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Mathematical analysis of average rates of return and investment decisions: The missing link

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Abstract

This paper expands Teichroew, Robichek and Montalbano's (TRM) (1965a,b) rate-of-return model into a complete and general model of economic profitability for investment decision-making. Specifically, TRM's assumptions are relaxed and a project rate of return is derived, expressing the project's overall economic profitability; direct relations among rates, costs of capital and net present value are supplied. The various value drivers are identified and isolated, and the NPV is decomposed into financing NPV and investment NPV. The approach allows for any pattern of financing rates, investment rates, and costs of capital. Relations with old literature and new literature on rates of return are shown: the link between them is obtained by making use of the *mean operator* (i.e., affine combinations of rates) and via the one-to-one correspondence between rates and invested capitals.

Keywords and phrases. Capital investment analysis, rate of return, net present value, financing, average.

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1 Introduction

Although the internal rate of return (IRR) model is not so robust a model, theoretically and practically,[1] it is nonetheless massively used by managers and professionals for investment decision-making. Soon after its inception (Fisher, 1930; Boulding 1935), scholars have long debated about difficulties of the IRR, some of them explicitly claiming that the problems are the sign of a general inadequacy of the IRR to capture the idea of a rate of return. For example, Hirshleifer (1958) wrote: "The fact that the use of [the IRR] leads to the correct decision in a particular case or a particular class of cases does not mean that it is correct in principle" (p. 348) and "These instances of failure of the multi-period internal-rate-of-return rule . . . are, of course, merely the symptom of an underlying erroneous conception" (p. 349). Bailey (1959) recognized that there exist infinitely many sequences of varying period rates, the IRR sequence (with constant rate) being only one of them, bearing no more importance than any other sequence: "This is an example of the "paradox" that have attracted so much attention in connection with investment decision criteria. It should be evident, however, that this paradox is merely an accident of the simplifying device of dealing with a single long-term rate of interest, and that it has no special importance" (pp. 478-479); he acknowledged that "recognition of the correct general solution of the investment problem has been hindered by the habit of thinking in terms of a single, long-term rate of interest" (p. 477). Herbst (1978) underscored that the mixed nature of IRR arising in mixed projects (rate of return/rate of cost) shows that IRR, in general, "is not a proper measure of return on investment. *This is a crucial criticism of the IRR − even though it may be unique, and real in the mathematical sense, this in itself is not a sufficient condition for it to be a correct measure of return on investment*" (p. 367, italics in original) and "a unique, real IRR is no assurance that the rate obtained is a proper measure of return on investment (or cost of financing)" (p. 369). Some scholars went as far as to propose alternative methods (e.g., Solomon 1956; Karmel 1959; Weingartner, 1966; Arrow and Levhari, 1969; Ramsey, 1970). Particularly relevant is Teichroew, Robichek and Montalbano's (1965a,b) proposal; though not widespread in finance, it is still well appreciated, both in academic papers (e.g., Chiu and Garza Escalante, 2012) and in textbooks (Herbst, 2002; Park, 2011; Blank and Tarquin, 2012). Teichroew, Robichek and Montalbanno (TRM, hereafter) presented an NPV-consistent model based on two rates,

[1] See Magni (2013) for a compendium of eighteen flaws of the IRR.

a financing rate and an investment rate, either of which is exogenously set equal to the cost of capital and the other is endogenously derived. By comparing the latter with the former, value creation (i.e., wealth increase) is signalled.

In the last decade, some important contributions have shed light on the issue and given fresh insights. Hazen (2003, 2009) set a direct relation between internal rate of return (IRR) and the net present value (NPV), the link being provided by IRR-implied investment balances. In particular, the IRR is a rate of return associated with an implied overall capital: if the capital is positive (negative), the project is interpreted as an investment (borrowing) and the IRR is a rate of return (cost); in such a way, any one IRR, associated with its own implied capital, can be interchangeably used for decision-making. Hartman and Schafrick (2004) made use of differential calculus and intervals of monotonicity of the NPV function to identify the "relevant" rate of return from a set of IRRs and to identify the nature of a project as an investment or a borrowing.[2] In the last paragraph of their paper, the authors called for future research bringing forth a link between the past tradition and the recent findings: "As for future research, it would be interesting to more formally tie this method to the methods of Hazen (2003) and Teichroew, Robichek, and Montalbano (1965). Perhaps the link may lie in investment balances" (p. 157). In this respect, Magni (2010) partially accomplished the task: consistently with Bailey (1959), who underscored the role of sequences of period rates (as opposed to a single, long-term return rate), Magni (2010) showed that the financial nature of a project depends on the choice of one among infinite sequences of project balances (capitals) and, therefore, on the choice of one among infinite sequences of period rates. The capital-weighted mean of a selected sequence of period rates, called Average Internal Rate of Return (AIRR), correctly signals value creation. This idea is made more explicit in Magni (2013), where a project is redefined as a vector of estimated cash flows and capitals,[3] which is associated with a unique rate of return.

However, the link to the TRM model, invoked by Hartman and Schafrick (2004), is a task

[2]Some interesting contributions have been devoted to complex-valued rates of return: Osborne (2010) showed that the product of the absolute values of all the IRRs (complex-valued as well as real-valued IRRs) can be used for accept/reject decisions. Pierru (2010) showed that complex-valued rates can be given economic significance. Hazen (2003) succeeded in extending his acceptability rule to complex-valued rates and capitals.

[3]The determination of capitals is a matter of estimation, analogous to the estimation of cash flows. As Magni (2013) underlies, estimation of cash flows often derives from estimation of capitals and incomes (see Titman and Martin, 2011. See also F11 and F15 in Magni, 2013).

which still waits to be carried out. This paper just supplies this missing link. We develop the TRM model, complete it and generalize it by making use of the same tool Magni (2010, 2013) used for developing the AIRR paradigm: the mean operator.[4] We then realize an average-based TRM model, which (i) removes the stringent assumption according to which either the project financing rate or the project investment rate is equal to the cost of capital, (ii) condenses the above-mentioned pair of rates into a rate of return capturing the whole project's economic profitability, (iii) provides direct relations among rates, NPV and cost of capital, (iv) highlights the role of both capitals and rates as value drivers, (v) divides the project's NPV into two shares, one generated in the investment periods (investment NPV), the other one generated in the financing periods (financing NPV), (vi) generalizes the approach allowing for any pattern of financing rates, investment rates, costs of capital, (vii) unearthes the existence of two project costs of capital: the financing cost of capital, applied in the financing periods, and the investment cost of capital, applied in the investment periods.

Not surprisingly, our average-based approach turns out to be equivalent to Magni's (2010, 2013) approach. The difference lies in the fact that we start from (financing and investment) rates to obtain project balances (i.e., capitals), whereas Magni (2010, 2013) starts from capitals to derive period rates. But there is a one-to-one correspondence between the vector of capitals and the vector of period rates, so the two approaches are but two sides of the same coin.

Old and new ideas, seemingly disparate, are naturally encompassed in this average-based TRM model. In particular, Bailey's (1959) suggestion of fixing all period rates equal to the equilibrium rates except one is naturally encompassed in our average-based model, and leads to a significant market-determined project rate of return, which is consistent with security pricing in an efficient market; Chiu and Garza Escalante's (2012) "Generalized Relative Rate of Return" is embodied in our average-based model as a particular case.

The remainder of the paper is structured as follows. Section 2 presents some preliminary notions and TRM's two fundamental theorems, one of which states the existence and unique-

[4]Strictly speaking, we will use *affine combination* (of rates), i.e. linear combinations with (positive or negative) weights summing to 1. However, if a weight is negative, one can always charge a rate with the opposite sign in order to keep positive signs for all the weights. In such a way, the affine combination can always be viewed as a convex combination, that is, a weighted mean.

ness of an invertible investment-rate function, the other one presents two NPV-consistent acceptability rules based on comparison of the project investment rate (or project financing rate) with the cost of capital. In section 3 we complete the TRM model; in particular, we remove the stringent assumption according to which either rate must be equal to the cost of capital and present an acceptability rule which include TRM's acceptability rules as particular cases. The financing and investment rates are condensed into a project rate of return, the NPV is decomposed into financing NPV and investment NPV, and the results are reframed in terms of markups and excess rates. Section 4 allows for varying costs of capital; we assume that financing rates and investment rates are variable but move in line with the market rates. We prove the existence and uniqueness of an invertible investment-markup function which generalizes the invertible investment-rate function introduced by TRM (1965a,b). Chiu and Garza Escalante's (2012) "Generalized Relative Rate of Return" is found to be a particular value taken on by the investment-markup function. Section 5 allows for any possible patterns of borrowing rates and investment rates, irrespective of the term structure of interest rates. Bailey's (1959) hint at value creation in one single period is studied in detail: it is shown that the result is the same as that which would be obtained in an efficient market, that is, in a market in which security prices quickly adjust to new information. Section 6 shows that the approach is logically equivalent to the AIRR approach and that the average-based TRM model can be interpreted as an 'AIRR-based' model. Some concluding remarks end the paper.

2 Preliminary results

Consider a market in equilibrium (i.e., no arbitrage opportunities exist) and let $\vec{r} = (r_1, \dots, r_n)$ describe the term structure of interest rates. The rate r_t is the forward rate (or equilibrium rate) holding between $t-1$ and t. Consider any asset a (e.g., a firm, a project, a security) whose cash flow stream is $\vec{a} = (a_1, \dots, a_q), q \in \mathbb{N}$. The value V_t^a of such an asset at time $t = 0, 1, \dots, q-1$ is defined as the price at which the asset (or a twin security) is traded in the market:

$$V_t^a = V_t^a(\vec{r}) = \sum_{j=t+1}^{q} \frac{a_j}{(1+r_{t+1})(1+r_{t+2}) \cdot \ldots \cdot (1+r_j)} \qquad V_q^a = 0. \tag{1}$$

Consider now a project, denoted as p, consisting of the cash-flow vector $\vec{f}=(f_0,f_1,\dots,f_n),\ n\in\mathbb{N}$. The project's net present value is the difference between the project's value at time 0 and the project cost $c_0=-f_0$:

$$NPV=NPV(\vec{r})=V_0^p-c_0=f_0+\sum_{t=1}^{n}\frac{f_t}{(1+r_1)(1+r_2)\cdot\ldots\cdot(1+r_t)}.$$

We assume that cash flows are expressed as certainty equivalents, but the formal analysis is unvaried if cash flows are thought of as expected values of stochastic variables; in such a case, r_t is a risk-adjusted rate of return which incorporates a premium for risk, and V_t^a represents the market price, at time t, of a portfolio which replicates the asset's cash flows in every state of nature (see Mason and Merton, 1985; Smith and Nau, 1995).

A project p is worth undertaking (economically profitable) if and only if $NPV(\vec{r})>0$. In this case, value is created (i.e., wealth is increased) for the firm's shareholders. The equilibrium rate r_t represents the cost of capital (COC, hereafter) in the t-th period. Throughout the paper, we will use the terms "equilibrium rate", "market rate", "cost of capital" as synonyms.

Vector $\vec{\imath}=(\imath_1,\dots,\imath_n)\in\mathbb{R}^n$ is a vector of internal rates of return if $NPV(\vec{\imath})=0$:

$$f_0+\sum_{t=1}^{n}\frac{f_t}{(1+\imath_1)(1+\imath_2)\ldots(1+\imath_t)}=0. \tag{2}$$

The vector $\vec{\jmath}=(\jmath,\jmath,\dots,\jmath)$ is a particular case of internal-rate-of-return vector with a constant period rate. The common value $\jmath$ is called "internal rate of return" (IRR). In this case, (2) reduces to

$$f_0+\sum_{t=1}^{n}\frac{f_t}{(1+\jmath)^t}=0. \tag{3}$$

The capital invested in the project at the beginning of the t-th period is recursively defined as

$$c_t=c_{t-1}(1+\imath_t)-f_t \qquad\qquad c_0=-f_0, \qquad c_n=0. \tag{4}$$

A positive capital is interpreted as an investment (i.e., lending money), a negative capital is interpreted as a financing (i.e. borrowing money).[5] Note that $\imath_t$ is the internal rate of return of the one-period project whose cash-flow vector is $(-c_{t-1},c_t+f_t)$:

$$-c_{t-1}+\frac{c_t+f_t}{1+\imath_t}=0.$$

[5]Project balance is project's capital changed in sign.

From (1), the value of any asset a fulfills the equilibrium relation

$$V_t^a = V_{t-1}^a(1+r_t) - a_t \qquad V_q = 0 \tag{5}$$

which is analogous to (4). Letting $T = \{1, \ldots, n\}$, we give the following

Definition 2.1. A project is *pure* if and only if either $c_{t-1} \geq 0$ for every $t \in T$ or $c_{t-1} \leq 0$ for every $t \in T$. A project is *mixed* if and only if there exists $j, k \in T$ such that $c_{j-1} \cdot c_{k-1} < 0$.

Teichroew, Robichek and Montalbano (1965a,b) (hereafter, TRM) made use of a *financing* rate and an *investment* rate to derive the sequence $\vec{c} = (c_0, c_1, \ldots, c_{n-1})$ of interim capitals. Letting x and y denote the two rates, the capital is recursively defined as

$$c_t(x,y) = \begin{cases} c_{t-1}(x,y)(1+x) - f_t & \text{if } c_{t-1}(x,y) \leq 0 \\ c_{t-1}(x,y)(1+y) - f_t & \text{if } c_{t-1}(x,y) > 0 \end{cases} \tag{6}$$

with $c_0(x,y) := -f_0$. This definition is a particular case of (4) where

$$\imath_t = \begin{cases} x & \text{if } c_{t-1}(x,y) \leq 0 \\ y & \text{if } c_{t-1}(x,y) > 0 \end{cases} \tag{7}$$

We summarize the first result of TRM in the following

Theorem 2.1 (TRM1)**.** *The equation $c_n(x,y) = 0$ implicitly defines an investment-rate function $y = y(x)$ which is continuous and strictly increasing. Given that $y(x)$ is strictly increasing, the same equation implicitly defines a financing-rate function $x = x(y)$, which is continuous and strictly increasing as well, and represents the inverse function of $y(x)$.*

(See, in particular, TRM, 1965a, Theorems III-IV and Figures I(a)-I(b)). Note that any pair (x,y) satisfying the terminal condition $c_n(x,y) = 0$ unequivocally determines the sequence $\vec{c} = \Big(c_0(x,y), c_1(x,y), \ldots, c_{n-1}(x,y)\Big)$. The authors assume a constant equilibrium rate (i.e., $r_t = r$ for every t) and use TRM1 to show two NPV-consistent decision criteria.

Theorem 2.2 (TRM2)**.** *The following acceptability rules hold:*

Rule 1. Assume $x = r$. Then, a project is acceptable (i.e., $NPV > 0$) if and only if $y(r) > r$.

Rule 2. Assume $y = r$. Then, a project is acceptable (i.e., $NPV > 0$) if and only if $x(r) < r$.

(See also TRM, 1965b, p. 176). We will henceforth refer to these theorems as TRM1 and TRM2.

Consistently with TRM's framework, we explicitly partition the project in an investment (lending) region $T_I = \{t \in T : c_{t-1} > 0\}$ and a financing (borrowing) region $T_B = \{t \in T : c_{t-1} \leq 0\}$, where $T_B \cup T_I = T$. In the investment region, a capital is interpreted as an investment (the firm is loaning to the project), whereas in the borrowing region, a capital is interpreted as a financing (the project is loaning to the firm). From this point of view, $c_{t-1}(x, y), t \in T_I$, is the amount "invested" in the t-th period, while $|c_{t-1}(x, y)|, t \in T_B$ is the amount "borrowed" in the t-th period.

Note that (7) can also be framed as

$$\imath_t = \begin{cases} x & \text{if } t \in T_B \\ y & \text{if } t \in T_I \end{cases}$$

or, in vectorial form,

$$\vec{\imath} = (\imath_t = x \text{ for } t \in T_B, \quad \imath_t = y \text{ for } t \in T_I).$$

Definition 2.2. A pair (x, y) is said to be *admissible* if $c_n(x, y) = 0$

It is worth highlighting the often neglected fact that TRM present (not one but) two acceptability rules (see TRM 2), derived from two different admissible pairs: the authors suggest to assume *either* $x = r$ *or* $y = r$, so that two pairs of financing rate and investment rate can be interchangeably used by the evaluator; the two pairs are $\big(r, y(r)\big)$ and $\big(x(r), r\big)$, which mean

$$\imath_t = \begin{cases} r & \text{if } t \in T_B \\ y(r) & \text{if } t \in T_I \end{cases} \qquad \text{or} \qquad \imath_t = \begin{cases} x(r) & \text{if } t \in T_B \\ r & \text{if } t \in T_I. \end{cases}$$

In vectorial terms,

$$\vec{\imath} = (\imath_t = r \text{ for } t \in T_B, \quad \imath_t = y(r) \text{ for } t \in T_I) \quad \text{or} \quad \vec{\imath} = (\imath_t = x(r) \text{ for } t \in T_B, \quad \imath_t = y \text{ for } t \in T_I)$$

which entail two different vectors of interim capitals (i.e., project balances):

$$\vec{c} = \Big(c_0, c_1\big(r, y(r)\big), \ldots, c_{n-1}\big(r, y(r)\big)\Big) \quad \text{or} \quad \vec{c} = \Big(c_0, c_1\big(x(r), r\big), \ldots, c_{n-1}\big(x(r), r\big)\Big).$$

It is up to the evaluator to choose either assumption.

3 Relaxing the assumptions and completing the TRM model

In this section we assume, consistently with TRM, $r_t = r$ for every $t \in T$. We first aggregate invested capitals on one side and borrowed capitals on the other side:

$$I := \sum_{t \in T_I} c_{t-1} v^t$$

is the total invested amount and

$$B := \sum_{t \in T_B} |c_{t-1}| v^t$$

is the total borrowed amount. The net capital invested is

$$I - B = \sum_{t \in T} c_{t-1} v^t$$

where $v^t := (1+r)^{-t}$ denotes the current value of one monetary unit available at time t. Evidently, if $I > B$, the firm loans the project more than it borrows from it. If, instead $I < B$, the firm borrows from the project more than it loans to it. This triggers the following

Definition 3.1. If $I > B$, the project is a net investment; if $I < B$, the project is a net financing (borrowing).

Theorem 3.1. *Let (x, y) be an admissible pair. Then, the NPV can be partitioned into an "investment NPV" and a "financing NPV":*

$$NPV(r) = \underbrace{\big(y - r\big)I}_{\textit{investment } NPV} + \underbrace{(r - x)B}_{\textit{financing } NPV} \tag{8}$$

where $I = I(x, y)$ and $B = B(x, y)$.

Proof.

$$\begin{aligned} NPV(r) = \sum_{t=0}^{n} f_t v^t &= \sum_{t=0}^{n} (f_t - c_t + c_t) v^t \\ &= \sum_{t=1}^{n} \Big(v^t (f_t + c_t) - v^{t-1} \cdot c_{t-1} \Big) \\ &= \sum_{t \in T_B} \Big(v^t \cdot c_{t-1}(1 + x) - v^{t-1} \cdot c_{t-1} \Big) + \sum_{t \in T_I} \Big(v^t \cdot c_{t-1}(1 + y) - v^{t-1} \cdot c_{t-1} \Big). \end{aligned} \tag{9}$$

Using the fact that $v^t \cdot c_{t-1}(1+y) - v^{t-1} \cdot c_{t-1} = y \cdot c_{t-1}v^t - r \cdot c_{t-1}v^t$ for $t \in T_I$ and, analogously, $v^t \cdot c_{t-1}(1+x) - v^{t-1} \cdot c_{t-1} = x \cdot c_{t-1}v^t - r \cdot c_{t-1}v^t$ for $t \in T_B$,

$$NPV(r) = \sum_{t \in T_B} xc_{t-1}v^t + \sum_{t \in T_I} yc_{t-1}v^t - \Big(\sum_{t \in T_B} rc_{t-1}v^t + \sum_{t \in T_I} rc_{t-1}v^t \Big) \tag{10}$$

whence

$$NPV(r) = yI - xB - r(I - B) \tag{11}$$

Equation (8) is just a reframing of (11). □

Theorem 3.1 enables the evaluator to isolate the drivers of value creation: $NPV_I(r) = I(y - r)$ is the investment NPV, while $NPV_B(r) = B(r - x)$ is the financing NPV:

$$NPV(r) = NPV_I(r) + NPV_B(r). \tag{12}$$

In other words, the project's overall economic profitability is the result of two joint effects: one captures the value created by the investment side, the second one captures the value created by the borrowing side. As for the former, investors earn $y-r$ per each unit of capital invested I in the lending periods. A positive (negative) sign of the excess rate indicates that the return rate is greater (smaller) than the COC. Analogously, in the borrowing periods, investors earn $r-x$ per each unit of borrowed capital B. A positive (negative) sign means that the investor is borrowing from the project at a rate that is smaller (greater) than the market rate. Thus, the equilibrium rate r acts as a benchmark for investment in the investment periods, whereas it acts as a benchmark for financing in the financing periods: in the former case, r is a foregone rate of return (investors might invest the same amount I at the rate r); in the latter case, r is a foregone rate of cost (investors might borrow the same amount B at the rate r). Therefore, comparison of y and r informs about economic profitability on the investment side, while comparison of x and r informs about economic profitability on the borrowing side.

Inspecting eq. (8), the following acceptability criterion is straightforward.

Corollary 3.1. *For any admissible pair (x, y), the following conditions hold:*

— if $y > r > x$, then the project should be undertaken;

— if $y < r < x$, then the project should be rejected.

Whenever the assumptions made in the above corollary are not fulfilled, the two NPVs (NPV_I) and NPV_B) have different signs and the net effect will depend not only on the respective excess rates $y-r$ and $r-x$ but also on the capital bases on which the excess rates are applied − i.e., on I and B.

The twofold information provided by the financing and investment sides can be condensed into a single relative measure of worth, capturing the whole project's economic profitability.

Theorem 3.2. *For any admissible pair, there exists a unique rate of return $\imath$, which is an affine combination of x and y, whose weights are represented by the proportion of investment and financing on the overall capital involved:*

$$\imath = \frac{yI - xB}{I - B} \tag{13}$$

where $I = I(x, y)$ and $B = B(x, y)$. Also,

$$\text{If the project is a net investment, then } NPV(r) > 0 \quad \text{if and only if} \quad \imath > r \tag{14a}$$

$$\text{If the project is a net financing, then } NPV(r) > 0 \quad \text{if and only if} \quad \imath < r. \tag{14b}$$

Proof. The amount $yI - xB$ represents the project's net return (difference between return accrued on invested capital and interest accrued on borrowed capital); dividing it by the net capital invested $I - B$, one obtains the net return per unit of net capital invested, which just expresses the overall project's *rate* of return (or rate of cost, if $I < B$). Using (13), eq. (11) becomes

$$NPV(r) = (I - B)(\imath - r) \tag{15}$$

whence (14). □

Practically, the steps for computing the project's rate of return and establish whether the project is worth undertaking are the following ones:

1. pick the financing rate and the investment rate and compute the invested capital I and the borrowed amount B
2. compute the capital-weighted mean of the two rates (this is the project rate of return)
3. compare the project rate of return with the cost of capital to determine acceptability.

Note that step 2. can be replaced by

$$\imath = r + \frac{NPV(r)}{I - B} \tag{16}$$

which, derived from (15), is equivalent to (13).

Remark 3.1. TRM2 can be directly derived from Theorem 3.1 by picking $x = r$ (Rule 1) or $y = r$ (Rule 2). Consider $x = r$, which implies that the project investment rate is $y(r)$. The invested amount is then $I = I\big(r, y(r)\big)$, and (8) reduces to

$$\begin{aligned} NPV(r) &= I\big(r, y(r)\big) \cdot \big(y(r) - r\big) + B\big(r(y(r)\big) \cdot 0 \\ &= I(r, y(r)) \cdot \big(y(r) - r\big). \end{aligned} \tag{17}$$

In such a way, the excess investment rate $y(r) - r$ has the same sign as the NPV (since $I(r, y(r)) > 0$ by definition). Assuming, by contrast, $y = r$, the project financing rate $x = x(r)$ is derived. In this case, (8) may be written as

$$\begin{aligned} NPV(r) &= I\big(x(r), r\big) \cdot 0 + B\big(x(r), y\big) \cdot \big(r - x(r))\big) \\ &= B(x(r), r) \cdot \big(r - x(r)\big). \end{aligned} \tag{18}$$

so the excess financing rate $r - x(r)$ has the same sign as $NPV(r)$ (since $B(x(r), r) > 0$ by definition).

It is now clear that, to impose the requirement that x be equal to r, means to exogenously impose that the financing side of the project become a zero-NPV course of action and that the rate of return on the investment side be $y(r)$. Alternatively, to impose the requirement that y be equal to r means to impose that the investment side of the project be a zero-NPV course of action and that the financing rate be $x(r)$. Formally, either choice can be made (resulting in (17) or (18)), depending on whether one aims to stress the financing side or the investment side of the project and consistently use the cost of capital as a benchmark *investment* rate or as a benchmark *financing* rate.

Therefore, the hub of TRM's model is that of nullifying (i.e., neutralizing) one of the two sides so as to shift the whole creation of value to the other side. The authors' *ad hoc* strategy is an ingenious device to obtain a pure IRR having the meaning of rate of cost (borrowing side) and a pure IRR having the meaning of a rate of return (investment side). One of the two IRRs is just the equilibrium rate r, the other one is endogenously derived from it. Armed with these two pure IRRs, one of which is irrelevant for value creation, NPV consistency is ensured by comparing the other IRR with the equilibrium rate.

More generally, value creation or destruction depends on both the financing side and the investment side: part of the wealth increase occurs in the investment periods, and part of it is created in the financing periods. However, in TRM's world the evaluator has only two (mandatory) choices: $x = r$ or $y = r$: if one removes these assumptions, the TRM model as such breaks down, for consistency with NPV is not guaranteed any more. In contrast, we have just shown that, by employing the mean operator, the financing rate and the investment rate can be freed from TRM's restrictive assumptions: x and y need not be equal to r. That is, both $x \neq r$ and $y \neq r$ are allowed, which means that both the investment and the financing region contribute to determine the investors' wealth increase or decrease. As a result, Theorems 3.1-3.2 (i) disengage x and y from r, (ii) quantify the contributions of financing and investment side, (iii) give account of the specific role of x, y and r, (iii) provide the evaluator with the project rate of return. As for the latter point, it should be clear that neither x nor y is the project rate of return; y represents the rate of return of the investment side and x is the rate of cost of the financing side, respectively. So, each of them only tells part of the story. Both x and y give a contribution, and Theorem 3.2 just supplies the important piece of information which is not present in the original TRM's model.

Remark 3.2. TRM's Rule 1 and Rule 2 bring about different quantifications of the investment and financing side:

$$B\big(r, y(r)\big) \neq B\big(x(r), r\big)$$
$$I\big(r, y(r)\big) \neq I\big(x(r), r\big)$$

as well as different investment rates and different financing rates: $(r, y(r)) \neq (x(r), r)$. This in turn entails the existence of two project rates of return, one for each choice: if x is fixed first ($x = r$), then the project rate of return is

$$\imath(r, y) = \frac{yI(r, y) - rB(r, y)}{I(r, y) - B(r, y)}$$

where $y = y(r)$; if y is fixed first ($y = r$), then the project rate of return is

$$\imath(x, r) = \frac{rI(x, r) - xB(x, r)}{I(x, r) - B(x, r)}$$

where $x = x(r)$. Consequently, from (15)

$$NPV(r) = \big(I(r,y) - B(r,y)\big) \cdot \big(\imath(r,y) - r\big) \tag{19a}$$

$$NPV(r) = \big(I(x,r) - B(x,r)\big) \cdot \big(\imath(x,r) - r\big). \tag{19b}$$

We can then state the following

Corollary 3.2. *In the TRM model, the project rate of return is either $\imath(r,y)$ or $\imath(x,r)$ depending on whether the evaluator assumes $x = r$ or $y = r$. Also,*

$$NPV(r) > 0 \quad \textit{if and only if} \quad \imath(r,y) > r \tag{20a}$$

$$NPV(r) > 0 \quad \textit{if and only if} \quad \imath(x,r) > r \tag{20b}$$

where $y = y(r)$ and $x = x(r)$ (if the project is a net borrowing, the sign of the second equalities of (20) *is reversed).*

Remark 3.3. We have claimed that x and y represent the IRR of the investment side and the IRR of the financing side, respectively. This derives from the fact that $NPV_I(r) = (y-r)I = 0$ if and only if $y = r$ and $NPV_B(r) = (r-x)B = 0$ if and only if $x = r$.[6] This makes $\imath$ an average of two internal rates of return. Therefore, it is an "average internal rate of return" (AIRR) and lies on the iso-value line (the graph of the AIRR function. See Magni, 2010, 2013).

Any pair (x, y) unequivocally defines a pair (ϱ, φ) of financing markup rate and investment

[6] Alternatively, in terms of cash flows, note that $c_n(x,y) = 0$, may be rewritten as

$$f_0 + \frac{f_1}{(1+x)^{\alpha_1}(1+y)^{\beta_1}} + \frac{f_2}{(1+x)^{\alpha_2}(1+y)^{\beta_2}} + \ldots + \frac{f_n}{(1+x)^{\alpha_n}(1+y)^{\beta_n}} = 0$$

where α_j represents the number of financing periods and β_j represents the number of investment periods between time 0 and time j, so that $\alpha_j + \beta_j = j$, $j = 1, \ldots, n$. Therefore,

$$f_0 + \frac{f_1^x}{(1+y)^{\beta_1}} + \frac{f_2^x}{(1+y)^{\beta_2}} + \ldots + \frac{f_n^x}{(1+y)^{\beta_n}} = 0$$

where $f_t^x := f_t(1+x)^{-\alpha_t}$ or, equivalently,

$$f_0 + \frac{f_1^y}{(1+x)^{\alpha_1}} + \frac{f_2^y}{(1+x)^{\alpha_2}} \ldots + \frac{f_n^y}{(1+x)^{\alpha_n}} = 0$$

where $f_t^y := f_t(1+y)^{-\beta_t}$.

markup rate (markdown if negative)

$$\varphi := \frac{y-r}{1+r} \tag{21a}$$

$$\varrho := \frac{x-r}{1+r}. \tag{21b}$$

such that

$$1+y = (1+r)(1+\varphi) \Longleftrightarrow y = r + \varphi(1+r)$$

$$1+x = (1+r)(1+\varrho) \Longleftrightarrow x = r + \varrho(1+r).$$

We can then reframe the project rate of return in terms of markup:

$$\imath = r + \vartheta(1+r) \iff 1+\imath = (1+r)(1+\vartheta) \tag{22}$$

where ϑ is a suitable weighted mean of the markups:

$$\vartheta = \vartheta(\varrho, \varphi) = \frac{\varphi \cdot I - \varrho \cdot B}{I - B}. \tag{23}$$

Therefore,

$$NPV(r) = \big(\varphi I - \varrho B\big)(1+r) \tag{24}$$

and (14) can be reframed as

$$\textit{If the project is a net investment, then } NPV(r) > 0 \quad \textit{if and only if} \quad \vartheta(\varrho, \varphi) > 0 \tag{25a}$$

$$\textit{If the project is a net financing, then } NPV(r) > 0 \quad \textit{if and only if} \quad \vartheta(\varrho, \varphi) < 0. \tag{25b}$$

TRM's investment-rate function $y(x)$ unequivocally defines an investment-markup function $\varphi(\varrho)$, while the financing-rate function $x(y)$ unequivocally defines a financing-markup function $\varrho(\varphi)$ which is the inverse function of $\varphi(\varrho)$. The two particular cases studied by TRM (i.e., $x = r$ and $y = r$) become $\varrho = 0$ and $\varphi = 0$; in the former case, $\varphi = \varphi(0)$ and $NPV(r) = \varphi(0) \cdot I(0, \varphi(0))$, so that Rule 1 becomes

$$NPV(r) > 0 \quad \textit{if and only if} \quad \varphi(0) > 0;$$

in the latter case, $\varrho = \varrho(0)$ and $NPV(r) = -\varrho(0) \cdot B(\varrho(0), 0)$ so that Rule 2 becomes

$$NPV(r) > 0 \quad \textit{if and only if} \quad \varrho(0) < 0.$$

Remark 3.4. Let $I' := \sum_{t \in T_I} c_{t-1} v^{t-1}$ and $B' := \sum_{t \in T_B} c_{t-1} v^{t-1}$. Then, the project markup can be reframed as

$$\vartheta = \frac{\varphi I' - \varrho B'}{I' - B'},$$

since $I' = I(1+r)$ and $B' = B(1+r)$, and (24) can be written as

$$NPV(r) = \varphi I' - \varrho B' \tag{26}$$

(We will make extensive use of I' and B' in section 4.)

Any pair (x, y) also unequivocally defines a pair (ρ, ϕ) of financing excess rate and investment excess rate:

$$\rho := x - r \tag{27a}$$

$$\phi := y - r. \tag{27b}$$

The project rate of return can be reframed as

$$\imath = r + \theta \tag{28}$$

where

$$\theta := \theta(\rho, \phi) = \frac{\phi I - \rho B}{I - B} = \frac{\phi I' - \rho B'}{I' - B'}$$

is the weighted mean of the two excess rates, so (14) may be reframed as

$$\textit{If the project is a net investment, then } NPV(r) > 0 \quad \textit{if and only if} \quad \theta > 0 \tag{29a}$$

$$\textit{If the project is a net financing, then } NPV(r) > 0 \quad \textit{if and only if} \quad \theta < 0. \tag{29b}$$

The investment-rate function $y(x)$ unequivocally defines an investment-excess-rate function $\phi(\rho)$ and the financing-rate function $x(y)$ unequivocally defines a financing-excess-rate function $\rho(\phi)$. The particular cases studied by TRM ($x = r$ and $y = r$) become $\rho = 0$ and $\phi = 0$; in the former case, $\phi = \phi(0)$ and $NPV(r) = \phi(0) \cdot I(0, \phi(0))$ so that Rule 1 becomes

$$NPV(r) > 0 \quad \textit{if and only if} \quad \phi(0) > 0;$$

in the latter case, $\rho = \rho(0)$ and $NPV(r) = -\rho(0) \cdot B(\rho(0), 0)$ so that Rule 2 becomes

$$NPV(r) > 0 \quad \textit{if and only if} \quad \rho(0) < 0.$$

Excess rates and markups supply the same information, albeit in different forms: an excess rate enters an arithmetic (i.e., additive) relation, whereas a markup enters a geometric (i.e. multiplicative) relation:

$$1 + \imath = (1 + r) + \theta \tag{30}$$

$$1 + \imath = (1 + r)(1 + \vartheta). \tag{31}$$

Markups and excess rates relate each other in the following way:

$$\rho = \varrho(1 + r)$$
$$\phi = \varphi(1 + r)$$
$$\theta = \vartheta(1 + r)$$

Therefore, assuming $x = r$, the excess return ϕ can be interpreted as the maximum financing rate over and above the equilibrium rate that investors can accept to fund the invested amounts in the lending periods, while φ can be viewed as the investment markup which is necessary to charge the equilibrium rate in order to generate y. Analogously, assuming $y = r$, ρ can be interpreted as the minimum lending rate over and above the equilibrium rate that investors can accept to invest the borrowed amounts, while ϱ can be viewed as the financing markup which is necessary to charge the equilibrium rate in order to give rise to x. When both x and y differ from r, θ represents the maximum *average* financing rate over the equilibrium rate r which is acceptable by the investors (if $I > B$) or the minimum *average* investment rate that is acceptable by the investors (if $B > I$). Analogously, ϑ represents the *average* investment or financing markup on the equilibrium rate.

Example 1. Table 1 supplies the data for a project such that $\vec{f} = (300, -280, -100, -330, 430)$. Suppose the financing rate is $x = 0.06$. This implies that the project investment rate is $y(0.06) = 0.0886$.[7] Also, $T_B = \{1, 2\}$ and $T_I = \{3, 4\}$, so $\vec{\imath} = (0.06, 0.06, 0.0886, 0.0886)$. Assuming $r = 8\%$, the net return is $0.0886 \cdot 337.8 - 0.06 \cdot 310.4 = 11.3$ and the net invested capital is $337.8 - 310.4 = 27.4$, so the project rate of return is $\imath = 11.3/27.4 = 0.4124 > 0.08$: the project is worth undertaking. The NPV can be computed as $NPV = 27.4 \cdot (0.04124 - 0.08) = 9.1$, which is decomposed as follows. Most of the created value is generated in the financing periods: $NPV_B = 310.4 \cdot (0.08 - 0.06) = 6.2$. In the investment periods, the value

[7] Note that the IRR of this project does not exist.

created is $NPV_I = 337.8 \cdot (0.0886 - 0.08) = 2.9$. This means that about 68% of the total value created comes from financing, while the remaining 32% is due to investment.

Consistently, the financing excess rate is $\rho = 0.06 - 0.08 = -0.02$, which means that the investor borrows at a financing rate which is smaller than the cost of capital by two percentage points. This favorable situation is enhanced by an investment rate which is greater than the cost of capital; in particular, the excess investment rate is $\phi = 0.0886 - 0.08 = 0.0086$. Accordingly with these figures, the financing markdown is $\varrho = -0.02/1.08 = -0.0185$ and the investment markup is $\varphi = 0.0086/1.08 = 0.0079$. The project's excess return and the project's markup are, respectively,

$$\theta = \frac{0.0086 \cdot 364.8 - (-0.02) \cdot 335.2}{29.6} = 0.3324$$

$$\vartheta = \frac{0.0079 \cdot 364.8 - (-0.0185) \cdot 335.2}{29.6} = 0.3077$$

and, as expected, $0.3324 = 0.3077 \cdot (1.08)$. The positive signs of θ and ϑ signal value creation.[8]

Table 1: Description of example 1

time t	cash flow f_t	financing $\lvert c_{t-1} \rvert$	investment c_{t-1}	rate (x, y)	COC r	excess rate (ρ, ϕ)	markup (ϱ, φ)
0	300	300		6%	8%	−2%	−1.85%
1	−280	38		6%	8%	−2%	−1.85%
2	−100		59.7	8.86%	8%	0.86%	0.79%
3	−330		395	8.86%	8%	0.86%	0.79%
4	430						
	NPV=9.1	B=310.4	I=337.8	$\imath$=41.24%		$\theta = 33.24\%$	$\vartheta = 30.77\%$

If one selected $x = r = 8\%$, as in TRM's Rule 1, one nullified the value which is created in the financing periods, shifting value off of the financing region onto the investment regions. In this case, $y(0.08) = 0.1078$ would be obtained, and, consequently, $I = 327, B = 315.5$. Therefore, the assumed net invested capital would be equal to $I - B = 11.45$, and the net

[8] We have used I' and B' for computing ϑ and θ: the same result is obtained by using I and B.

return turned out to be 10, whence the implied project rate of return: $\imath = 10/11.45 = 0.875 > 0.08$. Hence,

$$NPV = 11.45 \cdot (0.875 - 0.08) = 9.1 = 327 \cdot (0.1078 - 0.08) = NPV_I.$$

In terms of markup, TRM's Rule 1 assumes $\varrho = 0$, which means $\varphi(0) = 0.0258$, so that $y = 0.08 + 0.0258 \cdot (1.08) = 0.1078$.

In contrast, if one selected $y = r = 8\%$, as in TRM's Rule 2, one would shift the value off of the investment region onto the financing region, finding $x(0.08) = 0.05$, $I = 342.7$, $B = 307.9$, $\imath = 34.1\%$ and

$$NPV = 34.9 \cdot (0.0341 - 0.08) = 9.1 = 307.9 \cdot (0.08 - 0.05) = NPV_B.$$

In terms of markup, $y = 8\%$ means $\varphi = 0$, whence $\varrho(0) = -0.0274\%$, so that $x = 0.08 - 0.0274 \cdot (1.08) = 0.05$, as already seen.

Example 2. An investor is offered a transaction whereby he makes a down payment of \$2, another payment of \$5 at the end of two months, and a third payment of \$75 at the end of three months, in exchange for a payment of \$20 at the end of one month and a payment of \$70 at the end of four months. The borrowing rate is assumed to be $x = 7\%$, which implies $y(0.07) = 0.164$. The COC is $r = 20\%$. Table 2 describes the transaction.[9]

Table 2: Description of example 2

time t	cash flow f_t	financing $\lvert c_{t-1} \rvert$	investment c_{t-1}	rate (x, y)	COC r	excess rate (ρ, ϕ)	markup (ϱ, φ)
0	-2		2	16.4%	20%	-3.56%	-2.97%
1	20	17.7		7%	20%	-13%	-10.83%
2	-5	13.9		7%	20%	-13%	-10.83%
3	-75		60.1	16.4%	20%	-3.56%	-2.97%
4	70						
	NPV=1.55	B=20.3	I=30.7	$\imath$=35%		$\theta = 15\%$	$\vartheta = 12.5\%$

[9] Note that the IRR is equal to $\jmath = 834.61\%$.

The project's borrowing side creates value: $NPV_B = 20.3 \cdot (0.2 - 0.07) = 2.64$; the investment side destroys value: $NPV_I = 30.7 \cdot (0.164 - 0.2) = -1.09$. The former more than compensates the latter, so the net effect is favorable: $NPV = 2.64 - 1.09 = 1.55$. The net return is $0.164 \cdot 30.7 - 0.07 \cdot 20.3 = 3.6$; dividing by the net invested capital of 10.4, one gets the project rate of return: $\imath = 0.35 > 0.2 = r$. The NPV is found back as $NPV = 10.4 \cdot (0.35 - 0.20) = 1.55$. Consistently, the excess financing rate is $\rho = -0.13$, which means that the investor is able to borrow, at time 2 and 3, at a financing rate which is smaller than the market rate by 13 percentage points; the excess investment rate is $\phi = -0.0356$, which means that the investment made in the first and fourth periods are less profitable than they would be if funds were invested at the cost of capital by 3.56 percentage points. As seen, the net effect is positive: $\theta = 0.15 > 0$. In terms of marking up/down, one gets a financing markdown equal to $\varrho = -0.1083$ and an investment markdown equal to $\varphi = -0.0297$, so the project markup is $\vartheta = 0.125 > 0$.

Example 3. A firm, which does contract-based work, receives a purchase order from a customer, accompanied by an upfront payment of \$20,000 for the delivery of some goods after three months. The production begins after one month and involves expenditures of $\$22,000$ after one month, $\$80,000$ after two months. At the end of three months, the finished product is delivered. The customer settles the account with two installments of $\$90,000$ at the delivery and \$14,000 after one month. This implies $\vec{f} = (20, -22, -80, 90, 14)$ (in thousands). The upfront payment can be interpreted, from the perspective of the firm, as a non-interest-bearing liability, which means $x = 0$. This implies $y(0) = 0.2295$ and $T_B = \{1\}, T_I = \{2, 3, 4\}$, so that $\vec{\imath} = (0, 0.2295, 0.2295, 0.2295)$. Assuming a cost of capital of $r = 20\%$ the firm creates value in the first period, for the down payment made by the customer bears no interest payments: if the firm turned to the market to borrow the same amount, the firm would have to pay 20% interest; the excess financing rate is then $\rho = -0.2$ and the financing NPV turns out to be $NPV_B = 3,333$. The following periods are investment periods: the firm earns a 22.95% return, that is, it earns an excess rate equal to $\phi = 0.0295$ over and above the market rate; the investment NPV is NPV_I= 1,612, so the project is worth undertaking: $NPV = 3,333 + 1,612 = 4,946$. The project rate of return $\imath = 0.33 > 0.20$ and the excess rate of return is then $\theta = 0.13 > 0$. Given that ρ is negative, the firm gets a financing markdown, which is equal to $\varrho = -0.1667$; the investment markup is equal to $\varphi = 0.0246$. Note that two thirds of the NPV is generated by the upfront payment. If the latter were less than \$10,447, then the firm would not find it economically profitable to produce the product;[10] it should then find a different strategy to increase the NPV (e.g., by increasing the price, by anticipating the customer's last payment by one month, by reducing costs, etc.).

4 Generalizing the approach: varying rates and constant markups

In the TRM model x, y, r are constant. In this section, we generalize allowing for varying rates. Let $\vec{x} = (x_t, t \in T_B)$ and $\vec{y} = (y_t, t \in T_I)$ be vectors of borrowing rates and investment rates. The capital is recursively defined as

$$c_t(\vec{x}, \vec{y}) = \begin{cases} c_{t-1}(\vec{x}, \vec{y})(1 + x_t) - f_t & \text{if } c_{t-1}(\vec{x}, \vec{y}) \leq 0 \\ c_{t-1}(\vec{x}, \vec{y})(1 + y_t) - f_t & \text{if } c_{t-1}(\vec{x}, \vec{y}) > 0. \end{cases} \tag{32}$$

Let the (non-flat) term structure of interest rates be described by $\vec{r} = (r_1, ..., r_n)$ where r_t is the forward (equilibrium) rate between $t-1$ and t. Let $v^{t,s}$ be the value at s of a monetary unit available at time t: $v^{t,s} := \prod_{j=s+1}^{t}(1 + r_j)^{-1}$, $t \geq s$, with $v^{t,t} := 1$.

In a recent paper, Chiu and Garza Escalante (2012) (CG hereafter) retrieve the original TRM model and allow for varying COCs under the following assumptions:

(i) every financing rate x_t is set equal to the equilibrium rate r_t

(ii) the investment rate y_t is linked to the equilibrium rate; in particular, y_t is marked up by the same multiplicative amount φ

We summarize CG's result in the following

Theorem 4.1 (CG Theorem)**.** *If $x_t = r_t$ for every $t \in T_B$, then there exists a unique markup φ such that $y_t = r_t + \varphi(1 + r_t)$ for every $t \in T_I$ and*

$$NPV(\vec{r}) > 0 \qquad \textit{if and only if} \quad \varphi > 0. \tag{33}$$

CG call φ the "Generalized Relative Rate of Return". CG's citerion can be seen as a generalization of TRM's Rule 1, and it is consistent with it in that financing rates are made to coincide with the costs of capital in each period.

[10] With a down payment of \$10,447.09, the investment rate is $y(0) = 0.1758$, the NPV becomes zero and the project rate of return becomes 20%, equal to the COC.

In this section we keep CG's assumption (ii), but relax assumption (i) and allow for financing rates that vary in line with equilibrium rates as well; in particular, we assume that the financing rates are obtained by marking up the equilibrium rates by a constant multiplicative amount ϱ.

As a first step, we generalize TRM1 by showing that there exists a unique invertible markup function.

Theorem 4.2. *If $x_t = r_t + \varrho(1+r_t)$ for every $t \in T_B$ and $y_t = r_t + \varphi(1+r_t)$ for every $t \in T_I$, the equation $c_n(\vec{x}, \vec{y}) = 0$ implicitly defines a unique function $\varphi = \varphi(\varrho)$ which is continuous and strictly increasing. The same equation also defines a function $\varrho = \varrho(\varphi)$, itself strictly increasing.*

Proof. Consider a modified project p' with a cash-flow vector $\vec{f'} = (f'_0, f'_1, \ldots, f'_n)$ such that $f'_t := f_t / \prod_{k=1}^{t}(1 + r_k)$, $t = 1, \ldots, n$, $f'_0 = f_0$. Consider the recursive relation for p':

$$c'_t = c'_t(\varrho, \varphi) = \begin{cases} c'_{t-1}(1+\varrho) - f'_t & \text{if } c'_{t-1} \le 0 \\ c'_{t-1}(1+\varphi) - f'_t & \text{if } c'_{t-1} > 0 \end{cases}$$

with $c'_0 := -f'_0$. Remembering that the assumptions imply $1 + x_t = (1 + r_t)(1 + \varrho)$ and $1 + y_t = (1 + r_t)(1 + \varphi)$, the recursive relation for p is

$$c_t = c_t(\varrho, \varphi) = \begin{cases} c_{t-1}(1+r_t)(1+\varrho) - f_t & \text{if } c_{t-1} \le 0 \\ c_{t-1}(1+r_t)(1+\varphi) - f_t & \text{if } c_{t-1} > 0 \end{cases}$$

with $c_0 := -f_0$. We now show, by induction, that, $c'_t \prod_{k=1}^{t}(1+r_k) = c_t$ for every $t = 1, \ldots, n$. Let $t = 1$. We have $c'_0 = c_0 = -f_0$, so

$$c'_1 = \begin{cases} c'_0(1+\varrho) - f'_1 = -f_0(1+\varrho) - \frac{f_1}{1+r_1} & \text{if } c'_0 = c_0 \le 0 \\ c'_0(1+\varphi) - f'_0 = -f_0(1+\varphi) - \frac{f_1}{1+r_1} & \text{if } c'_0 = c_0 > 0. \end{cases}$$

Multiplying by $(1 + r_1)$, one gets $c'_1(1 + r_1) = c_1$. Let us assume that $c'_j \prod_{k=1}^{j}(1 + r_k) = c_j$ for some $j > 1$. Then,

$$c'_{j+1} = \begin{cases} c'_j(1+\varrho) - f'_{j+1} & \text{if } c'_j \le 0 \\ c'_j(1+\varphi) - f'_{j+1} & \text{if } c'_j > 0 \end{cases}$$

which implies

$$c'_{j+1} = \begin{cases} \frac{c_j}{\prod_{k=1}^{j}(1+r_k)}(1+\varrho) - f'_{j+1} & \text{if } c_j \leq 0 \\ \frac{c_j}{\prod_{k=1}^{j}(1+r_k)}(1+\varphi) - f'_{j+1} & \text{if } c_j > 0 \end{cases}$$

whence

$$c'_{j+1}\prod_{k=1}^{j+1}(1+r_k) = \begin{cases} c_j(1+r_{j+1})(1+\varrho) - f_{j+1} & \text{if } c_j \leq 0 \\ c_j(1+r_{j+1})(1+\varphi) - f_{j+1} & \text{if } c_j > 0 \end{cases}$$

which just means $c'_{j+1}\prod_{k=1}^{j+1}(1+r_k) = c_{j+1}$. Therefore, $c'_t\prod_{k=1}^{t}(1+r_k) = c_t$ for every t. In particular, $c'_n\prod_{k=1}^{n}(1+r_k) = c_n$ and, thus, the pairs (ϱ,φ) which satisfy $c'_n(\varrho,\varphi) = 0$ are exactly the pairs which satisfy $c_n(\varrho,\varphi) = 0$. Applying TRM1 to $\vec{f}'$, the thesis is obtained. □

Definition 4.1. A pair (ϱ,φ) is said to be *admissible* if and only if $c_n(\varrho,\varphi) = 0$.

We now show that a suitable average of the markups (affine combination of rates) captures the project's economic profitability, so generalizing both CG Theorem and eq. (26).

Theorem 4.3. *Let (ϱ,φ) be an admissible pair of markups. Then,*

$$NPV(\vec{r}) = \varphi I' - \varrho B' \tag{34}$$

where $I' := I'(\varrho,\varphi) = \sum_{t\in T_I} c_{t-1}v^{t-1,0}$, $B' := B(\varrho,\varphi) = \sum_{t\in T_B}|c_{t-1}|v^{t-1,0}$ and there is a unique project markup

$$\overline{\vartheta} = \overline{\vartheta}(\varrho,\varphi) = \frac{\varphi\cdot I' - \varrho\cdot B'}{I' - B'} \tag{35}$$

such that

$$NPV(\vec{r}) > 0 \quad \text{if and only if} \quad \overline{\vartheta} > 0 \tag{36}$$

(if $I' < B'$ the sign is reversed).

Proof. It suffices to note that (9) holds replacing r with r_t, $(1+x)$ with $(1+r_t)(1+\varrho)$, and $(1+y)$ with $(1+r_t)(1+\varphi)$, so that, after some algebraic manipulations,

$$NPV(\vec{r}) = \varphi\sum_{t\in T_I} c_{t-1}(1+r_t)v^{t,0} + \varrho\sum_{t\in T_B} c_{t-1}(1+r_t)v^{t,0} = \varphi I' - \varrho B'. \tag{37}$$

Dividing by $I' - B'$,

$$\frac{NPV(\vec{r})}{I' - B'} = \frac{\varphi I' - \varrho B'}{I' - B'} = \overline{\vartheta}.$$

The decision rule in (36) follows by noting that the right-hand side has the same sign as the NPV if and only if $I' > B'$. $\square$

The Generalized Rate of Return found by CG (2012) is the value taken on by the function $\varphi(\varrho)$ when $\varrho = 0$ (i.e., when $x_t = r_t$ in each period), so CG Theorem is embedded in Theorem 4.3. To show it, remember that $\varrho = 0$ implies $\varphi = \varphi(0)$, and (34) becomes

$$\varphi(0) \cdot I' = NPV(\vec{r}) > 0 \quad \textit{if and only if} \quad \varphi(0) > 0 \tag{38}$$

which is just eq. (33).

As seen, CG's choice $x_t = r_t$ generalizes TRM's Rule 1. Evidently, one can make the symmetric choice $y_t = r_t$ for every $t \in T_I$ and generalize TRM's Rule 2 by fixing the investment rate equal to the equilibrium rate in each investment period. Formally, this means $\varphi = 0$, so that $\varrho = \varrho(0)$ whence

$$-\varrho(0) \cdot B' = NPV(\vec{r}) > 0 \quad \textit{if and only if} \quad \varrho(0) < 0 \tag{39}$$

However, the choices $\varphi = 0$ and $\varrho = 0$ are only two possible cases. More generally, Theorem 4.3 allows the evaluator to cope with $\varphi \neq 0$ *and* $\varrho \neq 0$, which means $x_t \neq r_t$ for every $t \in T_B$ and $y_t \neq r_t$ for every $t \in T_B$. As Theorem 4.2 shows, there are infinitely many pairs (ϱ, φ) which plot on the implicit function defined by the equation $c_n(\varrho, \varphi) = 0$, and an appropriate mean of the two markups signals wealth creation.

Exploiting, again, the linearity of the mean operator, relations among rates, excess rates and markups are easy to obtain. The project financing rate and the project investment rate are the means of the period financing rates and the period investment rates, respectively:

$$\overline{x} = \frac{\sum_{t \in T_B} x_t \cdot |c_{t-1}| v^{t,0}}{\sum_{t \in T_B} |c_{t-1}| v^{t,0}} = \frac{\sum_{t \in T_B} x_t \cdot |c_{t-1}| v^{t,0}}{B} \tag{40a}$$

$$\overline{y} = \frac{\sum_{t \in T_I} y_t \cdot c_{t-1} v^{t,0}}{\sum_{t \in T_I} c_{t-1} v^{t,0}} = \frac{\sum_{t \in T_I} y_t \cdot c_{t-1} v^{t,0}}{I} \tag{40b}$$

where $c_{t-1} := c_{t-1}(\vec{x}, \vec{y})$, and, in turn, the project rate of return is the weighted mean of $\overline{x}$ and $\overline{y}$:

$$\bar{\imath} = \frac{\overline{y}I - \overline{x}B}{I - B}. \tag{41}$$

Analogously, we can define a *financing* cost of capital and an *investment* cost of capital by separately averaging out the r_t's in the borrowing periods ($t \in T_B$) and the r_t's in the investment periods ($t \in T_I$):

$$\overline{r}_B = \frac{\sum_{t \in T_B} r_t |c_{t-1}| v^{t,0}}{\sum_{t \in T_B} |c_{t-1}| v^{t,0}} = \frac{\sum_{t \in T_B} r_t |c_{t-1}| v^{t,0}}{B} \tag{42a}$$

$$\overline{r}_I = \frac{\sum_{t \in T_I} r_t c_{t-1} v^{t,0}}{\sum_{t \in T_I} c_{t-1} v^{t,0}} = \frac{\sum_{t \in T_I} r_t c_{t-1} v^{t,0}}{I}. \tag{42b}$$

In turn, the project cost of capital is the mean of $\overline{r}_B$ and $\overline{r}_I$:

$$\overline{r} = \frac{\overline{r}_I I - \overline{r}_B B}{I - B}. \tag{43}$$

Hence, the financing and investment excess rates are, respectively,

$$\overline{\rho} = \overline{x} - \overline{r}_B \tag{44}$$

$$\overline{\phi} = \overline{y} - \overline{r}_I \tag{45}$$

and the project excess return is

$$\overline{\theta} = \overline{\imath} - \overline{r} \tag{46}$$

which turns out to be, not surprisingly, a mean of excess financing rate and excess investment rate:

$$\overline{\theta} = \frac{\overline{\rho} I - \overline{\phi} B}{I - B}. \tag{47}$$

Further, after some algebraic passages, the following equalities are found: $I' = (1 + \overline{r}_I)$ and $B' = (1 + \overline{r}_B)$, so that

$$I' - B' = I(1 + \overline{r}_I) - B(1 + \overline{r}_B) = (I - B)(1 + \overline{r})$$

whence

$$(1 + \overline{x}) = (1 + \overline{r}_B)(1 + \varrho) \tag{48a}$$

$$(1 + \overline{y}) = (1 + \overline{r}_I)(1 + \varphi) \tag{48b}$$

$$(1 + \overline{\imath}) = (1 + \overline{r})(1 + \overline{\vartheta}) \tag{48c}$$

$$\overline{\rho} = \varrho(1 + \overline{r}_B) \tag{48d}$$

$$\overline{\phi} = \varphi(1 + \overline{r}_I) \tag{48e}$$

$$\overline{\theta} = \overline{\vartheta}(1 + \overline{r}) \tag{48f}$$

which generalize the relations found in section 3. So, one can interpret $\overline{\vartheta} = \overline{\vartheta}(\varrho, \varphi)$ as the rate which marks up the project cost of capital $\overline{r}$ to get the project rate of return $\overline{\imath}$. Equivalently, $\overline{\theta}$ can be interpreted as the maximum *average* borrowing rate (if $I > B$) or minimum *average* investment rate (if $I < B$) that the project can sustain without decreasing the investors' wealth. Owing to the above relations and Theorem 4.3, we can then state a generalized version of Theorems 3.1 and 3.2.

Theorem 4.4. *Let $(\vec{x}, \vec{y})$ be any pair of vectors satisfying $c_n(\vec{x}, \vec{y}) = 0$ and such that $x_t = r_t + \varrho(1 + r_t)$, $t \in T_B$, and $y_t = r_t + \varphi(1 + r_t)$, $t \in T_I$. Then, there exists a unique project rate of return $\overline{\imath}$ which is an affine combination of project investment rate $\overline{y}$ and project financing rate $\overline{x}$:*

$$\overline{\imath} = \frac{\overline{y}I - \overline{x}B}{I - B} \tag{49}$$

and a unique cost of capital, which is an affine combination of the investment COC and the financing COC:

$$\overline{r} = \frac{\overline{r}_I I - \overline{r}_B B}{I - B}. \tag{50}$$

The NPV is

$$NPV(r) = (I - B)(\overline{\imath} - \overline{r}) \tag{51}$$

whence the acceptability rule

$$\text{If the project is a net investment, then } NPV(r) > 0 \quad \text{if and only if} \quad \overline{\imath} > \overline{r} \tag{52a}$$

$$\text{If the project is a net financing, then } NPV(r) > 0 \quad \text{if and only if} \quad \overline{\imath} < \overline{r}. \tag{52b}$$

Also,

$$NPV(r) = (\overline{y} - \overline{r}_I)I + (\overline{r}_B - \overline{x})B. \tag{53}$$

Remark 4.1. Equation (42) reveals the existence of two different costs of capital and (53) highlights the major role played by them. When equilibrium rates are constant, one can appreciate the role of invested capital and borrowed capital, but only when equilibrium rates are variable the two costs of capital arise, one for the investment side and one for the borrowing side. They represent a foregone rate of return (I could be invested at $\overline{r}_I$) and a foregone rate of cost (B could be borrowed at $\overline{r}_B$), respectively. The first one affects the investment NPV and the second one affects the financing NPV:

$$NPV_I = (\overline{y} - \overline{r}_I)I, \qquad NPV_B = (\overline{r}_B - \overline{x})B.$$

Remark 4.2. One might reframe the project markup in (35) in terms of I and B, so that

$$\overline{\vartheta} := \frac{\varphi I - \varrho B}{I - B}.$$

However, so doing, $\overline{\vartheta}$ is not a markup of the project's cost of capital any more, but a markup of an "adjusted" cost of capital R:

$$R := \frac{\overline{r}_I \cdot \varphi I - \overline{r}_B \cdot \varrho B}{\varphi I - \varrho B} = \frac{\overline{r}_I \cdot NPV_I - \overline{r}_B \cdot NPV_B}{NPV}$$

whence

$$1 + \bar{\imath} = (1 + R)(1 + \overline{\vartheta}).$$

R and $\overline{r}$ coincide if $\overline{r}_B = \overline{r}_I$. In this case, $I' = I(1 + \overline{r})$ and $B' = B(1 + \overline{r})$, so

$$\overline{\vartheta} = \frac{\varphi I - \varrho B}{I - B} = \frac{\varphi I' - \varrho B'}{I' - B'}.$$

Remark 4.3. Note that $\overline{\phi}$ and $\overline{\rho}$ can be directly obtained from *period* excess rates. Letting $\phi_t := x_t - r_t$ and $\rho_t := y_t - r_t$, then

$$\overline{\phi} = \frac{\sum_{t \in T_I} \phi_t c_{t-1} v^{t,0}}{I}, \qquad \overline{\rho} = \frac{\sum_{t \in T_B} \rho_t |c_{t-1}| v^{t,0}}{B}.$$

It is worth noting that the assumption of varying equilibrium rates and constant markups implies that the ϕ_t's and ρ_t's are not constant over time: $\phi_t = r_t + \varphi(1 + r_t)$ and $\rho_t = r_t + \varrho(1 + r_t)$

Example 4. Table 3 revisits example 2. The cash-flow vector is unvaried, but we now assume that the term structure of interest rate is $\vec{r} = (0.13, 0.16, 0.19, 0.21)$. Suppose that the investor can borrow funds, at each date $t \in T_B$, at a financing rate which is equal to the forward rate marked down by a 1%. That is, $\varrho = -0.01$ so that

$$1 + x_t = (1 + r_t)(1 - 0.01).$$

This implies $\varphi(-0.01) = 0.016$, which means that, for every $t \in T_I$, the investment rate is equal to the forward rate marked up by a 1.6% return:

$$1 + y_t = (1 + r_t)(1 + 0.016).$$

This also implies that the internal rate-of-return vector is $\vec{\imath} = (0.1481, 0.1484, 0.1781, 0.2294)$, the first and fourth rate being investment rates, while the second and third rates being financing rates (i.e., $T_B = \{2,3\}$ and $T_I = \{1,4\}$). The project rate of return is greater than the project COC ($0.398 > 0.295$), so value is created. The same information is provided by the project excess rate ($\overline{\theta} = 0.1036$) and the project markup ($\vartheta = 0.08$), which are both positive. The financing COC is $\overline{r}_B = 0.173$ and the project financing rate is $\overline{x} = 0.161$, so the financing NPV is $NPV_B = 23.3 \cdot (0.173 - 0.161) = 0.27$. The investment COC is $\overline{r}_I = 0.206$ and the project investment rate is $\overline{y} = 0.225$. The investment NPV is $NPV_I = 31.9(0.225 - 0.206) = 0.62$ so that, overall, $NPV = 0.89$. Note that, while the markups are constant, the period excess rates vary over time.

Table 3: Example 2 with varying rates and constant markups

time t	cash flow f_t	financing $\lvert c_{t-1}\rvert$	investment c_{t-1}	rate (x_t, y_t)	COC r_t	excess rate (ρ_t, ϕ_t)	markup (ϱ, φ)
0	-2		2	14.81%	13%	1.81%	1.6%
1	20	17.7		14.84%	16%	-1.16%	-1%
2	-5	15.3		17.81%	19%	-1.19%	-1%
3	-75		56.9	22.94%	21%	1.9%	1.6%
4	70						
	NPV=0.89	B=23.3	I=31.9	$\bar{\imath}$=39.8%	$\overline{r} = 29.5\%$	$\overline{\theta} = 10.36\%$	$\vartheta = 8\%$

CG's Generalized Relative Rate of Return is the investment markup which corresponds to the choice $\varrho = 0$. With this choice, one would get the investment markup $\varphi(0) = 0.0233$ and the project markup would be $\vartheta = 0.0828$. Hence, the project rate of return would be $\bar{\imath} = 40.6\%$. Symmetrically, if one chose $\varphi = 0$, one would get a financing markdown equal to $\varrho(0) = -0.0329$ so that $\vartheta = 0.0741$, which would imply a project rate of return equal to $\bar{\imath} = 38.3\%$.

Example 5. Let us return to Example 3 and assume $\vec{r} = (0.15, 0.17, 0.25, 0.3)$ (see Table 4). Owing to the down payment, the financing rate in the first period is $x_1 = 0\%$, which implies $\varrho = (x_1 - r_1)/(1+r_1) = -0.15/1.15 = -0.1304$. This implies, in turn, $\varphi(-0.1304) = -0.0185$. Hence, $T_B = \{1\}$, $T_I = \{2,3,4\}$ and $\vec{\imath} = (0, 0.148, 0.227, 0.276)$. The project investment rate is $\overline{y} = 22.9\%$ and the project financing rate is $\overline{x} = x_1 = 0\%$. Therefore, the project rate of return is $\bar{\imath} = (0.229 \cdot 55{,}436 - 0 \cdot 17{,}391)/38{,}045 = 33.4\%$, which exceeds the project cost of capital $\overline{r} = 29.9\%$, so the project is worth undertaking. However, the investment side destroys value: the investment COC is $\overline{r}_I = 25.2\% > 22.9\% = \overline{y}$ so that $NPV_I = 55,436 \cdot (0.229 - 0.252) = -1,282$. This unfavorable result is more than compensated by the financing side: the financing COC is $\overline{r}_B = r_1 = 15\%$ and the upfront payment of \$20,000 creates value: $NPV_B = 20,000(0.15 - 0)/1.15 = 2,609$, so that, overall, $NPV = 1,327$. It is worth noting that, if the \$20,000 down payment were made at the delivery (time 3), the NPV would be negative ($NPV = -6,782$);[11] even if, additionally, \$14,000 were shifted from time 4 to time 3, the NPV would still be negative ($NPV = NPV_I = -4,861$). In other words, the negative value of the investment NPV signals a problem in the operating activity: given the term structure of interest rates, the structure of price and costs for this project is not adequate; the wealth increase occurs only because of a financing ability of the firm of collecting a sufficiently high upfront payment (note that it is just the decomposition of the NPV which reveals this problem).

Table 4: Example 3 with varying rates and constant markups

time t	cash flow f_t	financing $\lvert c_{t-1}\rvert$	investment c_{t-1}	rate (x_t, y_t)	COC r_t	excess rate (ρ_t, ϕ_t)	markup (ϱ, φ)
0	20,000	20,000		0%	15%	−15%	−13.04%
1	−22,000		2,000	14.8%	17%	−2.2%	−1.85%
2	−80,000		82,297	22.7%	25%	−2.3%	−1.85%
3	90,000		10,972	27.6%	30%	−2.4%	−1.85%
4	14,000						
	NPV=1,327	B=17,391	I=55,436	$\bar{\imath}$=33.4%	$\overline{r} = 29.9\%$	$\theta = 3.49\%$	$\vartheta = 2.68\%$

5 A further generalization: Varying rates and varying markups

As seen, TRM (1965a,b) rest on either choice $x = r$ or $y = r$ and derive an investment rate or a financing rate which signals wealth creation. In section 4, we have completed the TRM model and generalized it. In particular, (i) TRM's unique investment-rate function $y(x)$ (and

[11] In this case, the project would be a pure investment project, so $NPV = NPV_I$.

its inverse $x(y)$) has been extended into a unique investment-markup function $\varphi(\varrho)$ (and its inverse $\varrho(\varphi)$), (ii) a project rate of return has been derived and (iii) the NPV has been divided into financing NPV and investment NPV, each of which is associated with its own cost of capital. We also have shown that the same information may be reframed in terms of excess returns. However, Theorem 4.2-4.4 can be further generalized to allow for variable markups. Indeed, to assume that in *every* investment (financing) period either value is created or value is destroyed is unnecessarily restrictive; it may well occur that some periods are value-creating periods (i.e., $y_t > r_t$ or $x_t < r_t$), some other periods are value-destroying periods (i.e., $y_t < r_t$ or $x_t > r_t$), and some other ones are value-neutral periods ($y_t = r_t$ or $x_t = r_t$). In these cases, the borrowing rate and the investment rate can be written as $x_t = r_t + \varrho_t(1 + r_t)$ and $y_t = r_t + \varphi_t(1 + r_t)$ where ϱ_t and φ_t are period markups. Once again, we can make use of (9), with r_t replacing r, $v^{t,0}$ replacing v^t, x_t replacing x and y_t replacing y, and exploit the linearity of the mean operator, so that the following generalization of Theorems 4.3 and 4.4 is achieved.

Theorem 5.1. *Consider any vector $(\vec{x}, \vec{y})$ such that $c_n(\vec{x}, \vec{y}) = 0$. Then, the same thesis as Theorem 4.4 holds with $x_t = r_t + \varrho_t(1 + r_t)$ and $y_t = r_t + \varphi_t(1 + r_t)$. Also,*

$$NPV(\vec{r}) = \overline{\varphi} I' - \overline{\varrho} B'$$

where

$$\overline{\varphi} := \frac{\sum_{t \in T_I} \varphi_t c_{t-1} v^{t-1,0}}{\sum_{t \in T_I} c_{t-1} v^{t-1,0}} \qquad \overline{\varrho} := \frac{\sum_{t \in T_B} \varrho_t c_{t-1} v^{t-1,0}}{\sum_{t \in T_B} c_{t-1} v^{t-1,0}}$$

so that

$$NPV(\vec{r}) > 0 \qquad \textit{if and only if} \quad \overline{\vartheta} > 0$$

where

$$\overline{\vartheta} := \frac{\overline{\varphi} I' - \overline{\varrho} B'}{I' - B'}$$

(sign is reversed if $I' < B'$).

Example 6. An investor has the opportunity of depositing and withdrawing cash flows from an account balance with prefixed borrowing rates and lending rates which change period by period. The borrowing rates are activated when the account balance is negative, whereas the lending rates are activated when the account balance is positive.

period	borrowing rate	lending rate
1	23%	16%
2	13%	10%
3	8%	6%
4	20%	19%

Suppose the sequence of cash flows exchanged is $\vec{f} = (-2, 20, -5 - 75, 70)$. This implies $T_B = \{2, 3\}, T_I = \{1, 4\}$, and $\vec{\imath} = (0.16, 0.13, 0.08, 0.19)$. Assuming $\vec{r} = (0.21, 0.1, 0.16, 0.12)$, the investment excess rates are $\phi_1 = 16\% - 21\% = -5\%$, $\phi_4 = 19\% - 12\% = 7\%$, so the investment region consists of a wealth-creating period (the fourth one) and a wealth-destroying period (the first one). As for the borrowing region, one period destroys value ($\rho_2 = 13\% - 10\% = 3\%$) and the other one creates value ($\rho_3 = 8\% - 16\% = -8\%$). Overall, $NPV_I = 2.3$ and $NPV_B = 0.38$, so that $NPV = 2.68$ (the project is economically profitable). The project rate of return is $\bar{\imath} = 33.3\%$ and the project COC is $\bar{r} = 12.2\%$. (The activated markups/markdowns are $\varphi_1 = -4.13\%$, $\varrho_2 = 2.73\%$, $\varrho_3 = -6.9\%$, $\varphi_4 = 6.25\%$.)

We now turn to a particular case of Theorem 5.1 which restricts the choice of the rates so that all investment rates and all borrowing rates are equal to the equilibrium rates, *except one*. Essentially, we consider all-but-one periods to be value-neutral periods, while leaving one period as the lone determinant of wealth creation/destruction.[12] Far from being a bizarre unrealistic pattern, this idea has a deep economic content which is worth investigating. Historically, the idea traces back to Bailey (1959). The author wrote:

> the criterion for multiperiod investments can center on a short-term rate *when all other short-term rates are assumed to be equal to the equilibrium rates*" (Bailey, 1959, p. 479, italics supplied).

Formally, consider a project p such that $f_0 < 0$. Set $x_t = r_t$ for all $t \in T_B$ and $y_t = r_t$ for all $t \in T_I - \{b\}$ for some b. This implies that there exists a unique investment rate y_b such that

$$NPV(r) > 0 \quad \textit{if and only if} \quad y_b > r_b. \tag{54}$$

[12] Note that this pattern of rates cannot be coped with by Theorem 4.4.

The proof is straightforward: the financing NPV is zero, since $x_t = r_t$ for every $t \in T_B$ so that

$$NPV(r) = NPV_I = \sum_{t \in T_I - \{b\}} v^{t,0} c_{t-1}(y_t - r_t) + v^{b,0} c_{b-1}(y_b - r_b) \\ = v^{b,0} \cdot c_{b-1}(y_b - r_b). \tag{55}$$

whence (54). If $f_0 > 0$, set $y_t = r_t$ for $t \in T_I$ and $x_t = r_t$ i $t \in T_B - \{b\}$ for some b and the same reasoning leads to the symmetric result:

$$NPV(r) > 0 \quad \textit{if and only if} \quad x_b < r_b. \tag{56}$$

While Bailey (1959) did not indicate which rate should be allowed to differ from the cost of capital, we now show that such a choice is unequivocally provided by an efficient market.

In an efficient market, current market prices fully reflect available information. This means that prices adjust quickly whenever new information arrives. In particular, a firm's stock price will instantaneously change (rise or fall) when the decision of undertaking a project is made public by a firm (see Ross, Westerfield and Jordan, 2011, pp. 326-327). As Rubinstein (1973, p. 172) put it, "when a project is undertaken, the firm can be viewed in temporary disequilibrium", and for the project to be worth undertaking, "it is ... necessary to show that, given the market strives for equilibrium, the market value of the stock will increase by more than the investment outlay" (Bierman and Hass, 1973, p. 122). Formally, let us assume that the market is efficient and in equilibrium, and let a be a firm facing the opportunity of undertaking project p. The current value of the firm's equity V_0^a is determined by the number of outstanding shares N and the share price P:

$$V_0^a = NP.$$

Let $(a_1, a_2, \ldots, a_q)$ be the prospective equity cash flows in absence of the project, where $q \geq n$. Remembering the equilibrium relation (5), before acceptance of the project

$$a_1 + V_1^a = (1 + r_1) \cdot NP; \tag{57}$$

after acceptance of the project, the stock price adjusts to fully reflect new information. Letting $P + \Delta P$ be the new equilibrium price, the firm issues $\Delta N = c_0/(P + \Delta P)$ shares to finance the project. The new equilibrium relation is then

$$(a_1 + f_1) + (V_1^a + V_1^p) = (1 + r_1)(N + \Delta N)(P + \Delta P). \tag{58}$$

Equations (57) and (58) represent the equilibrium equations for firm a before and after acceptance of the project, respectively. Let $i := (f_1 + V_1^p)/c_0 - 1$; then, $f_1 + V_1^p = \Delta N(P + \Delta P)(1 + i)$, and, subtracting (57) from (58),

$$\Delta N(P + \Delta P)(i - r_1) = (1 + r_1)\Delta P \cdot N. \tag{59}$$

The project is worth undertaking if and only if "there is an immediate increase in the value of the company's stock as soon as the company is committed to the investment" (Robichek and Myers, 1965, p. 11. See also Ross, Westerfield and Jordan, 2011, p. 327, Figure 10.13). This means $\Delta P > 0$, which, owing to (59), holds if and only if $i > r_1$. Also, (59) can be written as $c_0(i - r_1)/(1 + r_1) = \Delta P \cdot N$. The right-hand side is the increase in the firm's equity value; the left-hand side is the project's NPV, since

$$NPV(\vec{r}) = \sum_{t=0}^{n} f_t v^{t,0} = -c_0 + \frac{f_1 + V_1^p}{1 + r_1} = \frac{c_0(i - r_1)}{1 + r_1}.$$

Therefore, the NPV is an instantaneous (excess) return, it is

> a 'windfall gain', which accrues to the owners of the firm as a result of their being able to invest in a project that is more profitable than the standard market rate. (Robichek and Myers, 1965, p. 11).

Note that, after equilibrium is restored, nothing else occurs, for "there is no tendency for subsequent increases and decreases" (Ross, Westerfield and Jordan, p. 327). In particular,

> after the windfall gain is realized through the increase in value of the owners' stock, income will continue to be realized at a rate of exactly $[r_t]$ ... for the remainder of the project's life (Robichek and Myers 1965, p. 11).

Formally, this means that the vector of project's return rates is $\vec{\imath} = (i, r_2, \ldots, r_n)$. This vector is then determined by the price behavior of an efficient market.[13] However, such a result is just the same as the one that would be obtained if the evaluator chose to set $x_t = r_t$ for $t \in T_B$ *and* $y_t = r_t$ for $t \in T_I - \{1\}$. Such a choice unequivocally determines y_1. In particular, $NPV_B = 0$, and, from (55) with $b = 1$,

$$NPV(\vec{r}) = NPV_I = v^{1,0} \cdot c_0(y_1 - r_1) \tag{60}$$

[13]We stress that this (return and) price behavior is valid not only for a firm's stock but also for any kind of traded asset (security or portfolio of securities) whenever a state of disequilibrium occurs.

whence $i = y_1$. Therefore, taking into account Theorem 5.1, we have proved the following

Theorem 5.2. *Assume all borrowing rates and investment rates are equal to the equilibrium rates except in the first period:* $x_t = r_t, t \in T_B$, $y_t = r_t, t \in T_I - \{1\}$. *Then, a unique rate of return* $y_1 = y_1(r_2, \ldots, r_n)$ *exists which is equal to the shareholder return* i *that would be generated as a windfall gain in an efficient market. The project investment rate is*

$$\overline{y} = \frac{ic_0 v^{1,0} + \sum_{t \in T_I - \{1\}} r_t c_{t-1} v^{t,0}}{I} \tag{61}$$

and the project financing rate is $\overline{x} = \overline{r}_B$. *Hence,*

$$NPV(\vec{r}) = \frac{c_0(i - r_1)}{1 + r_1}$$
$$NPV(\vec{r}) = \Delta P \cdot N$$

and the project rate of return can be written as

$$\overline{\imath} = \overline{r} + \frac{c_0}{I - B}(i - r_1)v^{1,0} \tag{62}$$

The project is worth undertaking if and only if $y_1 = i > r_1$.[14]

Remark 5.1. It is worth noting that the assumptions of Theorem 5.2 imply that all markups are zero except the first one: $1 + i = (1 + r_1)(1 + \varphi_1)$ whence $\varphi_1 = NPV/c_0$. The latter is just the well-known *profitability index* (PI). The project is worth undertaking if and only if $\varphi_1 > 0$.

Remark 5.2. The first-period rate of return y_1 can be rewritten as

$$y_1 = \frac{f_1 + V_1^p}{c_0} - 1 \tag{63}$$
$$= \frac{V_0^p(1 + r_1)}{c_0} - 1 \tag{64}$$
$$= \varphi_1 + \frac{r_1 V_0^p}{c_0}. \tag{65}$$

In such a way, y_1 is partitioned into two return components: φ_1 is the instantaneous shareholder return accrued as a consequence of the market reaction, while $r_1 \cdot V_0^p/c_0$ is the rate of return accrued to the investors after equilibrium has been restored.

[14] While the theorem implicitly assumes $f_0 < 0$, it holds for $f_0 > 0$ as well. In this case, the role of the borrowing rates and the investment rates is reversed. Also $x_1 < r_1$ as a condition for wealth creation replaces $y_1 > r_1$.

Example 7. Consider an efficient market where $\vec{r} = (0.25, 0.1, 0.18, 0.3)$ and suppose a firm undertakes a project whose cash flow stream is $\vec{f} = (-35, 20, 18, 39, -30)$. Setting $\imath_t = r_t$ for $t > 1$, the vector $\vec{c} = (35, 28.6, 13.5, -23.1)$ is found. This implies $y_1 = i = (20+28.6)/35-1 = 38.9\% > 25\%$. Therefore, the market reacts positively to the announcement of the project undertaking. In the first-period, the excess return is $\phi_1 = 13.9\%$ and the corresponding markup is $\varphi_1 = 0.139/1.25 = 11.16\%$, so the windfall gain determined by the market is

$$35 \cdot 0.1116 = 3.9 = NPV.$$

In the other periods, value is neither created nor destroyed: $\phi_2 = \varphi_2 = \phi_3 = \varphi_3 = \rho_4 = \varrho_4 = 0$. The internal rate-of-return vector is then $\vec{\imath} = (0.389, 0.1, 0.18, 0.3)$, where the last-period rate is the only financing rate, so it also represents the project financing rate: $x_4 = \overline{x}$. The latter coincides with the financing COC: $\overline{x} = \overline{r}_B = 30\%$. The investment rate (weighted average of the other period rates) is $\overline{y} = 25.3\%$, while the investment COC is $\overline{r}_I = 18.5\%$. The investment positions amount to $I = 57.1$ and the financing positions amount to $B = 10.9$. Therefore, the project rate of return is

$$\overline{\imath} = \frac{0.253 \cdot 57.1 - 0.3 \cdot 10.9}{46.2} = 24.2\%.$$

The financing NPV is evidently zero: $NPV_B = 10.9(0.3 - 0.3) = 0$ while the investment NPV is equal to the overall NPV: $NPV_I = 57.1 \cdot (0.253 - 0.185) = 3.9 = NPV$.

6 The average-based TRM model and the AIRR paradigm

The previous sections complete and fully generalize the TRM model. It is evident that the approach holds for pure projects as well as mixed projects: all theorems keep on holding with $I = 0$ (pure borrowing) or $B = 0$ (pure investment), which means $\overline{\imath} = \overline{y}$ or $\overline{\imath} = \overline{x}$, respectively.

Our model bears very strict relations to the AIRR paradigm (Magni, 2010, 2013) and can be interpreted as a link between the new literature (Hazen, 2003, 2009; Hartman and Schafrick, 2004; Magni, 2010, 2013) and that strand of the past literature which aimed at creating a reliable and economically meaningful theory of rate of return, alternative to the internal rate of return (e.g., Hirshleifer, 1958; Bailey, 1959; Teichroew, Robichek and Montalbano, 1965; Weingartner, 1966; Ramsey, 1970).

Some of Magni's (2010) results are summarized hereafter.

Definition 6.1. Let $\vec{c}$ a vector of capitals and $\vec{\imath}$ the corresponding vector of one-period internal rates of return. Any weighted mean $\overline{k}$ of such rates is called *Average Internal Rate of Return* (AIRR):

$$\overline{k} = w_1\imath_1 + \ldots + w_n\imath_n \tag{66}$$

where $w_t = c_{t-1}v^{t,0}/C$ and $C := \sum_{t=1}^{n} c_{t-1}v^t$. The difference between an AIRR and the cost of capital is the "excess AIRR" $\overline{\xi} := \overline{k} - \overline{r}$.

Theorem 6.1 (AIRR fundamental theorem)**.** *The project NPV is given by*

$$NPV(r) = C \cdot (\overline{k} - \overline{r}); \tag{67}$$

so, if the project is a net investment (borrowing), then it is acceptable if and only if $\overline{k} > \overline{r}$ (<). If $\overline{k} = \jmath$ and the cost of capital is constant ($r_t = r$), then the rule is equivalent to Hazen's (2003) acceptability rule.

The AIRR paradigm and our average-based TRM model are two sides of the same coin. In Magni (2010, 2013), the analysis starts from exogenous fixation of capitals c_t; in our approach, the analysis starts from exogenous fixation of the period rates x_t and y_t; then, we pick $\imath_t = x_t$ or $\imath_t = y_t$ depending on the sign of the capital (which is itself a function of the rate). There is a one-to-one correspondence between capitals and period rates:

$$\imath_t = \frac{c_t + f_t - c_{t-1}}{c_{t-1}} \Longleftrightarrow c_t = c_{t-1}(1 + \imath_t) - f_t$$

which implies that there is a one-to-one correspondence between vector $\vec{c}$ and vector $\vec{\imath}$. This has been explicitly recognized in Magni (2010): "It is important to underline that the internal return vector $\boldsymbol{k}$ and the investment stream $\boldsymbol{c}$ are in a biunivocal relation" (p. 155) Hence, $\bar{\imath} = \overline{k}$, $\overline{\xi} = \overline{\theta}$ and $C = I - B$, which implies that the AIRR paradigm and the average-based TRM model are logically equivalent.

Conceptually speaking, the average-based TRM model *reframes* the AIRR approach so as to give particular emphasis to financing and investment: the sequence of period rates $\vec{\imath}$ is divided into a sequence $\vec{x}$ of period rates associated with negative capitals and a sequence $\vec{y}$ associated with positive capitals. In such a way, the contribution of either side to value creation is found, as well as the subtle relations among the various groups of rates. The project rate of return in the TRM model is just an AIRR, that is, an average of internal rates of return: the project financing rate x and the project investment rate y. (Generalizing the TRM model, the project financing rate and project investment rate become themselves weighted averages of one-period IRRs, namely, the x_t's and the y_t's.)

The "Economic AIRR", introduced in Magni (2013), derives from a specific choice of $\vec{c}$, named "economic depreciation". Economic depreciation represents the change in an asset's present value: $V^p_{t-1} - V^p_t$, so the economic income is $f_t - (V^p_t - V^p_{t-1})$ (see Brealey, Myers and Allen, 2011, p. 331. See also Lindblom and Sjögren, 2009). The use of economic depreciation is equivalent to choosing $\vec{c} = (c_0, V^p_1, V^p_2, \ldots, V^p_{n-1})$. This entails that the economic rate of return is $(f_t + V^p_t - V^p_{t-1})/V_{t-1}$ for $t > 1$ and $(f_1 + V^p_1 - c_0)/c_0$ for $t = 1$. But $(f_t + V^p_t - V^p_{t-1})/V^p_{t-1} = r_t$ and $(f_1 + V^p_1 - c_0)/c_0 = i$. This means that the associated internal-rate-of-return vector is $\vec{\imath} = (i, r_2, \ldots, r_n)$ and the AIRR is

$$\overline{k} = \frac{ic_0v^{1,0} + r_2c_1v^{2,0} + \ldots + r_nc_{n-1}v^{n,0}}{C} \tag{68}$$

whence

$$\begin{aligned}\overline{k} = &= \frac{ic_0v^{1,0} + r_2c_1v^{2,0} + \ldots + r_nc_{n-1}v^{n,0} + r_1c_0v^{1,0} - r_1c_0v^{1,0}}{C}\\ &= \overline{r} + \frac{(i - r_1)c_0v^{1,0}}{C}.\end{aligned} \tag{69}$$

Remembering that $C = I - B$, eq. (62) and eq. (68) coincide. We have then proved the following result.

Proposition 6.1. *The economic AIRR, found by exogenously assuming a capital stream equal to economic values, is the rate of return which is obtained by exogenously assuming that the rates of return are those determined by an efficient market where investors quickly react and fully absorb the new information on the firm's project.*

7 Concluding remarks

This paper completes Teichroew, Robichek and Montalbano's (TRM) (1965a,b) model, generalizes it and turns it into a complete model of value (i.e., wealth) creation, where relations among the various value drivers are explicitly supplied.

We start from the original TRM model, which aimed at providing a more reliable model of economic profitability than that provided by the unsatisfactory internal rate of return

(IRR). An ingenious device, TRM model as such is, nonetheless, *restrictive* and *incomplete*. Restrictive, because

- either project investment rate or project financing rate are assumed to be equal to the cost of capital
- financing rate and investment rate are assumed to be constant over time
- cost of capital is assumed to be constant over time;

incomplete, because

- no project rate of return is supplied (investment rate and financing rate capture only part of the project's economic performance)
- no explicit relation is given among the project's NPV, the financing rate, the investment rate, and the cost of capital.

We relax TRM's assumptions, expand the model, and generalize it so as to enable the evaluator to accomplish a detailed analysis of the project's economic performance. To this end, we make use of an average-based approach. More precisely, we show that the project's rate of return is obtained as an affine combination of the financing rate and the investment rate, where the weights are the amount invested in the investment region and the amount borrowed in the borrowing region. The comparison between the project rate of return and the cost of capital signals value creation or value destruction. TRM's two acceptability rules are particular cases.

We also isolate the value drivers by decomposing the NPV into financing side and investment side, which enables the evaluator to understand how value is generated and how the operating activity and the financing policy interact to increase or decrease the investors' wealth.

We reframe the results in terms of financing and investment excess rates and in terms of financing and investment markups/markdowns, the latter being equal to the former discounted at the cost of capital. We then derive the project excess rate (project markup) as a capital-weighted mean of the financing excess rate (financing markup) and investment excess rate (investment markup): its sign correctly signals value creation.

Then, we allow for varying financing and investment rates and varying costs of capital. To this end, we employ a two-step generalizing process. As a first step, we assume that both financing and investment markups are constant over time and show that there exists a unique invertible investment-rate function that satisfies the project's terminal condition. In such a way, once the financing markup (investment markup) is selected, a unique investment markup (financing markup) is found. This is a generalization of TRM's unique investment-rate function. The sign of the weighted average of the two markups correctly captures value creation. Chiu and Garza Escalante (2012)'s "Generalized Relative Rate of Return" is shown to be a particular case of this approach, obtained when the financing markup is set equal to zero.

As a second step, we relax the assumption of constant markups and allow for any pattern of period markups, which implies that any pattern of period (financing or investment) rates is allowed. In such a way, Magni's (2013) "Economic AIRR" is found to be a particular case, when all period financing rates and investment rates are set equal to the costs of capital, except in the first period. We show that such a rate is related to Bailey's (1959) proposal of value creation in one period and is equal to the rate of return that would accrue to shareholders of a firm whose shares are traded in an efficient market, that is, a market where investors react quickly to the announcement of the project undertaking.

We show that this paper's approach is logically equivalent to the AIRR paradigm, the difference lying in the framing of the model: while Magni (2010, 2013) starts from interim capitals to compute the project's one-period rates, we start from (financing and investment) rates to estimate the (invested and borrowed) capitals. Given that there is a one-to-one correspondence between the vector of rates and the vector of capitals, the two approaches coincide formally and conceptually. In this regard, this paper links TRM model and AIRR paradigm and, therefore, constitutes a bridge between past and recent literature on rates of return.

Table 5: List of rates

constant rate		variable rates
x	project financing rate	$\overline{x}$
y	project investment rate	$\overline{y}$
$\imath$	project rate of return	$\overline{\imath}$
	financing cost of capital	$\overline{r}_B$
	investment cost of capital	$\overline{r}_I$
r	project cost of capital	$\overline{r}$
ϱ	financing markup	$\overline{\varrho}$
φ	investment markup	$\overline{\varphi}$
ϑ	project markup	$\overline{\vartheta}$
ρ	excess financing rate	$\overline{\rho}$
ϕ	excess investment rate	$\overline{\phi}$
θ	project excess return rate	$\overline{\theta}$

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