = max\{35 - [45; 60; 90],\\
&\qquad \frac{1}{1.02}([.1750; .3467; .5826]v_2(S_2^{uu}) + [.4174; .65333; .8250]v_2(S_2^{ud}))\}\\
&\qquad = [0; 3.2025; 15.5699]\\
&v_1(S_1^d) = max\{35 - [10.5; 15; 18],\\
&\qquad \frac{1}{1.02}([.1750; .3467; .5826]v_2(S_2^{ud}) + [.4174; .65333; .8250]v_2(S_2^{dd}))\}\\
&\qquad = [17; 20; 36.3315]\\
&v_0(30) = max\{35 - 30,\\
&\qquad \frac{1}{1.02}([.1750; .3467; .5826]v_1(S_1^u) + [.4174; .65333; .8250]v_1(S_1^d))\}\\
&\qquad = [6.96; 13.90; 38.28]
\end{aligned}
$$

In order to underline the difference between the present approach and the one taken in Yoshida (2003b) we note the following. Yoshida (2003b) considers a fuzzy-valued stock price whereby the fuzziness amount is described by a constant $0 < c < 1$ that represents the decision maker subjective estimate of

the volatility σ. The initial stock price S_0 is multiplied by the fuzzy factor $b = [b^-, b^+] = [1-(1-\alpha)c, 1+(1-\alpha)c], \alpha \in [0,1]$ and the up and down jump factors u and d are crisp. The fuzzy factor b is a triangular shaped fuzzy number with symmetrical spreads.
The present approach differs from Yoshida (2003b), in at least two aspects. First the triangular fuzzy numbers used are not restricted to be symmetrical as in Yoshida (2003b), but the left and right spread can have different lenght. This is an important feature to better capture the information on the volatility. For example the decision maker can be rather sure about the amount the stock will gain in case it will increase, but she can be rather uncertain about the amount the stock will loose in case it will decrease. Moreover the decision maker can have a more optimistic (pessimistic) view on the single jump factor, that can be modelled by a longer (shorter) right spread and a shorter (longer) left spread. Second it clearly illustrates how the assumption on fuzzy up and down jump factors changes the no arbitrage condition and in turn affects the risk-neutral probabilities derivation. In fact, in Yoshida (2003b) the fuzzy factor does not affect the no arbitrage condition and in turn the risk-neutral probabilities derivation. Besides, we notice that in Yoshida (2003b) the following condition should be verified in order to ensure no arbitrage: $db^+ < 1+r < ub^-$, i.e. $d(1+(1-\alpha)c) < 1+r < u(1-(1-\alpha)c)$, therefore the decision maker is not allowed to choose any value of $0 < c < 1$, but the interval of possible values should be restricted by the no arbitrage condition. Moreover, the risk-neutral probabilities shoud be accordingly derived, in order to take into account the fuzziness in the model. They can be easily obtained as a special case of our model when u and d are symmetrical triangular fuzzy numbers.

5 THE CASE OF TRAPEZOIDAL FUZZY NUMBERS

In this section we assume that the information about the possible values of the jump factors can be described by means of a nested set of intervals within which a most possible interval can be found. A trapezoidal fuzzy number is used to describe an interval whose lower and upper bounds are uncertain. A trapezoidal fuzzy number f is uniquely defined by the quartet (f_1, f_2, f_3, f_4) where f_1 and f_4 are the lower and the upper bounds of the interval of possible values and $[f_2, f_3]$ is interval of the most possible values. A trapezoidal fuzzy number may be seen as the representation of the statement "the value of a real variable is approximately in the interval $[f_1, f_4]$". The membership

function $\mu_{(f)}(x) = 0$ outside (f_1, f_4), and $\mu_{(f)}(x) = 1$ at $x \in [f_2, f_3]$, the graph of the membership function is a straight line from $(f_1, 0)$ to $(f_2, 1)$ and from $((f_3, 1)$ to $(f_4, 0))$.
Alternatively, one can write a trapezoidal fuzzy number in terms of its α-cuts, $f(\alpha)$, α in $[0, 1]$:

$$\begin{aligned} f(\alpha) &= [\underline{f}(\alpha), \overline{f}(\alpha)] \\ &= [f_1 + \alpha(f_2 - f_1), f_4 - \alpha(f_4 - f_3)]. \end{aligned}$$

For simplicity of the notations the α-cuts will also be noted by $[\underline{f}, \overline{f}]$. Since the α-cuts of a trapezoidal fuzzy number are compact intervals of the set of real numbers, the interval calculus of Moore (1966) can be applied on them. The up and down factors are represented by the trapezoidal fuzzy numbers: $u = (u_1, u_2, u_3, u_4)$ and $d = (d_1, d_2, d_3, d_4)$. Assumptions (A1), (A2) and (A3) are still valid, while assumption (A4) changes as follows: $d_1 \leq d_2 \leq d_3 \leq d_4 < 1 + r < u_1 \leq u_2 \leq u_3 \leq u_4$.
There is no fuzziness in the risk-free rate of interest, since it is given at time zero.
In this setting system (1) is a fuzzy linear system of the form:

$$Ax = b \tag{-59}$$

where some of the elements, a_{ij}, $i = 1, 2$ $j = 1, 2$ of the matrix A are trapezoidal fuzzy numbers and the elements, b_i, of the vector b are crisp. Note that the no arbitrage condition guarantees that the resulting fuzzy matrix:

$$\begin{bmatrix} 1 & 1 \\ (d_1, d_2, d_3, d_4) & (u_1, u_2, u_3, u_4) \end{bmatrix}$$

has always full rank for all $d \in [d_1, d_4]$ and for all $u \in [u_1, u_4]$.
The solution of the system is:

$$([\frac{(1+r) - \overline{d}}{\overline{u} - \overline{d}}, \frac{(1+r) - \underline{d}}{\underline{u} - \underline{d}}], [\frac{(\underline{u} - (1+r)}{\underline{u} - \underline{d}}, \frac{(\overline{u} - (1+r)}{\overline{u} - \overline{d}}]),$$

where

$$\begin{aligned} \underline{u} &= u_1 + \alpha(u_2 - u_1) \\ \overline{u} &= u_4 - \alpha(u_4 - u_3) \\ \underline{d} &= d_1 + \alpha(d_2 - d_1) \\ \overline{d} &= d_4 - \alpha(d_4 - d_3) \end{aligned}$$

In order to get the price of the American put option, the American algorithm should now be applied.

For simple and fast computation between fuzzy numbers a restriction to trapezoidal shaped fuzzy numbers is often preferable. Therefore we use the following approximations, let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers and $c \in \Re$ a crisp number:

$$A \circ B = (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3, a_4 \cdot b_4)$$

$$max(A, B) = (max(a_1, b_1), max(a_2, b_2), max(a_3, b_3), max(a_4, b_4))$$

$$max(A, c) = (max(a_1, c), max(a_2, c), max(a_3, c), max(a_4, c))$$

The risk-neutral probabilities are approximated by the following trapezoidal fuzzy numbers:

$$\begin{aligned} p_u &= (\frac{1+r-d_4}{u_4-d_4}, \frac{1+r-d_3}{u_3-d_3}, \frac{1+r-d_2}{u_2-d_2}, \frac{1+r-d_1}{u_1-d_1}) \\ p_d &= (\frac{u_1-(1+r)}{u_1-d_1}, \frac{u_2-(1+r)}{u_2-d_2}, \frac{u_3-(1+r)}{u_3-d_3}, \frac{u_4-(1+r)}{u_4-d_4}) \end{aligned}$$

Numerical example

Consider the two-period model where $S_0 = 30$, $K = 35$, $u = [1.50; 2; 2.50; 3]$, $d = [.35; .45; .50; .60]$ and $r = .02$.

The up and down probabilities are: $p_u = [.1750; .2600; .3467; .5826]$ and $p_d = [.4174; .6323; .7400; .8250]$.

The binomial tree for the price of the underlying asset is illustrated in Figure 7. By applying the American algorithm one obtains the American put option prices reported in Figure 8, as follows:

$$\begin{aligned} &v_2(S_2^{uu}) = max\{(35 - [67.5; 120; 187.5; 270]), 0\} = 0 \\ &v_2(S_2^{ud}) = max\{(35 - [15.75; 27; 37.5; 54]), 0\} = [0; 0; 8; 19.25] \\ &v_2(S_2^{dd}) = max\{(35 - [3.675; 6.075; 7.50; 10.8]), 0\} = [24.2; 27.50; 28.935; 31.325]\} \\ &v_1(S_1^u) = max\{35 - [45; 60; 75; 90], \\ &\quad \frac{1}{1.02}([.1750; .2600; .3667; .5826]v_2(S_2^{uu}) + [.4174; .6323; .7400; .8250]v_2(S_2^{ud}))\} \\ &\quad = [0; 0; 5.8039; 15.5699] \\ &v_1(S_1^d) = max\{35 - [10.5; 13.5; 15; 18], \\ &\quad \frac{1}{1.02}([.1750; .2600; .3667; .5826]v_2(S_2^{ud}) + [.4174; .6323; .7400; .8250]v_2(S_2^{dd}))\} \\ &\quad = [17; 20; 23.8687; 36.3315] \\ &v_0(30) = max\{35 - 30, \\ &\quad \frac{1}{1.02}([.1750; .2600; .3677; .5826]v_1(S_1^u) + [.4174; .6323; .7400; .8250]v_1(S_1^d))\}, \\ &\quad = [6.9567; 12.4118; 19.4088; 38.2789] \end{aligned}$$

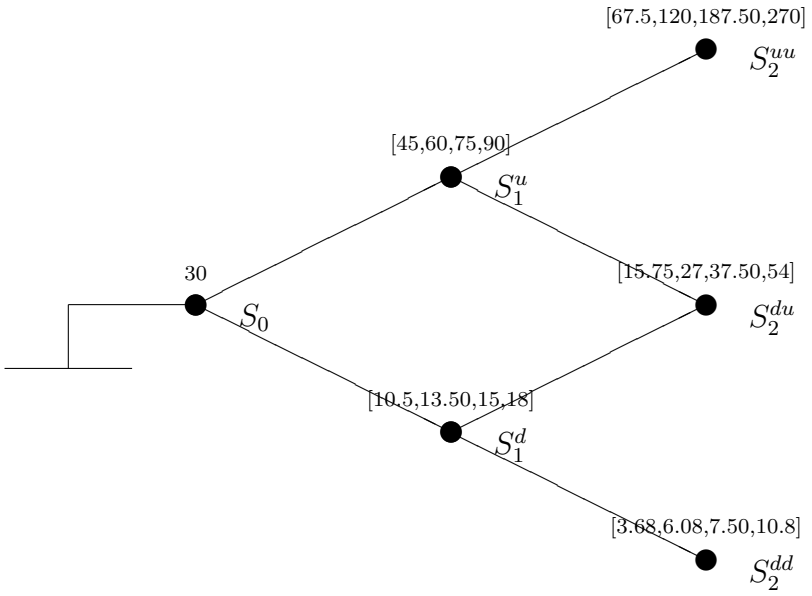

Figure 7: *Price for the underlying asset.*

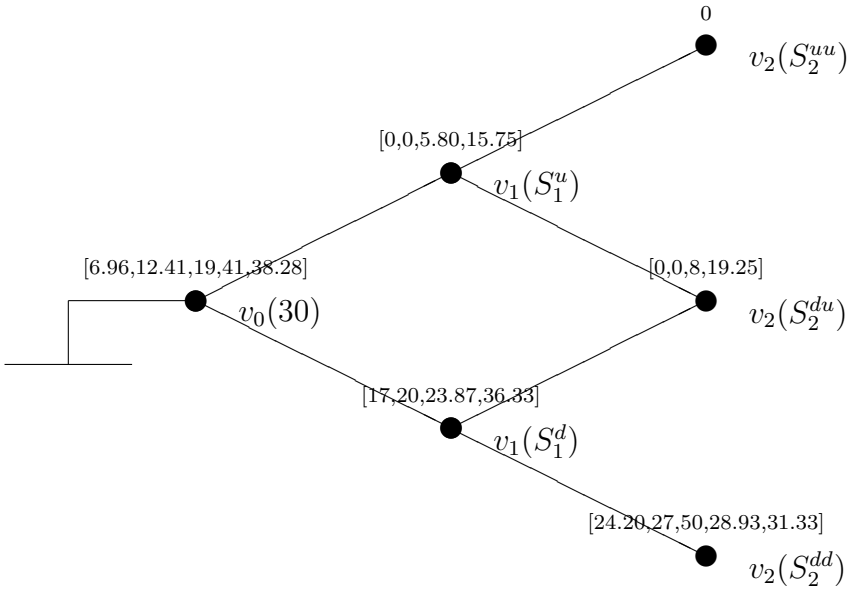

Figure 8: *American put option prices*

6 CONCLUSIONS

In this paper we have investigated the derivation of the price of an American put option written on a stock in the presence of uncertainty in the volatility. As in real markets it is usually hard to precisely estimate the volatility of the underlying asset, fuzzy sets and possibility distributions are a convenient tool for capturing this kind of imprecision. We started from the Cox Ross Rubinstein (1979) binomial model and we investigated which is the effect on the option price of assuming the volatility as an uncertain parameter. As a first step we assumed that the jump factors are represented by intervals. In such a framework we derived the risk-neutral probabilities by solving a linear system of equations with interval coefficients and we evaluated the option price by using interval computation. Then we analyzed the case in which the jump factors are represented by fuzzy numbers. Both the cases of triangular and trapezoidal fuzzy numbers are discussed. We derived the risk-neutral probabilities by solving a linear system of equations with fuzzy coefficients, by using the vector solution proposed by Buckley et al. (1991, 2002). Finally, the risk-neutral probabilities derived are used to evaluate the option price. The present paper improves over previous approaches in at least four aspects. First, by modelling the uncertainty in the volatility by means of both intervals and fuzzy numbers, deeply explores the relation between the two. Second, it uses two different types of fuzzy numbers: triangular and trapezoidal. Third, it clearly illustrates the no arbitrage condition and its role in the derivation of the risk-neutral probabilities. Finally, it provides a simple and fast computational algorithm for the derivation of the option price.

7 Acknowledgements

S. Muzzioli would like to acknowledge financial support from MIUR.
S. Muzzioli Department of Economics and CEFIN, University of Modena and Reggio Emilia, Italy
Tel.: +390592056771; Fax.: +390592056937
E-mail address: muzzioli.silvia@unimore.it

References

[1] R. Baker, R.B. Kearfott, Rigorous global search: continuous problems, Kluwer Academic Publishers The Netherlands (1996).

[2] J.J. Buckley, Y. Qu, Solving systems of linear fuzzy equations, Fuzzy sets and systems 43 (1991) 33-43.

[3] B.J. Christensen, N.R. Prabhala, The relation between implied and realized volatility. Journal of Financial Economics 50 (1998) 125-150.

[4] B.J. Christensen, C. Strunk, New evidence on the implied-realized volatility relation. The European Journal of Finance 8 (2002) 187-205.

[5] J.C. Cox, S.A. Ross, S. Rubinstein, Option pricing, a simplified approach. Journal of Financial Economics (1979) 229-263.

[6] E. Derman, I. Kani, Riding on a smile. Risk 7 (2) (1994) 32-39.

[7] D.Dubois, H. Prade, Fuzzy sets and systems: theory and applications. Academic Press NY (1980).

[8] D.Dubois, H. Prade, Possibility theory: an approach to computerized processing of uncertainty. Plenum Press NY (1988).

[9] B.Dupire, Pricing with a smile. Risk, 7 (1) (1994) 18-20.

[10] L. Ederington, W. Guan, The information frown in option prices. Journal of Banking and Finance 29 (2005) 1429-1457.

[11] J. Hull, A. White, The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance 42 (2) (1987) 281-300.

[12] R.E. Moore, Methods and applications of interval analysis. SIAM Studies in Applied Mathematics (1966).

[13] S. Muzzioli, C. Torricelli, A multiperiod binomial model for pricing options in a vague world. Journal of Economic Dynamics and Control 28 (2004) 861-887.

[14] S. Muzzioli, H. Reynaerts, Fuzzy linear systems of the form $A_1x + b_1 = A_2x + b_2$. Fuzzy sets and systems, article in press (2006a).

[15] S. Muzzioli and H. Reynaerts, The solution of fuzzy linear systems by non linear programming: a financial application. European Journal of Operational Research, article in press (2006b).

[16] H.T. Nguyen, V. Kreinovich, Nested intervals and sets: concepts, relations to fuzzy sets, and applications. In: R.B. Kearfott and V. Kreinovich (eds.), Applications of Interval Computations, Kluwer Academic Publishers, The Netherlands, (1996) 245-290.

[17] M. Rubinstein, Implied binomial trees. Journal of Finance 49 (3) (1994) 771-818.

[18] S. Shreve, Stochastic calculus for finance. Springer Verlag (2004).

[19] Hsien-Cung Wu. Pricing European options based on the fuzzy pattern of Black-Scholes formula. Computers & Operations Research 31 (2004) 1069-1081.

[20] Yuji Yoshida, The valuation of European options in uncertain environment. European Journal of Operational Research 145 (2003a) 221-229.

[21] Yuji Yoshida, A discrete-time model of American put option in an uncertain environment. European Journal of Operational Research 151 (2003b) 153-166.