Using Average Internal Rates of Return for investment performance measurement and attribution

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Introduction. In investment funds, two agents cooperate in creating value: the investment manager and the investor (client). The manager receives an amount of money for investment and manages the fund; he selects assets and allocates resources across assets. The client may periodically inject further capital in the fund or withdraw some amount of money from it. The decisions on timing and size of interim cash flows affect the overall performance of the investment, which is then the result of an interaction between manager's and investor's decisions. The literature on performance measurement is unanimous in regarding the IRR as the correct rate of return for the overall investment, while the Time-Weighted Rate of Return (TWRR) is regarded as the correct rate for capturing the "manager's rate of return", unaffected by the client's decisions (e.g. Gray and Drewer, 1971; Feibel, 2003; Spaulding, 2005; Kellison, 2009). We use the Average Internal Rate of Return (AIRR) approach, first introduced in Magni (2010), to introduce a pair of metrics, opposed to IRR and TWRR, which measure the manager's performance and the investor's performance on the basis of the market values of the fund. We also underscore relations between manager's AIRR and the TWRR.

The fund's return. Suppose an investor (client) invests a monetary amount in a fund which is managed by an investment manager. Let f_t denotes the investor's cash flow at time t. The project is characterized by the cash-flow stream $\boldsymbol{f} = (f_0, f_1, \dots, f_n)$; at time n the investment is liquidated. Obviously, $f_0 < 0$ and $f_n > 0$. For each date $t = 1,2, \dots, n$, let b_{t-1} denote the value of the fund at the beginning of period $[t-1, t]$ and let e_t represent the value of the fund at the end of the same period: $b_t = e_t - f_t$. The return in a generic period is $R_t = e_t - b_{t-1} = b_t + f_t - b_{t-1}$, so the rate of return i_t in period $[t-1, t]$ is $i_t = R_t / b_{t-1}$. Therefore, $b_t = b_{t-1} + R_t - f_t = b_{t-1}(1 + i_t) - f_t$ with the boundary conditions $b_0 = -f_0$ and $b_n = 0$. The latter implies $e_n = f_n$, whence $\boldsymbol{f} = (-b_0, f_1, \dots, f_{n-1}, e_n)$. For assessing profitability, it is necessary to compare the investment fund's performance with an alternative fund's performance: a *benchmark* fund or index is selected, whose rate of return represents the minimum required rate of return. To this end, let r_t be the rate of return of the benchmark in period $[t-1, t]$, $t = 1,2, \dots, n$, and suppose the investor replicates his policy of deposits and withdrawals by using the benchmark rather than the fund: this means that the investor contributes to (withdraws from) the benchmark fund the cash flow f_t at time $t = 0,1, \dots, n-1$. This replicating strategy implies a cash-flow stream equal to $\boldsymbol{f}$ with the exception of the terminal value, which is $e_n^* = -\sum_{t=0}^{n-1} f_t \, u^{n-t}$, where $u^{n-t} \coloneqq \prod_{k=t+1}^{n}(1 + r_k)$. The value added (VA) is obtained as the difference of the terminal values: $VA = e_n - e_n^* = f_n - e_n^* = \sum_{t=0}^{n} f_t \, u^{n-t}$. Wealth creation or destruction for the investor is supplied by the sign of VA. The rate r_t is a hurdle rate which represents an opportunity cost of capital in period $[t-1,t]$. If $i_t > r_t$ value is created in period $[t-1,t]$, if $i_t < r_t$ value is destroyed. The excess return generated by the fund investment in each period is $R_t - r_t b_{t-1} = b_{t-1}(i_t - r_t)$, whence $VA = \sum_{t=1}^{n} b_{t-1}(i_t - r_t)\, u^{n-t}$ which may be rewritten as $VA = B\,(i-r)$, where $B = \sum_{t=1}^{n} b_{t-1}\, u^{n-t}$, $i = \sum_{t=1}^{n} w_t i_t$, $r = \sum_{t=1}^{n} w_t r_t$, $w_t = b_{t-1}u^{n-t}/B$. The rate i is the mean of the fund's return rates, with weights given by the (discounted) market value of the fund investment. The rates i and r are particular cases of the "Average Internal Rate of Return" (AIRR) model, first introduced in Magni (2010). The client invests an overall capital B at an overall rate of return equal to i (the project's AIRR), and, so doing, he foregoes an overall rate of return equal to r. A computational shortcut (particularly useful if the required rate of return is constant) is available, which splits up the project's rate of return into the benchmark's overall return rate and an excess return rate: $i = r + \xi$, where $\xi = VA/B$. It is worth stressing that the fund investment's AIRR is the overall rate of return *on the true fund's values.* By contrast, the internal rate of return (IRR) internally devises its own interim capital values, which are disconnected from the true values of the fund: $h_t = h_{t-1}(1 + i^{IRR}) - f_t$, $h_0 = -f_0$. If the fund values grew at a constant rate, then h_t would be the correct capital investment at the beginning of period $[t-1,t]$ and the IRR would be the correct rate of return. But the fund's value is $b_t \neq h_t$, therefore i^{IRR} is not the rate of return on the capital actually invested and the IRR-implied return is different from the true return: $R_t = e_t - b_{t-1} = i_t b_{t-1} \neq i^{IRR} \cdot h_{t-1}$ (see also Altshuler and Magni, 2011).

The manager's return. In order to measure the manager's performance, it is necessary to suppress the effect of timing and magnitude of cash flows deposited and withdrawn by the client. To this end, assume the client invests $b_0 = -f_0$ and employs a buy-and-hold strategy so that the investment is characterized by the cash-flow stream $\hat{\boldsymbol{f}} = (f_0, 0, 0, \ldots, 0, \hat{e}_n)$ where $\hat{e}_n = -f_0 \prod_{t=1}^{n}(1 + i_t)$ is the liquidation value, which is entirely attributable to the manager's decisions about asset selection and allocation (given the client's initial contribution). Project $\hat{\boldsymbol{f}}$ is then interpretable as the "manager's project". Denote with $\hat{b}_{t-1}$ the beginning-of-period value of the fund in period $[t-1,t]$, $t = 1,2,\ldots,n$, with $\hat{b}_0 = -f_0$. It is evident that $\hat{b}_t = \hat{b}_{t-1}(1 + i_t)$ represents the market value of the manager's project, which grows at the rate i_t in each period. Given that the market value of the fund, b_t, depends on both manager's decisions and client's decisions, the capital $\hat{b}_t$ represents that part of b_t which is generated by the manager's decisions. To compute the value added by the manager one needs consider what the financial result would be if the client invested b_0 in the benchmark rather than in the fund managed by the fund manager, keeping the assumption of buy-and-hold strategy. Denoting with $\hat{e}_n^*$ such a value, one gets $\hat{e}_n^* = b_0 u^n$. The value added by the manager is then given by the difference $\hat{e}_n - \hat{e}_n^*$. The latter, denoted by $\widehat{VA}$, is not affected by the client's choices about interim deposits and withdrawals. Hence, $\widehat{VA} = \hat{e}_n - \hat{e}_n^* = \hat{e}_n + f_0 u^n = b_0[\prod_{t=1}^{n}(1 + i_t) - \prod_{t=1}^{n}(1 + r_t)]$. It is now easy to calculate the AIRR of the manager's project, which we call the "manager's rate of return", with the same reasoning employed for the fund: we get $\widehat{VA} = \hat{B}(\hat{\imath} - \hat{r})$, where $\hat{B} = \sum_{t=1}^{n} \hat{b}_{t-1}\, u^{n-t}$, $\hat{\imath} = \sum_{t=1}^{n} \widehat{w}_t i_t$, $\hat{r} = \sum_{t=1}^{n} \widehat{w}_t r_t$, $\widehat{w}_t = \hat{b}_{t-1} u^{n-t}/\hat{B}$. A shortcut for the manager's AIRR is available as well: $\hat{\imath} = \hat{r} + \hat{\xi}$, where $\hat{\xi} = \widehat{VA}/\hat{B}$ is an excess return rate. The literature on performance measurement makes use of the time-weighted rate of return (TWRR) to assess the manager's performance: it is defined as $i^{TWRR} = \sqrt[n]{(e_1/b_0)\cdot(e_2/b_1)\cdot\ldots\cdot(e_n/b_{n-1})} - 1$; but this means $i^{TWRR} = [\prod_{t=1}^{n}(1+i_t)]^{1/n} - 1$, so the TWRR is project $\hat{f}$ internal rate of return and, as such, it internally devises its own capitals $\hat{h}_t = \hat{h}_{t-1}(1+i^{TWRR}) = -f_0(1+i^{TWRR})^t$ as opposed to the correct value $\hat{b}_t = \hat{b}_{t-1}(1+i_t) = b_0 \prod_{t=1}^{n}(1+i_t)$.

The client's return. Analogously, one can measure the timing and size effect of the client's decisions. The value added by the value added by the client is $\widecheck{VA} = VA - \widehat{VA} = (e_n - e_n^*) - (\hat{e}_n - \hat{e}_n^*)$ which is the accumulated value of the cash flows of the "client's project" $\breve{\boldsymbol{f}} = \boldsymbol{f} - \hat{\boldsymbol{f}} = (0, f_1, f_2, \ldots, f_{n-1}, e_n - \hat{e}_n)$. The capital invested in $\breve{\boldsymbol{f}}$ at the beginning of period $[t-1, t]$, denoted by $\breve{b}_t$, the residual part of b_t which is not attributable to the manager: $\breve{b}_t = b_t - \hat{b}_t$, $t = 1,2,,\ldots,n$, so that $\breve{B} = B - \hat{B} = \sum_{t=1}^{n} \breve{b}_{t-1} u^{n-t}$. Hence, $\widecheck{VA} = \breve{B}(\breve{\imath} - \breve{r})$, where $\breve{\imath} = \sum_{t=1}^{n} \breve{w}_t i_t$ $\quad \breve{r} = \sum_{t=1}^{n} \breve{w}_t r_t$, and $\widehat{w}_t = \breve{b}_{t-1} u^{n-t}/\breve{B}$. The excess return is in this case $\breve{\xi} = \widecheck{VA}/\breve{B}$.

Decomposition of value added and rates. The value added as a whole is the sum of the value added by the manager and the value added by the client: $VA = \breve{B}(\breve{\imath} - \breve{r}) + \hat{B}(\hat{\imath} - \hat{r})$ and the fund's rate of return is a weighted average of the manager's rate of return and the investor's rate of return: $i = \breve{\alpha}\cdot\breve{\imath} + \hat{\alpha}\cdot\hat{\imath}$, $\breve{\alpha} := \breve{B}/B$, $\hat{\alpha} := \hat{B}/B$. Analogously, the fund's hurdle rate is a weighted average of the manager's hurdle rate and the investor's hurdle rate: $r = \breve{\alpha}\cdot\breve{r} + \hat{\alpha}\cdot\hat{r}$. Each of the three AIRRs (project's, manager's, investor's) may be in turn splitted up into a hurdle rate and an excess return rate and the fund's excess return rate ξ is a weighted average of the manager's and investor's excess return rates: $\xi = \breve{\alpha}\breve{\xi} + \hat{\alpha}\hat{\xi}$. It may well occur that $\hat{r} < \hat{\imath} < r$, which means that the manager has added value even if it occurs $i < r$. It would be incorrect to contrast the manager's rate of return $\hat{\imath}$ with r, for the latter is a function of the b_t's which are determined by the interaction of manager's decisions and client's decisions. Analogously, the choices of the investor may determine, ceteris paribus, an increase in value added if $\breve{\imath} > \breve{r}$. A twofold decomposition, by periods and by agent, is possible. Let $VA(t,h)$ the value added in the t-th period by the h-th agent, $t = 1,\ldots,n$, $h = 1,2$ where 1 is the manager (hat) and 2 is the investor (reverse hat), respectively. We get $VA(t,h) = b_{t-1}^h(i_t^h - r_t^h)$ and $VA = \sum_{h=1}^{2}\sum_{t=1}^{n} VA(t,h)$. The average excess return is $b(i-r)$ where $b := \sum_{t=1}^{n} b_{t-1} u^{n-t} / \sum_{t=1}^{n} u^{n-t}$ is the average capital invested, so that $VA = \sum_{t=1}^{n} b(i-r)u^{n-t}$. Analogously, the average excess return supplied by either agent is $b^h(i^h - r^h)$ and we can write $VA = \sum_{h=1}^{2}\sum_{t=1}^{n} b^h(i^h - r^h)\, u^{n-t}$.

AIRR and TWRR. Ranking performance of different managers is possible by scaling the initial investments in order to neutralize the client's decisions on the investment scale. With no loss of generality and for the mere sake of notational convenience, we assume that the benchmark's rate of return is constant over time. Let $\tilde{\boldsymbol{f}} = (1/b_0)\cdot\hat{\boldsymbol{f}} = (-1,0,\dots,0,\hat{e}_n/b_0)$ be the *normalized* manager's project. Project $\tilde{\boldsymbol{f}}$'s value added is $\widetilde{VA} = \widehat{VA}/b_0 = \hat{e}_n/b_0 - (1+r)^n/b_0$. Now, consider the function $\tilde{\xi}(x) = \widetilde{VA}/x$. It is project $\tilde{\boldsymbol{f}}$'s excess rate of return, relative to an imputed capital of x. Note that, by definition, $\widetilde{VA} = x\tilde{\xi}(x)$ for all $x \neq 0$. In particular, $\widetilde{VA} = \tilde{\xi}(1)$. Now, the rate of return $\tilde{\imath}(1) = r + \widetilde{VA}$ is the AIRR of project $\tilde{\boldsymbol{f}}$ referred to an aggregate capital of 1 euro. Manager's capabilities are summarized in $\tilde{\imath}(1)$, which offsets the investment scale. The ranking based on $\widetilde{VA}$ or on $\tilde{\imath}(1)$ is the same: the former is an absolute measure (value added by one euro of capital invested), the latter is a relative measure (a rate of return). The TWRR provides the same ranking as well, because $\tilde{\imath}(1) - r = (1+i^{TWRR})^n - (1+r)^n$, so TWRR-based ranking is equivalent to AIRR-based ranking. It is worth stressing that $\tilde{\imath}(1)$ is indeed a mean of the manager's holding period rates: $\tilde{\imath}(1) = \sum_{t=1}^{n}\tilde{w}_t i_t$ where $\tilde{w}_t = c_{t-1}u^{n-t}/\sum_{t=1}^{n} c_{t-1}u^{n-t}$, $c_t = c_{t-1}(1+i_t) - \tilde{f}_t$, $c_0 = 1$. Alternatively, the same result may be achieved without normalizing the manager's project: one just computes manager's excess return corresponding to an aggregate capital equal to the initial investment b_0. To this end, consider the manager's excess return $\hat{\xi}$ as a function of the aggregate capital: $\hat{\xi} = \hat{\xi}(x) = \widehat{VA}/x$. It may be shown that $\hat{\xi}(b_0) = \tilde{\xi}(1)$ so that the corresponding AIRRs are equal: $\tilde{\imath}(1) = \hat{\imath}(b_0)$. We stress that $\hat{\imath}(\hat{B})$ is the rate of return on the aggregate capital invested in $\hat{\boldsymbol{f}}$ under an assumption of buy-and-hold strategy, whereas $\hat{\imath}(\hat{b}_0)$ captures the return on each euro of capital *initially* invested under the same assumption. While $\hat{\imath}(\hat{B})$ may be used for computation of incentive-based fees (for they are linked to the value added), $\hat{\imath}(\hat{b}_0)$ may be used for ranking performance. The passage from $\hat{\xi}(\hat{B})$ to $\hat{\xi}(\hat{b}_0)$ enables one to offset the client's decisions on cash flows. It comes as no surprise that $\hat{\xi}(\hat{b}_0) = \hat{\xi}(\hat{B})\cdot(\hat{B}/b_0)$; that is, $\hat{\xi}(\hat{b}_0)$ scales down the aggregate investment, so producing the same results as $\tilde{\xi}(1)$. Finally, the scaled-down $\hat{\imath}(b_0)$ is a weighted average of the benchmark rate of return and the manager's AIRR: $\hat{\imath}(b_0) = \beta\cdot\hat{\imath} + (1-\beta)r$ where $\beta := \hat{B}/b_0$.

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