A census of genus two 3-manifolds up to 42 coloured tetrahedra *

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Abstract

We improve and extend to the non-orientable case a recent result of Karabas, Malicki and Nedela concerning the classification of all orientable prime 3-manifolds of Heegaard genus two, triangulated with at most 42 coloured tetrahedra.

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1 Introduction

In [20], Karabas, Malicki and Nedela show that there exist exactly 78 non-homeomorphic, closed, orientable, prime 3-manifolds with Heegaard genus two, admitting a coloured triangulation with at most 42 tetrahedra.

Each manifold M is identified by a suitable 6-tuple of non-negative integers, representing a minimal crystallization – hence a minimal coloured triangulation – of M. From such a 6-tuple, a presentation of the fundamental group and of the first homology group of M are easily obtained (see also [21]).

The result is performed first by generating all "admissible" 6-tuples, encoding genus two crystallizations up to order 42 ([19]) and then by using combinatorics, topology and group theory to subdivide them into 78 equivalence classes (after excluding $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$, lens spaces and connected sums), which are proved to be in one-to-one correspondence with the homeomorphism classes of the represented 3-manifolds.

In the present paper, we improve the previous result and extend it to the non-orientable case, by using a computer program which generates directly all (bipartite and non-bipartite) 3-manifold crystallizations of a given order.

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The procedure, restricted to graphs of regular genus two and order at most 42, produces as output 703 bipartite crystallizations (thus representing orientable 3-manifolds) and 82 non-bipartite crystallizations (thus representing non-orientable 3-manifolds).

A classification algorithm based on the concept of "dipole moves", implemented in a C++ program, enables us to partition the graphs previously generated into 175 classes in the bipartite case and into 9 classes in the non-bipartite case, which are proved to represent non-homeomorphic (orientable and non-orientable) 3-manifolds, of Heegaard genus ≤ 2 (with the given bound for the number of vertices of the crystallizations).

In the orientable case, 97 classes represent genus one 3-manifolds or connected sums. The remaining 78 classes, representing prime, genus two 3-manifolds, are listed in Table 2, by increasing number of vertices of the crystallizations, where for each class a geometric description, the representative 6-tuple and the position in Karabas-Malicki-Nedela's list are presented. This completes the identification of all still unknown manifolds of [20].

In the non-orientable case, two classes represent $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ (the twisted 2-sphere bundle over $\mathbb{S}^1$, of Heegaard genus one) and a connected sum respectively. Hence, there exist exactly 7 prime, non-orientable 3-manifolds with genus two, all listed and identified in Table 3, again by increasing number of vertices of the crystallizations.

2 Preliminaries

Throughout this paper, spaces and maps will be in PL-category, for which we refer to [31]. Manifolds will be closed and connected, when not otherwise specified. The symbol $\cong$ will mean *PL-homeomorphism.*

Crystallization theory provides an useful tool for representing manifolds by means of edge-coloured graphs ([30]). In this section, we limit ourselves to give definitions and results which are necessary to understand our work. For an exhaustive look on the theory, we refer to [1] and [15]. For the basic facts about graph theory see [17].

An $(n+1)$-*coloured graph* is a pair (Γ, γ), where Γ is a graph, regular of degree $n+1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0, \ldots, n\}$ a map which is injective on each pair of adjacent edges of Γ. In the following, we will often write Γ instead of (Γ, γ).

For each $B \subseteq \Delta_n$, we call *B-residues* of (Γ, γ) the connected components of the coloured graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; given an integer $m \in \{1, \ldots, n\}$, we call *m-residue* of Γ each B-residue of Γ with $\#B = m$. Moreover, for each $i \in \Delta_n$, we set $\hat{\imath} = \Delta_n \setminus \{i\}$.

An isomorphism $\phi : \Gamma \to \Gamma'$ is called a *coloured isomorphism* between the $(n+1)$-coloured graphs (Γ, γ) and (Γ', γ') if there exists a permutation φ of Δ_n such that $\varphi \circ \gamma = \gamma' \circ \phi$.

A coloured n-complex is a pseudocomplex K of dimension n with a labelling of its vertices by $\Delta_n = \{0, \ldots, n\}$, which is injective on the vertex-set of each simplex of K.

For each $(n+1)$-coloured graph Γ, a coloured n-complex $K(\Gamma)$ can be obtained by the following rules:

- for each vertex v of Γ, take an n-simplex $\sigma(v)$ and label its vertices by Δ_n;

- if v and w are vertices of Γ joined by an c-coloured edge ($c \in \Delta_n$), then identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertices labelled c.

If M is a manifold of dimension n and Γ an $(n+1)$-coloured graph such that $|K(\Gamma)| \cong M$ (here $|K(\Gamma)|$ denotes the space of the complex $K(\Gamma)$), then, following Lins ([23]), we say that Γ is a *gem* (graph-encoded-manifold) representing M.

If, for each $i \in \Delta_n$, $\Gamma_{\hat{\imath}}$ is connected (equivalently if the corresponding coloured triangulation $K(\Gamma)$ has exactly one vertex labelled i, for each $i \in \Delta_n$), then Γ and $K(\Gamma)$ are called *contracted*; furthermore, a contracted gem representing an n-manifold M is called a *crystallization* of M. Note that M is orientable iff Γ is bipartite.

Given two $(n+1)$-coloured graphs Γ' and Γ'' representing the manifolds M' and M'' respectively, we can easily construct an $(n+1)$-coloured graph $\Gamma = \Gamma' \# \Gamma''$ representing $M' \# M''$. Let x be a vertex of Γ' and y a vertex of Γ''; then we obtain Γ by removing x from Γ' , y from Γ'' and by gluing the "hanging" edges according to their colours (see [15]).

It is well-known that, if both manifolds are orientable (i.e. Γ' and Γ'' are both bipartite) and do not admit orientation-reversing automorphisms, there exist two non-homeomorphic connected sums. In this case, by the above construction, we can obtain two $(n+1)$-coloured graphs, each corresponding to fix x in $V(\Gamma')$ and choose y in one of the two different bipartition classes of $V(\Gamma'')$.

Let Γ be an $(n+1)$-coloured graph representing an n-manifold M and suppose that Γ satisfies the following condition (which in the following will be referred to as "condition $(\#)$"):

$(\#)$ Γ has $n+1$ edges $\{e_0, \dots, e_n\}$, one for each colour $i \in \Delta_n$, such that $\Gamma - \{e_0, \dots, e_n\}$ splits into two connected components.

Then it is easy to reverse the connected sum construction and, starting from Γ, obtain two $(n+1)$-coloured graphs Γ' and Γ'', representing two n-manifolds M' and M'' respectively, such that $\Gamma = \Gamma' \# \Gamma''$. Hence Γ represents $M' \# M''$ (more precisely, Γ represents one of the two possibly non-homeomorphic connected sums).

Coloured graphs appearing in our catalogues are always represented by a numerical "string", which is called the *code*; it describes completely the combinatorial structure of the coloured graph (see [10] for definition and description of the related *rooted numbering algorithm*) and, since two $(n+1)$-coloured graphs are colour-isomorphic iff they have the same code ([10]), by representing each coloured graph by its code, we can easily reduce any catalogue of crystallizations to one containing only non-colour-isomorphic graphs. Moreover the code is easy to be handled by computer and starting from a code c there is a standard way to contruct a coloured graph having c as its code.

Main tools of our work are combinatorial moves (*dipole moves*) which transform a gem representing an n-manifold into another (usually non-colour-isomorphic) gem, representing the same manifold.

If x, y are two vertices of a $(n+1)$-coloured graph (Γ, γ) joined by k edges $\{e_1, \dots, e_k\}$ with $\gamma(e_h) = i_h$, for $h = 1, \dots, k$, then we call $\theta = \{x, y\}$ a *k-dipole* or a *dipole of type* k in Γ, *involving colours* $i_1, \dots, i_k$, iff x and y belong to different $(\Delta_n - \{i_1, \dots, i_k\})$-residues of Γ.

In this case a new $(n+1)$-coloured graph (Γ', γ') can be obtained from Γ by deleting x, y and all their incident edges and joining, for each $i \in \Delta_n - \{i_1, \dots, i_k\}$, the vertex i-adjacent to x to the vertex i-adjacent to y; (Γ', γ') is said to be obtained from (Γ, γ) by *deleting the k-dipole* θ. Conversely (Γ, γ) is said to be obtained from (Γ', γ') by *adding the k-dipole.*

From now on, we restrict ourselves to 3-manifolds; in this context, we can introduce further moves.

Let (Γ, γ) be a 4–coloured graph. Let Θ be a subgraph of Γ formed by a $\{i,j\}$-coloured cycle C of length $m+1$ and a $\{h,k\}$-coloured cycle C' of length $n+1$, having only one common vertex x_0 and such that $\{i,j,h,k\} = \{0,1,2,3\}$. Then Θ is called an *(m,n)–dipole.*

If x_1, x_m, y_1, y_n are the vertices respectively i, j, h, k-adjacent to x_0, we define the 4-coloured graph (Γ', γ') *obtained from* Γ *by cancelling the (m,n)–dipole*, in the following way:

1) delete Θ from Γ and consider the product Ξ of the subgraphs $C - \{x_0\}$ and $C' - \{x_0\}$;

2) for each $s, s' \in \{1, \ldots, n\}$ (resp. for each $r, r' \in \{1, \ldots, m\}$), let e be the edge joining y_s and $y_{s'}$ (resp. x_r and $x_{r'}$) in Γ. If $\gamma(e) = c \in \{0,1,2,3\}$, then, for each $t \in \{1, \ldots, m\}$ (resp. for each $t \in \{1, \ldots, n\}$), join the vertices (x_t, y_s) and $(x_t, y_{s'})$ (resp. (x_r, y_t) and $(x_{r'}, y_t)$) by a c-coloured edge in Ξ;

3) for all $r \in \{1, \ldots, m\}$, $s \in \{1, \ldots, n\}$, if a vertex z of $\Gamma - \Theta$ is joined to y_s (resp. x_r) by a i or j (resp. h or k)–coloured edge in Γ, then z is joined to $(x_1,\, y_s)$, $(x_m,\, y_s)$ (resp. $(x_r,\, y_1)$, $(x_r,\, y_n)$) by a i or j (resp. h or k)–coloured edge in Γ'.

Cancellation or addition of a (m,n)-dipole is called a *generalized dipole move.*

If two i-coloured edges $e, f \in E(\Gamma)$ belong to the same $\{i,j\}$-coloured cycle and to the same $\{i,k\}$-coloured cycle of Γ, with $j, k \in \Delta_3 - \{i\}$ (resp. to the same $\{i,h\}$-coloured cycle of Γ, for each $h \in \Delta_3 - \{i\}$), then (e,f) is called a *ρ_2-pair* (resp. a *ρ_3-pair*). Usually, we will write *ρ-pair* instead of ρ_2-pair or ρ_3-pair.

A graph Γ is a *rigid crystallization* of a 3-manifold M^3 if it is a crystallization of M^3 and contains no ρ-pairs. A non-rigid crystallization Γ of a 3-manifold M can be always transformed into a rigid one by *switching ρ-pairs* (see [23]) and cancelling the dipoles which might be created in the process.

The effects of cancellation/addition of a dipole, a generalized dipole and of switching of a ρ-pair are described in the following Proposition.

Proposition 1 ([14], [23])

(i) If Γ and Γ' are 4-coloured graphs representing two 3-manifolds M and M' respectively, and Γ' is obtained from Γ by a dipole move or a generalized dipole move or by switching a ρ_2-pair, then $M \cong M'$.

(ii) Let Γ be a 4-coloured graph representing a 3-manifold M, containing a ρ_3-pair. If Γ', obtained from Γ by switching it, is connected, then it represents a 3-manifold M', such that $M = M' \# H$, where either $H = \mathbb{S}^1 \times \mathbb{S}^2$ or $H = \mathbb{S}^1 \tilde{\times} \mathbb{S}^2$.

Note that, if Γ is a crystallization of M, then Γ' is always connected.

Remark 1 In the case of dipole moves, statement (i) of the above Proposition is actually stronger. In fact the main theorem of [14] proves that M ad M' are homeomorphic iff Γ and Γ' are obtained from each other by a sequence of dipole moves.

A different set of moves is defined in [24]

Remark 2 Each closed connected 3-manifold admits a rigid crystallization (see [7] for a detailed proof). Moreover, an easy consequence of Proposition 1 proves that every closed connected 3-manifold M different from $M'\#H$ admits a rigid crystallization of minimal order (see [7]). Hence, since we are interested mainly in prime manifolds, in the generation and analysis of our catalogues we will restrict ourselves to rigid crystallizations.

Note that, each orientable genus two 3-manifolds is the 2-fold covering of $\mathbb{S}^3$, branched over a knot or link L ([4]). The construction described in [13] allows to obtain a crystallization Γ of M, starting from a 3-bridge presentation of L. As a consequence of the construction, Γ belongs to a particular class of crystallizations, called *2-symmetric* in [11], which can be codified by suitable 6-tuples (called *admissible*) of non-negative integers ([8]). Hence, admissible 6-tuples are a representation tool for orientable genus two 3-manifolds.

In [16], the authors describe an equivalence relation on the set of admissible 6-tuples, whose equivalence classes consist only of 6-tuples representing 2-symmetric crystallizations of the same manifold. In [18] a catalogue was presented of the representatives of the equivalence classes of admissible 6-tuples, whose associated 2-symmetric crystallizations have at most 42 vertices. Subsequently in [20] the catalogue was reduced to 6-tuples all representing distinct manifolds.

3 Seifert manifolds and coloured triangulations

Let $M = (S, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ be the Seifert fibered space whose orbit space is the surface S and having n exceptional fibers, with non-normalized parameters (α_i, β_i), $i = 1, \ldots, n$.

Let us consider a triple of integers (p, q, r) such that:

(i) $(|p|, |q|) = 1$;

(ii) $p + q + r = 0$;

In [5] and [6], the author describes a triangulation of the solid torus, called *layered solid torus of type (p,q,r)* and denoted by $LST(p, q, r)$, which is used to construct triangulations of Seifert manifolds.

We recall briefly the main steps of the construction, which is done recursively. First of all, we point out that, at each step, we obtain a layered solid torus (from now on, LST for short) having exactly two 2-simplexes on the boundary, which will be called its *boundary faces*. Moreover, it has exactly three boundary edges with a labelling by means of integers satisfying conditions $(i) - (ii)$. The labelling is defined by recursion, too.

The main point of the procedure is the possibility of performing a *layering* on an edge e' of a layered solid torus $LST(p, q, r)$ in the following way:
suppose that e' is labelled $i \in \{p, q, r\}$, then a new tetrahedron is considered and two adjacent faces of it, say F and F', are identified with the boundary faces of $LST(p, q, r)$ so that the common edge of F and F' coincides with the edge e'. Let f and g be the boundary edges

of $LST(p,q,r)$ labelled j and k respectively $(j,k \in \{p,q,r\} - \{i\})$. Then the new boundary edge identified with f (resp. g) inherits the label j (resp. $-k$). The obtained complex is the layered solid torus whose set of related integers is $\{j,-k,k-j\}$.

Remark 3 The integers p,q are actually the intersection numbers of the boundary of a meridinal disk of $LST(p,q,r)$, which is a simple oriented curve on the boundary torus T of $LST(p,q,r)$, with the basis of $\pi_1(T)$ formed by the (suitably oriented) edges labelled p and q. It is not difficult to see that $LST(-p,-q,-r)$ is the same triangulation of the solid torus and describes the same curves with reversed orientations. Since for our aims we don't need to distinguish the two layered solid tori, from now on we suppose that two of the elements of the set $\{p,q,r\}$ are positive. Moreover, each permutation of the set $\{p,q,r\}$ doesn't change the related LST; therefore, in the following construction we need only to specify the set of integers we are working on, without imposing an ordering.

In order to construct the LST with set of parameters $\{p,q,-p-q\}$, we perform the following procedure:

- the initial step is the LST whose parameters are $\{1,2,-3\}$: it is obtained from the tetrahedron, with labelled edges, in Figure 1, by identifying the "back" faces according to the arrows.

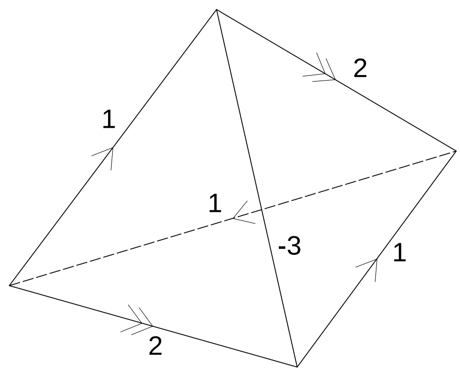

Figure 1:

- the LST with parameters $\{i,j,k\}$ $(0<j<k)$ is obtained from the LST with parameters $\{j,-k,k-j\}$, by a layering on the edge labelled $k-j$.

For more details about the construction and its geometric meanings, see [5] and [6].

We are now ready to describe how to construct a coloured triangulation of the Seifert fiber space $M = (\mathbb{S}^2, (\alpha_1,\beta_1), (\alpha_2,\beta_2), (\alpha_3,\beta_3))$.

Let us fix a point $A \in \mathbb{S}^2$ and let $\mathbb{D}_1,\ \mathbb{D}_2,\ \mathbb{D}_3$ be 2-disks in $\mathbb{S}^2$ such that $\mathbb{D}_1 \cap \mathbb{D}_2 \cap \mathbb{D}_3 = \partial\mathbb{D}_1 \cap \partial\mathbb{D}_2 \cap \partial\mathbb{D}_3 = \{A\}$. The pseudocomplex P of Figure 2 is a planar realization of $\mathbb{S}^2 \setminus \bigcup_{i=1}^{3} int\, D_i$.

Figure 3 shows the boundary surface of $P \times I$, with the identifications of the faces $P \times \{0\}$ and $\{a_1\} \times I$ with $P \times \{1\}$ and $\{a_2\} \times I$ respectively. Moreover, we marked the subdivision of

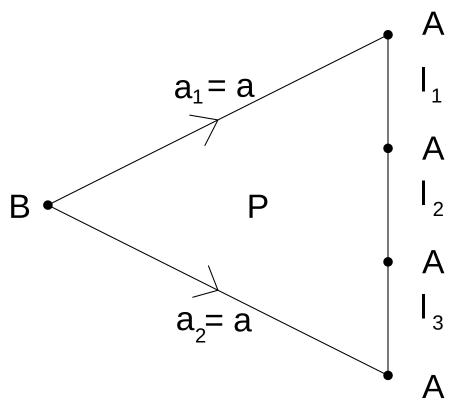

Figure 2:

the faces $\{l_i\} \times I$ $(i = 1, 2, 3)$ which will be necessary for the construction of a triangulation of M.

We point out that for each $i = 1, 2, 3$, the boundary face $\{l_i\} \times I$ is the usual representation of a torus by a square. Therefore, after the identifications of $P \times \{0\}$ and $\{a_1\} \times I$ with $P \times \{1\}$ and $\{a_2\} \times I$ respectively, we obtain $\mathbb{S}^1 \times \mathbb{S}^2$ with three solid tori removed.

The last step is the identification, for each $i = 1, 2, 3$, of the two triangles of $\{l_i\} \times I$ (see Figure 3) with the boundary faces of $LST(\alpha_i, \theta_i, \sigma_i)$, where either $\theta_i = \beta_i$ or $\sigma_i = -\beta_i$, so as to identify the edges $\{A\} \times I$ (resp. $l_i \times \{0\}$ and $l_i \times \{1\}$) with the edges labelled α_i (resp. either θ_i or σ_i). In this way we obtain a triangulation K of M.

Note that, in order to define precisely the gluing of the three layered solid tori, it is necessary to specify not only the sets of parameters but also their ordering. Hence the notation with triples $(\alpha_i, \theta_i, \sigma_i)$, $\;i = 1, 2, 3$.

Finally, in order to obtain a coloured triangulation, we consider the first barycentric subdivision K' of K and colour h $(h \in \Delta_3)$ the barycenters of h-dimensional cells of K'.

4 Generating and analysing genus two crystallizations

The *regular genus* of an n-manifold is a combinatorial invariant which extends to dimension n the classical concepts of genus of a surface and Heegaard genus of a 3-manifold. For the precise definition of the invariant, we refer to [1], but to suit our present aim it is sufficient to recall that, if Γ is a crystallization of a 3-manifold, then the regular genus of Γ is the integer

$$\rho(\Gamma) = min\{g_{01}(\Gamma), g_{02}(\Gamma), g_{03}(\Gamma)\} - 1$$

where for each $i = 1, 2, 3$, $\;g_{0i}(\Gamma)$ is the number of $\{0, i\}$-residues of Γ.

Therefore, the regular genus of a 3-manifold M can be defined as the minimum $\rho(\Gamma)$ among all crystallizations of M.

Combinatorial encoding of closed 3-manifolds by crystallizations, together with the restriction to rigid ones, allows us to construct essential catalogues of all contracted triangulations

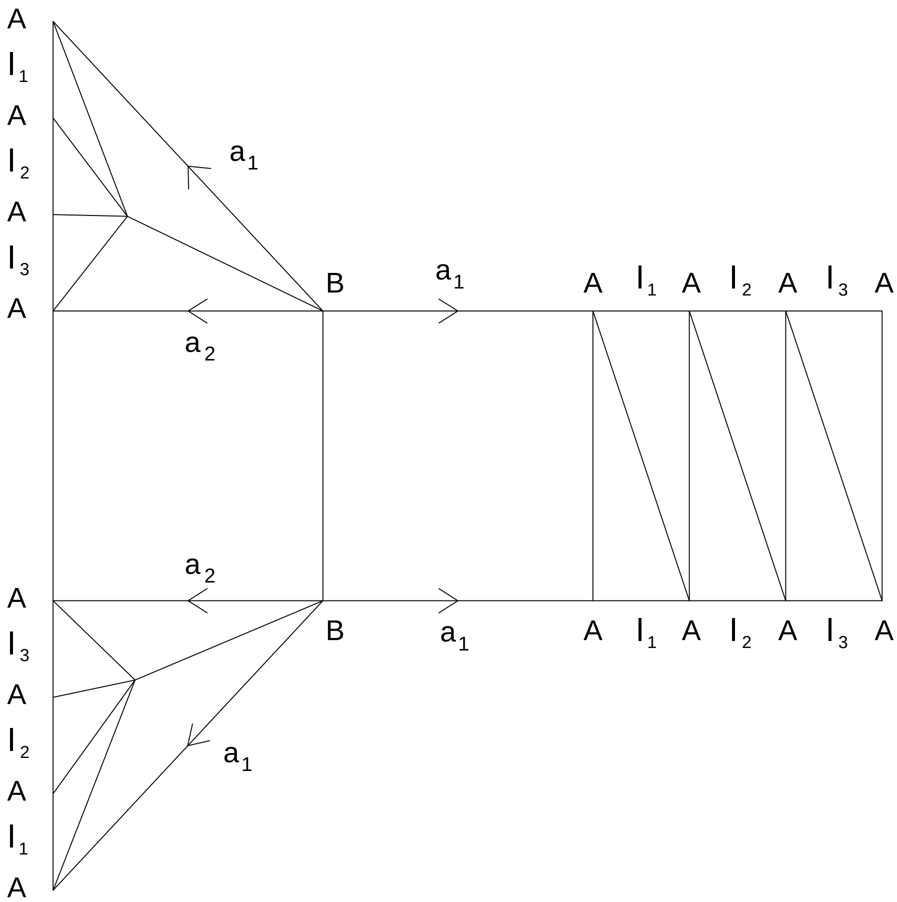

Figure 3:

of closed 3-manifolds up to a certain number of vertices. By using the codes, we can easily avoid isomorphic graphs, too.

The general algorithm, which is fully described in [7] and [9], runs as follows:

- The starting point is the set $\mathcal{S}^{(2p)}$ of all (connected) rigid[1] and planar 3-coloured graphs with $2p$ vertices. The construction makes use of Lins's results in [22] and is performed by induction on p. More precisely every rigid and planar 3-coloured graph with $2p$ vertices is obtained from an analogous one with $2p-2$ vertices, by a suitable operation (*antifusion*), with the possible exception of the "prism" with p-gonal base (with p even);

- to each element of $\mathcal{S}^{(2p)}$ 3-coloured edges are added in all possible ways so as to produce rigid crystallizations of 3-manifolds. By checking bipartition, crystallizations of orientable and non-orientable manifolds can be separated.

In this way we obtain the catalogues of all non-isomorphic rigid bipartite and non-bipartite crystallizations with $2p$ vertices.

[1]here "rigid" means without ρ_2-pairs

It is possible to modify the general algorithm in order to construct only crystallizations having a fixed regular genus. In particular, we modified Lins's construction in order to obtain the set $\tilde{\mathcal{S}}^{(2p)}$ of all rigid 3-coloured graphs $\tilde{\Sigma}$ with $2p$ vertices representing $\mathbb{S}^2$ and such that $g_{ij}(\tilde{\Sigma}) = 3$, for at least one pair of distinct colours $i, j \in \{0, 1, 2\}$ and $g_{hk}(\tilde{\Sigma}) \geq 3$ for the remaining pairs. Since Lins's procedure is inductive on the number of vertices and each step increases the number of bicoloured cycles of a given pair of colours by at most one, it was sufficient, at each step, to perform the required transformations only on graphs having at most two $\{i, j\}$-residues for at least one pair $i, j \in \{0, 1, 2\}$ and finally to eliminate the resulting graphs with $2p$ vertices which satisfy the same property.

By adding p 3-coloured edges to each element of $\tilde{\mathcal{S}}^{(2p)}$ in all possible ways so as not to destroy planarity of the $\hat{j}$-residues ($j \in \{0, 1, 2\}$) and rigidity, and by eliminating the resulting graphs Γ which are not contracted, we obtained the catalogues $\mathcal{C}_2^{(2p)}$ and $\tilde{\mathcal{C}}_2^{(2p)}$ of all rigid bipartite and non-bipartite, respectively, non-isomorphic crystallizations with $2p$ vertices having regular genus two.

The restriction about the genus allows us to obtain a reduction of time in the generation procedure and, consequently, we could obtain catalogues for higher number of vertices than in the general case where no genus bounds are imposed. More precisely, we generated these catalogues up to 42 vertices.

The above algorithm was implemented in a C++ program, whose output data are presented in Table 1 according to the number of vertices. We point out that there are no rigid genus two crystallizations with less than 14 vertices.

2p	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
$\#\mathcal{C}_2^{(2p)}$	1	2	4	6	8	14	18	23	38	47	58	79	118	128	159
$\#\tilde{\mathcal{C}}_2^{(2p)}$	1	1	1	1	2	2	3	2	6	7	9	7	12	12	16

Table 1: Genus two rigid crystallizations up to 42 vertices.

In [9] for the orientable case and [2] and [7] for the non-orientable case, catalogues of all rigid crystallizations with at most 30 vertices have been analysed and the represented manifolds identified.

In this paper we follow the same line with respect to the catalogues $\mathcal{C}_2^{(2p)}$ and $\tilde{\mathcal{C}}_2^{(2p)}$ with $p \leq 21$, i.e. we manipulate crystallizations through generalized dipole moves and subdivide them into classes according to the equivalence defined by the moves.

In fact, the procedure is completely general and requires only a given list X of rigid crystallizations and a fixed set $\mathcal{S}$ of sequences (called *admissible*) of generalized dipoles moves, dipole moves and ρ-pairs switching, such that each element of $\mathcal{S}$ transforms rigid crystallizations into rigid crystallizations (see [9] for details). For each $\Gamma \in X$ and for each $\epsilon \in \mathcal{S}$, we denote by $\theta_\epsilon(\Gamma)$ the (rigid) crystallization obtained by applying the sequence ϵ to Γ.

Note that, by Proposition 1, if Γ represents a 3-manifold M then $\theta_\epsilon(\Gamma)$ represents the 3-manifold M', such that $M = M' \#_r H$ where $\#_r H$ denotes the connected sum of r copies either of the orientable or of the non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$; more precisely $H = \mathbb{S}^1 \times \mathbb{S}^2$ iff Γ and $\theta_\epsilon(\Gamma)$ are both bipartite or both non-bipartite. Obviously r is the number of ρ_3-pairs, which have been eventually deleted while applying the sequence ϵ (usually we denote this number by $h_\epsilon(\Gamma)$).

As a consequence, by using the elements of $\mathcal{S}$, it is possible to subdivide X into disjoint classes $\{c_1, \dots, c_s\}$ such that, for each $i \in \{1, \dots, s\}$ and for each $\Gamma, \Gamma' \in c_i$, there exist two integers $h, k \geq 0$ and a 3-manifold M such that $|K(\Gamma)| = M \#_h H$ and $|K(\Gamma')| = M \#_k H$.

More precisely, for each $\Gamma \in X$, we define the *class* of Γ as the set

$$\begin{aligned} cl(\Gamma) = \{\Gamma' \in X \mid \exists\, \epsilon, \epsilon' \in \mathcal{S} \ \text{ s.t.} \\ \theta_\epsilon(\Gamma) \text{ and } \theta_{\epsilon'}(\Gamma') \text{ have the same code}\}. \end{aligned}$$

In [9] and [2] it is shown how to construct the set $cl(\Gamma)$, for each $\Gamma \in X$, and also how to compute a non-negative number denoted by $h(\Gamma)$; it defines a function $h \ : \ X \to \mathbb{N} \cup \{0\}$ inducing a natural subdivision of each class c_i ($i = 1, \dots, s$) into subclasses $c_{i,k} = \{\Gamma' \in c_i \mid h(\Gamma') = k\}$ such that all crystallizations of a given subclass represent the same manifold.

Moreover, for each $\Gamma \in X$ such that $cl(\Gamma) = c_i$, if the elements of $c_{i,0}$ represents a 3-manifold M, then Γ represents the 3-manifold $M' = M \#_{h(\Gamma)} H$, with H as above.

For the precise description of the algorithm yielding $cl(\Gamma)$ and $h(\Gamma)$, we refer to the already cited papers. Moreover, we point out that the choice of the set $\mathcal{S}$, which is used in our implementation, is exactly the same as described in [2].

Note that, by the above definitions and results, if known catalogues of crystallizations are inserted in X, all classes of X containing at least one known crystallization are completely identified, with respect to the manifolds represented by all their subclasses.

Moreover, condition (#) can be checked to recognize connected sums.

According to these ideas, the classification algorithm has been implemented in the C++ program Γ-*class* [2]: its input data are a list X of rigid crystallizations and the informations about already known crystallizations in X (possibly none), i.e the identification of their represented manifolds through suitable "names"; the output is the list of classes of X, together with their representatives and, if possible, their names.

In the following sections we present the results of Γ-class applied to catalogues $\mathbf{C}_2^{42} = \bigcup_{p=1}^{21} \mathcal{C}_2^{(2p)}$ and $\widetilde{\mathbf{C}}_2^{42} = \bigcup_{p=1}^{21} \widetilde{\mathcal{C}}_2^{(2p)}$.

5 Genus two orientable 3-manifolds

The catalogue $\mathbf{C}_2^{42}$ is partitioned by the program Γ-class into 175 classes, 93 of which are known through former results in [23] and [9]; moreover the program recognized 23 connected sums, which didn't appear in the cited papers.

In particular, we have the following result.

Lemma 2 *For each $(\Gamma, \gamma) \in \mathbf{C}_2^{(42)}$ satisfying condition (#) with summands $(\Gamma^{(1)}, \gamma^{(1)})$ and $(\Gamma^{(2)}, \gamma^{(2)})$, then*

[2]developed by M.R. Casali and P. Cristofori and available at WEB page http://cdm.unimo.it/home/matematica/casali.mariarita/CATALOGUES.htm where a detailed description of the program can be found, too.

- *if at least one of* $|K(\Gamma^{(1)})|$, $|K(\Gamma^{(2)})|$ *admits orientation-reversing self-homeomorphisms, then* $cl(\Gamma)$ *is the only class representing a connected sum with summands* $|K(\Gamma^{(1)})|$ *and* $|K(\Gamma^{(2)})|$*;*
- *otherwise there are exactly two classes* $cl(\Gamma)$, $cl(\Gamma')$ *representing a connected sum with summands* $|K(\Gamma^{(1)})|$ *and* $|K(\Gamma^{(2)})|$*; in this case* $|K(\Gamma)| \ncong |K(\Gamma')|$.

Proof. Our program results show that, whenever at least one of $|K(\Gamma^{(1)})|$, $|K(\Gamma^{(2)})|$ admits orientation-reversing self-homeomorphisms, there is only one class as in the statement.

On the other hand, in the case of neither $|K(\Gamma^{(1)})|$ nor $|K(\Gamma^{(2)})|$ admitting orientation-reversing self-homeomorphisms, there are exactly two classes $cl(\Gamma)$, $cl(\Gamma')$ representing a connected sum with $|K(\Gamma^{(1)})|$ and $|K(\Gamma^{(2)})|$ as summands.

Furthermore, we point out that, if Γ is the connected sum of the graphs (Γ_1, γ_1) and (Γ_2, γ_2) with respect to the vertices $v_1 \in V(\Gamma^{(1)})$ and $v_2 \in V(\Gamma^{(2)})$, then, by choosing in $V(\Gamma^{(2)})$ a vertex v_2' belonging to a different bipartition class from v_2, we can construct a connected sum Γ'' of (Γ_1, γ_1) and (Γ_2, γ_2) (with $\#V(\Gamma') = \#V(\Gamma'') \leq 42$) such that $|K(\Gamma)| \ncong |K(\Gamma'')|$. Moreover, by a suitable choice of v_2', we can obtain that Γ'' has regular genus two. Actually Γ and Γ'' represent the two non-homeomorphic connected sums of $|K(\Gamma^{(1)})|$ and $|K(\Gamma^{(2)})|$.

Since Γ'' must belong to $\mathbf{C}_2^{(42)}$ too, we have necessarily $\Gamma'' \in cl(\Gamma')$. ∎

Remark 4 The identification of the summands $|K(\Gamma^{(1)})|$ and $|K(\Gamma^{(2)})|$ involved in the above lemma, has been done directly by the program for a large number of classes; namely those having $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with less than 32 vertices. All other summands had cyclic fundamental groups. Therefore, we constructed crystallizations of genus one of lens spaces with the required groups and inserted them in the list handled by Γ-class. The program identified all unknown summands as lens spaces in our list.

Further 29 classes of $\mathbf{C}_2^{42}$, having cyclic fundamental groups, were recognized by applying the same procedure as described in the above remark. They all turned out to represent lens spaces, which don't appear in catalogues $\mathcal{C}^{(2p)}$, $1 \leq p \leq 15$ or among the manifolds of [3]. Furthermore, as we will see in the following, no lens space is left among the still unidentified manifolds.

The main consequence of the output results of Γ-class and the above lemma, is that a bjiective correspondence exists between the already identified subclasses and the represented manifolds. As we will prove, the same holds for the whole catalogue.

The classes of $\mathbf{C}_2^{42}$, which have not been identified by Γ-class, are 30. The problem of their recognition will be discussed and wholly solved in the following sections.

A comparison of codes yields that there is a bijective correspondence between our still unknown classes of crystallizations of $\mathbf{C}_2^{42}$ and 30 of the 78 6-tuples which in [20] are proved to represent all distinct prime orientable genus two 3-manifolds admitting a coloured triangulation with $2p$ tetrahedra, with $p \leq 21$ and having acyclic fundamental groups. Among these classes three are already identified by the results of [3] up to 34 tetrahedra.

The 48 manifolds which have been already identified by Γ-class, by means of their admitting at least one coloured triangulation with less than 32 tetrahedra (see [9]), are mostly Seifert spaces with base $\mathbb{S}^2$ and three exceptional fibers (for the complete list see Appendix A).

The remaining unidentified manifolds fall into two cases: those with finite and those with infinite fundamental group. As pointed out in [20], all finite groups are Milnor's groups, therefore the correspoding manifolds are elliptic and completely known. In Appendix A, besides explicitly writing down the groups, we specified also the Seifert structure of these manifolds.

The problem of identifying the manifolds with infinite fundamental group is left open by the authors of [20]. We are solving it by manipulating group presentations and by constructing coloured triangulations of Seifert spaces with base $\mathbb{S}^2$ and three exceptional fibres. In fact all manifolds under examination turn out to belong to this family.

Our starting point is the following Proposition, which enables us to recognize all groups in Karabas-Malicky-Nedela's list (in the following "KMN-list" for short) corresponding to our still unknown manifolds, as fundamental groups of Seifert spaces of the above described type.

Proposition 3

(i) the group $G(\alpha_1, \alpha_2, \alpha_3)$ defined by the presentation

$$< a, b \ / \ a^{\alpha_1} = b^{\alpha_2} = (ab)^{\alpha_3} >, \quad \alpha_i > 0, \ \textit{for each } i = 1, 2$$

is isomorphic to the fundamental group of the Seifert manifold

$$(\mathbb{S}^2, (\alpha_1, 1), (\alpha_2, 1), (|\alpha_3|, \varepsilon)), \quad \textit{where} \ \ \varepsilon = -\alpha_3/|\alpha_3|;$$

(ii) the group

$$G'(\alpha_1, \alpha_2, \alpha_3) =< a, b \ / \ a^{\alpha_3} = b^{\alpha_2} = (a^{-\varepsilon} b^{\varepsilon})^{\alpha_1} >,$$

with $\alpha_i > 0$, *for each* $i = 1, 2, 3$, $\varepsilon = \pm 1$, *is isomorphic to the fundamental group of the Seifert manifold*

$$(\mathbb{S}^2, (\alpha_1, 1), (\alpha_2, -\varepsilon), (\alpha_3, \varepsilon)).$$

(iii) the group

$$G'' =< a, b \ / \ a^5 = b^3 = (ab^{-2})^{-3} >$$

is isomorphic to the fundamental group of the Seifert manifold

$$(\mathbb{S}^2, (3, 1), (3, 1), (5, -4)).$$

Proof.

(i) Let us set $q_1 = a,\ q_2 = b,\ q_3 = (ab)^{-1},\ h = (ab)^{-\alpha_3}$, then from the relations of G, we have

$$q_1^{\alpha_1} = q_2^{\alpha_2} = h^{-1},\ q_3^{|\alpha_3|}h^{\varepsilon} = 1,\ q_1q_2q_3 = 1$$

Moreover, it easy to see that, for each $i = 1, 2, 3$, q_i and h commute.

Therefore G admits the presentation

$$\begin{gathered}<\ q_1,\ q_2,\ q_3,\ h\ /\ q_1h = hq_1,\ q_2h = hq_2,\ q_3h = hq_3,\ q_1^{\alpha_1}h = 1,\\ q_2^{\alpha_2}h = 1,\ q_3^{|\alpha_3|}h^{\varepsilon} = 1,\ q_1q_2q_3 = 1 >\end{gathered}$$

which is a well-known presentation of $\pi_1((\mathbb{S}^2, (\alpha_1, 1), (\alpha_2, 1), (|\alpha_3|, \varepsilon)))$ (see [29]).

(ii) Let us consider the case $\varepsilon = 1$, then

$$G'(\alpha_1, \alpha_2, \alpha_3) =< a, b\ /\ a^{\alpha_3} = b^{\alpha_2} = (a^{-1}b)^{\alpha_1} >$$

Set $q_1 = a^{-1}b,\ q_2 = b^{-1},\ q_3 = a,\ h = (a^{-1}b)^{-\alpha_1}$

We have

$$q_1^{\alpha_1} = q_2^{-\alpha_2} = q_3^{\alpha_3} = h^{-1},\ q_1q_2q_3 = 1$$

and again each q_i $(i = 1, 2, 3)$ commutes with h.

Therefore $G' \cong \pi_1((\mathbb{S}^2, (\alpha_1, 1), (\alpha_2, -1), (\alpha_3, 1)))$. The case $\varepsilon = -1$ is analogous.

(iii) If we set $q_1 = ab^{-2},\ q_2 = b^{-1},\ q_3 = a^{-1},\ h = (ab^{-2})^{-3}$, then we have the relations

$$q_1^3 = q_2^3 = q_3^5 = h^{-1},\ q_2^2 = q_3q_1$$

Hence $h^{-1} = q_2^3 = q_3q_1q_2$. Since each q_i $(i = 1, 2, 3)$ commutes with h, we can write

$$q_3^{-1}h^{-1} = h^{-1}q_3^{-1} \quad \Longrightarrow \quad q_1q_2 = q_3q_1q_2q_3^{-1} \quad \Longrightarrow \quad q_1q_2q_3 = q_3q_2q_1$$

By comparing the relations we have

$$h^{-1} = q_3q_1q_2 = q_1q_2q_3$$

Therefore

$$\begin{gathered}G'' \cong<\ q_1,\ q_2,\ q_3,\ h\ /\ q_1h = hq_1,\ q_2h = hq_2,\ q_3h = hq_3,\ q_1^3h = 1,\\ q_2^3h = 1,\ q_3^5h = 1,\ q_1q_2q_3 = h^{-1} >\end{gathered}$$

which is the fundamental group of

$$(\mathbb{S}^2, (3, 1), (3, 1), (5, 1), (1, -1)) = (\mathbb{S}^2, (3, 1), (3, 1), (5, -4)).$$

■

We point out that all unknown 6-tuples in KMN-list corresponding to manifolds which were not identified by our former results have fundamental group admitting a presentation of one of the above types.

For each of these 6-tuples f, by means of the algorithm described in section 3, we constructed a coloured triangulation $\Gamma(f)$ of the Seifert manifold $M = (\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, with parameters α_i, β_i $(i = 1, 2, 3)$ determined by the presentation of the fundamental group of $\Gamma(f)$ given in KMN-list and by Proposition 3. In the following table, we present for each of these Seifert manifolds, the triples of parameters $(\alpha_i, \theta_i, \sigma_i)$ $(i = 1, 2, 3)$ of the three layered solid tori which have been glued to the three boundary components of $P \times I$, in order to obtain the required triangulation of M.

As a consequence, by cancellation of dipoles and switching of ρ-pairs in the above coloured triangulations, we obtain easily a list Y of crystallizations of the Seifert manifolds which could match our unknown classes.

Seifert manifold	**Layered Solid tori**
$(\mathbb{S}^2, (3,1), (3,2), (4,-3))$	$(3,1,-4), (3,2,-5), (4,-7,3)$
$(\mathbb{S}^2, (2,1), (4,1), (4,-1))$	$(2,1,-3), (4,1,-5), (4,-5,1)$
$(\mathbb{S}^2, (2,1), (4,1), (5,-4))$	$(2,1,-3), (4,1,-5), (5,-9,4)$
$(\mathbb{S}^2, (3,1), (3,1), (3,1))$	$(3,-2,-1), (3,1,-4), (3,1,-4)$
$(\mathbb{S}^2, (3,1), (3,1), (4,-1))$	$(3,1,-4), (3,1,-4), (4,-5,1)$
$(\mathbb{S}^2, (2,1), (3,1), (7,-6))$	$(2,1,-3), (3,1,-4), (7,-13,6)$
$(\mathbb{S}^2, (3,1), (3,1), (4,-3))$	$(3,1,-4), (3,1,-4), (4,-7,3)$
$(\mathbb{S}^2, (2,1), (3,2), (6,-5))$	$(2,1,-3), (3,2,-5), (6,-11,5)$
$(\mathbb{S}^2, (3,1), (3,2), (5,-4))$	$(3,1,-4), (3,2,-5), (5,-9,4)$
$(\mathbb{S}^2, (2,1), (3,1), (6,-1))$	$(2,1,-3), (3,1,-4), (6,-7,1)$
$(\mathbb{S}^2, (2,1), (4,1), (5,-3))$	$(2,1,-3), (4,1,-5), (5,-8,3)$
$(\mathbb{S}^2, (2,1), (4,3), (5,-4))$	$(2,1,-3), (4,3,-7), (5,-9,4)$
$(\mathbb{S}^2, (2,1), (4,1), (6,-5))$	$(2,1,-3), (4,1,-5), (6,-11,5)$
$(\mathbb{S}^2, (3,1), (3,2), (3,-1))$	$(3,1,-4), (3,2,-5), (3,-4,1)$
$(\mathbb{S}^2, (2,1), (4,1), (4,1))$	$(2,1,-3), (4,-3,1), (4,1,-5)$
$(\mathbb{S}^2, (3,2), (4,1), (4,-3))$	$(3,2,-5), (4,1,-5), (4,-7,3)$
$(\mathbb{S}^2, (2,1), (4,1), (5,-1))$	$(2,1,-3), (4,1,-5), (5,-6,1)$
$(\mathbb{S}^2, (2,1), (5,1), (5,-4))$	$(2,1,-3), (5,1,-6), (5,-9,4)$
$(\mathbb{S}^2, (3,1), (3,1), (4,1))$	$(3,1,-4), (3,1,-4), (4,-3,-1)$
$(\mathbb{S}^2, (3,1), (4,1), (4,-1))$	$(3,1,-4), (4,1,-5), (4,-5,1)$
$(\mathbb{S}^2, (3,1), (3,1), (5,-1))$	$(3,1,-4), (3,1,-4), (5,-6,1)$
$(\mathbb{S}^2, (2,1), (3,1), (8,-7))$	$(2,1,-3), (3,1,-4), (8,-15,7)$
$(\mathbb{S}^2, (2,1), (3,1), (7,-5))$	$(2,1,-3), (3,1,-4), (7,-12,5)$
$(\mathbb{S}^2, (3,1), (3,1), (5,-4))$	$(3,1,-4), (3,1,-4), (5,-9,4)$

The following Proposition and its corollary solve the recognition problem both for the crystallizations of $\mathbf{C}_2^{42}$ and for the 6-tuples of [20].

Proposition 4 *There are exactly 78 genus two prime orientable 3-manifolds admitting a coloured triangulation with at most 42 tetrahedra and regular genus two. They are:*

- *seventy-three Seifert manifolds* [3]*;*
- *three Dehn-fillings (of the complement of link* 6_1^3*)* [4]*;*
- *two non-geometric graph-manifolds;*

Proof. 48 manifolds appeared already in catalogues $\mathcal{C}^{(2p)}$ with $p \leq 15$ and program Γ-class proved that at least one of their genus two crystallizations is equivalent (by dipole and generalized dipole moves and ρ-pair switchings) to a crystallization with less than 32 vertices. Among them there are the three Dehn-fillings and the two non-geometric graph-manifolds. Further three manifolds are listed in [3]. Of the remaining ones, 11 admit finite fundamental group and could be recognized through Milnor's list of groups. In all cases, by the results in [20], the group identifies univocally the manifold.

We remark once more that the remaining 16 classes represent manifolds with infinite fundamental groups of the type (i), (ii) or (iii) in Proposition 3 (see Appendix A of [20]). Therefore, we added to the set Y, which we described above, the crystallizations of the unknown classes and we applied program Γ-class to the resulting list. The output results proved that the suspected identifications were true. ■

Table 2 of Appendix A contains KMN-list of 6-tuples together with their represented manifolds according to the results summarized in the above Proposition.

6 Genus two non-orientable 3-manifolds

Γ-class, applied to $\widetilde{\mathbf{C}}_2^{(42)}$, produced nine classes, which were all recognized by the program by means of the inserted catalogues $\widetilde{\mathcal{C}}^{(2p)}$ with $1 \leq p \leq 15$ (see (resp. [2])) and correspond to nine distinct manifolds, including $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ (of genus one) and the connected sum $L(2,1)\#(\mathbb{S}^1 \tilde{\times} \mathbb{S}^2)$. As a consequence we can state the following Proposition[5].

Proposition 5 *There exist exactly seven non-orientable prime genus two 3-manifolds admitting a coloured triangulation with at most 42 tetrahedra and regular genus two. They are*

- $\mathbb{RP}^2 \times \mathbb{S}^1$
- *the two flat manifolds* $\mathbb{E}^3/Bb$ *and* $\mathbb{E}^3/Pna2_1$
- *the three torus bundles* $TB\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $TB\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $TB\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ *with Sol geometry;*
- *the Seifert manifold* $(\mathbb{RP}^2; (2,1), (3,1))$ *with geometry* $\mathbb{H}^2 \times \mathbb{R}$.

[3] thirty-nine elliptic, four flat, ten with Nil and twenty with $\widetilde{SL}_2(\mathbb{R})$ geometry

[4] two of these manifolds admit also a torus bundle structure with Sol geometry, the remaining one is hyperbolic

[5] for notations see Appendix A

All these manifolds, excepting $\mathbb{RP}^2 \times \mathbb{S}^1$*, also admit coloured triangulations of strictly higher genus with 30 or less tetrahedra.*

More precisely, in Table 3 of Appendix A we present the list of the above manifolds according to the number of vertices of their minimal genus two crystallization.

A Appendix A

Table 2 (resp. Table 3) presents the catalogue of Heegaard genus two prime orientable (resp. non-orientable) 3-manifolds admitting a crystallization (with regular genus two) with at most 42 vertices. The manifolds are identified via their JSJ decomposition or fibering structure and, possibly, via a further structure as quotient of $\mathbb{S}^3$ or $\mathbb{E}^3$. The second and last column of Table 2 contain the informations about the 6-tuple which represent the manifold in KMN-list and its position in the same list.

As far as the identification of a manifold is concerned, the following notations are used:

- $\mathbb{S}^3/G$ is the quotient space of $\mathbb{S}^3$ by the action of the group G; the involved groups are groups of type

$$
\begin{aligned}
&Q_{4n} =< x, y \,|\, x^2 = (xy)^2 = y^n >, (n \geq 2)\\
&D_{2^k(2n+1)} =< x, y \,|\, x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} >,\ \ (k \geq 3,\ n \geq 1),\\
&P_{24} =< x, y \,|\, x^2 = (xy)^3 = y^3,\ x^4 = 1 >,\\
&P_{48} =< x, y \,|\, x^2 = (xy)^3 = y^4,\ x^4 = 1 >,\\
&P_{120} =< x, y \,|\, x^2 = (xy)^3 = y^5,\ x^4 = 1 >,\\
&P'_{3^k8} =< x, y, z \,|\, x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^{3^k} = 1 >, (k \geq 2)
\end{aligned}
$$

  or direct products of the above with cyclic groups $\mathbb{Z}_n$ $(n \in \mathbb{Z}^+)$;

- $\mathbb{E}^3/G$ is the quotient space of $\mathbb{E}^3$ by the action of the group G; the notations for groups G are those of the International Tables for Crystallography (see also [33] and [34], where the alternative notations, used in [2] and [7], were introduced, too).

- as in section 3, $(S, (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$ is the (orientable or non-orientable according to the context) Seifert fibered space whose orbit space is the surface S and having n exceptional fibers, with non-normalized parameters (α_i, β_i), $i = 1, \dots, n$;

- for each matrix $A \in GL(2; \mathbb{Z})$, $TB(A) = (T \times I)/A$ is the torus bundle over $\mathbb{S}^1$ with monodromy induced by A;

- $H_1 \bigcup_A H_2$ is the graph manifold obtained by gluing a Seifert manifold H_1, with $\partial H_1 \cong T$, and a Seifert manifold H_2, with $\partial H_2 \cong T$, along their boundary tori by means of the attaching map associated to matrix A;

- following [28], $Q_i(p,q)$ denotes the closed manifold obtained as Dehn filling with parameters (p,q) of the compact manifold Q_i, whose interior is one of the 11 hyperbolic manifolds of finite volume with a single cusp and complexity at most three (see [12] and [27]).

In Table 2 we wrote in italics the 6-tuples representing manifolds which don't appear in former catalogues of crystallizations ([2],[3],[9]).

tetrahedra	**6-tuple**	**3-manifold**	**position in [20]**
18	$(3,3,3,2,2,2)$	$\mathbb{S}^3/Q_8 = (\mathbb{S}^2,(2,1),(2,1),(2,-1))$	**P.6**
22	$(3,3,5,2,2,4)$	$\mathbb{S}^3/Q_{12} = (\mathbb{S}^2,(2,1),(2,1),(3,-2))$	**P.14**
24	$(4,4,4,1,1,1)$	$\mathbb{S}^3/Q_8 \times Z_3 = (\mathbb{S}^2,(2,1),(2,1),(2,1))$	**P.25**
	$(4,4,4,1,1,5)$	$\mathbb{S}^3/D_{24} = (\mathbb{S}^2,(2,1),(2,1),(3,-1))$	**P.29**
	$(4,4,4,3,3,3)$	$\mathbb{S}^3/P_{24} = (\mathbb{S}^2,(2,1),(3,1),(3,-2))$	**P.11**
26	$(3,3,7,2,2,6)$	$\mathbb{S}^3/Q_{16} = (\mathbb{S}^2,(2,1),(2,1),(4,-3))$	**P.7**
28	$(4,4,6,1,1,1)$	$\mathbb{S}^3/D_{48} = (\mathbb{S}^2,(2,1),(2,1),(3,1))$	**P.50**
	$(4,4,6,1,1,7)$	$\mathbb{S}^3/P'_{72} = (\mathbb{S}^2,(2,1),(3,1),(3,-1))$	**P.34**
	$(4,4,6,1,5,1)$	$\mathbb{S}^3/Q_{16} \times Z_3 = (\mathbb{S}^2,(2,1),(2,1),(4,-1))$	**P.26**

(Table 2 continues...)

tetrahedra	**6-tuple**	**3-manifold**	**position in [20]**
28	$(4,4,6,3,3,5)$	$\mathbb{S}^3/P_{48} = (\mathbb{S}^2,(2,1),(3,1),(4,-3))$	**P.3**
30	$(3,3,9,2,2,8)$	$\mathbb{S}^3/Q_{20} = (\mathbb{S}^2,(2,1),(2,1),(5,-4))$	**P.15**
	$(5,5,5,2,2,2)$	$\mathbb{E}^3/P2_12_12_1 = (\mathbb{RP}^2,(2,1),(2,-1))$	**P.18**
	$(5,5,5,4,4,4)$	$\mathbb{S}^3/P_{120} = (\mathbb{S}^2,(2,1),(3,1),(5,-4))$	**P.1**
32	$(4,4,8,1,1,1)$	$\mathbb{S}^3/Q_{16} \times Z_5 = (\mathbb{S}^2,(2,1),(2,1),(4,1))$	**P.40**
	$(4,4,8,1,1,9)$	$\mathbb{S}^3/P_{48} \times Z_5 = (\mathbb{S}^2,(2,1),(3,2),(4,-3))$	**P.38**
	$(4,4,8,1,5,1)$	$\mathbb{S}^3/D_{80} = (\mathbb{S}^2,(2,1),(2,1),(5,-1))$	**P.51**
	$(4,6,6,1,1,1)$	$\mathbb{S}^3/P_{24} \times Z_7 = (\mathbb{S}^2,(2,1),(3,1),(3,1))$	**P.59**
	$(4,6,6,1,1,9)$	$TB\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix} = (\mathbb{S}^2,(3,1),(3,1),(3,-1))$	**P.13**
	$(4,6,6,1,7,1)$	$\mathbb{S}^3/P_{48} \times Z_7 = (\mathbb{S}^2,(2,1),(3,1),(4,-1))$	**P.46**
	$(4,6,6,5,5,3)$	$\mathbb{E}^3/P4_1 = TB\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = (\mathbb{S}^2,(2,1),(4,1),(4,-3))$	**P.74**
34	$(3,3,11,2,2,4)$	$\mathbb{S}^3/D_{40} = (\mathbb{S}^2,(2,1),(2,1),(5,-3))$	**P.30**
	$(3,3,11,2,2,10)$	$\mathbb{S}^3/Q_{24} = (\mathbb{S}^2,(2,1),(2,1),(6,-5))$	**P.8**
	$(3,7,7,2,2,2)$	$\mathbb{S}^3/P_{24} \times Z_5 = (\mathbb{S}^2,(2,1),(3,2),(3,-1))$	**P.48**
	$(5,5,7,2,4,2)$	$(\mathbb{RP}^2,(2,1),(2,1))$	**P.19**
	$(5,5,7,2,6,6)$	$\mathbb{E}^3/P3_1 = TB\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix} = (\mathbb{S}^2,(3,1),(3,1),(3,-2))$	**P.75**
	$(5,5,7,4,4,6)$	$\mathbb{E}^3/P6_1 = TB\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} = (\mathbb{S}^2,(2,1),(3,1),(6,-5))$	**P.72**

(Table 2 continues...)

tetrahedra	**6-tuple**	**3-manifold**	**position in [20]**
36	*(4, 4, 10, 1, 1, 1)*	$\mathbb{S}^3/D_{40} \times Z_3 = (\mathbb{S}^2, (2,1), (2,1), (5,1))$	**P.63**
	(4, 4, 10, 1, 1, 7)	$\mathbb{S}^3/Q_{12} \times Z_5 = (\mathbb{S}^2, (2,1), (2,1), (3,2))$	**P.57**
	(4, 4, 10, 1, 1, 11)	$\mathbb{S}^3/P_{120} \times Z_{11} = (\mathbb{S}^2, (2,1), (3,2), (5,-4))$	**P.42**
	(4, 4, 10, 1, 5, 1)	$\mathbb{S}^3/Q_{24} \times Z_5 = (\mathbb{S}^2, (2,1), (2,1), (6,-1))$	**P.41**
	(4, 4, 10, 1, 5, 7)	$\mathbb{S}^3/Q_{20} \times Z_3 = (\mathbb{S}^2, (2,1), (2,1), (5,-2))$	**P.43**
	(4, 4, 10, 3, 3, 3)	$\mathbb{S}^3/P_{120} \times Z_7 = (\mathbb{S}^2, (2,1), (3,1), (5,-3))$	**P.28**
	(4, 6, 8, 1, 1, 1)	$\mathbb{S}^3/P_{48} \times Z_{13} = (\mathbb{S}^2, (2,1), (3,1), (4,1))$	**P.68**
	(4, 6, 8, 1, 1, 11)	$(\mathbb{S}^2, (3,1), (3,2), (4,-3))$	**P.35**
	(4, 6, 8, 1, 7, 1)	$\mathbb{S}^3/P_{120} \times Z_{19} = (\mathbb{S}^2, (2,1), (3,1), (5,-1))$	**P.56**
	(4, 6, 8, 3, 9, 13)	$(\mathbb{S}^2, (2,1), (4,1), (4,-1))$	**P.33**
	(4, 6, 8, 5, 5, 11)	$(\mathbb{S}^2, (2,1), (4,1), (5,-4))$	**P.4**
	(6, 6, 6, 1, 1, 1)	$(\mathbb{S}^2, (3,1), (3,1), (3,1))$	**P.37**
	(6, 6, 6, 1, 1, 9)	$(\mathbb{S}^2, (3,1), (3,1), (4,-1))$	**P.49**
	(6, 6, 6, 1, 7, 7)	$TB\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix} = (K, (1,1))$	**P.76**
	(6, 6, 6, 5, 5, 5)	$TB\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = (T, (1,1))$	**P.78**
38	*(3, 3, 13, 2, 2, 12)*	$\mathbb{S}^3/Q_{28} = (\mathbb{S}^2, (2,1), (2,1), (7,-6))$	**P.16**
	(5, 5, 9, 2, 2, 2)	$(\mathbb{RP}^2, (2,1), (3,-1))$	**P.66**
	(5, 5, 9, 4, 4, 8)	$(\mathbb{S}^2, (2,1), (3,1), (7,-6))$	**P.2**
	(5, 7, 7, 4, 6, 12)	$(\mathbb{S}^2, (3,1), (3,1), (4,-3))$	**P.12**

(Table 2 continues...)

tetrahedra	**6-tuple**	**3-manifold**	**position in [20]**
40	$(\mathit{4},\mathit{4},\mathit{12},\mathit{1},\mathit{1},\mathit{1})$	$\mathbb{S}^3/Q_{24}\times Z_7 = (\mathbb{S}^2,(2,1),(2,1),(6,1))$	**P.47**
	$(4,4,12,1,1,5)$	$\mathbb{S}^3/Q_8\times Z_5 = (\mathbb{S}^2,(2,1),(2,1),(2,3))$	**P.39**
	$(\mathit{4},\mathit{4},\mathit{12},\mathit{1},\mathit{1},\mathit{13})$	$(\mathbb{S}^2,(2,1),(3,2),(6,-5))$	**P.44**
	$(\mathit{4},\mathit{4},\mathit{12},\mathit{1},\mathit{5},\mathit{1})$	$\mathbb{S}^3/D_{56}\times Z_3 = (\mathbb{S}^2,(2,1),(2,1),(7,-1))$	**P.64**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{1},\mathit{1},\mathit{1})$	$\mathbb{S}^3/P_{120}\times Z_{31} = (\mathbb{S}^2,(2,1),(3,1),(5,1))$	**P.70**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{1},\mathit{1},\mathit{13})$	$(\mathbb{S}^2,(3,1),(3,2),(5,-4))$	**P.36**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{1},\mathit{7},\mathit{1})$	$(\mathbb{S}^2,(2,1),(3,1),(6,-1))$	**P.65**
	$(4,6,10,3,5,3)$	$(\mathbb{S}^2,(2,1),(4,1),(5,-3))$	**P.23**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{3},\mathit{9},\mathit{15})$	$(\mathbb{S}^2,(2,1),(4,3),(5,-4))$	**P.55**
	$(4,6,10,5,1,1)$	$\mathbb{S}^3/P_{48}\times Z_{11} = (\mathbb{S}^2,(2,1),(3,2),(4,-1))$	**P.61**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{5},\mathit{5},\mathit{13})$	$(\mathbb{S}^2,(2,1),(4,1),(6,-5))$	**P.10**
	$(4,6,10,5,9,3)$	$(\mathbb{S}^2,(3,1),(3,2),(3,-1))$	**P.27**
	$(\mathit{4},\mathit{6},\mathit{10},\mathit{7},\mathit{1},\mathit{1})$	$\mathbb{S}^3/P'_{216} = (\mathbb{S}^2,(2,1),(3,1),(3,2))$	**P.69**
	$(4,6,10,7,3,15)$	$\mathbb{S}^3/P_{120}\times Z_{13} = (\mathbb{S}^2,(2,1),(3,1),(5,-2))$	**P.45**
	$(\mathit{4},\mathit{8},\mathit{8},\mathit{1},\mathit{1},\mathit{1})$	$(\mathbb{S}^2,(2,1),(4,1),(4,1))$	**P.53**
	$(\mathit{4},\mathit{8},\mathit{8},\mathit{1},\mathit{1},\mathit{13})$	$(\mathbb{S}^2,(3,2),(4,1),(4,-3))$	**P.32**
	$(\mathit{4},\mathit{8},\mathit{8},\mathit{1},\mathit{9},\mathit{1})$	$(\mathbb{S}^2,(2,1),(4,1),(5,-1))$	**P.62**
	$(4,8,8,5,5,13)$	$(\mathbb{S}^2,(2,1),(5,1),(5,-4))$	**P.20**
	$(\mathit{6},\mathit{6},\mathit{8},\mathit{1},\mathit{1},\mathit{1})$	$(\mathbb{S}^2,(3,1),(3,1),(4,1))$	**P.71**
	$(\mathit{6},\mathit{6},\mathit{8},\mathit{1},\mathit{1},\mathit{11})$	$(\mathbb{S}^2,(3,1),(4,1),(4,-1))$	**P.52**
	$(\mathit{6},\mathit{6},\mathit{8},\mathit{1},\mathit{9},\mathit{1})$	$(\mathbb{S}^2,(3,1),(3,1),(5,-1))$	**P.60**

(Table 2 continues...)

tetrahedra	**6-tuple**	**3-manifold**	**position in [20]**
40	$(6,6,8,5,5,7)$	$TB\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} = Q_2(0,1)$	**P.73**
	$(6,6,8,5,11,7)$	$TB\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} = Q_1(1,1)$	**P.77**
42	$(3,3,15,2,2,6)$	$\mathbb{S}^3/D_{56} = (\mathbb{S}^2,(2,1),(2,1),(7,-5))$	**P.31**
	$(\mathit{3},\mathit{3},\mathit{15},\mathit{2},\mathit{2},\mathit{14})$	$\mathbb{S}^3/Q_{32} = (\mathbb{S}^2,(2,1),(2,1),(8,-7))$	**P.9**
	$(3,7,11,4,2,2)$	$\mathbb{S}^3/P_{120} \times Z_{17} = (\mathbb{S}^2,(2,1),(3,2),(5,-3))$	**P.54**
	$(5,5,11,2,4,2)$	$(\mathbb{RP}^2,(2,1),(3,1))$	**P.67**
	$(5,5,11,4,4,10)$	$(\mathbb{S}^2,(2,1),(3,1),(8,-7))$	**P.5**
	$(5,5,11,4,8,4)$	$(\mathbb{S}^2,(2,1),(3,1),(7,-5))$	**P.21**
	$(5,7,9,2,4,4)$	$(\mathbb{D},(2,1),(2,-3)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (\mathbb{D},(2,1),(3,-2))$	**P.58**
	$(\mathit{5},\mathit{7},\mathit{9},\mathit{4},\mathit{6},\mathit{14})$	$(\mathbb{S}^2,(3,1),(3,1),(5,-4))$	**P.24**
	$(7,7,7,2,2,2)$	$Q_1(2,-3)$	**P.22**
	$(7,7,7,2,6,10)$	$(\mathbb{D},(2,1),(2,1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (\mathbb{D},(2,1),(3,1))$	**P.17**

TABLE 2: Prime genus two 3-manifolds represented by crystallizations of $\mathbf{C}_2^{(42)}$

tetrahedra	3-manifold
16	$\mathbb{RP}^2 \times \mathbb{S}^1$
32	$\mathbb{E}^3/Bb = TB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\mathbb{E}^3/Pna2_1 = (\mathbb{RP}^2; (2,1), (2,1))$
34	$TB\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
36	$TB\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$
40	$TB\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ $(\mathbb{RP}^2; (2,1), (3,1))$

TABLE 3: Prime genus two 3-manifolds represented by crystallizations of $\tilde{\mathbf{C}}_2^{(42)}$

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