LE MATEMATICHE
Vol. LVIII (2003) – Fasc. I, pp. 51–65

GENERALIZED REGULAR GENUS FOR MANIFOLDS WITH BOUNDARY

PAOLA CRISTOFORI

We introduce a generalization of the regular genus, a combinatorial invariant of PL manifolds ([10]), which is proved to be strictly related, in dimension three, to generalized Heegaard splittings defined in [12].

1. Introduction.

Throughout this paper we consider only compact, connected, PL-manifolds and PL-maps.

The regular genus of a manifold is an invariant defined by Gagliardi in [7] (for closed manifolds) and [10] (for manifolds with boundary), by using 2-cells embeddings of "edge-coloured" graphs representing the manifold and satisfying some conditions of regularity.

More precisely, in the general case of non-empty boundary, the graphs are required to be "regular with respect to one colour", i.e. they become regular after deleting the edges of one fixed colour .

Entrato in redazione il 26 Novembre 2003.

2000 *Mathematics Subject Classification*: Primary 57M15 - 57N10; Secondary 57Q15.
Keywords: Heegaard splittings, embeddings of coloured graphs, gems of 3-manifold.

Work performed under the auspices of G.N.S.A.G.A. of C.N.R. of Italy and supported by M.I.U.R. of Italy (project "Strutture geometriche delle varietà a reali e complesse").

In this paper, by introducing the weaker concept of ”regularity with respect to a cyclic permutation”, we extend the definition of the regular genus to a larger class of coloured graphs.

This generalized regular genus is always bounded by the regular one, but it turns out to be generally strictly less than it; this happens for example in the case of $T_g \times \mathbb{D}^1$, (resp. $U_g \times \mathbb{D}^1$), for each $g \geq 1$. In fact we construct coloured graphs representing these manifolds and regularly embedding into the orientable (resp. non orientable) surface with two holes and genus g.

Moreover we prove, as in the case of the regular genus, that a punctured 3-sphere (i.e. a 3-sphere with holes) is characterized by having generalized regular genus zero.

For the case of 3-manifolds, it is known (see [2] and [3]) that the regular genus coincides with the classical Heegaard one. This result highly depends on the fact that a coloured graph, regular with respect to a colour and representing a 3-manifold M, defines a Heegaard splitting of M (see [3] for details).

Montesinos, in [12], defined a generalization of the concepts of Heegaard splittings and Heegaard genus for orientable 3-manifolds; they coincide with the classical ones in the case of connected boundary. Later the constructions were extended to the non orientable case in [3].

In section 3 we investigate the relationship between coloured graphs representing a 3-manifold and satisfying our ”weaker” condition of regularity and generalized Heegaard splittings of the same manifold; as a consequence we establish an inequality between the generalized Heegaard genus and the generalized regular genus of a 3-manifold with boundary.

2. Coloured graphs and the regular genus of a manifold.

An $(n+1)$*-coloured graph (with boundary)* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph and $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \dots, n\}$ a map, injective on each pair of adjacent edges of Γ.

For each $B \subseteq \Delta_n$, we call $B - residues$ the connected components of the multigraph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; we set $\hat{\imath} = \Delta_n \setminus \{i\}$ for each $i \in \Delta_n$.

The vertices of Γ whose degree is strictly less than $n+1$ are called *boundary vertices*; if (Γ, γ) has no boundary vertices is called *without boundary*. We denote by $\partial V(\Gamma)$ the set of boundary vertices of Γ.

If K is an n-dimensional homogeneous pseudocomplex, and $V(K)$ its set of vertices, we call *coloured n-complex* the pair (K, ξ) where $\xi : V(K) \longrightarrow \Delta_n$ is a map which is injective on every simplex of K.

If σ^h is an h-simplex of K then the *disjoint star* $std(\sigma^h, K)$ *of* σ^h *in* K

is the pseudocomplex obtained by taking the disjoint union of the simplexes of K containing σ^h and identifying the $(n-1)$-simplexes containing σ^h together with all their faces.

The *disjoint link* $lkd(\sigma^h, K)$ *of* σ^h *in* K is the subcomplex of $std(\sigma^h, K)$ formed by the simplexes which don't intersect σ^h.

From now on we shall restrict our attention to the coloured complexes K, such that:

- each $(n-1)$-simplex is a face of exactly two n-simplexes of K;
- for each simplex σ of K, $std(\sigma, K)$ is strongly connected.

Coloured graphs are an useful tool for representing manifolds (see [6] for a survey on this topic), due to the existence of a bijective correspondence between coloured graphs and coloured complexes which triangulate manifolds.

Given a coloured complex K, a direct way to see this correspondence is to consider a coloured graph (Γ, γ) imbedded in $K = K(\Gamma)$ as its dual 1-skeleton, i.e. the vertices of Γ are the barycenters of the n-simplexes of $K(\Gamma)$ and the edges of Γ are the 1-cells dual of the $(n-1)$-simplexes of $K(\Gamma)$. Of course the $(n-1)$-simplex dual to an edge e with $\gamma(e) = i$ has its vertices labelled by $\hat{\imath}$. Furthermore, there is a bijective correspondence between the h-simplexes $(0 \leq h \leq dim\ K(\Gamma))$ of $K(\Gamma)$ and the $(n-h)$-residues of Γ, in the sense that, if σ^h is an h-simplex of $K(\Gamma)$, whose vertices are labelled by $\{i_0, \ldots, i_h\}$, there is a unique $(n-h)$-residue Ξ of Γ whose edges are coloured by $\Delta_n \setminus \{i_0, \ldots, i_h\}$ and such that $K(\Xi) = lkd(\sigma^h, K)$.

See [6] for a more precise description of the constructions involved.

If M is a manifold (with boundary) of dimension n and (Γ, γ) a $(n+1)$-coloured graph (with boundary) such that $|K(\Gamma)| \cong M$, we say that M is *represented* by (Γ, γ). In this case M is orientable iff (Γ, γ) is bipartite.

Let (Γ, γ) be a $(n+1)$-coloured graph such that the set of its boundary vertices is $\partial V(\Gamma) = V^{(0)} \cup V^{(1)} \cup \ldots \cup V^{(n)}$ where, for each $i \in \Delta_n$, $V^{(i)}$ is formed by the vertices missing the i-coloured edge (of course it can occur that $V^{(i)} = \emptyset$).

We call *extended graph associated to* (Γ, γ) the $(n+1)$-coloured graph (Γ^*, γ^*) obtained in the following way:

- for each $v \in V^{(i_1)} \cap \ldots \cap V^{(i_h)}$ add to $V(\Gamma)$ the vertices $v_{i_1}, \ldots, v_{i_h}$; we call V^* the set of these new vertices;
- for each $v \in V^{(i_1)} \cap \ldots \cap V^{(i_h)}$ and for each $j = 1, \ldots, h$ add to $E(\Gamma)$ an edge e_{i_j} with endpoints v and v_{i_j} and the obvious coloration.

A *regular imbedding* of (Γ, γ) into a surface (with boundary) F, is a cellular imbedding of (Γ^*, γ^*) into F, such that:

(a) the image of a vertex of Γ^* lies on ∂F iff the vertex belongs to V^*;

(b) the boundary of any region of the imbedding is either the image of a cycle of (Γ^*, γ^*) (*internal region*) or the union of the image α of a path in (Γ^*, γ^*) and an arc of ∂F, the intersection consisting of the images of two (possibly coincident) vertices belonging to V^* (*boundary region*);

(c) there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of Δ_n such that for each internal region (resp. boundary region), the edges of its boundary (resp. of α) are alternatively coloured ε_i and ε_{i+1} $(i \in \mathbb{Z}_{n+1})$.

From now on, to avoid long notations, we write Γ for a $(n + 1)$-coloured graph instead of (Γ, γ).

For each $i, j \in \Delta_n$, let us denote by $\dot{g}_{ij}(\Gamma)$ the number of cycles of $\Gamma_{i,j}$, by $p(\Gamma)$ (resp. $q(\Gamma)$) the number of vertices (resp. of edges) of Γ.

Given a cyclic permutation ε of Δ_n, a $(n + 1)$-coloured graph Γ is *regular with respect to* ε, if for each $i \in \mathbb{Z}_{n+1}$, $v \in V^{(\varepsilon_i)}$ and $w \in V^{(\varepsilon_{i+1})}$, v and w don't belong to the same connected component of $\Gamma_{\{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i-1}\}}$.

In particular, since it can be $v = w$, each vertex of Γ can't miss two colours which are consecutive in ε.

Remark 1. Note that, if there exists $i \in \Delta_n$ such that $V^{(j)} = \emptyset$, for each $j \neq i$ (i.e. Γ is *regular with respect to the colour* i in the sense of [10]), then Γ is regular with respect to any cyclic permutation of Δ_n.

For each $i \in \Delta_n$, let us denote by ${}^{\partial}g_{\varepsilon_i}(\Gamma)$ the number of closed walks in Γ defined by starting from a vertex belonging to $V^{(\varepsilon_i)}$, following first the ε_{i+1}-coloured edge and going on by the following rules:

- if we arrive in a vertex w by a ε_{i+1}- (resp. ε_{i-1}-) coloured edge, then we follow the ε_{i-1}- (resp. ε_{i+1}-) or the ε_i-coloured edge whether $w \in V^{(\varepsilon_i)}$ or $w \notin V^{(\varepsilon_i)}$;
- if we arrive in a vertex by a ε_i-coloured edge e, then we follow the ε_{i+1}- or the ε_{i-1}-coloured edge whether the edge we met before e is ε_{i+1}- or the ε_{i-1}-coloured.

Proposition 1. *Given a* $(n + 1)$*-coloured bipartite (resp. non bipartite) graph* Γ*, and a cyclic permutation* $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ *of* Δ_n *such that* Γ *is regular with respect to* ε*, there exists a regular embedding of* Γ^* *into the orientable (resp.*

non orientable) surface with boundary F_ε with Euler characteristic:

$$\chi(F_\varepsilon) = \sum_{i\in\mathbb{Z}_{n+1}} \dot{g}_{\varepsilon_i\varepsilon_{i+1}}(\Gamma) - q(\Gamma) + p(\Gamma)$$

and hole number:

$$\lambda_\varepsilon(F_\varepsilon) = \sum_{i\in\mathbb{Z}_{n+1}} {}^\partial g_{\varepsilon_i}(\Gamma)$$

Proof. Let us write $\varepsilon_{\hat\iota_1\dots\hat\iota_h}$ for the cyclic permutation of Δ_{n-h} obtained from ε by deleting $\varepsilon_{\iota_1},\dots,\varepsilon_{\iota_h}$.

We shall prove first the orientable case.

We can define a 2-cell embedding of Γ into a closed surface S_ε by means of a rotation system Φ (see [14]) on Γ as follows:

let B, N be the two bipartition classes of Γ, for each $v \in V(\Gamma)$ let us set

$$\text{if } v\in B \quad \Phi_v = \begin{cases} \varepsilon_{\hat\iota_1\dots\hat\iota_h} & \text{if } v \in V^{(\varepsilon_{\iota_1})}\cup\dots\cup V^{(\varepsilon_{\iota_h})} \\ \varepsilon & \text{otherwise}\end{cases}$$

$$\text{if } v\in N \quad \Phi_v = \begin{cases} \varepsilon^{-1}_{\hat\iota_1\dots\hat\iota_h} & \text{if } v \in V^{(\varepsilon_{\iota_1})}\cup\dots\cup V^{(\varepsilon_{\iota_h})} \\ \varepsilon^{-1} & \text{otherwise}\end{cases}$$

As a consequence of the condition of regularity on Γ, the 2-cells of the regular immersion of Γ, defined by the above rotation system, can only be of two types: either the cell is bounded by edges coloured alternatively ε_i and ε_{i+1} $(i \in \mathbb{Z}_{n+1})$, or it is bounded by edges coloured ε_{i-1}, ε_i and ε_{i+1}.

In the first case the boundary of the cell contains no vertices belonging to $V^{(\varepsilon_i)}$, in the other case it contains vertices belonging to $V^{(\varepsilon_i)}$, but, by the regularity conditions, not to $V^{(\varepsilon_{i+1})}$.

Let us call $A^1_{\varepsilon_i},\dots,A^{r_i}_{\varepsilon_i}$ the cells whose boundary contains vertices of $V^{(\varepsilon_i)}$. Obviously $r_i = {}^\partial g_{\varepsilon_i}(\Gamma)$. For each $i \in \Delta_n$ and $j = 1,\dots,r_i$, let us consider a disk $D^j_{\varepsilon_i}$ in the interior of $A^j_{\varepsilon_i}$. We can add to Γ the vertices v^* on the boundary of $D^j_{\varepsilon_i}$ and the "missing" ε_i-coloured edges (in a suitable order) in the interior of $A^j_{\varepsilon_i}$, thus obtaining a regular embedding of Γ^* into the surface F_ε obtained by deleting from S_ε the interiors of the disks $D^j_{\varepsilon_i}$.

The formulas for the Euler characteristic and hole number are straightforward.

If Γ is not bipartite we use, instead of a rotation system, a generalized embedding scheme (see [13]) (ϕ, λ) associated to ε, where ϕ is the rotation system defined for each $v \in V(\Gamma)$ as

$$\phi_v = \begin{cases} \varepsilon_{\hat\iota_1\dots\hat\iota_h} & \text{if } v \in V^{(\varepsilon_{\iota_1})}\cup\dots\cup V^{(\varepsilon_{\iota_h})} \\ \varepsilon & \text{otherwise}\end{cases}$$

and $\lambda : E(\Gamma) \longrightarrow \mathbb{Z}_2$ is defined $\lambda(e) = 1$ for each $e \in E(\Gamma)$.

The (bipartite) derived $(n+1)$-coloured graph Γ^λ has vertices $V(\Gamma)\times\{0, 1\}$ and for each $v, w \in V(\Gamma),\ i, j \in \mathbb{Z}_2,\ k \in \Delta_n$ the vertices (v, i) and (w, j) are k-adjacent in Γ^λ iff v and w are k-adjacent in Γ and $i + j = 1$.

Note that Γ^λ is regular with respect to ε, since Γ is.

Moreover ϕ induces a rotation system ϕ^λ on Γ^λ as $\phi^\lambda_{(v,0)} = \phi_v$ and $\phi^\lambda_{(v,1)} = \phi_v^{-1}$ (see [10]).

Let ι_ε (resp. $\iota^\lambda_\varepsilon$) be the regular embedding of Γ (resp. of Γ^λ) into the non-orientable (resp. orientable) closed surface S_ε (resp. S^λ_ε) associated to (ϕ, λ) (resp. to ϕ^λ).

An easy calculation shows that the number of 2-cells of $\iota^\lambda_\varepsilon$ is double of the number of 2-cells of ι_ε, therefore $\chi(S^\lambda_\varepsilon) = 2\chi(S_\varepsilon)$ and we can use the same arguments as in the orientable case to obtain the formulas for the surface with boundary F_ε. □

Let us define $\chi_\varepsilon(\Gamma) = \chi(F_\varepsilon),\ \lambda_\varepsilon(\Gamma) = \lambda(F_\varepsilon)$ and

$$\varrho_\varepsilon(\Gamma) = \begin{cases} 1 - \frac{\chi_\varepsilon(\Gamma)+\lambda_\varepsilon(\Gamma)}{2} & \text{if } \Gamma \text{ is bipartite} \\ 2 - \chi_\varepsilon(\Gamma) - \lambda_\varepsilon(\Gamma) & \text{if } \Gamma \text{ is not bipartite.} \end{cases}$$

The *generalized regular genus* $\varrho(\Gamma)$ of Γ is the minimum $\varrho_\varepsilon(\Gamma)$ among all cyclic permutations ε of Δ_n such that Γ is regular with respect to ε.

Given a n-manifold M the *generalized regular genus* of M is the non-negative integer $\overline{\mathcal{G}}(M)$ defined as the minimum $\varrho(\Gamma)$ among all $(n + 1)$-coloured graphs Γ representing M and regular with respect to at least one cyclic permutation ε of Δ_n.

Given a n-manifold M, we denote by $\mathcal{G}(M)$ the regular genus of M ([10]).

As a direct consequence of the above definition, Remark 1 and the definition of regular genus, we have:

Proposition 2. *For each n-manifold M,*

$$\overline{\mathcal{G}}(M) \leq \mathcal{G}(M).$$

Now we are going to prove that the generalized regular genus is generally strictly less than the regular one.

In [11] a 4-coloured graph is shown which represents $\mathbb{T}_1 \times \mathbb{D}^1$ and regularly embeds into the bordered surface of genus 1, while the regular genus is known to be 2 (see [10]).

In the following, for each $g \geq 1$ (resp. $h \geq 1$) we shall construct a bipartite (resp. non bipartite) 4-coloured graph Γ_g (resp. Γ_h) representing $T_g \times \mathbb{D}^1$, where T_g is the closed orientable surface of genus g (resp. $U_h \times \mathbb{D}^1$, where U_h is the closed non orientable surface of genus h) and regularly embedding into the orientable (resp. non orientable) surface with two holes and genus g (resp. h). In both cases the graph is such that $\partial V = V^{(2)} \cup V^{(3)}$ and $V^{(2)} \cap V^{(3)} = \emptyset$.

The graphs are as follows:

- Γ_g (resp. Γ_h) has $6(2g+1)$ (resp. $6(h+1)$) vertices labeled as $A_1, \ldots, A_{2(2g+1)}, a_1, \ldots, a_{2(2g+1)}, B_1, \ldots, B_{2(2g+1)}$ (resp. $A_1, \ldots, A_{2(h+1)}$, $a_1, \ldots, a_{2(h+1)}$, $B_1, \ldots, B_{2(h+1)}$)
- for each $i = 1, \ldots, 2(2g+1)$ (resp. for each $i = 1, \ldots, 2(h+1)$) $A_i \in V^{(2)}$ and $B_i \in V^{(3)}$
- the 0- , 1- and 2-adjacency are drawn in Figure 1 for the orientable case; the non orientable is analogous;

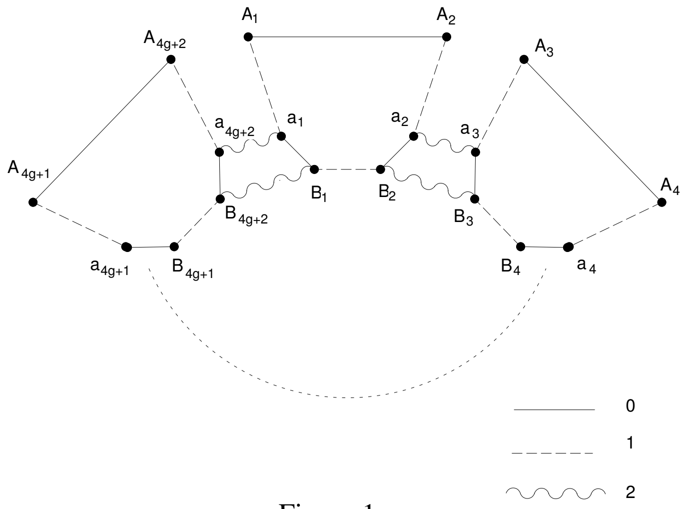

Figure 1.

- the 3-adjacency are:
  for each $i = 1, \ldots, g$, A_{2i} with $A_{4g-2i+3}$, A_{2i-1} with $A_{4g-2i+2}$ and A_{2g+1} with $A_{2(2g+1)}$ (resp. for each $i = 1, \ldots, h$, A_i with A_{2h-i+2} and A_{h+1} with $A_{2(h+1)}$) The 3-adjacency of the a_i's are analogous.

We claim that Γ_g represents $T_g \times \mathbb{D}^1$ (resp. Γ_h represents $U_h \times \mathbb{D}^1$). In fact the above construction comes from an easy generalization of the one in [8] for $T_1 \times \mathbb{D}^1$ and $U_1 \times \mathbb{D}^1$, together with a permutation of the colours on one of the boundary components.

Let $\varepsilon = (0132)$, then for each $g \geq 1$ (resp. $h \geq 1$), Γ_g (resp. Γ_h) is regular with respect to ε and it is easy to see that:

$g_{01} = g_{02} = g_{23} = 2g + 1$, $g_{03} = g_{12} = g_{13} = 1$
(resp. $g_{01} = g_{02} = g_{13} = h + 1$, $g_{03} = g_{12} = g_{23} = 1$).

Since $\chi_\varepsilon(\Gamma_g) = -2g$ (resp. $\chi_\varepsilon(\Gamma_h) = -h$) and the number of holes is 2 both in the orientable and the non orientable case, we have $\varrho_\varepsilon(\Gamma_g) = g$ (resp. $\varrho_\varepsilon(\Gamma_h) = h$).

Therefore $\overline{\mathcal{G}}(T_g \times \mathbb{D}^1) \leq g < \mathcal{G}(T_g \times \mathbb{D}^1) = 2g$ and $\overline{\mathcal{G}}(U_h \times \mathbb{D}^1) \leq h < \mathcal{G}(U_h \times \mathbb{D}^1) = 2h$ (see [1]); actually the first are equalities, since we can establish the following theorem:

Theorem 3. $\overline{\mathcal{G}}(T_g \times \mathbb{D}^1) = \overline{\mathcal{G}}(U_g \times \mathbb{D}^1) = g$

Before proving the theorem let us fix some notations.

Let $\varepsilon = (\alpha\alpha'\beta\beta')$ be a cyclic permutation of Δ_3 and Γ a 4-coloured graph representing a 3-manifold M and regular with respect to ε. We denote by $\partial_i K(\Gamma)$ $(i = 1, \ldots, r)$ the i-th boundary component of $K(\Gamma)$ and by $V_i(\Gamma)$ the subset of $\partial V(\Gamma)$ formed by those vertices whose dual 3-simplices have a face on $\partial_i K(\Gamma)$.

Note that, since Γ is regular with respect to ε, then for each $i = 1, \ldots, r$, $V_i(\Gamma) \subseteq V^{(\alpha)}(\Gamma) \cup V^{(\beta)}(\Gamma)$ or $V_i(\Gamma) \subseteq V^{(\alpha')}(\Gamma) \cup V^{(\beta')}(\Gamma)$.

The proof of Theorem 3 requires two lemmas.

Lemma 4. *Given a 3-manifold with r boundary components M, a cyclic permutation ε of Δ_3 and a 4-coloured graph Γ representing M and regular with respect to ε, then there exists a 4-coloured graph Γ′, representing M, and satisfying the following conditions:*

(1) $\varrho_\varepsilon(\Gamma') = \varrho_\varepsilon(\Gamma)$;

(2) $\forall v \in V(\Gamma')$, *deg* $v \geq 3$ *and* $\forall i = 1, \ldots, r$, $V_i(\Gamma') \cap (V^{(\beta)}(\Gamma') \cup V^{(\alpha')}(\Gamma') \cup V^{(\beta')}(\Gamma')) = \emptyset$ *or* $V_i(\Gamma') \cap (V^{(\alpha)}(\Gamma') \cup V^{(\beta)}(\Gamma') \cup V^{(\beta')}(\Gamma')) = \emptyset$.

Proof. Let $i \in \{1, \ldots, r\}$ be such that $V_i(\Gamma) \cap V^{(\alpha)}(\Gamma) \neq \emptyset$ and $V_i(\Gamma) \cap V^{(\beta)}(\Gamma) \neq \emptyset$, and let w be a α-coloured vertex of $\partial_i K(\Gamma)$.

Let us consider the 4-coloured graph $b\Gamma$ obtained by performing a *bisection of type* (α, β) around w (see [9]) i.e. we perform a stellar subdivision on each edge having as endpoints w and a β-coloured vertex and colour w by β and the new vertices by α, keeping the colours of $K(\Gamma)$ for the remaining vertices (see [9]).

Note that $card\ (V_i(b\Gamma) \cap V^{(\alpha)}(b\Gamma)) = card\ (V_i(\Gamma) \cap V^{(\alpha)}(\Gamma)) - 1$.

We claim that $\varrho_\varepsilon(b\Gamma) = \varrho_\varepsilon(\Gamma)$.

In fact, let Λ_w be the $\widehat{\alpha}$-residue of Γ representing the disjoined link of w in $K(\Gamma)$.

We have:

$$\forall j \neq \beta\,, \qquad \dot{g}_{\alpha j}(b\Gamma) = \dot{g}_{\alpha j}(\Gamma) + \dot{g}_{\beta j}(\Lambda_w)$$
$$\forall i \neq \alpha\,, \qquad \dot{g}_{\beta i}(b\Gamma) = \dot{g}_{\beta i}(\Gamma) - \dot{g}_{\beta i}(\Lambda_w) + q^{(i)}(\Lambda_w)$$

where $q^{(i)}(\Lambda_w)$ is the number of i-coloured edges of Λ_w.

$$p(b\Gamma) = p(\Gamma) + p(\Lambda_w) \qquad q(b\Gamma) = q(\Gamma) + q^{(\alpha')}(\Lambda_w) + q^{(\beta')}(\Lambda_w) + p(\Lambda_w)$$

Therefore:

$$\begin{aligned}\chi_\varepsilon(b\Gamma) &= \dot{g}_{\alpha\alpha'}(b\Gamma) + \dot{g}_{\alpha'\beta}(b\Gamma) + \dot{g}_{\beta\beta'}(b\Gamma) + \dot{g}_{\beta'\alpha}(b\Gamma) - q(b\Gamma) + p(b\Gamma) = \\ &= \dot{g}_{\alpha\alpha'}(\Gamma) + \dot{g}_{\beta\alpha'}(\Lambda_w) + \dot{g}_{\alpha'\beta}(\Gamma) - \dot{g}_{\alpha'\beta}(\Lambda_w) + q^{(\alpha')}(\Lambda_w) + \dot{g}_{\beta\beta'}(\Gamma) - \\ &- \dot{g}_{\beta\beta'}(\Lambda_w) + q^{(\beta')}(\Lambda_w) + \dot{g}_{\beta'\alpha}(\Gamma) + \dot{g}_{\beta\beta'}(\Lambda_w) - q(\Gamma) - q^{(\alpha')}(\Lambda_w) - \\ &- q^{(\beta')}(\Lambda_w) - p(\Lambda_w) + p(\Gamma) + p(\Lambda_w) = \chi_\varepsilon(\Gamma).\end{aligned}$$

Moreover, note that, for each $i \in \Delta_3$, if j is the colour non consecutive to i in ε, ${}^{\partial}g_i(\Gamma)$ equals the number of j-coloured vertices in the components of $\partial K(\Gamma)$ missing colour i.

Therefore

$$^{\partial}g_\alpha(b\Gamma) = {}^{\partial}g_\alpha(\Gamma) + 1 \qquad {}^{\partial}g_{\alpha'}(b\Gamma) = {}^{\partial}g_{\alpha'}(\Gamma)$$
$$^{\partial}g_\beta(b\Gamma) = {}^{\partial}g_\beta(\Gamma) - 1 \qquad {}^{\partial}g_{\beta'}(b\Gamma) = {}^{\partial}g_{\beta'}(\Gamma)$$

and $\lambda_\varepsilon(b\Gamma) = \lambda_\varepsilon(\Gamma)$.

Finally we have that $\varrho_\varepsilon(b\Gamma) = \varrho_\varepsilon(\Gamma)$.

By performing a finite number of bisection of type (α, β) on the components of $\partial K(\Gamma)$ missing α and β and, similarly a finite number of bisection of type (α', β') on the components missing α' and β', we obtain the graph Γ'.

□

Suppose now that Γ is a 4-coloured graph satisfying condition (2) of Lemma 4, with respect to a cyclic permutation ε of Δ_3 and suppose that $\partial|K(\Gamma)|$ has r connected components. Let us choose one of them, say $\partial_i K(\Gamma)$. Then there exists $j \in \Delta_3$ such that for each $k \in \Delta_3 - \{j\}$, $V_i(\Gamma) \cap V^{(k)}(\Gamma) = \emptyset$.

Let us denote by $\Gamma_i^{(j)}$ the 4-coloured graph obtained from Γ by the following rule:

- $\forall v, w \in V_i(\Gamma) \cap V^{(j)}(\Gamma)$, join the vertices v and w by a j-coloured edge iff v and w belong to the same $\{j, j+1\}$-residue of Γ.

It is easy to see that, if Γ represents a 3-manifold M with r boundary components, $\Gamma_i^{(j)}$ represents the singular 3-manifold obtained from M by capping off the i-th boundary component by a cone over it.

Moreover, we have

Lemma 5. $\varrho_\varepsilon(\Gamma_i^{(j)}) = \varrho_\varepsilon(\Gamma)$

Proof. We have

$$p(\Gamma_i^{(j)}) = p(\Gamma) \qquad q(\Gamma_i^{(j)}) = q(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2}$$
$$\dot{g}_{kk+1}(\Gamma_i^{(j)}) = \dot{g}_{kk+1}(\Gamma) \qquad \forall k \in \Delta_3 - \{j-1, j+1\}$$
$$\dot{g}_{jj+1}(\Gamma_i^{(j)}) = \dot{g}_{jj+1}(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2} \qquad \dot{g}_{j-1j}(\Gamma_i^{(j)}) = \dot{g}_{j-1j}(\Gamma) + {}^{\partial}g_i^{(j)}(\Gamma)$$

where $p^{(j)}(\Gamma) = card\ (V_i(\Gamma) \cap V^{(j)}(\Gamma))$ and ${}^{\partial}g_i^{(j)}(\Gamma)$ is the number of closed walks defined as for ${}^{\partial}g_i(\Gamma)$, whose boundary vertices belong only to $V_i(\Gamma)$.

Then

$$\begin{aligned}\chi_\varepsilon(\Gamma_i^{(j)}) &= \sum_{k\in\mathbb{Z}_4} \dot{g}_{kk+1}(\Gamma_i^{(j)}) - q(\Gamma_i^{(j)}) + p(\Gamma_i^{(j)}) \\ &= \sum_{k\in\mathbb{Z}_4} \dot{g}_{kk+1}(\Gamma) + \frac{p_i^{(j)}(\Gamma)}{2} + {}^{\partial}g_i^{(j)}(\Gamma) - q(\Gamma) - \frac{p_i^{(j)}(\Gamma)}{2} + p(\Gamma) \\ &= \chi_\varepsilon(\Gamma) + {}^{\partial}g_i^{(j)}(\Gamma).\end{aligned}$$

Moreover $\lambda_\varepsilon(\Gamma_i^{(j)}) = \lambda_\varepsilon(\Gamma) - {}^{\partial}g_i^{(j)}(\Gamma)$. Therefore $\varrho_\varepsilon(\Gamma_i^{(j)}) = \varrho_\varepsilon(\Gamma)$. □

Proof. (Theorem 3) Let $M = T_g \times \mathbb{D}^1$ or $M = U_g \times \mathbb{D}^1$. Suppose $\overline{\mathcal{G}}(M) < g$.

Let Γ be a 4-coloured graph representing M such that Γ is regular with respect to a cyclic permutation ε of Δ_3 and $\varrho_\varepsilon(\Gamma) < g$.

By Lemma 4, we can suppose, without loss of generality, that Γ satisfy condition (2) of the Lemma. Moreover we can also suppose, up to a change of colours, that $V_2(\Gamma) \subseteq V^{(3)}(\Gamma)$ (i.e. the vertices corresponding to one of the boundary components miss colour 3).

If also $V_1(\Gamma) \subseteq V^{(3)}(\Gamma)$, then the graph is regular with respect to the colour 3 and $\mathcal{G}(M) \leq \varrho_\varepsilon(\Gamma) < g$, which is clearly impossible.

If, on the contrary, $V_1(\Gamma) \subseteq V^{(2)}(\Gamma)$, let us consider the graph $\Gamma_1^{(2)}$. Then $\widetilde{M} = |K(\Gamma_1^{(2)})|$ is obtained from M by capping off one boundary component by a cone, i.e. it is a cone over the surface T_g or U_g.

Since $\Gamma_1^{(2)}$ is regular with respect to the colour 3, by Lemma 5, we have $\mathcal{G}(\widetilde{M}) \leq \varrho_\varepsilon(\Gamma_1^{(2)}) < g$; on the other hand it is well-known ([10]) that $\mathcal{G}(\widetilde{M}) \geq \mathcal{G}(\partial\widetilde{M}) = g$, since $\partial\widetilde{M} = T_g$ or $\partial\widetilde{M} = U_g$. □

If $g = 1$ the previous result is a corollary of the following theorem, which gives a characterization of punctured 3-spheres.

Theorem 6. *Let M be a 3-manifold with boundary and let r be the number of its boundary components, then*

$$\overline{\mathcal{G}}(M) = 0 \iff M \text{ is a sphere with } r \text{ holes (punctured 3-sphere).}$$

Proof. If M is a punctured 3-sphere, its generalized regular genus is clearly zero since its regular genus is zero (see [4]). Conversely let M be a 3-manifold such that $\overline{\mathcal{G}}(M) = 0$, ε a cyclic permutation of Δ_3 and Γ a 4-coloured graph representing M such that Γ is regular with respect to ε and $\varrho_\varepsilon(\Gamma) = 0$.

Again by Lemma 4, we can suppose, without loss of generality, that Γ satisfy condition (2) of the Lemma. Therefore we can consider the 4-coloured graph (without boundary) $\widetilde{\Gamma}$ obtained from Γ by joining, $\forall j \in \Delta_3$ and $\forall v, w \in V^{(j)}(\Gamma)$, the vertices v and w by a j-coloured edge iff v and w belong to the same $\{j, j+1\}$-residue of Γ, i.e. $\widetilde{\Gamma}$ is obtained by performing r times the operation involved in Lemma 5.

Therefore $\widetilde{\Gamma}$ represents the singular 3-manifold $\widehat{M}$ obtained from M by capping each component of ∂M by a cone.

By Lemma 5 we have that $\varrho_\varepsilon(\widetilde{\Gamma}) = \varrho_\varepsilon(\Gamma) = 0$ and by [4] (Corollary 3_3), $\widehat{M} \cong \mathbb{S}^n$; therefore for each $i = 1, \dots, r$, $\partial_i M$ is a sphere and M is a punctured 3-sphere. $\square$

Remark 2. The proof of Lemma 4 tells us that, as far as 3-manifolds are concerned, we can always suppose that the generalized regular genus is obtained by a 4-coloured graph satisfying condition (2). Let us denote by $\overline{G_4}$ the class of such graphs.

For each $\Gamma \in \overline{G_4}$ we can define a "boundary graph" $\partial\Gamma$ in the following way:

- $V(\partial\Gamma) = \partial V(\Gamma)$;
- $\forall i = 1, \dots, r$, $j \in \Delta_3$ and $\forall v, w \in V_i \cap V^{(j)}$ join v and w by a c-coloured edge ($c \in \Delta_3$) iff v and w belong to the same $\{c, j\}$-residue of Γ.

Note that $\partial\Gamma$ is not a 3-coloured graph, but has r connected components $\partial_1\Gamma, \dots, \partial_r\Gamma$ each of them being a 3-coloured graph with colour set $\Delta_3 - \{j\}$ for some $j \in \Delta_3$. Of course, for each $i = 1, \dots, r$, $\partial_i\Gamma$ represents $\partial_i M$.

Remark 3. Note that, as proved by the graphs we constructed in this section for $T_g \times \mathbb{D}^1$ and $U_h \times \mathbb{D}^1$, the generalized regular genus, still unlike the regular one (see [10]), is generally strictly less the sum of the genera of the boundary components.

3. Regular embeddings of 4-coloured graphs and generalized Heegaard splittings.

In this section we shall show that there exists a correspondence between regular embeddings of 4-coloured graphs in $\overline{G}_4$, representing a 3-manifold, and generalized Heegaard splittings of the same manifold. We briefly recall the basic concepts about generalized Heegaard splittings.

We shall denote by S_g either the orientable closed surface of genus g or the closed non orientable surface of genus $2g$.

A *hollow handlebody of genus* g is a 3-manifold with boundary X_g, obtained from $S_g \times [0, 1]$ by attaching 2- and 3-handles along $S_g \times \{1\}$. We call $S_g \times \{0\}$ the *free boundary* of X_g.

Note that the orientability of X_g depends on that of S_g and conversely.

A *generalized Heegaard splitting of genus* g of a 3-manifold with boundary M is a pair (X_g, Y_g) of hollow handlebodies of genus g, such that $X_g \cup Y_g = M$ and $X_g \cap Y_g$ is the free boundary of both X_g and Y_g.

The *generalized Heegard genus* of a 3-manifold M is the non negative integer

$$\overline{\mathcal{H}}(M) = \min\{g \,|\, \text{there exists a generalized Heegaard splitting of genus } g \text{ of M}\}.$$

Let Γ be a 4-coloured graph of $\overline{G}_4$, regular with respect to a cyclic permutation ε of Δ_3 and such that the "boundary" colours are consecutive in ε. Then, up to a change of colours, we can suppose that

$(*)$ $V^{(\varepsilon_0)} = V^{(\varepsilon_1)} = \emptyset$

We can state the following

Proposition 7. *Let M be a connected 3-manifold, $\Gamma \in \overline{G}_4$ a 4-coloured graph representing M, regular with respect to a cyclic permutation ε of Δ_3 and satisfying condition $(*)$, then there exists a generalized Heegaard splitting for M of genus $\varrho_\varepsilon(\Gamma)$.*

Proof. To avoid long notations let us suppose $\varepsilon = id$.

Given the graph Γ, representing M and regular with respect to ε, we know, from the proof of Theorem 6, that there exists a 4-coloured graph without boundary $\widetilde{\Gamma}$ such that $\varrho_\varepsilon(\widetilde{\Gamma}) = \varrho_\varepsilon(\Gamma)$ and $\widetilde{\Gamma}$ represents the singular 3-manifold $\widehat{M}$ obtained from M by capping off each boundary component by a cone.

$\widetilde{\Gamma}$ is obtained from Γ by adding a 3-coloured edge (resp. 2-coloured edge) between two vertices $v, w \in V^{(3)}$ (resp. $v, w \in V^{(2)}$) iff v and w belong to the same connected component of $\Gamma_{\{0,3\}}$ (resp. $\Gamma_{\{1,2\}}$).

Let K' (resp. K'') the 1-dimensional subcomplex of $K(\widetilde{\Gamma})$ generated by its 0- and 2-coloured (resp. 1- and 3-coloured) vertices and H the largest subcomplex of $SdK(\widetilde{\Gamma})$ (where Sd means first barycentric subdivision) disjoint from $SdK' \cup SdK''$; then H splits $SdK(\widetilde{\Gamma})$ into two subcomplexes N' and N'' such that $N' \cap N'' = \partial N' \cap \partial N'' = H$. Set $\mathcal{A}' = |N'|$, $\mathcal{A}'' = |N''|$ and $\mathcal{S} = |H|$. $\mathcal{A}'$ and $\mathcal{A}''$ are handlebodies, $\mathcal{S}$ is a closed connected surface of genus $\varrho_\varepsilon(\widetilde{\Gamma})$, where $\widetilde{\Gamma}$ regularly embeds.

Let C be a collar of $\mathcal{S}$ in $\mathcal{A}'$; let C_0, C_1 be the surfaces corresponding to $\mathcal{S} \times \{0\}$ and $\mathcal{S} \times \{1\}$ respectively. For ech 1-simplex e of $K(\widetilde{\Gamma})$ whose endpoints are 0- and 2-coloured, let H_e^{02} be a regular neighbourhood in $\mathcal{A}'$ of the 2-cell dual of e (see Figure 2).

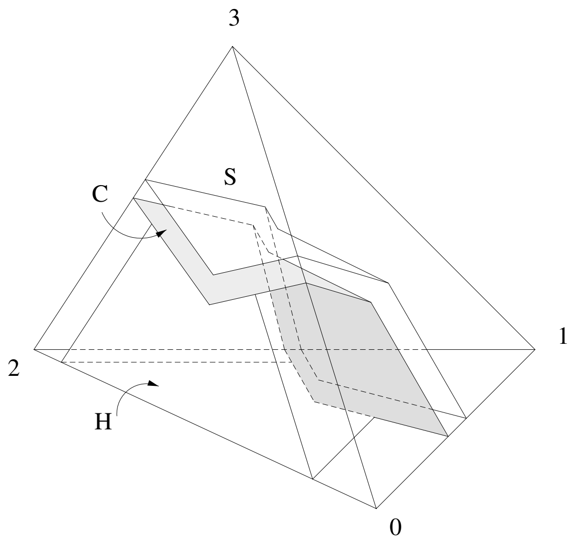

Figure 2.

Set $X = C \cup (\bigcup_e H_e^{02})$. X is a hollow handlebody, since the H_e^{02}'s are 2-handles attached along $C_1 \cong \mathcal{S} \times \{1\}$.

Moreover $\overline{\mathcal{A}' - X}$ is the union of regular neighbourhoods of the 0- and 2-coloured vertices of $K(\widetilde{\Gamma})$.

Let $\widetilde{X}$ be the hollow handlebody obtained by adding to X the neighbourhoods corresponding to non singular vertices.

Similarly we can define a hollow handlebody $\widetilde{Y}$ by starting from a collar of $\mathcal{S}$ in $\mathcal{A}''$ and attaching on it:

- the 2-handles H_e^{13} dual to the 1-simplexes of $K(\widetilde{\Gamma})$ having endpoints coloured by 1 and 3;
- the 3-handles corresponding to the neighbourhoods of the non singular 1- and 3-coloured vertices.

We have that $\widetilde{X} \cup \widetilde{Y} = M$ and $\widetilde{X} \cap \widetilde{Y} = \mathcal{S}$.

Therefore $(\widetilde{X}, \widetilde{Y})$ is a generalized Heegaard splitting for M of genus $g(\mathcal{S}) = \varrho_\varepsilon(\widetilde{\Gamma}) = \varrho_\varepsilon(\Gamma)$. □

As a consequence of Proposition 7 and Lemma 4, we have the following:

Corollary 8. *For each 3-manifold* M, $\overline{\mathcal{H}}(M) \leq \overline{\mathcal{G}}(M)$.

Proof. Let Γ be a 4-coloured graph representing M and ε a cyclic permutation of Δ_3 such that Γ is regular with respect to ε and $\overline{\mathcal{G}}(M) = \varrho_\varepsilon(\Gamma)$.

By Lemma 4 we know that we can always suppose that Γ misses at most two colours.

If these colours are non consecutive in ε, then, by means of suitable bisections, we can obtain a new graph, still representing M, with the same genus as Γ and missing only one colour, i.e. a graph regular with respect to a colour, that we can always suppose to be 3.

In this case by Lemma 1 of [2] there exists a proper ([2]) Heegaard splitting of M of genus $\overline{\mathcal{G}}(M) = \mathcal{G}(M)$.

On the other hand, if the "boundary" colours are consecutive in ε, we can apply Proposition 7 to get a Heegaard splitting of M of the required genus. □

Note that the splitting is always proper in the case of M having connected boundary. In this case $\overline{\mathcal{G}}(M) = \mathcal{G}(M) = \mathcal{H}(M) = \overline{\mathcal{H}}(M)$ (see [3]).

REFERENCES

[1] P. Bandieri - M. Rivi, *Some bounds for the genus of* $M^n \times I$, Note di Matematica, 18 (1998), pp. 175–190.

[2] P. Cristofori - C. Gagliardi - L. Grasselli, *Heegaard and regular genus of 3-manifolds with boundary,* Revista Matematica de la Universidad Complutense de Madrid, 8 - 2 (1995), pp. 221–235.

[3] P. Cristofori, *Heegaard and regular genus agree for compact 3-manifolds,* Cahiers de Topologie et Geometrie Differentielle Categoriques 39 - 3 (1998), pp. 221–235.

[4] M. Ferri - C. Gagliardi, *The only genus zero n-manifold is* $\mathbb{S}^n$, Proc. Amer. Math. Soc., 85 (1982), pp. 638–642.

[5] M. Ferri - C. Gagliardi, *A characterization of punctured n-spheres,* Yokohama Math. J., 33 (1985), pp. 29–38.

[6] M. Ferri - C. Gagliardi - L. Grasselli, *A graph-theoretical representation of PL-manifolds - A survey on crystallizations,* Aeq. Math., 31 (1986), pp. 121–141.

[7] C. Gagliardi, *Regular embeddings of edge-coloured graphs,* Geom. Dedicata, 11 (1981), pp. 397–414.

[8] C. Gagliardi, *Cobordant crystallizations,* Discrete Math., 45 (1983), pp. 61–73.

[9] C. Gagliardi, *On a class of 3-dimensional polyhedra,* Ann. Univ. Ferrara, 33 (1987), pp. 51–88.

[10] C. Gagliardi, *Regular genus: the boundary case,* Geom. Dedicata, 22 (1987), pp. 261–281.

[11] C. Gagliardi - G. Volzone, *Una osservazione sul genere delle varietà con bordo,* Atti Sem. Mat. Fis. Univ. Modena, 47 - 2 (1999), pp. 473–478.

[12] J.M. Montesinos, *Representing 3-manifolds by a universal branching set,* Math. Proc. Camb. Phil. Soc., 94 (1983), pp. 109–123.

[13] S. Stahl, *Generalized embedding schemes,* J. Graph Theory, 2 (1978), pp. 41–52.

[14] S. Stahl, *The embedding of a graph – a survey,* J. Graph Theory, 2 (1978), pp. 275–298.

Dipartimento di Matematica Pura ed Applicata
via Campi 213/B
41100 Modena (ITALY)
e-mail: cristofori.paola@unimo.it