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# On the long-standing issue of the Internal Rate of Return: A complete resolution

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Consider the cash-flow stream  $\mathbf{f} = (f_0, f_1, f_2, \dots, f_T) \in \mathbb{R}^{T+1}$ . The net present value (NPV) of  $\mathbf{f}$  is  $PV(\mathbf{f} | r) = \sum_{t=0}^T f_t \cdot (1+r)^{-t}$ , where  $r > -1$  is the opportunity cost of capital (often, the market rate). Any number  $x$  such that of  $PV(\mathbf{f} | x) = 0$  is called *Internal Rate of Return* (IRR).

**The IRR problem.** As widely known, the IRR has serious flaws:

- (i) multiple real-valued IRRs may arise
- (ii) the meaning of each IRR may be ambiguous (rate of return or rate of cost?)
- (iii) complex-valued IRRs may arise
- (iv) the IRR is, in general, incompatible with the net present value (NPV) in accept/reject decisions and the IRR ranking is, in general, different from the NPV ranking
- (v) the IRR decision criterion is not applicable with variable costs of capital.

Since the origins of the notions (Boulding 1935, 1936; Keynes 1936), the IRR drawbacks have stimulated an immense bulk of contributions over the decades investigating this issue and searching for some solution (see references in Gronchi 1987 and in Magni 2010a).

We here present two autonomous solutions which solve the IRR problems completely. As a pleasant byproduct, a further problem is solved:

**The accounting problem.** Accounting rates of return are period rates applied to book values of firms or projects. In accounting, an accounting rate of return (ARR) is considered a “false” return rate as opposed to the “true” yield, represented by the IRR. Numerous accounting scholars have been focusing for decades on the relations between ARR and the IRR and, in particular, between some mean of ARR and the IRR (see Magni 2009a). The search for a significant relation is still under scrutiny in the accounting literature.

**First solution.** For a one-period project  $\mathbf{f} = (f_0, f_1)$  the rate of return is unambiguously defined as

$$x := \frac{f_0 + f_1}{-f_0}.$$

It may be viewed as the unique solution of  $f_0 + f_1(1+y)^{-1} = PV(\mathbf{f} | y) = 0$ . The IRR equation for a multi-period project is just a generalization of the latter. Unfortunately, while unanimously accepted for 75 years, this generalization is ill-fated, for all the problems stem from the very equation. We dismiss the IRR equation and propose a more “natural” way of generalizing the notion of rate of return. Note that

$$x = \frac{I_1}{c_0} \tag{1}$$

with  $I_1 := f_0 + f_1$  and  $c_0 := -f_0$ . That is: rate of return means “interest on capital”. In a project, the capital  $c_{t-1}$  invested in  $[t-1, t]$  and interest  $I_t$  are related in the following way:  $c_t = c_{t-1} + I_t - f_t$ ,  $t = 1, 2, \dots, T$ ,  $c_T = 0$ . The total capital in the span of the project life is  $\sum_1^T c_{t-1}$ , and the total interest is  $\sum_1^T I_t$ . Therefore, a natural generalization of (1) is just “total interest on total capital”:

$$\bar{x} := \frac{I}{C} \quad (2)$$

with  $I := \sum_1^T I_t$  and  $C := \sum_1^T c_{t-1}$ . We call eq. (2) *Purely Internal Rate of Return* (PIRR). To check for project acceptability, the investor should compare the PIRR with an appropriate cost of capital. To this end, note that the investor may replicate  $\mathbf{f}$  by investing  $c_0$  in the market and withdrawing  $f_t$  in each period. In this case, the capital invested in the interval  $[t-1, t]$  is  $c_t^* = c_{t-1}^* + I_t^* - f_t$  with  $c_0^* = -f_0$  and  $I_t^* = r c_{t-1}^*$ . The total capital invested is  $C^* := \sum_1^T c_{t-1}^*$ , which differs, in general, from  $C$ . This implies that if the project is undertaken, the investor incurs two opportunity costs: she renounces to investing  $C$  at the rate  $r$  and to investing the extra capital  $C^* - C$  at the rate  $r$ . The cost of capital  $r$  must then be corrected to account for the excess capital foregone. For any euro of capital invested in the project, the investor renounces to invest  $C^*/C$  euro in the market. The comprehensive cost of capital (CCOC) is then

$$\bar{\rho} := \underbrace{r}_{\text{market rate}} \cdot \underbrace{\frac{C^*}{C}}_{\text{correction factor}} = \frac{I^*}{C} \quad (3)$$

where  $I^* := \sum_1^T I_t^*$ . Note that both PIRR and COCC do not depend on capitals but on *total* capital:  $\bar{x} = \bar{x}(C)$  and  $\bar{\rho} = \bar{\rho}(C)$ . Using (2) and (3) and making appropriate algebraic manipulations one finds  $\bar{x} = \bar{\rho} + PV(\mathbf{f} | r)(1+r)^T/C$ , so that the following theorem holds:

**Theorem 1** (PIRR 1). *For any  $\mathbf{c} = (c_0, c_1, \dots, c_{T-1}) \in \mathbb{R}^T$  such that  $c_0 = -f_0$ , if  $C > 0$  ( $<$ ) the project should be accepted (i.e. NPV  $> 0$ ) if and only if  $\bar{x} > \bar{\rho}$  ( $<$ ).*

**Remark.** In his famous book, Keynes (1936, 1967) unconsciously offers the reader a clue to solve the problems he himself has generated. The author introduces the notion of *user cost*: denoting user cost with  $u_t$  and carefully formalizing Keynes's sentences, it may be shown that user cost is just  $u_t = c_t^* - c_t$ . This implies that depreciation of user cost is  $u_{t-1} - u_t = (c_t^* - c_t) - (c_{t-1}^* - c_{t-1})$ . Summing over  $t$ , one finds  $\sum_1^T (u_{t-1} - u_t) = \sum_1^T I_t - \sum_1^T I_t^* = PV(\mathbf{f} | r)(1+r)^T$ . Dividing by  $C$ , PIRR 1 is found back (Magni 2009b, 2010b).

Given that the NPV is unvaried under changes in  $\mathbf{c}$ , one may choose *whatever* sequence  $\mathbf{c}$  of outstanding capitals to economically describe the project. Hence, by (PIRR 1) and eq. (3), the market rate itself may serve as the COCC.

**Corollary 1.** Choose any vector  $\mathbf{c}$  such that  $C = C^*$ . Then, if  $C^* > 0$  ( $<$ ) the project should be accepted (i.e. NPV  $> 0$ ) if and only if  $\bar{x} > r$  ( $<$ ).

Let  $\mathbf{f}_A$  and  $\mathbf{f}_B$  be two competing projects whose length is  $T_A$  and  $T_B$  respectively. We have  $PV(\mathbf{f}_A | r) = C^A(\bar{x}^A - \bar{\rho}^A)(1+r)^{-T_A}$  and  $PV(\mathbf{f}_B | r) = C^B(\bar{x}^B - \bar{\rho}^B)(1+r)^{-T_B}$ , with obvious meaning of the symbols. By choosing outstanding capitals  $c_t$  so that  $C^A(1+r)^{-T_A} = C^B(1+r)^{-T_B}$ , one finds

**Theorem 2.** (PIRR 2). *Consider competing projects  $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ . Choosing the same discounted capital for all projects, project ranking via the margin  $(\bar{x} - \bar{\rho})$  is equal to the NPV ranking.*

An IRR is actually nothing more than a particular case of PIRR. Let  $c_t(x) = c_{t-1}(x)(1+x) - f_t$  be the capital associated to an IRR and let  $H := \sum_1^T c_{t-1}(x)$  be the total capital invested corresponding to that IRR. We have  $x = \bar{x}(H)$ . There are infinitely many sequences  $\mathbf{c}$  such that the total capital is  $H$ . All such sequences constitute a *Hotelling depreciation class* (after Hotelling 1925). Given that the PIRR is unvaried under changes in  $\mathbf{c}$ , as long as total capital is unvaried, the following result holds.

**Theorem 3** (PIRR 3). *An IRR is a PIRR corresponding to a Hotelling depreciation class.*

(See also Magni, 2009a). The PIRR theorems contribute to solve the accounting problem as well. Let  $b_{t-1}$  be the book value of a project or a firm. The ARR is  $a_t = (b_t + f_t - b_{t-1})/b_{t-1}$  and is known in finance and accounting as *Return On Equity* or *Return On Asset*, depending on whether  $b_t$  is the book value of equity or the book value of total assets. Weighing the ARRs by the book values, the average accounting rate of return (AARR) is

$$\text{AARR} = \frac{a_1 b_0 + a_2 b_1 + \dots + a_T b_{T-1}}{b_0 + b_1 + \dots + b_{T-1}}.$$

Given that the PIRR theorems hold for *any* sequence of capitals outstanding, it is obvious that the AARR is a PIRR where the outstanding capital is the book value ( $c_t = b_t$ ) and interest is the accounting income of the firm ( $I_t = a_t b_{t-1}$ ). That is:  $\text{AARR} = \bar{x}(B)$ ,  $B := \sum_1^T b_{t-1}$ . Thus, (PIRR 3) tells us that an IRR is only an average of ARRs obtained from a Hotelling depreciation class. Therefore, while accounting scholars are involved in finding the relations between ARRs and the IRR, such a line of research is misdirected: the appropriate line of research involves finding the relations between ARRs and AARR.

**Second solution.** Consider Makeham's (1874) formula, conceived for loans: the original formula is  $c_0 = \left[ \frac{j}{x}(N - N(1+x)) \right] + N(1+x)^{-T}$ , where  $c_0$  is the amount financed,  $N$  is the capital receivable (nominal value including any bonus),  $j$  is the rate of interest on  $N$ ,  $x$  is the yield to the purchaser (i.e., IRR). As known, such a formula may be modified to account for any cash flow stream  $\mathbf{f}$ . In particular, let  $K_t := c_{t-1} - c_t$  be the capital repayment,  $\mathcal{K}(r) := \sum_1^T K_t(1+r)^{-t}$  be the value of capital,  $\mathcal{J}(r) := \sum_1^T I_t(1+r)^{-t}$  be the value of interest. Then, the value of  $\mathbf{f}$  is  $V(r) = \mathcal{J}(r) + \mathcal{K}(r)$  and the value of interest is  $\mathcal{J}(r) = \frac{x}{r}(c_0 - \mathcal{K}(r))$ . But such a formula is inapplicable if an IRR does not exist. A fruitful generalization is the following one. Let  $\bar{x}_r := r\mathcal{J}(r)/(c_0 - \mathcal{K}(r))$  so that

$$PV(\mathbf{f} | r) = V(r) - c_0 = (c_0 - \mathcal{K}(r)) \left( \frac{\bar{x}_r}{r} - 1 \right). \quad (4)$$

The difference  $c_0 - \mathcal{K}(r)$  is the capital sacrificed by the investor who undertakes project  $\mathbf{f}$ , so the difference  $\left( \frac{\bar{x}_r}{r} - 1 \right)$  represents a (scaled) excess return. It may be shown that  $(c_0 - \mathcal{K}(r))/r = C_r/(1+r)$ , where  $C_r := c_0 + c_1(1+r)^{-1} + \dots + c_{T-1}(1+r)^{-T+1}$ , which implies

$$\bar{x}_r = \frac{I_r}{C_r} \quad (5)$$

where  $I_r := I_1 + I_2(1+r)^{-1} + \dots + I_T(1+r)^{-T+1}$ . Equation (5) is a rate of return which we name *Average Internal Rate of Return* (AIRR). From (4) and (5),  $\bar{x}_r = r + PV(\mathbf{f} | r)(1+r)/C_r$ , whence

**Theorem 4** (AIRR 1). *For any  $\mathbf{c} = (c_0, c_1, \dots, c_{T-1}) \in \mathbb{R}^T$  such that  $c_0 = -f_0$ , if  $C_r > 0$  ( $<$ ) the project should be accepted (i.e. NPV  $>$  0) if and only if  $\bar{x}_r >$  ( $<$ )  $r$ .*

Using the same argument as for Theorem 2 and Theorem 3, one finds

**Theorem 5.** (AIRR 2) *Consider competing projects  $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ . Choosing the same discounted capital  $C_r$  for all projects, the ranking of the projects via  $\bar{x}_r$  is equal to the NPV ranking.*

**Theorem 6.** (AIRR 3) *An IRR is an AIRR corresponding to a Hotelling depreciation class.*

(See also Magni 2009a). The AIRR is evidently a function of  $C_r$ :  $\bar{x}_r = \bar{x}_r(C_r)$ . Also, if the outstanding capital is the book value, then, weighing the ARR's by the present value of the book values, the average accounting rate of return is

$$\text{AARR}_r = \frac{a_1 b_0 (1+r)^{-1} + a_2 b_1 (1+r)^{-2} + \dots + a_T b_{T-1} (1+r)^{-(T-1)}}{b_0 + b_1 (1+r)^{-1} + \dots + b_{T-1} (1+r)^{-(T-1)}}.$$

Then,  $\text{AARR}_r = \bar{x}_r(B_r)$ ,  $B_r := \sum_1^T b_{t-1}$ . Moreover, AIRR 3 warrants again that the IRR is but an average of ARR's. Furthermore, we have  $I = I_0$  and  $C = C_0$  so that  $\bar{x} = \bar{x}_0$ . Note that the AIRR depends on the market rate as well as on capital, and the market rate affects each term of the triplet  $(C_r, \bar{x}_r, r)$ ; conversely, the PIRR is genuinely internal and only the last term in the triplet  $(C_0, \bar{x}_0, \bar{\rho})$  is affected by the market rate. However, in order to rank projects, the PIRR must be used in association with the COCC, whereas the ranking of AIRR suffices to rank projects. Note that, choosing  $C := c_0$  and using the fact that  $I = \sum_0^T f_t$ , a (seemingly heretic) heuristic may be drawn from the PIRR function to compute a project's rate of return:

*Subtract outflows from inflows and divide by initial capital invested.*

All results may be generalized for a variable market rate  $r_t, t = 1, 2, \dots, T$ . Let  $v_t := [(1+r_1)(1+r_2) \cdot \dots \cdot (1+r_t)]^{-1}$ . As for the PIRR, it suffices to define COCC as  $\bar{\rho} := \sum_1^T r_t c_{t-1}^* / \sum_1^T c_{t-1}$ ; as for the AIRR, it suffices to replace  $\bar{x}_r/r$  in (AIRR 1) with  $\bar{x}_r/\bar{r}$ , where  $\bar{r} := \sum_1^T r_t c_{t-1} v_{t-1} / \sum_1^T c_{t-1} v_{t-1}$  is an average of market rates.

The results obtained radically revise the way we think about investments and rates of return, and finally unlock the chains economists have imposed upon themselves: (i) the PIRR and the AIRR represent a major link between economics, finance and accounting: these indexes evidence that the "true" rates of return are means of *accounting* rates; (ii) a rate of return in itself is uninformative: each term of the triplet (*total capital, rate of return, cost of capital*) is essential in conveying information about economic profitability; (iii) any IRR is only one among infinitely many AIRR's or PIRR's: decision makers may safely use it (if at least one exists), provided that the PIRR theorems and the AIRR theorems are complied with; (iv) an unambiguous (and useful) definition of investment/financing does not relate to cash flows, but to (the sign of) total capital: surprisingly as it may be, any project may be seen as either an investment or a financing at the decision maker's discretion.

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