Generalized Makeham's formula and economic profitability*

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Abstract. This paper generalizes Makeham's formula, allowing for varying interest rates and for a non-flat structure of valuation rates. An *average interest rate* (AIR) is introduced, as well as an *average valuation rate* (AVR), which exist and are unique for any asset. They can be computed either as principal-weighted arithmetic means or as interest-weighted harmonic means of period rates. Economic profitability of an asset or a portfolio of assets is captured by the spread between AIR and AVR, which has the same sign as the Net Present Value. This makes (i) AIR a more reliable tool for valuation and decision than the venerable Internal Rate of Return, and (ii) AVR a natural generalization of the cost-of-capital notion.

Keywords. Makeham's formula, net present value, average interest rate.

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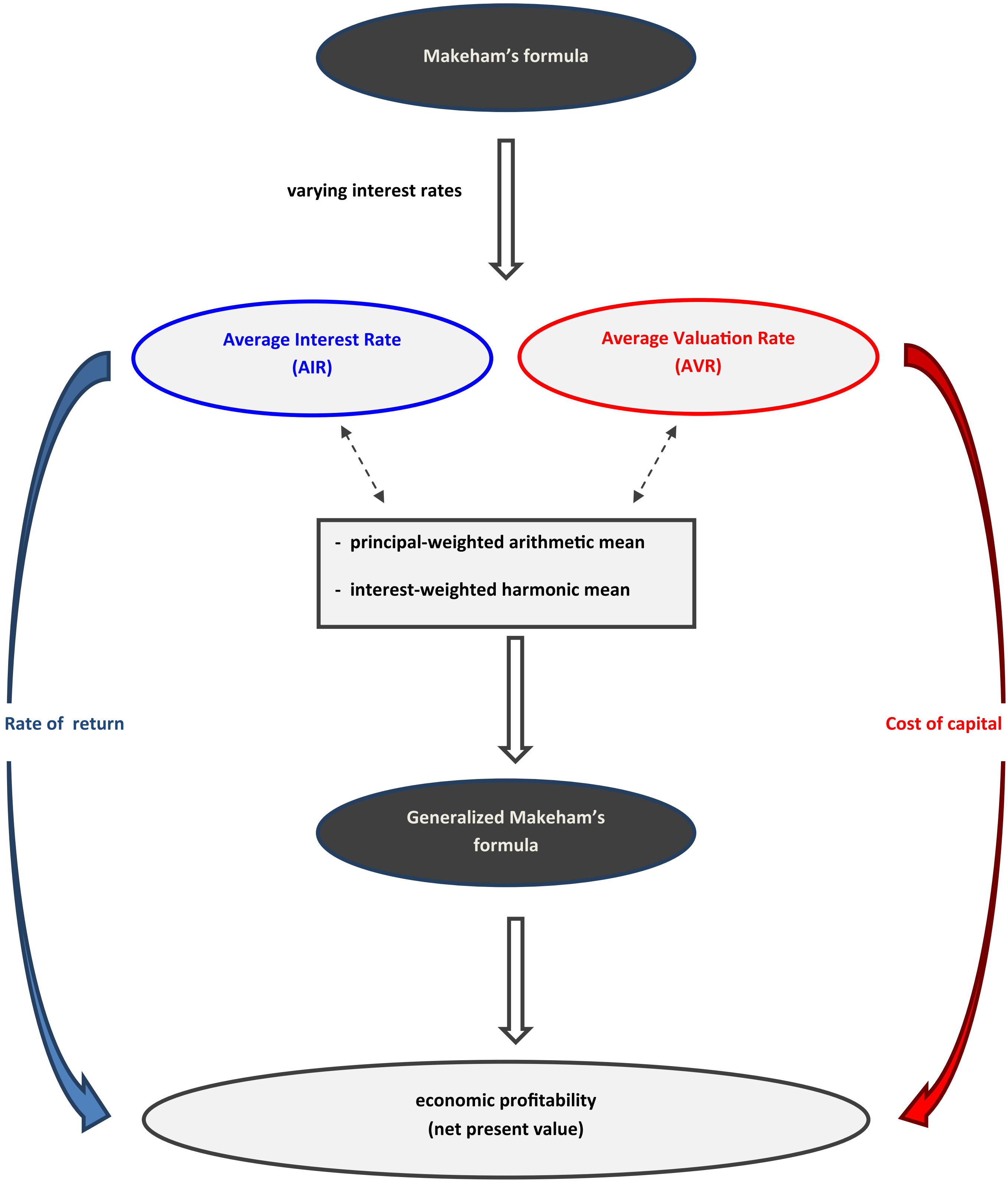
Graphical abstract



Introduction

One of the most interesting relations in the theory of interest has been proposed in nineteenth century by the English actuary and mathematician William Makeham. Named after its begetter, his formula states that the price of a bond can be divided into two components: present value of redemption value plus present value of interest, where the latter is obtained as the ratio of the (possibly modified) coupon rate to the valuation rate times the difference between the redemption value and its present value (Makeham, 1874; Glen, 1893). A more general version of Makeham's formula can be applied to any type of loan, where the coupon rate is replaced by the interest rate of the loan and the difference between redemption value and its present value is replaced by the difference between the borrowed amount and the present value of capital repayments (see Broverman, 2008; Kellison, 2009). While occasionally used in the relatively recent past (Hossack and Taylor, 1975; Ramlau-Hansen, 1988; Astrup Jensen, 1999a,b), Makeham's formula is nowadays essentially neglected in finance and actuarial science, although it directly provides important connections among an asset's value, overall interest and economic profitability expressed as the ratio of two relative measures of worth (interest rate vs. valuation rate). Admittedly, the formula only copes with traditional assets bearing constant interest rate and supplies the above mentioned connections only when the valuation rate is constant. Also, it only copes with financial assets, not with real investments. These features makes it only moderately useful for valuation and decision-making. This paper just aims at generalizing the formula, in such a way that the above mentioned connections are made valid for any kind of assets in any circumstance. In particular, we (i) allow for assets with varying interest rates and consider the more realistic situation where valuation rates vary across time (i.e., the term structure of interest rates is not flat), (ii) extend the application of the formula to any kind of economic activity, including real assets and portfolios of (financial or real) assets, (iii) provide a valuation/decision tool which is consistent with the net present value (NPV). We find that a suitable weighted mean of the interest rates and a suitable weighted mean of the valuation rates can be used to decompose an asset's value into interest and capital; we call the means 'Average Interest Rate' (AIR) and 'Average Valuation Rate' (AVR), respectively. The term "average" is in a twofold sense: both the AIR and the AVR are principal-weighted arithmetic means of period rates and, at the same time, interest-weighted harmonic means of period rates. While the internal-rate-of-return (IRR) notion suffers from problems of existence and uniqueness and does not guarantee value additivity, the AIR (as well as the AVR) exists and is unique, and the comparison of AIR and AVR correctly captures an asset's economic profitability, while at the same time complying with value additivity.

The paper is structured as follows. Section 1 introduces Makeham's formula and supplies the main definitions. Section 2 generalizes the formula by allowing for varying interest rates: the *average interest rate* (AIR) is introduced, which is shown to exist and be unique. Economic profitability is captured by the yield spread, which is the difference between the AIR and the valuation rate. Section 3 further generalizes the formula by allowing for varying valuation rates: an *average valuation rate* (AVR) is shown to be the correct benchmark for assessment of economic profitability. Section 4 provides a third generalization of Makeham's formula: portfolios of assets are considered, and the portfolio's AIR and AVR are variously expressed as arithmetic or harmonic means of interest rates with the proper weights; it is also shown that the two means enjoy a twofold commutative property. Some concluding remarks end the paper.

1. Makeham's formula

Let $t = 0$ be the current date and let $T_0 = \{0,1,2,\dots,n\}$ and $T_1 = \{1,2,\dots,n\}$.[1] Consider a sequence of cash flows $\{f_t\}_{t\in T_0}$ describing any financial transaction involving two parties which exchange a sequence of monetary amounts by pre-determining an (assumed constant) interest rate i. Following are the well-known relations of an amortization schedule, for $t \in T_1$:

$$f_t = K_t + I_t \qquad (1a)$$

$$P_t = P_{t-1} - K_t \qquad f_0 := -P_0 \quad P_n := 0 \qquad (1b)$$

$$I_t = i \cdot P_{t-1}. \qquad (1c)$$

P_t is the principal outstanding, also known as capital (outstanding) or outstanding balance, f_t is the payment/disbursement, K_t is the capital payment (principal repayment), I_t is the interest payment, i is the interest rate. All variables are real numbers, with $i > 0$. Let V be the (present) value of cash-flow stream $\{f_t\}_{t\in T_1}$, computed at a valuation rate $r > 0$: $V = V(r) = \sum_{t\in T_1} f_t(1+r)^{-t}$. The valuation rate r is the investor's minimum desired rate of return. Assuming that the cash flows are (certain or) expressed as certainty equivalents, r is the *risk-free* rate. Certainty equivalents are the theoretically correct way of dealing with risky cash flows and V represents the asset's arbitrage-free value in a complete market; alternatively, it is possible to discount the asset's expected cash flows at a discount rate that reflects the asset's risk. The latter is often measured by the so-called beta derived from the well-known Capital Asset Pricing Model, so the valuation rate is the return rate of equal-risk (i.e., equal beta) alternatives traded in the market, which means that V is the mean-variance value of the asset.[2]

The *internal rate of return* (IRR) is a discount rate x such that the present value of payments equals the present value of disbursements. Note that the interest rate i is the IRR of $\{f_t\}_{t\in T_0}$, since (1) implies $\sum_{t\in T_0} f_t(1+i)^{-t} = 0$. Likewise, the valuation rate r is the IRR of the asset $(-V, f_1, \dots, f_n)$.

By (1a), one may divide the value of the asset into an interest portion $\mathcal{I}$ and a capital portion $\mathcal{K}$:

$$V = \mathcal{I} + \mathcal{K} \qquad (2)$$

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[1] Throughout the paper, we use the set notations $\sum_{t\in T_0}, \sum_{t\in T_1}$ etc. for in-text summations.

[2] For relations between mean-variance pricing and arbitrage-free pricing see Dybvig and Ingersoll (1982) and Magni (2009).

where $\mathcal{I} = \mathcal{I}(i,r) := \sum_{t \in T_1} i \cdot P_{t-1}(1+r)^{-t} = \sum_{t \in T_1} I_t v^t$ is the (present) value of the interest portion and $\mathcal{K} = \mathcal{K}(i,r) := \sum_{t \in T_1} (f_t - i \cdot P_{t-1}) \cdot (1+r)^{-t} = \sum_{t \in T_1} K_t v^t$ is the (present) value of the capital portion, and $v := (1+r)^{-1}$ is the discount factor. Makeham's formula relates $\mathcal{I}$ and $\mathcal{K}$ in the following way:

$$\mathcal{I} = \mathcal{I}(i,r) = \frac{i}{r}(P_0 - \mathcal{K}) \tag{3a}$$

so that

$$V = V(i,r) = \frac{i}{r}(P_0 - \mathcal{K}) + \mathcal{K}. \tag{3b}$$

Economic profitability of an asset depends on a comparison between value and borrowed amount or, equivalently, on the sign of the Net Present Value (NPV). An asset is economically profitable (i.e., wealth is increased) if

$$V > P_0 \Leftrightarrow NPV(r) > 0. \tag{4}$$

The NPV measures the investor's wealth increase, with respect to the preference rate, which is also known as *cost of capital* in corporate finance. It is evident that $i = r$ implies $V = P_0$, which means $NPV = 0$.

While NPV is sufficient to capture economic profitability, rates of return are often used in place of (or in conjunction with) the NPV for various reasons:

(i) a relative information such as, say, 10% return is considered more intuitive than an absolute information such as €150

(ii) to compare two rates (the asset's rate of return and the cost of capital) is considered more natural than verifying the sign of an absolute amount (NPV) which in turn depends on a relative one (r)

(iii) an absolute amount such as the NPV is inappropriate for assessing a manager's performance: for example, a fund manager has no control over interim cash flows and makes decisions about asset selection and allocation, not on withdrawals or deposits.

For these reasons, the use of a rate of return is often required (see Gray and Dewar, 1971; Jaffe, 1977; Evans and Forbes, 1993; Graham and Harvey, 2001; Sandahl and Sjögren, 2003; Brounen, de Jong and Koedijk, 2004). In all this, the IRR notion has a privileged role in practical applications as well as in the literature, owing to its respectable ancestry (Fisher, 1930; Boulding, 1935; Keynes, 1936). It is worth noting that Makeham's formula supplies a direct link between value and rate of return: other things unvaried, the ratio i/r determines wealth increase/decrease. If the asset is a constant-interest loan, then i is the IRR, and, from (3), the asset is economically profitable if and only if $x = i > r$. However, from this point of view, (3) has a limited scope: first, it cannot cope with assets with varying interest rates and/or varying valuation rates, which are most common in capital markets; secondly, it is well-know that the IRR is reliable only if $\partial V/\partial r < 0$ for every $r > -1$. In this case, the IRR exists and is unique, and the asset can be interpreted as an investment (i.e., a lending opportunity), in which case the comparison of the IRR and the valuation rate r correctly captures the investment's economic profitability: $V > P_0$ if and only if $x > r$. However, in particularly complex financial transactions or real assets such as corporate projects, $\partial V/\partial r$ may change sign, which means that the asset at hand is interpretable as an investment or as a borrowing depending on the value of r (see Hartman and Schafrick 2004). In these cases, there may be more than one IRR or no one at all, so the comparison of i and r is ambiguous or impossible.[3] In addition, the criterion $i > r$ cannot be applied if the valuation rate is not constant. The economic literature has produced an enormous amount of contributions on this issue, proposing several different solutions.[4] The following sections overcome this issue by presenting an unambiguous pair of rate of return and cost of capital which enlarge the scope of application of the formula and provide a reliable, sufficiently general tool for capturing any asset's economic profitability.

2. Generalizing Makeham's formula – first step: varying interest rates

In this section we generalize Makeham's formula allowing for varying interest rates. In particular, we denote as i_t the interest rate holding in period t (i.e., between date $t-1$ and t), $t \in T_1$. In this case, (1c) becomes

$$I_t = i_t P_{t-1}. \tag{5}$$

Given that $f_t = K_t + i_t P_{t-1}$, the value of the loan depends on the entire sequence $\{i_t\}_{t \in T_1}$ as well as r: $V = V(i_1, i_2, \dots, i_n, r) = \sum_{t \in T_1}(K_t + i_t P_{t-1})(1+r)^{-t}$. In the amortization schedule, the ratio $I_t/P_{t-1} = i_t$ represents the interest in period $t \in T_1$ on a unit of principal accrued. $\mathcal{I}$ and $\mathcal{K}$ are now functions of the entire sequence $\{i_t\}_{t \in T_1}$ as well, so eq. (3) cannot be applied. One might consider the IRR as a candidate for replacing the missing i, given that the IRR just aims at summarizing information conveyed by the interest rates i_t. Unfortunately, the use of the IRR leads to incorrect results, since, in general,

$$\mathcal{I} = \sum_{t=1}^{n} I_t v^t \neq \frac{x}{r}(P_0 - \mathcal{K}).$$

We now show that a principal-weighted average of the interest rates correctly generalizes Makeham's formula.

Proposition 2.1. *Consider the following convex combination of interest rates:*

$$\bar{\imath} := \alpha_1 i_1 + \alpha_2 i_2 + \dots + \alpha_1 i_n \tag{6}$$

[3] Multiple IRRs may occur when considering investment funds, where the investor's choices about deposits and withdrawals can determine several changes in sign. Corporate projects may have a considerable length and several changes in sign may occur in the cash flow stream (e.g., investments with disposal/remediation costs, phased expansion, natural resource extraction). The problem may also be encountered when ex post economic performance is assessed for an ongoing activity (such as a firm or a business unit) in a given interval of time, if dividends and new investments alternate. Further, multiple IRRs can easily occur even in the most regular circumstances, when a levered project is studied or a portfolio of investments and borrowings is considered.

[4] In the last decade, important results have been obtained by Hazen (2003), Hartman and Schafrick (2004), Magni (2010), Pierru (2010).

where $\alpha_t := P_{t-1}v^{t-1}/\sum_{t\in T_1} P_{t-1}v^{t-1}$. Then, the following generalized Makeham's formula holds:

$$\mathcal{I} = \mathcal{I}(\bar{\iota}, r) = \frac{\bar{\iota}}{r}(P_0 - \mathcal{K}), \tag{7a}$$

which implies

$$V = V(\bar{\iota}, r) = \mathcal{K} + \frac{\bar{\iota}}{r}(P_0 - \mathcal{K}) \tag{7b}$$

Proof: Using (1b), $P_t = \sum_{h=t+1}^{n} K_h$ whence

$$P_1v^2 + P_2v^3 + \ldots + P_{n-1}v^n = (K_2 + \ldots + K_n)v^2 + (K_3 + \ldots + K_n)v^3 + \ldots + K_nv^n$$

Letting $\mathcal{P} := P_0v + P_1v^2 + \ldots + P_{n-1}v^n$ and reminding that $P_0 = K_1 + K_2 + \ldots + K_n$,

$$\begin{aligned}
\mathcal{P} &= v[K_1 + (K_2 + K_2v) + (K_3 + K_3v + K_3v^2) + \ldots + (K_n + K_nv + \ldots + K_nv^{n-1})] \\
&= v\left[K_1 \cdot \frac{1-v}{1-v} + K_2 \cdot \frac{1-v^2}{1-v} + \cdots + K_n \cdot \frac{1-v^n}{1-v}\right] \\
&= K_1 \cdot \frac{1-v}{r} + K_2 \cdot \frac{1-v^2}{r} + \cdots + K_n \cdot \frac{1-v^n}{r} \\
&= \frac{K_1 + K_2 + \cdots + K_n - (K_1v + K_2v^2 + \cdots + K_nv^n)}{r}
\end{aligned}$$

which leads to

$$\mathcal{P} = \frac{P_0 - \mathcal{K}}{r}. \tag{8}$$

Given that $i_1P_0v + i_2P_1v^2 + \cdots + i_nP_{n-1}v^n = \sum_{t\in T_1} I_tv^t = \mathcal{I}$, (6) and (8) imply

$$\frac{\bar{\iota}}{r} = \frac{\mathcal{I}}{P_0 - \mathcal{K}} \tag{9}$$

which is equivalent to (7a).■

The above proposition says that if the interest rate of the financial transaction is not constant, then one may nonetheless employ a generalized Makeham's formula by making use of the weighted average of the interest rates. Given that P_t expresses the borrowing position at time t, $\mathcal{P}$ represents the overall (discounted) value of the borrowed amounts. Therefore, α_t is the amount borrowed in period t as a proportion of the aggregate borrowed amount.

Evidently, interpreting $\mathcal{P}$ as the overall borrowed capital, $\bar{\iota}$ can be viewed as an intuitive generalization of the notion of interest rate, for it represents (overall) interest on (overall) principal: (7a) and (8) imply

$$\bar{\iota} = \frac{\mathcal{I}}{\mathcal{P}}. \tag{7c}$$

From the point of view of the lender, $\mathcal{P}$ is the overall invested capital and (7c) is the "return on capital".

We call $\bar{\imath}$ the *Average Interest Rate* (AIR). It is worth noting that Proposition 2.1 guarantees *existence* and *uniqueness* of the AIR, for $\bar{\imath}$ is, simply, a mean of interest rates weighted by (unambiguous) capitals.

Remark 2.1. Proposition 2.1 implicitly assumes that the outstanding principal is nonzero in every period (i_t is not defined if $P_{t-1} = 0$). This is certainly the case of a loan contract. However, the proposition holds for any financial transactions, even if some outstanding principal is zero; one just need rewrite (6) as

$$\bar{\imath} := \beta_1 + \beta_2 + \cdots + \beta_n \tag{10}$$

with $\beta_t := I_t v^t / \mathcal{P}$, so that eq. (7) continues to hold. Equation (10) decomposes the investment's rate of return into period shares; each share is the interest earned by the investor in a period per unit of (overall) invested capital.

The following corollary is straightforward from eqs. (8) and (9).

Corollary 2.1. *If the value of the overall principal is known, the value of the principal repayments can be computed as*

$$\mathcal{K} = P_0 - r \cdot \mathcal{P}. \tag{11a}$$

If the value of overall interest is known, the value of the principal repayments can be computed as

$$\mathcal{K} = P_0 - \frac{r}{\bar{\imath}} \cdot \mathcal{I}(\bar{\imath}, r). \tag{11b}$$

The generalized Makeham's formula may be restated highlighting the role of the capital payments.

Corollary 2.2. *The value of interest is*

$$\mathcal{I} = \sum_{t=1}^{n} K_t \, (i_1 v + i_2 v^2 + \cdots + i_t v^t) \tag{12}$$

Proof: from (6) and the equality $P_t = \sum_{h=t+1}^{n} K_h$ one gets

$$\bar{\imath} = \frac{v(K_1 + \cdots + K_n) i_1 + v^2 (K_2 + .. + K_n) i_2 + \cdots + v^n K_n i_n}{\mathcal{P}} = \frac{\sum_{t=1}^{n} K_t (i_1 + i_2 v + \cdots + i_t v^t)}{\mathcal{P}}.$$

Using (8),

$$\frac{\bar{\imath}}{r} = \frac{\sum_{t=1}^{n} K_t (i_1 v + i_2 v^2 + \cdots + i_t v^t)}{P_0 - \mathcal{K}}$$

which leads, owing to (7a), to the thesis.■

It is easy to show that the AIR is a reliable rate of return: it correctly captures economic profitability, in the sense that it is NPV-consistent.

Proposition 2.2*. Consider a cash-flow stream $\{f_t\}_{t\in T_0}$. Then, the asset is economically profitable if and only if $\bar{\imath} > r$.*

Proof: From (7) and the definition of *NPV*,

$$NPV(r) = \mathcal{K} + \frac{\bar{\imath}}{r}(P_0 - \mathcal{K}) - P_0 \quad = (P_0 - \mathcal{K})\left(\frac{\bar{\imath}}{r} - 1\right). \tag{13}$$

As $P_0 = \sum_{t\in T_1} K_t > \sum_{t\in T_1} K_t v^t = \mathcal{K}$, then $NPV > 0$ if and only if $\bar{\imath} > r$. ■

*Remark 2.2.*The economic rationale of Proposition 2.2 is rather intuitive: the investor lends P_0 and gets back a sequence of capital payments whose present value is $\mathcal{K}$. The latter is (the value of) the capital that the investor recovers from the borrower, so the difference $P_0 - \mathcal{K}$ is the *unrecovered* capital, that is, the capital which the investor sacrifices. The financial transaction is economically profitable if and only if the capital sacrificed is more than compensated by the total interest accrued $\mathcal{I}$. However, as previously shown, the latter is a multiple of the unrecovered capital, with $\bar{\imath}/r$ being the multiplier. So, ultimately, it is the comparison of $\bar{\imath}$ and r that determines economic profitability. Also, (11a) tells us that the unrecovered capital coincides with the interest $r\mathcal{P}$ foregone by the lender, which implies

$$NPV(r) = \mathcal{I} - r \cdot \mathcal{P}. \tag{14}$$

Economic profitability is then signaled by the comparison of the total interest accrued to the lender and the foregone interest (the lender might lend the overall amount $\mathcal{P}$ at the market rate r). A different but equivalent interpretation is obtained by noting that (13) may be rewritten as

$$(P_0 - \mathcal{K})\left(\frac{\bar{\imath}}{r} - 1\right) = \sum_{t=1}^{\infty} \frac{[(P_0 - \mathcal{K})(\bar{\imath} - r)]}{(1+r)^t}. \tag{15}$$

The product $(P_0 - \mathcal{K})(\bar{\imath} - r)$ may be interpreted as a measure of excess return: $(P_0 - \mathcal{K})$ is the unrecovered capital and $(\bar{\imath} - r)$ is the excess rate of return on this capital. Therefore, the finite sequence of cash flows $\{f_t\}_{t\in T_0}$ is financially equivalent to a perpetuity of the excess return earned on the capital sacrificed.

Remark 2.3. It is worth noting that (13) generalizes Astrup Jensen's (1999a, p. 5) Theorem 1. The author deals with bonds and assumes a coupon rate equal to i. He shows that, for a given rate r,

$$P_0 - V(r) = (r - i) \cdot \sum_{t=1}^{n} K_t \frac{1 - v^t}{r}. \tag{16}$$

From (13), we get $V(r) - P_0 = (\bar{\imath} - r)(P_0 - \mathcal{K})/r$. Reminding that $P_0 = \sum_{t\in T_1} K_t$ and $\mathcal{K} = \sum_{t\in T_1} K_t v^t$, if one assumes $i_t = i$ for every $t \in T_1$, then $\bar{\imath} = i$ and (13) boils down to (16).

Remark 2.4. The profitability condition in Proposition 2.2 may be expressed in terms of *yield spread* , defined as the difference between the AIR and the valuation rate. Denoting it as $\bar{\sigma} \coloneqq \bar{\imath} - r$, its sign is the same as the NPV's.

3. Generalizing Makeham's formula – second step: varying valuation rates

In this section we allow for a structure of valuations rates varying over time. Let r_t be valuation rate in period t and let $v_{t,k} \coloneqq (1 + r_{k+1})^{-1} \cdot (1 + r_{k+2})^{-1} \cdot \ldots \cdot (1 + r_t)^{-1}$ be the corresponding discount factor for the interval $[k, t], k, t = 0,1, \ldots n, t \geq k,\ v_{k,k} := 1$.[5]

Proposition 3.1 *Consider the convex combination of interest rates*

$$\bar{\imath} := \alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n \tag{17a}$$

and the corresponding convex combination of forward rates

$$\bar{r} := \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n \tag{17b}$$

where α_t is generalized as $\alpha_t := P_{t-1} v_{t,0} / \sum_{t\in T_1} P_{t-1} v_{t,0}$. Then, the following generalized Makeham's formula holds:

$$\mathcal{I} = \mathcal{I}(\bar{\imath}, \bar{r}) = \frac{\bar{\imath}}{\bar{r}}(P_0 - \mathcal{K}) \tag{18a}$$

where $\mathcal{K}$ is generalized as $\mathcal{K} = \sum_{t\in T_1} K_t v_{t,0}$. This implies

$$V = V(\bar{\imath}, \bar{r}) = \mathcal{K} + \frac{\bar{\imath}}{\bar{r}}(P_0 - \mathcal{K}). \tag{18b}$$

Proof: Let $\mathcal{P}$ be generalized as $\mathcal{P} \coloneqq \sum_{t\in T_1} P_{t-1} v_{t,0}$. Using $P_t = \sum_{h=t+1}^{n} K_h$,

$$\mathcal{P} = (K_1 + K_2 + \cdots + K_n) v_{1,0} + (K_2 + K_3 \ldots + K_n) v_{2,0} + \cdots + K_n v_{n,0}$$

$$= \mathcal{K} + (K_2 + K_3 \ldots + K_n) v_{1,0} + (K_3 + K_4 + \cdots + K_n) v_{2,0} + \cdots + K_n v_{n-1,0}.$$

Hence, $\mathcal{P} = \mathcal{K} + P_1 v_{1,0} + P_2 v_{2,0} + \cdots + P_{n-1} v_{n-1,0}$, so that

$$\mathcal{K} = P_0 + \sum_{t=1}^{n} P_{t-1}(v_{t,0} - v_{t-1,0}).$$

Exploiting the relation $v_{t-1,0} - v_{t,0} = r_t v_{t,0}$, one gets

$$\mathcal{K} = P_0 - \sum_{t=1}^{n} r_t\, P_{t-1} v_{t,0} = P_0 - \bar{r} \cdot \mathcal{P} \tag{19}$$

whence

$$\mathcal{P} = \frac{P_0 - \mathcal{K}}{\bar{r}}. \tag{20}$$

However, $\mathcal{I} = \sum_{t\in T_1} I_t v_{t,0} = \bar{\imath} \cdot \mathcal{P}$, so that $(P_0 - \mathcal{K})/\bar{r} = \mathcal{I}/\bar{\imath}$, which implies (18a).■

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[5] If the valuation rate is selected equal to the market rate (as usual in finance), then r_t is the forward rate of the term structure of interest rate and $v_{t,k}$ is the market value, at time k, of €1 available at time t.

We call $\bar{r}$ the 'Average Valuation Rate' (AVR). The AVR is to r_t what the AIR is to i_t. Proposition 3.1 makes two sets of rates (interest rates and valuation rates) collapse into two single metrics, each of which representing the weighted mean of the rates with the same weights α_t.

Proposition 2.2. is also immediately generalized, since

$$NPV(\bar{r}) = \mathcal{K} + \frac{\bar{\imath}}{\bar{r}}(P_0 - \mathcal{K}) - P_0 = (P_0 - \mathcal{K})\left(\frac{\bar{\imath}}{\bar{r}} - 1\right) \tag{21}$$

We can then state the following

Proposition 3.2. *Consider a cash-flow stream $\{f_t\}_{t\in T_0}$. Then, $NPV(\bar{r}) > 0$ if and only if $\bar{\imath} > \bar{r}$, or, in terms of (average) yield spread, if and only if $\bar{\sigma} = \bar{\imath} - \bar{r} > 0$.*

The usefulness of AIR should now be clear as opposed to the usefulness of the IRR as a tool for capturing economic profitability. As already seen, the use of the IRR in Makeham's formula supplies an incorrect valuation of the interest. Also, the IRR cannot be used for assessing economic profitability: given that, in general,

$$NPV(r) = (P_0 - \mathcal{K})\left(\frac{\bar{\imath}}{\bar{r}} - 1\right) \neq (P_0 - \mathcal{K})\left(\frac{x}{\bar{r}} - 1\right).$$

the comparison of IRR and $\bar{r}$ is misleading.

As noted in the previous section the AIR is, literally, a rate of return, that is, an amount of return per unit of invested capital, for (17a) is equivalent to $\bar{\imath} = \mathcal{I}/\mathcal{P}$, which is just the ratio of the investor's overall return to overall invested capital. Likewise, $\bar{r}$ represents the return foregone by the investor per unit of invested capital $\mathcal{P}$, so it represents an intuitive generalization of the cost-of-capital notion. Equivalently, $\bar{\sigma}$ represents the *excess* return per unit of invested capital.

Remark 3.1. The generalizations so far presented assume that the cash flow stream fulfills $(1a)$, that is, cash flow is explicitly divided into a capital component K_t and an interest component I_t, which is typical of a loan. However, even if the asset is not a loan (e.g., a common stock or a real asset), the division can be naturally accomplished by making recourse to the notion of *economic depreciation*. Economic depreciation represents the change in an asset's present value: letting $V_t = \sum_{k=t+1}^{n} f_k\, v_{k,\mathrm{t}}$ be the asset's value as of time t, economic depreciation is formally computed as $V_{t-1} - V_t$. The corresponding *economic income* is $r \cdot V_{t-1}$ (see Brealey, Myers and Allen, 2011, pp. 331-332. See also Lindblom and Sjögren, 2009). Therefore, cash flow is naturally divided into a capital component (economic depreciation) and an interest component (economic income):

$$f_t = \underbrace{rV_{t-1}}_{\text{economic income}} + \underbrace{V_{t-1} - V_t}_{\text{economic depreciation}}$$

whence

$$P_t = \begin{cases} -f_0 \text{ if } t = 0 \\ V_t \text{ if } 1 < t < n \\ 0 \text{ if } t = n \end{cases} \qquad I_t = i_t \, P_{t-1}$$

where $i_t := (f_t + P_t - P_{t-1})/P_{t-1}$.[6] In this way, all results hold for any kind of assets (security, corporate investment etc.).

Remark 3.2. Throughout the paper, we assume cash-flow streams are discrete. However, all results apply to continuous cash-flow streams. In this case, let $f(t)$ and $P(t)$ be the asset's cash flow and the invested capital, respectively. The (instantaneous) interest rate is $i(t) = f(t)/P(t)$ and $-P'(t)$ is the amount of capital depreciation. Denoting the principal repayment and the interest component as $K(t)$ and $I(t)$ respectively, (1) boils down to the following set of relations:

$$\begin{aligned} f(t) &= I(t) + K(t) \\ K(t) &= -P'(t) \\ I(t) &= i(t)P(t). \end{aligned}$$

Equation (6) becomes

$$\bar{\imath} = \frac{\int_0^n i(t) \cdot P(t) \cdot e^{-rt} dt}{\int_0^n P(t) \cdot e^{-rt} dt}$$

where r denotes the (assumed constant) instantaneous valuation rate, and the acceptability criterion is, consistently, $\bar{\imath} > r$. If valuation rate is a function $r(t)$ of time t, then $(17a)$-$(17b)$ become

$$\bar{\imath} = \frac{\int_0^n i(t) \cdot P(t) \cdot e^{-\int_0^t r(s)ds} dt}{\int_0^n P(t) \cdot e^{-\int_0^t r(s)ds} dt} \qquad \bar{r} = \frac{\int_0^n r(t) \cdot P(t) \cdot e^{-\int_0^t r(s)ds} dt}{\int_0^n P(t) \cdot e^{-\int_0^t r(s)ds} dt}.$$

4. Generalizing Makeham's formula – third step: portfolio of assets

In this section we further generalize the formula to account for a portfolio of assets. This means that the investor can simultaneously lend funds to some borrower and borrow funds from some creditor. Financially speaking, this means that the investor can take long and short positions at the same time. Consider a portfolio of m assets. The symbols we previously used for a single asset will now denote the portfolio's financial variables;[7] superscripts will be used for single assets. So, i_t^s, P_t^s, K_t^s, I_t^s, f_t^s will denote, respectively, the interest rate, the outstanding principal, the capital payment, the interest payment, the cash flow of asset s at time t. Let $\mathcal{I}^s, \mathcal{K}^s, \mathcal{P}^s, V^s, NPV^s$ denote, respectively, the value of interest, the value of principal repayments, the overall invested capital, the present value, and the NPV of asset s. Let n_s be the length of asset $s \in T_m = \{1,2,\dots,m\}, m \in \mathbb{N}$. Then, $\bar{\imath}^s = \sum_{t=1}^{n_s} \alpha_t^s \cdot i_t^s$ denotes the

[6] It is easily seen that $i_1 = (f_1 + V_1 + f_0)/(-f_0)$ and $i_t = (f_t + V_t - V_{t-1})/V_{t-1} = r_t$ for $t > 1$.

[7] That is, $\mathcal{I}$ is the portfolio value of interest, V is the portfolio value etc.

AIR of asset s and $\bar{r}^s = \sum_{t=1}^{n_s} \alpha_t^s \cdot r_t$ is the corresponding AVR, with $\alpha_t^s \coloneqq P_{t-1}^s v_{t,0} / \sum_{s \in T_m} P_{t-1}^s v_{t,0}$. The portfolio AIR can be found as the (unique) solution $\bar{\imath}$ of the following linear equation:

$$\sum_{s=1}^{m} \mathcal{I}^s = \sum_{s=1}^{m} \frac{\bar{\imath}^s}{\bar{r}^s} (P_0^s - \mathcal{K}^s) = \sum_{s=1}^{m} \frac{\bar{\imath}}{\bar{r}^s} (P_0^s - \mathcal{K}^s) \tag{22}$$

whence

$$\bar{\imath} = \frac{\sum_{s=1}^{m} \bar{\imath}^s \cdot \frac{P_0^s - \mathcal{K}^s}{\bar{r}^s}}{\sum_{s=1}^{m} \frac{P_0^s - \mathcal{K}_s}{\bar{r}^s}}. \tag{23}$$

Therefore, the portfolio AIR is a weighted average of the various assets' AIRs. More specifically, we have previously shown that $\mathcal{P}^s = (P_0^s - \mathcal{K}) / \bar{r}^s$, where $\mathcal{P}^s \coloneqq \sum_{t=1}^{n_s} P_{t-1}^s \cdot v_{t,0}$ can now be positive or negative. If it is positive, it represents the overall amount invested; if it is negative, it expresses the overall amount borrowed. Letting $\mathcal{P} \coloneqq \sum_{s \in T_m} \mathcal{P}^s$, eq. (23) boils down to

$$\bar{\imath} = w^1 \, \bar{\imath}^1 \, + w^2 \, \bar{\imath}^2 + \cdots + w^m \, \bar{\imath}^m \qquad\qquad w^s \coloneqq \frac{\mathcal{P}^s}{\mathcal{P}}. \tag{24}$$

Equivalently, from $\sum_{s \in T_m} \bar{\imath} \cdot (P_0^s - \mathcal{K}^s) / \bar{r}^s = \sum_{s \in T_m} \bar{\imath} \cdot (P_0^s - \mathcal{K}^s) / \bar{r}$ one gets the AVR:

$$\bar{r} = \frac{\sum_{s=1}^{m} \mathcal{I}'^s}{\sum_{s=1}^{m} \frac{\mathcal{I}'^s}{\bar{r}^s}} \qquad\qquad \mathcal{I}'^s \coloneqq P_0^s - \mathcal{K}^s. \tag{25}$$

Multiplying and dividing each summand in the numerator by $\bar{r}^s$, the portfolio AVR can reframed as

$$\bar{r} = w^1 \bar{r}^{\,1} \, + w^2 \bar{r}^{\,2} + \cdots + w^m \bar{r}^{\,m}. \tag{26}$$

Remark 4.1. The use of the symbol $\mathcal{I}'^s$ in (25) to denote the unrecovered capital of asset s is justified by the fact that the unrecovered capital coincides with the interest foregone by the investor. The amount $P_0^s - \mathcal{K}^s$ is equal to $\bar{r}^s \cdot \mathcal{P}^s$, which is the overall interest that the investor might earn if he invested the amount $\mathcal{P}^s$ at the average rate $\bar{r}^s$ rather than at the average rate $\bar{\imath}$. In such a way, the NPV of asset s can be expressed as the difference between the overall interest earned and the overall interest given up: $NPV^s = \mathcal{I}^s - \mathcal{I}'^s$.

It is possible to derive the AIR by averaging out the portfolio's interest rates i_t as well. First, note that every i_t is itself the weighted mean of the period interest rates of the various portfolio's assets:

$$i_t = \frac{\sum_{s=1}^{m} i_t^s \cdot P_{t-1}^s}{P_{t-1}} \tag{27}$$

where $P_{t-1} \coloneqq \sum_{s \in T_m} P_{t-1}^s$ is the portfolio principal outstanding at time $t-1$. From (24), exploiting additivity and letting $n = \max\, [n_1, n_2, \ldots, n_m\,]$,

$$
\begin{aligned}
\bar{\imath} &= \frac{\sum_{s=1}^{m} \bar{\imath}^s \cdot \mathcal{P}^s}{\mathcal{P}} \\
&= \frac{\sum_{s=1}^{m} \sum_{t=1}^{n_s} i_t^s \cdot P_{t-1}^s v_{t,0}}{\sum_{s=1}^{m} \mathcal{P}^s} \\
&= \frac{\sum_{t=1}^{n} \sum_{s=1}^{m} i_t^s \cdot P_{t-1}^s v_{t,0}}{\sum_{s=1}^{m} \mathcal{P}^s} \\
&= \frac{\sum_{t=1}^{n} \sum_{s=1}^{m} i_t^s \cdot P_{t-1}^s v_{t,0}}{\sum_{s=1}^{m} \sum_{t=1}^{n_s} P_{t-1}^s\, v_{t,0}} \\
&= \frac{\sum_{t=1}^{n} i_t \cdot P_{t-1}\, v_{t,0}}{\sum_{t=1}^{n} P_{t-1} v_{t,0}} \qquad (28)
\end{aligned}
$$

where $i_t^s := 0$ for $t > n_s$. Eq. (28) just means

$$\bar{\imath} = W_1 i_1 + \ldots + W_n i_n \qquad W_t := \frac{\mathcal{P}_t}{\mathcal{P}} \qquad (29)$$

where $\mathcal{P}_t := P_{t-1} v_{t,0}$ is the discounted value of the portfolio principal in period t. Equivalently, one gets the AVR:

$$\bar{r} = W_1 r_1 + \ldots + W_n r_n. \qquad (30)$$

We have then proved the following

Proposition 4.1. *The AIR of a portfolio of m assets is the arithmetic mean of the valuation rates weighted by the principal amounts (eq. (24)). The portfolio AVR can be computed as the harmonic mean of the valuation rates, where the weights are the unrecovered capitals (eq. (25)) and, at the same time, as the arithmetic mean of valuation rates weighted by the principal amounts (eq. (26)). Alternatively, the portfolio AIR and the portfolio AVR can be both seen as arithmetic means of the portfolio's interest (valuation) rates weighted by the portfolio's outstanding principals (eqs. (29)-(30)).* The *portfolio average yield spread is $\bar{\sigma} = \bar{\imath} - \bar{r} = \sum_{s=1}^{m} w^s \bar{\sigma}^s = \sum_{t=1}^{n} W_t \sigma_t$, where $\bar{\sigma}^s = \bar{\imath}^s - \bar{r}^s$ is asset s's average yield spread and $\sigma_t = i_t - r_t$ is the portfolio yield spread in period t. Further, the following generalized Makeham's formula holds:*

$$\mathcal{I} = \sum_{t=1}^{n} \sum_{s=1}^{m} I_t v_{t,0} = \frac{\bar{\imath}}{\bar{r}} (P_0 - \mathcal{K}) \qquad (31)$$

where $\mathcal{K} = \sum_{t \in T_1} \sum_{s \in T_m} K_t^s \cdot v_{t,0}$, so the portfolio value is

$$V = \mathcal{K} + \frac{\bar{\imath}}{\bar{r}} (P_0 - \mathcal{K}). \qquad (32)$$

Proposition 4.2. *Let $\bar{\xi}$ denote the harmonic mean of the assets' average yield spreads $\bar{\sigma}^s = \bar{\imath}^s - \bar{r}^s$, where the weights are the asset's NPVs. The portfolio AIR can be obtained as*

$$\bar{\imath} = \bar{\xi} + \bar{r} = \frac{NPV^1 + \cdots + NPV^m}{\frac{NPV^1}{\bar{\sigma}^1} + \cdots + \frac{NPV^2}{\bar{\sigma}^2}} + \frac{\mathcal{I}'^1 + \cdots + \mathcal{I}'^m}{\frac{\mathcal{I}'^1}{\bar{r}^1} + \cdots + \frac{\mathcal{I}'^m}{\bar{r}^m}} \qquad (33)$$

Proof:

$$\frac{NPV^1 + \cdots + NPV^m}{\frac{NPV^1}{\bar{\sigma}^1} + \cdots + \frac{NPV^m}{\bar{\sigma}^m}} + \frac{\mathcal{I}'^1 + \cdots + \mathcal{I}'^m}{\frac{\mathcal{I}'^1}{\bar{r}^1} + \cdots + \frac{\mathcal{I}'^m}{\bar{r}^m}} = \frac{NPV}{\mathcal{P}^1 + \cdots + \mathcal{P}^m} + \frac{\sum_{s=1}^{m} P_0^s - \sum_{s=1}^{m} \mathcal{K}^s}{\mathcal{P}^1 + \cdots + \mathcal{P}^m}$$

$$= \frac{NPV + P_0 - \mathcal{K}}{\mathcal{P}}$$
$$= \frac{\mathcal{I}}{\mathcal{P}}.$$

The latter ratio is just $\bar{\imath}$.■

Proposition 4.2 shows that the portfolio AIR is the sum of two (weighted) harmonic means: the first one averages out the assets' yield spreads by the respective NPVs, whereas the second one averages out the assets' unrecovered capitals by the assets' average valuation rates. Evidently, the proposition also shows that $\bar{\xi} = \bar{\sigma}$; that is, the portfolio yield spread $\bar{\sigma}$ can be obtained as the harmonic means of the assets' yield spreads weighted by the assets' NPVs.

One might ask whether $\bar{\imath}$, being the sum of the two harmonic means $\bar{\sigma}$ and $\bar{r}$, is itself a harmonic mean. The answer is positive, as the following proposition shows.

Proposition 4.3. The portfolio AIR is the harmonic mean of the assets' AIRs weighted by the assets' overall interests:

$$\bar{\imath} = \frac{\mathcal{I}^1 + \mathcal{I}^2 + \cdots + \mathcal{I}^m}{\frac{\mathcal{I}^1}{\bar{\imath}^1} + \frac{\mathcal{I}^2}{\bar{\imath}^2} + \cdots + \frac{\mathcal{I}^m}{\bar{\imath}^m}}. \tag{34}$$

Proof: from the definition of $\bar{\imath}$,

$$\begin{aligned}
\bar{\imath} &= \frac{\mathcal{P}^1}{\mathcal{P}} \cdot \bar{\imath}^1 + \frac{\mathcal{P}^2}{\mathcal{P}} \cdot \bar{\imath}^2 + \cdots + \frac{\mathcal{P}^m}{\mathcal{P}} \cdot \bar{\imath}^m \\
&= \frac{\mathcal{P}^1}{\mathcal{P}} \cdot \frac{\mathcal{I}^1}{\mathcal{P}^1} + \frac{\mathcal{P}^2}{\mathcal{P}} \cdot \frac{\mathcal{I}^2}{\mathcal{P}^2} + \cdots + \frac{\mathcal{P}^m}{\mathcal{P}} \cdot \frac{\mathcal{I}^m}{\mathcal{P}^m} \\
&= \frac{\mathcal{I}^1 + \mathcal{I}^2 + \cdots + \mathcal{I}^m}{\mathcal{P}} \\
&= \frac{\mathcal{I}^1 + \mathcal{I}^2 + \cdots + \mathcal{I}^m}{\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_m}
\end{aligned}$$

The thesis follows reminding that $\mathcal{I}^s = \bar{\imath}^s \cdot \mathcal{P}^s$.■

Reminding that $NPV^s = \mathcal{I}^s - \mathcal{I}'^s$, (33) and (34) prove the following (additivity) property.

Proposition 4.4*. The harmonic mean of the assets' average yield spreads is equal to the difference between the harmonic means of AIR and AVR:*

$$\bar{\sigma} = \frac{(\mathcal{I}^1 - \mathcal{I}'^1) + \cdots + (\mathcal{I}^m - \mathcal{I}'^m)}{\frac{\mathcal{I}^1 - \mathcal{I}'^1}{\bar{\imath}^1 - \bar{r}^1} + \cdots + \frac{\mathcal{I}^m - \mathcal{I}'^m}{\bar{\imath}^m - \bar{r}^m}} = \frac{\mathcal{I}^1 + \mathcal{I}^2 + \cdots + \mathcal{I}^m}{\frac{\mathcal{I}^1}{\bar{\imath}^1} + \frac{\mathcal{I}^2}{\bar{\imath}^2} + \cdots + \frac{\mathcal{I}^m}{\bar{\imath}^m}} - \frac{\mathcal{I}'_1 + \cdots + \mathcal{I}'_m}{\frac{\mathcal{I}'_1}{\bar{r}^1} + \cdots + \frac{\mathcal{I}'_m}{\bar{r}^m}}. \tag{35}$$

The very line of argument employed above for deriving harmonic means of m assets' AIRs and AVRs can be employed for deriving harmonic means of the portfolio interest rates and valuation rates. By replacing asset s's value of interest $\mathcal{I}^s$ with period t's portfolio value of interest $\mathcal{I}_t \coloneqq I_t v_{t,0}$ one gets

$$\bar{\iota} = \frac{\mathcal{I}_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_n}{\dfrac{\mathcal{I}_1}{i_1} + \dfrac{\mathcal{I}_2}{i_2} + \cdots + \dfrac{\mathcal{I}_n}{i_n}}. \tag{36}$$

Analogously,

$$\bar{r} = \frac{\mathcal{I}'_1 + \cdots + \mathcal{I}'_1}{\dfrac{\mathcal{I}'_1}{r_1} + \dfrac{\mathcal{I}'_2}{r_2} + \cdots + \dfrac{\mathcal{I}'_n}{r_n}} \tag{37}$$

$$\bar{\sigma} = \frac{(\mathcal{I}_1 - \mathcal{I}'_1) + (\mathcal{I}_2 - \mathcal{I}'_2) + \cdots + (\mathcal{I}_n - \mathcal{I}'_n)}{\dfrac{\mathcal{I}_1 - \mathcal{I}'_1}{\sigma_1} + \dfrac{\mathcal{I}_2 - \mathcal{I}'_2}{\sigma_2} + \cdots + \dfrac{\mathcal{I}_n - \mathcal{I}'_n}{\sigma_n}} \tag{38}$$

where $\mathcal{I}'_t$ is the value of the foregone interest, $\mathcal{I}'_t = r_t \cdot P_{t-1}v_{t,0}$. Equations (36), (37), (38) are the counterparts of (34), (25), (35) respectively.

Table 1 summarizes the twelve (weighted) means of the portfolio rates: average interest rate, average valuation rate, and average yield spread. Inspecting the table, it becomes evident that the algebraic structure of the means are the same: in the harmonic means, the rates are weighted by the corresponding interests; in the arithmetic means, the rates are weighted by the invested capitals.

Both arithmetic and harmonic means incorporate two kinds of averages: average by periods and average by asset, and the result is invariant with respect to the order in which the averages are taken (first by assets, then by periods or vice versa). In particular, let $R = [a_t^s]$ be the $(n \times m)$ rate matrix, where a may denote interest rate (i), valuation rate (r) or yield spread (σ), and where subscript and superscript denote, respectively, row and column, so that a_t^s is asset s's rate in period t. Denote as $\boldsymbol{a_{1,n}} = (a_1, \dots, a_n)$ the vector of the portfolio rates and $\overline{\boldsymbol{a}}^{\boldsymbol{1,m}} = (\overline{a}^1, \overline{a}^2, \dots, \overline{a}^m)$ the vector of the assets' rates, $a \coloneqq i, r, \sigma$. Let $\boldsymbol{a_t^*} = (a_1^1, a_1^2, \dots, a_t^m)$ be the t-th row of R and $\boldsymbol{a_*^s} = (a_1^s, a_2^s, \dots, a_n^s)^T$ be the s-th column of R (see Table 2). Let $H(\cdot)$ and $A(\cdot)$ denote, respectively, the interest-weighted harmonic mean and principal-weighted arithmetic mean of rates. Then, for any $t \in T_1$, $A(\boldsymbol{a_t^*}) = a_t$ and, for any $s \in T_m$, $A(\boldsymbol{a}_*^s) = \overline{a}^s$, and the following equalities hold:

$$A\big(A(\boldsymbol{a_1^*}), A(\boldsymbol{a_2^*}), \dots, A(\boldsymbol{a_n^*})\big) = A\big(\boldsymbol{a_{1,n}}\big) = \bar{a} = A\Big(\overline{\boldsymbol{a}}^{1,m}\Big) = A\big(A(\boldsymbol{a}_*^1), A(\boldsymbol{a}_*^2), \dots, A(\boldsymbol{a}_*^m)\big). \tag{39}$$

Analogously, $H(\boldsymbol{a_t^*}) = a_t$, $H(\boldsymbol{a}_*^s) = \overline{a}^s$, so that

$$H\big(H(\boldsymbol{a_1^*}), H(\boldsymbol{a_2^*}), \dots, H(\boldsymbol{a_n^*})\big) = H\big(\boldsymbol{a_{1,n}}\big) = \bar{a} = H\Big(\overline{\boldsymbol{a}}^{1,m}\Big) = H\big(H(\boldsymbol{a}_*^1), H(\boldsymbol{a}_*^2), \dots, H(\boldsymbol{a}_*^m)\big). \tag{40}$$

This also implies

$$\begin{aligned} A\big(H(\boldsymbol{a_1^*}), H(\boldsymbol{a_2^*}), \dots, H(\boldsymbol{a_n^*})\big) &= H\big(A(\boldsymbol{a_1^*}), A(\boldsymbol{a_2^*}), \dots, A(\boldsymbol{a_n^*})\big) = \\ &= A\big(H(\boldsymbol{a}_*^1), H(\boldsymbol{a}_*^2), \dots, H(\boldsymbol{a}_*^m)\big) = H\big(A(\boldsymbol{a}_*^1), A(\boldsymbol{a}_*^2), \dots, A(\boldsymbol{a}_*^m)\big). \end{aligned} \tag{41}$$

Table 1. Weighted arithmetic and harmonic means of interest rates, market rates, yield spreads

Weighted mean	Averaging out by assets	Averaging out by period
Average interest rate ($\bar{\imath}$)		
Arithmetic	$\frac{\bar{\imath}^1\mathcal{P}^1+\bar{\imath}^2\mathcal{P}^2+\cdots+\bar{\imath}^m\mathcal{P}^m}{\mathcal{P}^1+\mathcal{P}^2+\cdots+\mathcal{P}^n}$	$\frac{i_1\mathcal{P}_1+i_2\mathcal{P}_2+\cdots+i_n\mathcal{P}_n}{\mathcal{P}_1+\mathcal{P}_2+\cdots+\mathcal{P}_n}$
Harmonic	$\frac{\mathcal{I}^1+\mathcal{I}^2+\cdots+\mathcal{I}^m}{\frac{\mathcal{I}^1}{\bar{\imath}^1}+\frac{\mathcal{I}^2}{\bar{\imath}^2}+\cdots+\frac{\mathcal{I}^m}{\bar{\imath}^m}}$	$\frac{\mathcal{I}_1+\mathcal{I}_2+\cdots+\mathcal{I}_n}{\frac{\mathcal{I}_1}{i_1}+\frac{\mathcal{I}_2}{i_2}+\cdots+\frac{\mathcal{I}_n}{i_n}}$
Average valuation rate ($\bar{r}$)		
Arithmetic	$\frac{\bar{r}^1\mathcal{P}^1+\bar{r}^2\mathcal{P}^2\ldots+\bar{r}^m\mathcal{P}^m}{\mathcal{P}^1+\mathcal{P}^2+\cdots+\mathcal{P}^m}$	$\frac{r_1\mathcal{P}_1+r_2\mathcal{P}_2+\ldots+r_n\mathcal{P}_n}{\mathcal{P}_1+\mathcal{P}_2+\ldots+\mathcal{P}_n}$
Harmonic	$\frac{\mathcal{I}'^1+\mathcal{I}'^2+\cdots+\mathcal{I}'^m}{\frac{\mathcal{I}'^1}{\bar{r}^1}+\frac{\mathcal{I}'^2}{\bar{r}^2}+\cdots+\frac{\mathcal{I}'^m}{\bar{r}^m}}$	$\frac{\mathcal{I}'_1+\mathcal{I}'_2+\cdots+\mathcal{I}'_n}{\frac{\mathcal{I}'_1}{r_1}+\frac{\mathcal{I}'_2}{r_2}+\cdots+\frac{\mathcal{I}'_n}{r_n}}$
Average yield spread ($\bar{\sigma}$)		
Arithmetic	$\frac{\bar{\sigma}^1\mathcal{P}^1+\bar{\sigma}^2\mathcal{P}^2+\cdots+\bar{\sigma}^m\mathcal{P}^m}{\mathcal{P}^1+\mathcal{P}^2+\cdots+\mathcal{P}^m}$	$\frac{\sigma_1\mathcal{P}_1+\sigma_2\mathcal{P}_2+\ldots+\sigma_n\mathcal{P}_n}{\mathcal{P}_1+\mathcal{P}_2+\ldots+\mathcal{P}_n}$
Harmonic	$\frac{(\mathcal{I}^1-\mathcal{I}'^1)+\cdots+(\mathcal{I}^m-\mathcal{I}'^m)}{\frac{\mathcal{I}^1-\mathcal{I}'^1}{\bar{\sigma}^1}+\cdots+\frac{\mathcal{I}^m-\mathcal{I}'^m}{\bar{\sigma}^m}}$	$\frac{(\mathcal{I}_1-\mathcal{I}'_1)+(\mathcal{I}_2-\mathcal{I}'_2)+\cdots+(\mathcal{I}_n-\mathcal{I}'_n)}{\frac{(\mathcal{I}_1-\mathcal{I}'_1)}{\sigma_1}+\frac{(\mathcal{I}_2-\mathcal{I}'_2)}{\sigma_2}+\cdots+\frac{(\mathcal{I}_n-\mathcal{I}'_n)}{\sigma_n}}$

As a result, a twofold commutative property holds: (i) arithmetic means and harmonic means of the rates commute, and (ii) assets and periods commute. In other words, it is irrelevant whether one average out rates first by assets and then by periods or first by periods and then by assets and, likewise, it is irrelevant whether one averages out arithmetically or harmonically, or even arithmetically *and* harmonically (in either order): the result is invariant and represents the portfolio average rate (interest rate, valuation rate, yield spread). Practically, the portfolio average rate is computed via two averaging steps (averaging out by period and averaging out by asset) whose order is irrelevant; in each step the evaluator can employ, interchangeably, either the arithmetic mean or the harmonic one. (This means that value additivity is fulfilled. See also numerical example below).

Table 2. The rate matrix

	asset 1	asset 2	$\cdots$	asset s	$\cdots$	asset m	either $A(\cdot)$ or $H(\cdot)$
period 1	a_1^1	a_1^2	$\cdots$	a_1^s	$\cdots$	a_1^m	a_1
period 2	a_2^1	a_2^2	$\cdots$	a_2^s	$\cdots$	a_2^m	a_2
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
period t	a_t^1	a_t^2	$\cdots$	a_t^s	$\cdots$	a_t^m	a_t
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
period n	a_n^1	a_n^2	$\cdots$	a_n^s	$\cdots$	a_n^m	a_n
either $A(\cdot)$ or $H(\cdot)$	$\bar{a}^1$	$\bar{a}^2$		$\bar{a}^s$		$\bar{a}^m$	$\bar{a}$

The assessment of economic profitability for a portfolio requires the evaluator to first ascertain whether, overall, the portfolio is, financially, a borrowing or a lending opportunity. For a single asset such as a loan, a bond, a fixed-income security, this problem does not arise: the investor can take either a long position (lending) or a short position (borrowing) on an asset.[8] In contrast, for a portfolio, an investor can simultaneously take a long position on some asset and a short position on some other asset. The following definition supplies an unambiguous definition of the financial nature of a portfolio.

Definition 4.1. *Consider a portfolio of assets* $\{\boldsymbol{f}^s\}_{s\in T_m}$*, with* $\boldsymbol{f}^s = (f_0^s, f_1^s, \dots, f_{n_s}^s)$*. The portfolio is a net investment if* $P_0 - \mathcal{K} > 0$*. The portfolio is a net borrowing if* $P_0 - \mathcal{K} < 0$*.*

The definition acknowledges the fact that if the portfolio's unrecovered capital is positive, then the capital lent is greater than the value of the capital repayments. This means that the economic agent "invests" funds, from which total interest $\mathcal{I}$ is earned. If, by contrast, the unrecovered capital is negative, then the agent is, overall, borrowing funds, on which he pays an overall interest equal to $\mathcal{I}$. This definition reverberates on the financial nature of the AIR: it is a rate of return (i.e., a lending rate) if the portfolio is a net investment, it is a rate of cost (i.e., borrowing rate) if the portfolio is a net borrowing. Likewise, the role of the AVR is that of a benchmark with the same nature as the AIR: in case of net investment (borrowing), the AVR is a lending (borrowing) rate. As a result, Definition 4.1, along with the equality $NPV = (P_0 - \mathcal{K})\left(\frac{\bar{\imath}}{\bar{r}} - 1\right)$, warrants the following criterion, which generalizes the previous ones.

Proposition 4.5. *Consider a portfolio of assets* $\{f_{s,t}\}_{s\in T_m, t\in T_0}$*. Then,*

- *If the portfolio is a net investment, it is economically profitable if and only if* $\bar{\imath} > \bar{r}$ *(i.e.,* $\bar{\sigma} > 0$*)*
- *If the portfolio is a net borrowing, it is economically profitable if and only if* $\bar{\imath} < \bar{r}$ *(i.e.,* $\bar{\sigma} < 0$*).*

[8] For a single *real* asset such as a project this problem does arise (see Hazen, 2003). In this case, Remark 3.1 and Definition 4.1 can be applied picking $m = 1$.

5. A numerical example

Suppose an insurance company undertakes three financial transactions, say A, B, and C, whose cash flow streams are $\boldsymbol{f^A} = (-1800, 1000, 500, 2300)$ $\boldsymbol{f^B} = (-2050, 1200, 0, 1100)$, $\boldsymbol{f^C} = (2850, 1600, -5235, -1465)$. Therefore, the company owns a portfolio of three assets. The portfolio net cash flows are collected in the vector $\boldsymbol{f} = \boldsymbol{f^A} + \boldsymbol{f^B} + \boldsymbol{f^C} = (-1000, 3800, -4735, 1935)$. The assumed interest rates are collected in Table 3.

Table 3. The *interest rate* matrix for A, B, and C ($\boldsymbol{a_t^s} = \boldsymbol{i_t^s}$)

	asset A	asset B	asset C	either $A(\cdot)$ or $H(\cdot)$
period 1	40.30%	9.50%	10.00%	63.52%
period 2	20.50%	4.60%	33.00%	55.51%
period 3	71.88%	0.66%	37.88%	41.41%
either $A(\cdot)$ or $H(\cdot)$	42.67%	6.10%	25.68%	11.41%

We assume the structure of valuation rates is such that $r_1^s = r_1 = 10\%, r_2^s = r_2 = 6\%, r_3^s = r_3 = 2\%$ (see Table 4).

Table 4. The *valuation rate* matrix for A, B, and C ($\boldsymbol{a_t^s} = \boldsymbol{r_t}$)

	asset A	asset B	asset C	either $A(\cdot)$ or $H(\cdot)$
period 1	10%	10%	10%	10%
period 2	6%	6%	6%	6%
period 3	2%	2%	6%	2%
either $A(\cdot)$ or $H(\cdot)$	6.5%	7.03%	6.9%	1.24%

This implies that the values of interest of the three assets are $\mathcal{I}^A = 1736.4$, $\mathcal{I}^B = 224.3$, $\mathcal{I}^C = -1937.6$ (the latter is negative, for C is a borrowing).[9] The assets AIRs are $\bar{\imath}^A = 42.67\%$, $\bar{\imath}^B = 6.1\%$, $\bar{\imath}^C = 25.68\%$ and the assets' AVRs are $\bar{r}^A = 6.5\%$, $\bar{r}^B = 7.03\%$, $\bar{r}^C = 6.9\%$. With the generalized Makeham's formula one find back the interest values:

$$\mathcal{I}^A(\bar{\imath}^A, \bar{r}^A) = \frac{0.4267}{0.065} \cdot (1800 - 1535.4) = 1736.4$$
$$\mathcal{I}^B(\bar{\imath}^B, \bar{r}^B) = \frac{0.061}{0.073} \cdot (2050 - 1791.5) = 224.3$$
$$\mathcal{I}^C(\bar{\imath}^C, \bar{r}^C) = \frac{0.2568}{0.069} \cdot (-2850 - (-2329) = -1937.6.$$

[9] C is a net borrowing, for $P_0 - \mathcal{K} = -2850 - (-2329) = -521 < 0$ (see Definition 4.1). Therefore, $\mathcal{I}^C$ represents interest expenses (negative value); conversely, A and B are net investment, so $\mathcal{I}^A$ and $\mathcal{I}^B$ express interest incomes (positive value).

The portfolio rates of return are $i_1 = 63.52\%$, $i_2 = 55.51\%$, $i_3 = 41.41\%$. Averaging out (arithmetically or harmonically) the period rates of return or the asset's rates of return above seen, one gets the portfolio AIR, which is $\bar{\imath} = 11.41\%$. Likewise, the portfolio AVR is obtained by averaging out (arithmetically or harmonically) either $r_1 = 10\%, r_2 = 6\%, r_3 = 2\%$ or the assets' AVRs, which leads to $\bar{r} = 1.24\%$. The portfolio value of interest is obtained by summing the three assets' value of interest: $\mathcal{I} = \mathcal{I}^A + \mathcal{I}^B + \mathcal{I}^C = 23.15$, or by summing the portfolio value of interest in the various periods: from $\mathcal{I}_1 = 577.41, \mathcal{I}_2 = -1030.7, \mathcal{I}_3 = 476.44$ one gets back to $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = 23.15$. Consistently, the generalized Makeham's formula leads to

$$\mathcal{I} = \frac{0.1141}{0.0124} \cdot (1000 - 997.48) = 23.15.$$

This also means that the additivity principle (i.e., no arbitrage principle) is fulfilled: the sum of the assets' interest values equals the portfolio interest value.

so value additivity is fulfilled. As for economic profitability, the portfolio is a net investment, for $P_0 - \mathcal{K} = 1000 - 997.48 > 0$, and the investor's wealth is increased, since $\bar{\imath} = 11.41\% > 1.24\% = \bar{r}$. The average yield spread is then $\overline{\sigma} = 10.17\%$, so the wealth increase is $NPV = 203 \cdot 0.1017/0.0124 = 20.6$.

The *original* Makeham's formula is evidently not applicable, given that the interest rates and valuations rates are not constant. Also, even assuming a constant valuation rate, the idea of using IRR in place of the AVR leads to incorrect results. For example, assuming $r_t = r = 10\%$, the AIRs would be $\bar{\imath}^A = 42.04\%, \bar{\imath}^B = 6.26\%, \bar{\imath}^C = 25.37\%$, $\bar{\imath} = 6.7\%$ so that

$$\begin{aligned}
\mathcal{I}^A(\bar{\imath}^A, r) &= \frac{0.4204}{0.1} \cdot (1800 - 1409.76) = 1640.57 \\
\mathcal{I}^B(\bar{\imath}^B, r) &= \frac{0.0626}{0.1} \cdot (2050 - 1695.19) = 222.16 \\
\mathcal{I}^C(\bar{\imath}^C, r) &= \frac{0.2537}{0.1} \cdot (-2850 - (-2119.76) = -1852.82 \\
\mathcal{I}(\bar{\imath}, r) &= 1640.57 + 222.16 - 1852.82 \\
&= \frac{0.067}{0.1}(1000 - 985.2) = 9.92.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}(\bar{\imath}, r) &= 1640.57 + 222.16 - 1852.82 \\
&= \frac{0.067}{0.1}(1000 - 985.2) = 9.92.
\end{aligned}$$

In contrast, the use of the assets' IRRs would lead to

$$\mathcal{I}^A(x^A, r) = \frac{0.4028}{0.1} \cdot (1800 - 1409.76) = 1572.06$$
$$\mathcal{I}^B(x^B, r) = \frac{0.0615}{0.1} \cdot (2050 - 1695.19) = 218.35$$
$$\mathcal{I}^C(x^C, r) = \frac{0.2452}{0.1} \cdot (-2850 - (-2119.76) = -1790.72$$

Summing the three amounts above, one obtains -0.31, which is negative (instead of positive). As for the portfolio, note that it has three IRRs: 0%, 24.19%, 55.81%, all of which generate incorrect results, which are even inconsistent with the amount -0.31 just found:

$$\mathcal{I}(0\%, 10\%) = \frac{0}{0.1} \cdot (1000 - 985.2) = 0$$
$$\mathcal{I}(24.19\%, 10\%) = \frac{0.2419}{0.1} \cdot (1000 - 985.2) = 35.8$$
$$\mathcal{I}(55.81\%, 10\%) = \frac{0.5581}{0.1} \cdot (1000 - 985.2) = 85.6.$$

This means that the use of IRR not only leads to incorrect results, but also does not guarantee the additivity property: indeed, any of the above portfolio interest value differs from the sum of the assets' interest values:

$$\mathcal{I}^A(x^A, r) + \mathcal{I}^B(x^B, r) + \mathcal{I}^C(x^C, r) \neq \mathcal{I}(x, 10\%)$$

for $x = 0\%, 24.19\%, 55.81\%$.

Concluding remarks

Makeham's formula enables one to divide the value of a financial transaction into interest and capital components. Unfortunately, Makeham's formula is nowadays neglected in the literature and in the practice. This paper aims at resurrecting the formula by:

(i) generalizing the formula for varying interest rates and varying valuation rates (e.g., non-flat term structure of interest rates)

(ii) generalizing the formula so as to cope with portfolios of assets

(iii) showing that the generalized Makeham's formula can be used for assessing an asset's economic profitability.

The task is equivalently accomplished by showing that a principal-weighted arithmetic mean of the (interest and valuation) rates or an interest-weighted harmonic mean of the (interest and valuation) rates successfully copes with the problem of computing the value of interest and the value of principal repayments. We consider a portfolio of assets and show that the arithmetic mean and the harmonic mean used in the generalized Makeham's formula are commutative; analogously, thanks to additivity, the rates can be averaged out by assets and by periods in either order leading to the same result. As a result, the new notions of 'Average Interest Rate' (AIR) and Average Valuation Rate (AVR) are introduced, which replace the interest and the valuation rate. The ratio of the AIR to the AVR, multiplied by the difference between principal and value of capital repayments supplies the value of interest of any financial transaction. We also show that, contrary to the venerable Internal Rate of Return, the AIR exists, is unique, ensures fulfillment of value additivity (no-arbitrage principle) and always provides a correct answer (i.e., it is aligned with net-present-value criterion) when compared to the AVR, which constitutes a natural generalization of the cost-of-capital notion when valuation rates are not constant.

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