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(Article begins on next page)

Lower bounds for regular genus and gem-complexity of PL 4-manifolds

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Abstract

We prove that, for any closed connected PL 4-manifold M, its gem-complexity $k(M)$ and its regular genus $\mathcal{G}(M)$ satisfy:

$$k(M) \geq 3\chi(M) + 10m - 6 \quad \text{and} \quad \mathcal{G}(M) \geq 2\chi(M) + 5m - 4,$$

where $rk(\pi_1(M)) = m$. These lower bounds enable to strictly improve previously known estimations for regular genus and gem-complexity of product 4-manifolds.

Moreover, the class of *semi-simple crystallizations* is introduced, so that the represented PL 4-manifolds attain the above lower bounds. The additivity of both gem-complexity and regular genus with respect to connected sum is also proved for such a class of PL 4-manifolds, which comprehends all ones of "standard type", involved in existing crystallization catalogues, and their connected sums.

1 Introduction

A *simplicial cell complex* K of dimension d is a poset isomorphic to the face poset $\mathcal{X}$ of a d-dimensional simplicial CW-complex X. The topological space X is called the *geometric carrier* of K and is also denoted by $|K|$. If a topological space M is homeomorphic to $|K|$, then K is said to be a *pseudo-triangulation* of M.

If a pseudo-triangulation K of a d-manifold M $(d \geq 1)$ contains exactly $d + 1$ vertices, then K is said to be *contracted*; its dual graph gives rise to a *crystallization* of M, i.e. a $(d+1)$-colored contracted graph $\Gamma = (V, E)$ with an edge coloring $\gamma : E \to \{0, \dots, d\}$, so that the vertices of K have one to one correspondence with the colors $0, \dots, d$ and the facets of K

1

have one to one correspondence with the vertices in V (for details see [19], or the following Section 2).

The existence of crystallizations for every closed connected PL-manifold is ensured by a classical Theorem due to Pezzana (see [24], or [19] for subsequent generalizations). Hence, every closed connected PL d-manifold M admits a contracted pseudo-triangulation, which is also called a *colored triangulation* of M, because of the edge-coloring of the associated crystallization.

Within crystallization theory, two interesting PL invariants for PL d-manifolds have been introduced, namely *regular genus* and *gem-complexity*.

First, let us recall that, if (Γ, γ) is a $(d+1)$-colored graph, an embedding $i : \Gamma \hookrightarrow F$ of Γ into a closed surface F is called *regular* if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d)$ of the color set $\Delta_d = \{0, \ldots, d\}$, such that the boundary of each face of $i(\Gamma)$ is a bi-colored cycle with colors $\varepsilon_j, \varepsilon_{j+1}$ for some j (where the addition is performed modulo $d+1$).

Then, we have the following:

Definition 1. The *regular genus* $\rho(\Gamma)$ of (Γ, γ) is the least genus (resp. half of genus) of the orientable (resp. non-orientable) surface into which Γ embeds regularly; the *regular genus* $\mathcal{G}(M)$ of a closed connected PL d-manifold M is defined as the minimum regular genus of its crystallizations.

Note that the notion of regular genus extends classical notions to arbitrary dimension: in fact, the regular genus of a closed connected orientable (resp. non-orientable) surface coincides with its genus (resp. half of its genus), while the regular genus of a closed connected 3-manifold coincides with its Heegaard genus (see [20, 21]). The invariant regular genus has been intensively studied, yielding some important general results: for example, regular genus zero characterizes the d-sphere among all closed connected PL d-manifolds ([18]). In particular, in dimension $d \in \{4, 5\}$, a lot of classifying results in PL-category have been obtained, both for closed and bounded PL d-manifolds (via suitable extensions of the involved notions): they concern the case of "low" regular genus, the case of "restricted gap" between the regular genus of the manifold and the regular genus of its boundary, and the case of "restricted gap" between the regular genus and the rank of the fundamental group of the manifold (see, for example, [14], [7] and [15]).

The second definition is quite natural, and is directly related to the combinatorial "complicatedness" of the representing tool via crystallization theory:

Definition 2. Given a PL d-manifold M, its *gem-complexity* is the non-negative integer $k(M) = p - 1$, where $2p$ is the minimum order of a crystallization of M.

It is easy to check that, for any dimension $d \geq 2$, gem-complexity zero characterizes the d-sphere among all closed connected PL d-manifolds. Moreover, gem-complexity is the natural invariant used to create automatic catalogues of PL-manifolds via crystallizations. This approach has been successfully followed in dimension three and four, where the choice and implementation of suitable sets of combinatorial moves preserving the represented manifold allowed the development of an effective "classifying algorithm" up to PL-homeomorphism: all 3-manifolds up to gem-complexity 14 have been identified (see [22], [10] and [11] for the

orientable case and [8], [9] and [1] for the non-orientable one), together with all PL 4-manifolds up to gem-complexity 8 (see [12]).

Obviously, any crystallization of a given d-manifold M yields an upper bound both for the regular genus and for the gem-complexity of M; on the contrary, the problem of finding lower bounds is generally more difficult. In [2], a lower bound for gem-complexity of a closed connected PL 3-manifold is obtained, by means of the *weight* of the fundamental group of M, while a lower bound for simply-connected PL 4-manifold, involving the second Betti of M, easily follows from [16, Proposition 2][1].

In the present paper, we present lower bounds both for gem-complexity and for regular genus, for the whole class of closed connected PL 4-manifold:[2]

Theorem 1. *Let M be a (closed connected) PL 4-manifold with $rk(\pi_1(M)) = m$. Then,*

$$k(M) \geq 3\chi(M) + 10m - 6,$$

$$\mathcal{G}(M) \geq 2\chi(M) + 5m - 4.$$

The concept of simple crystallizations was introduced in [3]. By definition, PL 4-manifolds represented by simple crystallizations are simply-connected; moreover, the characterization of the class of 4-manifolds admitting simple crystallizations (see [13]) easily proves that all elements of that class attain both bounds of Theorem 1 (see Remark 2). Hence, in order to investigate the possible sharpness of the above bounds, also in the not simply-connected case, the present paper introduces the concept of *semi-simple crystallizations*, which comprehend and generalize simple crystallizations.

Definition 3. A crystallization (Γ, γ) of a PL 4-manifold M is called a *semi-simple crystallization of type m* if the 1-skeleton of the associated colored triangulation contains exactly $m + 1$ 1-simplices for each pair of 0-simplices, where m is the rank of the fundamental group of M. Semi-simple crystallizations of type 0 are called *simple crystallizations*, according to [3, 13].

Note that all PL 4-manifolds involved in the existing crystallization catalogues turn out to admit a semi-simple crystallization (Proposition 7); moreover, the class of PL 4-manifolds admitting semi-simple crystallizations is proved to be closed under connected sum (Proposition 6). Hence, PL 4-manifolds admitting semi-simple crystallizations actually constitutes a huge class (Remark 4), which comprehends all PL 4-manifolds "of standard type" (i.e. $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{RP}^4$, the orientable and non-orientable $\mathbb{S}^3$-bundles over $\mathbb{S}^1$ and the $K3$-surface, together with their connected sums, possibly by taking copies with reversed orientation, too).

We prove that each PL 4-manifold in the above class attains both the bounds of Theorem 1.

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[1] Actually, [16, Proposition 2] yields also a lower bound for closed connected orientable PL 4-manifold, but in the not simply-connected case it is not so significant.

[2] From now on, for sake of simplicity, we will simply write "PL manifold" instead of "closed connected PL manifold".

Theorem 2. *Let M be a PL 4-manifold with $rk(\pi_1(M)) = m$. If M admits semi-simple crystallizations, then:*

$$\begin{aligned} k(M) &= 3\chi(M) + 10m - 6; \\ \mathcal{G}(M) &= 2\chi(M) + 5m - 4; \\ k(M) &= \tfrac{3\mathcal{G}(M)+5m}{2}. \end{aligned}$$

We also prove additivity of both gem-complexity and regular genus for the class of PL 4-manifolds admitting semi-simple crystallizations.

Theorem 3. *Let M_1 and M_2 be two PL 4-manifolds admitting semi-simple crystallizations. Then,*

$$k(M_1 \# M_2) = k(M_1) + k(M_2) \quad and \quad \mathcal{G}(M_1 \# M_2) = \mathcal{G}(M_1) + \mathcal{G}(M_2).$$

Note that the inequality $k(M\#M') \le k(M) + k(M')$ (resp. $\mathcal{G}(M\#M') \le \mathcal{G}(M) + \mathcal{G}(M')$) can be stated for all PL d-manifolds by direct estimation of $k(M\#M')$ (resp. of $\mathcal{G}(M\#M')$) on the crystallization $(\Gamma\#\Gamma', \gamma\#\gamma')$ obtained by graph-connected sum (see Subsection 2.1), when (Γ, γ) and (Γ', γ') are assumed to be crystallizations of M and M' respectively, realizing gem-complexity (resp. regular genus) of the represented d-manifolds. Moreover, we point out that the additivity of regular genus under connected sum has been conjectured, and the associated (open) problem is significant especially in dimension four. In fact, in dimension four, additivity of regular genus, at least in the simply-connected case, would imply the 4-dimensional Smooth Poincaré Conjecture, in virtue of a well-known Wall's Theorem ([26]).

Finally, as an application of Theorem 1, we provide lower bounds for regular genus and gem-complexity of product 4-manifolds, which strictly improve previous results (see Section 5).

2 Preliminaries

2.1 Colored graphs

A multigraph $\Gamma = (V(\Gamma), E(\Gamma))$ is a finite connected graph, with vertex-set $V(\Gamma)$ and edge-set $E(\Gamma)$, which can have multiple edges but no loops. A multigraph Γ is called $(d+1)$-*regular* if the number of edges adjacent to each vertex is $(d+1)$. An *edge coloring* of a $(d+1)$-regular multigraph $\Gamma = (V, E)$ is a map $\gamma \colon E \to \Delta_d := \{0, 1, \ldots, d\}$ such that $\gamma(e) \ne \gamma(f)$ whenever e and f are adjacent. The elements of the set Δ_d are called the *colors* and (Γ, γ) is said to be a $(d+1)$-*colored graph*.

Let (Γ, γ) be a $(d+1)$-colored graph with color set Δ_d. For each $B \subseteq \Delta_d$ with h elements, then the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ is a h-colored graph with coloring $\gamma|_{\gamma^{-1}(B)}$. If $\Gamma_{\Delta_d \setminus \{c\}}$ is connected for all $c \in \Delta_d$, then (Γ, γ) is called *contracted*.

Let (Γ_1, γ_1) and (Γ_2, γ_2) be two disjoint $(d+1)$-colored graphs with the same color set Δ_d, and let $v_i \in V_i$ ($1 \le i \le 2$). The *connected sum* of Γ_1, Γ_2 with respect to vertices v_1, v_2 (denoted by $(\Gamma_1 \#_{v_1 v_2} \Gamma_2, \gamma_1 \# \gamma_2)$, or simply $(\Gamma_1 \# \Gamma_2, \gamma_1 \# \gamma_2)$) is the graph obtained from $(\Gamma_1 \setminus \{v_1\}) \sqcup (\Gamma_2 \setminus \{v_2\})$ by adding $d+1$ new edges $e_0, \ldots, e_d$ with colors $0, \ldots, d$ respectively, such that the end points of e_j are $u_{j,1}$ and $u_{j,2}$, where v_i and $u_{j,i}$ are joined in (Γ_i, γ_i) with an edge of color j for $0 \le j \le d$, $1 \le i \le 2$.

For standard terminology on graphs, we refer to [5].

2.2 Crystallizations

A CW-complex X is said to be *regular* if the attaching maps which define the incidence structure of X are homeomorphisms. Given a regular CW-complex X, let $\mathcal{X}$ be the set of all closed cells of X together with the empty set. Then $\mathcal{X}$ is a poset, where the partial ordering is the set inclusion. This poset $\mathcal{X}$ is said to be the *face poset* of X. Clearly, if X and Y are two finite regular CW-complexes with isomorphic face posets, then X and Y are homeomorphic. A regular CW-complex X is said to be *simplicial* if the boundary of each cell in X is isomorphic (as a poset) to the boundary of a simplex of same dimension. A *simplicial cell complex* K of dimension d is a poset isomorphic to the face poset $\mathcal{X}$ of a d-dimensional simplicial CW-complex X. The topological space X is called the *geometric carrier* of K and is also denoted by $|K|$. If a topological space M is homeomorphic to $|K|$, then K is said to be a *pseudo-triangulation* of M. A simplicial cell complex K is said to be *connected* if the topological space $|K|$ is path connected (see [4, 23] for more details).

Let K be a simplicial cell complex with partial ordering $\leq$. If $\beta \leq \alpha \in K$ then we say β is a *face* of α. For $\alpha \in K$, the set $\partial \alpha := \{\gamma \in K \ : \ \alpha \neq \gamma \leq \alpha\}$ is a subcomplex of K with induced partial order and is said to be the *boundary* of α. If $\partial \alpha$ is isomorphic to the boundary complex of an i-simplex then we say that α is an *i-cell* or a cell of *dimension i*. We denote $f_i(K) = \#\{\alpha \in K \ : \ \alpha \text{ is a } i\text{-cell of } K\}$.

If all the maximal cells of a d-dimensional simplicial cell complex K are d-cells, then it is called *pure*. Maximal cells in a pure simplicial cell complex K are called the *facets* of K. If K is pure of dimension d and α is an i-cell, then $\mathrm{lk}_K(\alpha)$ is obviously $(d-i-1)$-dimensional and pure. The 0-cells in a simplicial cell complex K are said to be the *vertices* of K. If u is a face of α and u is a vertex then we say u is a *vertex of α*. Clearly, a d-dimensional simplicial cell complex $\mathcal{X}$ has at least $d+1$ vertices. If a d-dimensional simplicial cell complex $\mathcal{X}$ has exactly $d+1$ vertices then $\mathcal{X}$ is called *contracted*.

Let $\mathcal{X}$ be a pure d-dimensional simplicial cell complex. The *dual graph* of $\mathcal{X}$ is the graph $\Lambda(\mathcal{X})$ whose vertices are the facets of $\mathcal{X}$ and edges are the ordered pairs $(\{\sigma_1, \sigma_2\}, \tau)$, where σ_1, σ_2 are facets and τ is a $(d-1)$-cell which is a common face of σ_1, σ_2. Observe that $\Lambda(\mathcal{X})$ is in general a multigraph without loops. On the other hand, for $d \geq 1$, if (Γ, γ) is a $(d+1)$-colored graph with color set $\Delta_d = \{0, \ldots, d\}$, then we define a d-dimensional simplicial cell complex $\mathcal{K}(\Gamma)$ as follows: for each $v \in V(\Gamma)$, we take a d-simplex σ_v and label its vertices by $0, \ldots, d$; if $u, v \in V(\Gamma)$ are joined by an edge e and $\gamma(e) = i$, then we identify the $(d-1)$-faces of σ_u and σ_v opposite to the vertices labelled by i, so that equally labelled vertices are identified together. Since there is no identification within a d-simplex, a simplicial CW-complex W of dimension d is obtained. So, the face poset (denoted by $\mathcal{K}(\Gamma)$) of W is a pure d-dimensional simplicial cell complex. We say that (Γ, γ) *represents* the simplicial cell complex $\mathcal{K}(\Gamma)$. It is easy to check that the number of i-labelled vertices of $\mathcal{K}(\Gamma)$ is equal to the number of components of $\Gamma_{\Delta_d \setminus \{i\}}$ for each $i \in \Delta_d$; hence, the simplicial cell complex $\mathcal{K}(\Gamma)$ is contracted if and only if Γ is contracted (see [19]).

A *crystallization* of a (closed connected) PL d-manifold M is a $(d+1)$-colored contracted graph (Γ, γ) such that the simplicial cell complex $\mathcal{K}(\Gamma)$ is a pseudo-triangulation of M. Thus,

if (Γ, γ) is a crystallization of a PL d-manifold M, then the number of vertices in $\mathcal{K}(\Gamma)$ is $d+1$. On the other hand, if K is a contracted pseudo-triangulation of M, then the dual graph $\Lambda(K)$ gives a crystallization of M.

The following proposition collects some classical results of crystallization theory, which will be useful in the present paper. For details, see [19], together with its references.

Proposition 4. *Let (Γ, γ) be a crystallization of a PL d-manifold M, with $d \geq 3$. Then:*

(a) *M is orientable if and only if Γ is bipartite.*

(b) *For each $B \subseteq \Delta_d$ with h elements, there is a bijection between $(d-h)$-simplices of $\mathcal{K}(\Gamma)$ whose vertices are labelled by $\Delta_d - \{B\}$ and connected components of Γ_B.*

(c) *For any distinct $r, s \in \Delta_d$ (resp. $i, j, k \in \Delta_d$), let g_{rs} (resp. g_{ijk}) denote the number of connected components of $\Gamma_{\{r,s\}}$ (resp. $\Gamma_{\{i,j,k\}}$). Then, $2g_{ijk} = g_{ij} + g_{ik} + g_{jk} - \frac{\#V(\Gamma)}{2}$ for any distinct $i, j, k \in \Delta_d$.*

(d) *For any distinct $i, j \in \Delta_d$, the set of connected components of $\Gamma_{\Delta_d \setminus \{i,j\}}$, but one, is in bijection with a set of generators of the fundamental group $\pi_1(M)$.*

(e) *If (Γ', γ') is a crystallization of a PL d-manifold M', then the graph connected sum $(\Gamma \# \Gamma', \gamma \# \gamma')$ is a crystallization of $M \# M'$.*

2.3   The regular genus of PL d-manifolds

As already briefly recalled in Section 1, the notion of *regular genus* is strictly related to the existence of *regular embeddings* of crystallizations into closed surfaces, i.e. embeddings whose regions are bounded by the images of bi-colored cycles, with colors consecutive in a fixed permutation of the color set.

More precisely, according to [20], if (Γ, γ) is a crystallization of an orientable (resp. non-orientable) PL d-manifold M ($d \geq 3$), for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$ of Δ_d, a regular embedding $i_\varepsilon : \Gamma \hookrightarrow F_\varepsilon$ exists, where F_ε is the closed orientable (resp. non-orientable) surface with Euler characteristic

$$\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} g_{\varepsilon_i \varepsilon_{i+1}} + (1-d)\, \frac{\#V(\Gamma)}{2}.$$

In the orientable (resp. non-orientable) case, the integer

$$\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2$$

is equal to the genus (resp. half of the genus) of the surface F_ε.

Then, by Definition 1, the regular genus $\rho(\Gamma)$ of (Γ, γ) and the regular genus $\mathcal{G}(M)$ of M are:

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon \text{ is a cyclic permutations of } \Delta_d\};$$

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ is a crystallization of } M\}.$$

Note that $\mathcal{G}(M) \geq rk(\pi_1(M))$ is known to hold, for any PL d-manifold ($d \geq 3$). In the 4-dimensional settings, the following results about the "gap" between the regular genus and the rank of the fundamental group of a PL 4-manifold have been obtained (see [15, 7]).

Proposition 5. *Let M be a PL 4-manifold. Then:*

(a) *If $\mathcal{G}(M) = rk(\pi_1(M)) = \rho$, then M is PL-homeomorphic to $\#\rho(\mathbb{S}^1 \otimes \mathbb{S}^3)$, where $\mathbb{S}^1 \otimes \mathbb{S}^3$ denotes either the orientable or non-orientable $\mathbb{S}^3$-bundle over $\mathbb{S}^1$, according to the orientability of M.*

(b) *No PL 4-manifold M exists with $\mathcal{G}(M) = rk(\pi_1(M)) + 1$.*

(c) *If $\mathcal{G}(M) = rk(\pi_1(M)) + 2$ and $\pi_1(M) = *m\mathbb{Z}$, then M is PL-homeomorphic to $\mathbb{CP}^2 \#_m(\mathbb{S}^1 \otimes \mathbb{S}^3)$.*

Moreover, if (Γ, γ) is a crystallization of a 4-manifold and $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_4 = 4)$ is a cyclic permutation of the color set Δ_4, then [15, relations (1_j) and (2_j)] yields:

$$g_{(i-1)(i+1)} = g_{(i-1)(i)(i+1)} + \rho - \rho_{\hat{i}} \qquad \text{and} \qquad g_{(i-1)(i+1)(i+2)} = 1 + \rho - \rho_{\hat{i}} - \rho_{i\hat{+}3},$$

where ρ denotes $\rho_\varepsilon(\Gamma)$ and, for any $i \in \mathbb{Z}_5$, $\rho_{\hat{i}}$ denotes $\rho_\varepsilon(\Gamma_{\Delta_4 \backslash \{i\}})$.

3 Lower bounds for regular genus and gem-complexity in dimension 4

Let us now prove the general result (Theorem 1, already stated in Section 1) yielding lower bounds for gem-complexity and regular genus of any PL 4-manifold. In Section 4 it will be of fundamental importance in order to analyze the properties of PL 4-manifolds admitting semi-simple crystallizations, while in Section 5 it will enable to obtain new estimations for both invariants in the case of product 4-manifolds. On the other hand we believe that, thanks to its generality, it could be useful to investigate PL 4-manifolds also in wider contexts.

Proof of Theorem 1. Let (Γ, γ) be a crystallization of M. If $2p = \#V(\Gamma)$, then $X = \mathcal{K}(\Gamma)$ is an $2p$-facet contracted pseudo-triangulation of the PL 4-manifold M. The Dehn-Sommerville equations in dimension four yield:

$$
\begin{array}{rcrcrcrcrcl}
f_0(X) & - & f_1(X) & + & f_2(X) & - & f_3(X) & + & f_4(X) & = & \chi(M), \\
& & 2f_1(X) & - & 3f_2(X) & + & 4f_3(X) & - & 5f_4(X) & = & 0, \\
& & & & & & 2f_3(X) & - & 5f_4(X) & = & 0.
\end{array}
$$

Since $f_0(X) = 5$ by construction and $f_4(X) = \#V(\Gamma) = 2p$, the following equality holds:

$$2p = 6\chi(M) + 2f_1(\mathcal{K}(\Gamma)) - 30. \tag{1}$$

Since $rk(\pi_1(M)) = m$, Proposition 4(d) implies $g_{ijk} \geq m + 1$ for any distinct $i, j, k \in \Delta_4$. Therefore, $f_1(\mathcal{K}(\Gamma)) = \sum_{0 \leq i < j < k \leq 4} g_{ijk} \geq 10(m + 1)$. Hence, by equation (1):

$$2p \geq 6\chi(M) + 20(m+1) - 30 \;=\; 6\chi(M) + 20m - 10. \tag{2}$$

The first inequality now follows from equation (2) and Definition 2.

Let us now prove the second inequality. From equation (2), we have that $2\bar{p} = 6\chi(M) + 10(2m-1)$ is the minimal possible order of a crystallization of M.

Let (Γ, γ) be a crystallization of M. Then, $\#V(\Gamma) = 2\bar{p} + 2q$ for some non-negative integer q. This implies $6\chi(M) + 2f_1(\mathcal{K}(\Gamma)) - 30 = 6\chi(M) + 10(2m-1) + 2q$. Thus, $2\sum_{0 \leq i < j < k \leq 4} g_{ijk} - 30 = 10(2m-1) + 2q$. Again, $g_{ijk} \geq m+1$ for any distinct $i, j, k \in \Delta_4$. So, let us assume $g_{ijk} = (m+1) + t_{ijk}$ where $t_{ijk} \in \mathbb{Z}$, $t_{ijk} \geq 0$. Thus, $20(m+1) + 2\sum_{0 \leq i < j < k \leq 4} t_{ijk} - 30 = 20m - 10 + 2q$ and hence $q = \sum_{0 \leq i < j < k \leq 4} t_{ijk}$. Now, for any cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 = 4)$ of the colors we have $\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_5} g_{\varepsilon_i \varepsilon_{i+1}} - 3(\bar{p} + q)$.

On the other hand, by Proposition 4(c), we know that $2g_{ijk} = g_{ij} + g_{ik} + g_{jk} - \frac{\#V(\Gamma)}{2}$ for any distinct $i, j, k \in \Delta_4$. Thus, we have $g_{ij} + g_{ik} + g_{jk} = 2g_{ijk} + (\bar{p} + q)$ for $0 \leq i < j < k \leq 4$. This gives ten linear equations which can be written in the following form.

$$AX = B,$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} g_{01} \\ g_{02} \\ g_{03} \\ g_{04} \\ g_{12} \\ g_{13} \\ g_{14} \\ g_{23} \\ g_{24} \\ g_{34} \end{bmatrix} \text{ and } B = \begin{bmatrix} 2g_{012} + \bar{p} + q \\ 2g_{013} + \bar{p} + q \\ 2g_{014} + \bar{p} + q \\ 2g_{023} + \bar{p} + q \\ 2g_{024} + \bar{p} + q \\ 2g_{034} + \bar{p} + q \\ 2g_{123} + \bar{p} + q \\ 2g_{124} + \bar{p} + q \\ 2g_{134} + \bar{p} + q \\ 2g_{234} + \bar{p} + q \end{bmatrix}.$$

Therefore,

$$X = A^{-1}B,$$

where

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1/3 \\ 1/3 & -1/6 & -1/6 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & 1/3 & -1/6 \\ -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & -1/6 \\ -1/6 & -1/6 & 1/3 & -1/6 & 1/3 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 \\ 1/3 & -1/6 & -1/6 & -1/6 & -1/6 & 1/3 & 1/3 & 1/3 & -1/6 & -1/6 \\ -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 & 1/3 & -1/6 \\ -1/6 & -1/6 & 1/3 & 1/3 & -1/6 & -1/6 & -1/6 & 1/3 & 1/3 & -1/6 \\ -1/6 & -1/6 & 1/3 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & -1/6 & 1/3 \\ -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & -1/6 & 1/3 \\ 1/3 & -1/6 & -1/6 & -1/6 & -1/6 & 1/3 & -1/6 & -1/6 & 1/3 & 1/3 \end{bmatrix}.$$

Since $g_{ijk} = (m+1) + t_{ijk}$ for any distinct $i, j, k \in \Delta_4$, we have

$$B = M + \sum_{0 \le l < j < k \le 4} T_{ijk},$$

where

$$M = \begin{bmatrix} 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \\ 2(m+1) + \bar{p} + q \end{bmatrix} \text{ and } T_{ijk} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2t_{ijk} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$X = A^{-1}M + \sum_{0 \le l < j < k \le 4} A^{-1}T_{ijk}.$$

Therefore, $g_{\varepsilon_i \varepsilon_{i+1}} = \frac{2(m+1) + \bar{p} + q}{3} + 2 \sum_{0 \le l < j < k \le 4} c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} t_{ljk}$, where c_{ljk}^{rs} is the element of A^{-1} corresponding to $\{r, s\}$-row and $\{l, j, k\}$-column of A^{-1}. Now observe that, for any fixed $\{l, j, k\}$-column of A^{-1}, there exist r, s, v such that $c_{ljk}^{rs} = c_{ljk}^{rv} = c_{ljk}^{sv} = 1/3$. Thus, for any cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 = 4)$, at most three elements from the set $\{c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} \mid i \in \mathbb{Z}_5\}$ are $1/3$ and remaining elements are $-1/6$. Therefore, $\sum_{i \in \mathbb{Z}_5} c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} \le 1/3 + 1/3 + 1/3 - 1/6 - 1/6 = 2/3$. Thus,

$$
\begin{aligned}
\chi_\varepsilon(\Gamma) &= \sum_{i \in \mathbb{Z}_5} g_{\varepsilon_i \varepsilon_{i+1}} - 3(\bar{p} + q) \\
&= \sum_{i \in \mathbb{Z}_5} \left(\frac{2(m+1) + \bar{p} + q}{3} + 2 \sum_{0 \le l < j < k \le 4} c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} t_{ljk} \right) - 3(\bar{p} + q) \\
&= 5 \frac{2(m+1) + \bar{p} + q}{3} + 2 \sum_{0 \le l < j < k \le 4} t_{ljk} \sum_{i \in \mathbb{Z}_5} c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} - 3(\bar{p} + q) \\
&\le 5 \frac{2(m+1) + \bar{p} + q}{3} + \frac{4}{3} \sum_{0 \le l < j < k \le 4} t_{ljk} - 3(\bar{p} + q) \\
&= 5 \frac{2(m+1) + \bar{p} + q}{3} + \frac{4}{3} q - 3(\bar{p} + q) \\
&= \frac{10(m+1) - 4\bar{p}}{3}.
\end{aligned}
$$

Therefore, $\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2 \geq 1 - \frac{5(m+1)-2\bar{p}}{3} = \frac{2(\bar{p}-1)-5m}{3}$. Since this is true for any permutation ε, we have:

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \,|\, \varepsilon \text{ is a permutation of } \Delta_4\} \geq \frac{2(\bar{p}-1)-5m}{3}.$$

Since the crystallization (Γ, γ) is arbitrary:

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \,|\, \Gamma \text{ is a crystallization of } M\} \geq \frac{2(\bar{p}-1)-5m}{3}.$$

Finally, since $2\bar{p} = 6\chi(M) + 10(2m-1)$ we have:

$$\mathcal{G}(M) \geq \frac{(6\chi(M)+20m-12)-5m}{3} = 2\chi(M) + 5m - 4.$$

$\square$

Remark 1. As a consequence of the inequality involving regular genus in Theorem 1, we have

$$\chi(M) \leq 2 + \frac{\mathcal{G}(M)}{2} - \frac{5m}{2},$$

which improves the inequality $\chi(M) \leq 2 + \mathcal{G}(M)/2$ in [21, Corollary 6.5]. On the other hand, additivity of regular genus is proved in [21, Corollary 6.8 (b)] for the class of PL 4-manifolds characterized by $\mathcal{G}(M) = 2\chi(M) - 4$. Now, in virtue of Theorem 1, the above class of PL 4-manifolds turns out to consist of simply-connected PL 4-manifolds.

Remark 2. According to [13], (simply-connected) PL 4-manifolds admitting simple crystallizations are characterized by $k(M) = 3\beta_2(M)$; moreover, equality $\mathcal{G}(M) = 2\beta_2(M)$ holds for any PL 4-manifold admitting simple crystallizations. Hence, they constitute a class of (simply-connected) PL 4-manifolds which attain both the bounds of Theorem 1.

4 PL 4-manifolds admitting semi-simple crystallizations

In Section 1, the notion of *semi-simple crystallization* has been introduced, in terms of the 1-skeleton of the associated colored triangulation. By Proposition 4(b), it is easy to check that Definition 3 may be re-stated as follows, in terms of the number of components of the subgraph restricted to any triple of colors.

Definition 4. A crystallization (Γ, γ) of a PL 4-manifold M is called a *semi-simple crystallization of type m* if $g_{ijk} = m + 1$ for any distinct $i, j, k \in \Delta_4$, where m is the rank of the fundamental group of M.

The following definition is quite natural.

Definition 5. If (Γ, γ) is a semi-simple crystallization of type m of a PL 4-manifold M, then we will say that *M admits semi-simple crystallizations of type m* (or simply, that *M admits semi-simple crystallizations*).

Remark 3. The first part of the proof of Theorem 1 (in particular, equation (1)) immediately implies that (Γ, γ) is a semi-simple crystallization of type m (that is, $f_1(\mathcal{K}(\Gamma)) = 10(m+1)$) if and only if $\#V(\Gamma) = 6\chi(M) + 20m - 10$. Hence, *PL 4-manifolds admitting semi-simple crystallizations are characterized by $k(M) = 3\chi(M) + 10m - 6$, where m is the rank of the fundamental group of M.*

Proposition 6. *Let M and M' be two PL 4-manifolds admitting a semi-simple crystallization. Then, $M\#M'$ admits a semi-simple crystallization, too.*

Proof. Let (Γ, γ) (resp. (Γ', γ')) be a semi-simple crystallization of M (resp. M'), with $g_{ijk} = m + 1$ (resp. $g'_{ijk} = m' + 1$ for any distinct $i, j, k \in \Delta_4$. From the definition of graph connected sum (see Subsection 2.1), then the crystallization $(\bar\Gamma, \bar\gamma) = (\Gamma\#\Gamma', \gamma\#\gamma')$ of $M\#M'$ has $\bar g_{ijk} = m + m' + 1$ for all distinct $i, j, k \in \Delta_4$. The thesis now easily follows. $\square$

Proposition 7. *Let M be a PL 4-manifold with gem-complexity less than nine. Then, M admits semi-simple crystallizations.*

Proof. If M is a prime simply-connected PL 4-manifold then, by [12, Proposition 15], M is PL-homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$. Thus, M admits simple crystallizations (see [3]), i.e. it admits semi-simple crystallizations of type 0. If M is a prime and not simply-connected PL 4-manifold then, by [12, Proposition 15], M is PL-homeomorphic to either the orientable $\mathbb{S}^3$-bundle $\mathbb{S}^1 \times \mathbb{S}^3$, or the non-orientable $\mathbb{S}^3$-bundle $\mathbb{S}^1 \tilde\times \mathbb{S}^3$, or the 4-dimensional real projective space $\mathbb{RP}^4$. It is easy to check that the (well-known) crystallizations of $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \tilde\times \mathbb{S}^3$ depicted in Figure 1 are semi-simple crystallizations, as well as the unique (see [12]) order 14 crystallization of $\mathbb{RP}^4$ depicted in Figure 2. Now the result follows from the additivity of semi-simple crystallizations (Proposition 6). $\square$

Remark 4. Note that (semi-)simple crystallizations of all simply-connected PL 4-manifolds of "standard type" are known (see [3], where simple crystallizations of $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and the $K3$-surface are presented); moreover, Proposition 7 yields semi-simple crystallizations of the non-simply-connected PL 4-manifolds $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \tilde\times \mathbb{S}^3$ and $\mathbb{RP}^4$. Thus, additivity of semi-simple crystallizations (Proposition 6) gives a huge class of PL 4-manifolds which admit semi-simple crystallizations.

The following Proposition proves that semi-simple crystallizations are "minimal" both with respect to the invariant gem-complexity and with respect to the invariant regular genus. Moreover, a lot of details about their combinatorial structure are obtained.

Proposition 8. *Let (Γ, γ) be a semi-simple crystallization of type m. If M denotes the PL 4-manifold (with $rk(\pi_1(M^4)) = m$) represented by Γ, then:*

$$\begin{aligned} k(M) &= 3\chi(M) + 10m - 6; \\ \mathcal{G}(M) &= 2\chi(M) + 5m - 4; \\ k(M) &= \frac{3\mathcal{G}(M) + 5m}{2}. \end{aligned}$$
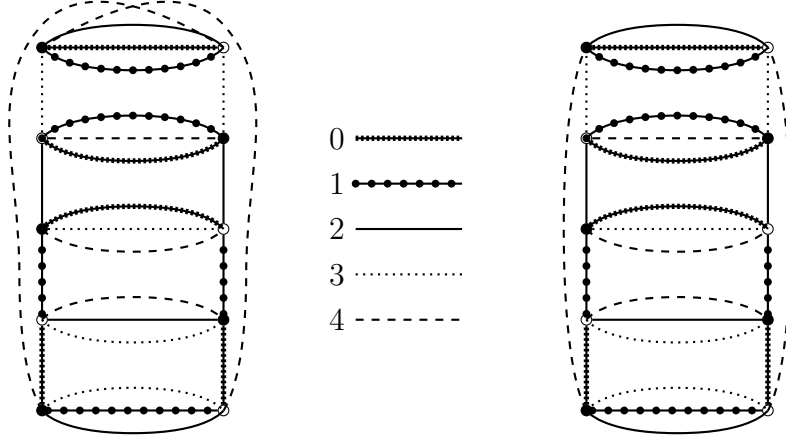
Moreover:

Figure 1: Semi-simple crystallizations of $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$

(i) $\rho_\varepsilon(\Gamma) = \mathcal{G}(M) = 2\chi(M) + 5m - 4$ *for any cyclic permutation ε of Δ_4,*

(ii) $\#V(\Gamma) = 2(k(M) + 1) = 6\chi(M) + 20m - 10,$

(iii) $g_{ij} = \chi(M) + 4m - 1$ *for any pair $i, j \in \Delta_4$,*

(iv) $\rho_\varepsilon(\Gamma_{\Delta_4 \setminus \{i\}}) = \frac{\mathcal{G}(M) - m}{2} = \chi(M) + 2m - 2$ *for any cyclic permutation ε of Δ_4 and for any color $i \in \Delta_4$.*

Proof. Let (Γ, γ) be a crystallization of M, with $\#V(\Gamma) = 2p$. From the proof of Theorem 1, we have $2p = 6\chi(M) + 2f_1(\mathcal{K}(\Gamma)) - 30$. Thus, if (Γ, γ) is semi-simple, $f_1(\mathcal{K}(\Gamma)) = m + 1$ and hence $2p = 6\chi(M) + 10(2m - 1)$. This proves $k(M) = \frac{\#V(\Gamma)}{2} - 1 = 3\chi(M) + 10m - 6$.

On the other hand, from Theorem 1, we get

$$\mathcal{G}(M) \geq 2\chi(M) + 5m - 4 = \frac{(\#V(\Gamma) - 2) - 5m}{3} = \frac{2k(M) - 5m}{3}.$$

Now, let $\bar{p}, q, t_{ijk}$ be as in the proof of Theorem 1. In this case $q = 0$, $t_{ijk} = 0$ for any i, j, k and $\#V(\Gamma) = 2\bar{p}$; hence $g_{\varepsilon_i \varepsilon_{i+1}} = \frac{2(m+1)+\bar{p}}{3}$ for any cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_4 = 4)$. Therefore, $\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_5} g_{\varepsilon_i \varepsilon_{i+1}} - 3\bar{p} = \frac{10(m+1)-4\bar{p}}{3}$. This implies $\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2 = \frac{2(\bar{p}-1)-5m}{3} = \frac{2k(M)-5m}{3}$. Therefore, $\mathcal{G}(M) = \frac{2k(M)-5m}{3}$. This proves both relation $\mathcal{G}(M) = 2\chi(M) + 5m - 4$ and relation $k(M) = \frac{3\mathcal{G}(M)+5m}{2}$.

As pointed out in the proof of Theorem 1,

$$g_{\varepsilon_i \varepsilon_{i+1}} = \frac{2(m+1) + \bar{p} + q}{3} + 2 \sum_{0 \leq l < j < k \leq 4} c_{ljk}^{\varepsilon_i \varepsilon_{i+1}} t_{ljk}$$

where c_{ljk}^{rs} is the element of A^{-1} corresponding to $\{r, s\}$-row and $\{l, j, k\}$-column of A^{-1}. Since in this case both q and all t_{ljk}'s are zero, we have $g_{\varepsilon_i \varepsilon_{i+1}} = \frac{2(m+1)+\bar{p}}{3}$. Since the same argument holds for any cyclic permutation ε of Δ_4, relation $g_{ij} = \chi(M) + 4m - 1$ is proved to be true for any pair $i, j \in \Delta_4$.
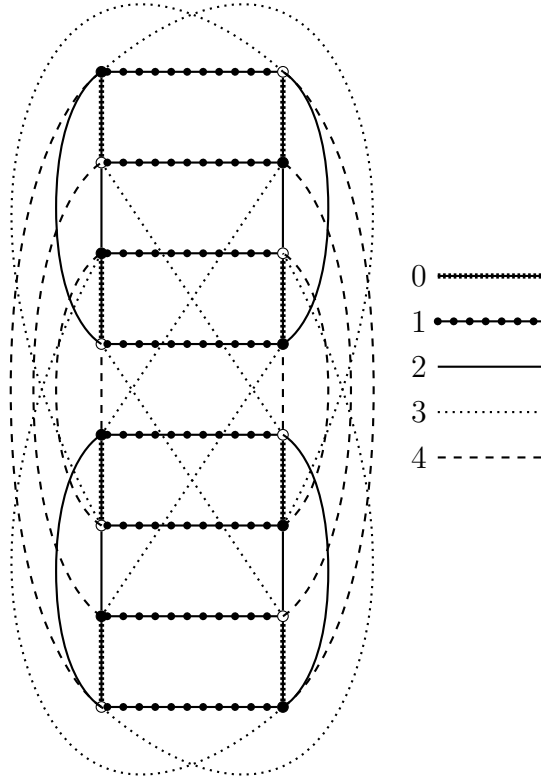
12

Figure 2: A semi-simple crystallization of $\mathbb{RP}^4$

Again, from Subsection 2.3 we know that, if ρ denotes $\rho_\varepsilon(\Gamma)$ and $\rho_{\hat{i}}$ denotes $\rho_\varepsilon(\Gamma_{\Delta_4\setminus\{i\}})$ then $g_{(i-1)(i+1)} = g_{(i-1)(i)(i+1)} + \rho - \rho_{\hat{i}}$ and $g_{(i-1)(i+1)(i+2)} = 1 + \rho - \rho_{\hat{i}} - \rho_{i\hat{+}3}$ for any $i \in \mathbb{Z}_5$. In the case of a semi-simple crystallization of type m, $\rho - \rho_{\hat{i}} - \rho_{\hat{j}} = m$ for any pair $i, j \in \Delta_4$. As a consequence, $\rho_{\hat{i}} = \frac{\rho - m}{2}$ holds for any $i \in \Delta_4$; the proof is completed, since $\rho_{\hat{i}} = \frac{(2\chi(M)+5m-4)-m}{2} = \chi(M) + 2m - 2$ directly follows. $\qquad\square$

Theorem 2 is now a direct consequence of Proposition 8.

Proof of Theorem 2. It is sufficient to consider an arbitrary semi-simple crystallization of M, and to apply Proposition 8. $\qquad\square$

Remark 5. If M admits a simple crystallization then, by Theorem 2:

$$k(M) = 3\chi(M) - 6 = 3(2 + \beta_2(M)) - 6 = 3\beta_2(M)$$

and

$$\mathcal{G}(M) = 2\chi(M) - 4 = 2(2 + \beta_2(M)) - 4 = 2\beta_2(M),$$

13

as already proved in [13, Theorem 1].

Corollary 9. *Let M be an orientable PL 4-manifold with $\pi_1(M) = *m\mathbb{Z}$. If M admits semi-simple crystallizations, then:*

$$k(M) = 3\beta_2(M) + 4m,$$
$$\mathcal{G}(M) = 2\beta_2(M) + m.$$

Moreover, for any semi-simple crystallization Γ of M:

(i) $\rho_\varepsilon(\Gamma) = 2\beta_2(M) + m$ *for any cyclic permutation ε of Δ_4;*

(ii) $\#V(\Gamma) = 2(k(M) + 1) = 6\beta_2(M) + 8m + 2$;

(iii) $g_{ij} = \beta_2(M) + 2m + 1$ *for any pair $i, j \in \Delta_4$;*

(iv) $\rho_\varepsilon(\Gamma_{\Delta_4 \setminus \{i\}}) = \beta_2(M)$ *for any cyclic permutation ε of Δ_4 and for any color $i \in \Delta_4$.*

Proof. Since M is assumed to be orientable with free fundamental group of rank m, $\chi(M) = 2 - 2m + \beta_2(M)$ holds. Hence, all statements follow from the analogue ones in Proposition 8. □

The following proposition gives a generalization of Proposition 5(b), within the class of PL 4-manifolds admitting semi-simple crystallizations.

Proposition 10. *No PL 4-manifold M with odd difference $\mathcal{G}(M) - rk(\pi_1(M))$ admits semi-simple crystallizations. In particular, no simply-connected PL 4-manifold M with odd regular genus admits simple crystallizations.*

Proof. It is sufficient to recall that, by Proposition 8, $\rho_\varepsilon(\Gamma_{\Delta_4 \setminus \{i\}}) = \frac{\mathcal{G}(M) - m}{2}$ holds for any semi-simple crystallization (Γ, γ) of M, for any cyclic permutation ε of Δ_4 and for any color $i \in \Delta_4$. □

Under the assumption of free fundamental group, we have also the following result about the PL classification of orientable PL 4-manifolds admitting semi-simple crystallizations.

Corollary 11. *Let M be an orientable PL 4-manifold with $\pi_1(M) = *m\mathbb{Z}$ and $\beta_2 = 1$. If M admits semi-simple crystallizations, then M is PL-homeomorphic to $\mathbb{CP}^2 \#_m (\mathbb{S}^1 \times \mathbb{S}^3)$.*

Proof. In virtue of Corollary 9, together with the assumption that $\beta_2(M) = 1$, we have $\mathcal{G}(M) - rk(\pi_1(M)) = 2$. Hence, the corollary directly follows from Proposition 5(c). □

Remark 6. As a consequence of the relation between regular genus and gem-complexity for PL 4-manifolds admitting semi-simple crystallizations, it is possible to yield new results about the PL classification via regular genus, within that class of PL 4-manifolds. For example, if M admits semi-simple crystallizations, and $\mathcal{G}(M) = 3$, $m = 1$ hold (resp. $\mathcal{G}(M) = 4$, $m = 0$ hold), then M turns out to be PL-homeomorphic to either $\mathbb{CP}^2 \# (\mathbb{S}^1 \otimes \mathbb{S}^3)$ or $\mathbb{RP}^4$ (resp. to either $\mathbb{CP}^2 \# \mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$). In fact, by Theorem 2, $k(M) = \frac{3\mathcal{G}(M) + 5m}{2}$ holds for any PL 4-manifold admitting semi-simple crystallizations; so, the assumption $\mathcal{G}(M) = 3$, $m = 1$ (resp. $\mathcal{G}(M) = 4$, $m = 0$) implies $k(M) = 7$ (resp. $k(M) = 6$). Hence, the PL-classification of the involved PL 4-manifolds follows from [12, Proposition 15].

We conclude the paragraph yielding from Theorem 2 the additivity of both the invariants regular genus and gem-complexity under connected sum, within the class of PL 4-manifolds admitting a semi-simple crystallization.

Proof of Theorem 3. For $1 \leq i \leq 2$, let (Γ_i, γ_i) be a semi-simple crystallization of the PL 4-manifold M_i with $rk(\pi_1(M_i)) = m_i$. Then, by additivity of semi-simple crystallizations, $(\Gamma_1 \# \Gamma_2, \gamma_1 \# \gamma_2)$ is also a semi-simple crystallization of $M_1 \# M_2$. Since $rk(\pi_1(M_1 \# M_2)) = rk(\pi_1(M_1)) + rk(\pi_1(M_2))$ and $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$, Theorem 3 now follows from Theorem 2. $\square$

Remark 7. In [21, Corollary 4], two classes of closed (not necessarily orientable) 4-manifolds have been detected, for which additivity of regular genus holds. It has been already pointed out (see [13]) that the first one (characterized by relation $\mathcal{G}(M) = 1 - \frac{\chi(M)}{2}$) consists of connected sums of $\mathbb{S}^3$-bundles over $\mathbb{S}^1$, while the second one (characterized by relation $\mathcal{G}(M) = 2\chi(M) - 4$, and consisting of simply-connected PL 4-manifolds, as pointed out in Remark 1) includes all PL 4-manifolds admitting simple crystallizations, i.e. semi-simple crystallizations of type zero. Hence, Theorem 3 strictly enlarges the set of PL 4-manifolds for which additivity of regular genus is known to hold.

5 Some consequences about regular genus and gem-complexity of product 4-manifolds

Theorem 1 enables to significantly improve some lower bounds for the regular genus of PL 4-manifolds, which have been proved by various authors via different techniques. Meanwhile, similar lower bounds are obtained also for gem-complexity.

Proposition 12. *For any 3-manifold M such that $\pi_1(M)$ is a finitely generated abelian group, we have:*

$$\mathcal{G}(M \times \mathbb{S}^1) \geq 5rk(\pi_1(M)) + 1 \quad and \quad k(M \times \mathbb{S}^1) \geq 10rk(\pi_1(M)) - 6.$$

In particular,
$$\mathcal{G}(L(p,q) \times \mathbb{S}^1) \geq 6 \quad and \quad k(L(p,q) \times \mathbb{S}^1) \geq 4.$$

Proof. It is well-known that $\chi(M) = 0$ for any 3-manifold M and $\chi(P \times Q) = \chi(P) \cdot \chi(Q)$ for any pair P, Q of polyhedra. Moreover, by the fundamental theorem of finitely generated abelian groups, $rk(\pi_1(M \times \mathbb{S}^1)) = rk(\pi_1(M)) + 1$ holds for any 3-manifold M such that $\pi_1(M)$ is a finitely generated abelian group. The statements are now direct consequences of the inequalities proved in Theorem 1. $\square$

Remark 8. The statement $\mathcal{G}(L(p,q) \times \mathbb{S}^1) \geq 6$ already appears in [25], by making use of heavy calculations performed in [17]. On the other hand, the genus six crystallization of $L(2,1) \times \mathbb{S}^1$ produced in [25] allows to prove the equality $\mathcal{G}(L(2,1) \times \mathbb{S}^1) = 6$. The inequality $\mathcal{G}(L(p,q) \times \mathbb{S}^1) \leq 6(p-1)$ is also proved in [25].
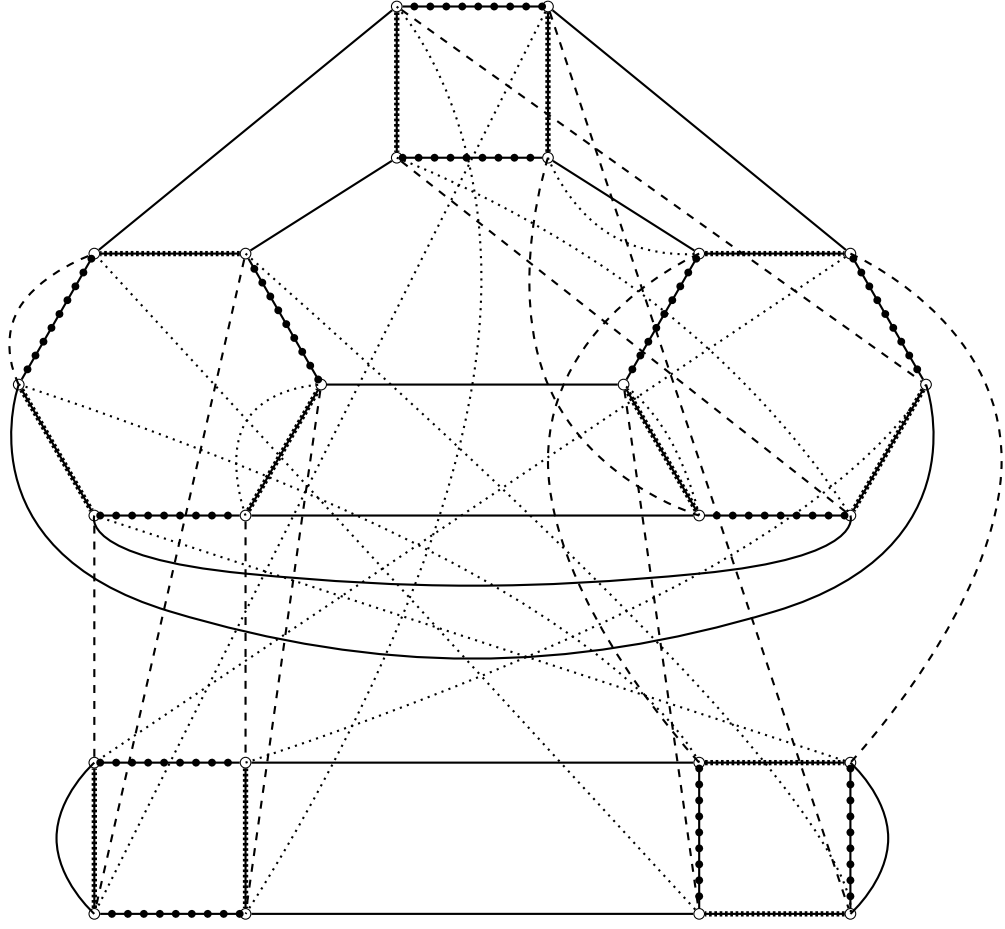
Figure 3: A crystallization of $\mathbb{S}^2 \times \mathbb{RP}^2$ (with genus five and order 24).

Proposition 13. *Let T_g (resp. U_h) denote the orientable (resp. non-orientable) surface of genus $g \geq 0$ (resp. $h \geq 1$). Then,*

$$\mathcal{G}(T_g \times T_r) \geq 8gr + 2g + 2r + 4 \quad and \quad k(T_g \times T_r) \geq 12gr + 8g + 8r + 6,$$

$$\mathcal{G}(T_g \times U_h) \geq 4gh + 2g + h + 4 \quad and \quad k(T_g \times U_h) \geq 6gh + 8g + 4h + 6,$$

$$\mathcal{G}(U_h \times U_k) \geq 2hk + h + k + 4 \quad and \quad k(U_h \times U_k) \geq 3hk + 4h + 4k + 6.$$

In particular,

$$\mathcal{G}(\mathbb{S}^2 \times T_g) \geq 2g + 4 \quad and \quad k(\mathbb{S}^2 \times T_g) \geq 8g + 6,$$

$$\mathcal{G}(\mathbb{S}^2 \times U_h) \geq h + 4 \quad and \quad k(\mathbb{S}^2 \times U_h) \geq 4h + 6.$$

Proof. The following facts are well-known: $\chi(T_g) = 2 - 2g$, $\chi(U_h) = 2 - h$, $rk(\pi_1(T_g)) = 2g$ and $rk(\pi_1(U_h)) = h$. Moreover, $\chi(P \times Q) = \chi(P) \cdot \chi(Q)$ for any pair P, Q of polyhedra. On the other hand, it is not difficult to prove that $rk(\pi_1(P \times Q)) = rk(\pi_1(P)) + rk(\pi_1(Q))$ for any pair P, Q of surfaces: in fact, $rk(G \times H) \leq rk(G) + rk(H)$ holds for any pair G, H of groups, while the equality for the fundamental groups of surfaces is a consequence of the equality regarding the first Betti numbers (with integer coefficients in the orientable case and with $\mathbb{Z}_2$

16

coefficients in the non-orientable case). The statements are now direct consequences of the inequalities proved in Theorem 1. □

Remark 9. The inequalities concerning regular genus in Proposition 13 strictly improve the similar ones obtained in [21, Corollary 6.6].

Finally, for $h = 1$, the last inequality of Proposition 13 concerning regular genus (resp. gem-complexity), together with the existence of the genus five (resp. order 24) crystallization of $\mathbb{S}^2 \times \mathbb{RP}^2$ depicted in Figure 3, allows the exact calculation of the regular genus (resp. an estimation with "strict range" of the gem-complexity) of the involved PL 4-manifold.

Proposition 14.

$$\mathcal{G}(\mathbb{S}^2 \times \mathbb{RP}^2) = 5 \qquad and \qquad k(\mathbb{S}^2 \times \mathbb{RP}^2) \in \{10, 11\}.$$

<div align="right">□</div>

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