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On Liouville-type theorems
and the uniqueness of the positive Cauchy problem
for a class of hypoelliptic operators

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Abstract

The paper contains a representation formula for positive solutions of linear degenerate second-order equations of the form

$$\partial_t u(x,t) = \sum_{j=1}^{m} X_j^2 u(x,t) + X_0 u(x,t) \qquad (x,t) \in \mathbb{R}^N \times \,]-\infty, T[,$$

where the X_j are smooth vector fields satisfying the Hörmander condition. It is assumed that X_j are invariant under left translations of a Lie group and the corresponding paths satisfy a local admissibility criterion.

The representation formula is established by an analytic approach based on Choquet theory. As a consequence we obtain Liouville-type theorems and uniqueness results for the positive Cauchy problem.

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1 Introduction

In this article we consider second-order partial differential operators of the form

$$\mathscr{L}u := \partial_t u - \sum_{j=1}^{m} X_j^2 u - X_0 u \qquad \text{in } \mathbb{R}^{N+1}. \tag{1.1}$$

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Points $z \in \mathbb{R}^{N+1}$ are denoted by $z = (x, t)$, where $x \in \mathbb{R}^N, t \in \mathbb{R}$. For $j = 0, \ldots, m$, the X_j are vector fields which are given by first-order linear partial differential operators in $\mathbb{R}^N$ with smooth coefficients

$$X_j(x) := \sum_{k=1}^{N} b_{jk}(x) \partial_{x_k} \qquad j = 0, \ldots, m.$$

We denote by Y the *drift*

$$Y := X_0 - \partial_t. \tag{1.2}$$

We recall that the class of operators of the form (1.1) has been studied by many authors. In particular, we refer to the monographs [6, 8, 9], and to the references therein.

The aim of the article is to prove a representation formula for nonnegative solutions of $\mathscr{L}u = 0$ in the set

$$\mathbb{R}^N \times \mathbb{R}_T := \mathbb{R}^N \times \,]-\infty, T[, \tag{1.3}$$

where $0 < T \leq +\infty$. In the sequel we use the following notation

$$\mathscr{H} := \left\{ u \in C^\infty(\mathbb{R}^N \times \mathbb{R}_T) \mid \mathscr{L}u = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_T \right\}, \tag{1.4}$$

$$\mathscr{H}_+ := \left\{ u \in \mathscr{H} \mid u \geq 0 \right\}. \tag{1.5}$$

We use a functional analytic approach based on Choquet theory that allows us to represent all functions belonging to the convex cone $\mathscr{H}_+$ in terms of its extremal rays. Moreover, we prove a *separation principle* for the extremal rays. The separation principle, in the nondegenerate case, says that (under certain conditions) nonnegative extremal solutions of the heat equations have the form $u(x, t) = e^{\beta t} u_\beta(x)$, with $\beta \in \mathbb{R}$. In our degenerate setting the separation principle has a different form that depends on $\mathscr{L}$. However, we prove in Theorem 3.4 that, under some additional assumptions, any nonnegative extremal solution of $\partial_t u = \sum_{j=1}^m X_j^2 u$ in $\mathbb{R}^N \times \mathbb{R}_T$, does not depend on the 'degenerate' variables. From the representation theorem it plainly follows that under the additional assumptions also any function in $\mathscr{H}_+$ does not depend on the 'degenerate' variables. A similar result is proved in Theorem 4.1 for degenerate stationary operators $\sum_{j=1}^m X_j^2 u = 0$, and in Corollary 8.2 for Kolmogorov equations. We refer to this kind of results as *Liouville-type theorems* because of the very specific form of any point in $\mathscr{H}_+$.

Let us informally explain this remarkable phenomenon. We assume in theorems 3.4 and 4.1 that $\mathscr{L}$ is invariant with respect to the *left translations* of a nilpotent stratified Lie group. On the other hand, the proof of our separation principle relies on Harnack inequalities that are invariant with respect to the *right translations* of the group. Both these two properties are satisfied in the particular case of the last layer of the nilpotent Lie group. In this case, we can prove our separation principle, that yields our claim. Let us also note that this fact is not completely unexpected. Indeed, Danielli, Garofalo

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and Petrosyan consider in [15] the subelliptic obstacle problem in Carnot groups of step two, and prove that the non-horizontal derivatives of any solution vanish continuously on the free boundary. For an extensive treatment on sub-Laplacians on Carnot groups we refer to the book [6] by Bonfiglioli, Lanconelli and Uguzzoni.

We also give a simple proof of a known uniqueness result for the positive Cauchy problem. We note that this integral representation theory approach was previously used to prove the uniqueness of the positive Cauchy problem and Liouville-type theorems for locally *uniformly* parabolic and elliptic operators [29, 33, 38, 39, 40, 41, and references therein].

We next focus on Mumford and degenerate Kolmogorov operators. Their drift term X_0 is nontrivial, and plays a crucial role in the regularity properties of the solutions. In Section 7 we prove a uniqueness result for the positive Cauchy problem for Mumford operators. In Section 8 we consider a family of degenerate Kolmogorov operators, and prove in Corollary 8.2 that any nonnegative solution of this partial differential equation in $\mathbb{R}^N \times \mathbb{R}_T$ does not depend on the 'degenerate' variables, and hence, the uniqueness of the positive Cauchy problem holds true for such operators.

We list below our assumptions on $\mathscr{L}$ that will be used to accomplish this project. We assume that $\mathscr{L}$ satisfies the celebrated Hörmander condition:

(H0) $\operatorname{rank} \operatorname{Lie}\{X_1, \ldots, X_m, Y\}(z) = N + 1$ for every $z \in \mathbb{R}^{N+1}$.

Under this condition Hörmander proved in [20] that $\mathscr{L}$ is hypoelliptic, that is, any distributional solution u of the equation $\mathscr{L}u = f$ is a smooth classical solution, whenever f is smooth. In particular, $\mathscr{H}$ contains *all* distributional solutions of the equation $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$.

Our second hypothesis is as follows:

(H1) there exists a Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ such that the vector fields $X_1, \ldots, X_m, Y$ are invariant with respect to the left translation of $\mathbb{G}$. That is, for every $z, \zeta \in \mathbb{R}^{N+1}$ we have

$$(X_j u)\,(\zeta \circ z) = X_j\,(u(\zeta \circ z)) \qquad j = 1, \ldots, m, \text{ and}$$
$$(Y u)\,(\zeta \circ z) = Y\,(u(\zeta \circ z)).$$

In particular, it follows from (H1) that

$$\left(\mathscr{L}u\right)(z) = f(z) \quad \Leftrightarrow \quad \mathscr{L}\big(u(\zeta \circ z)\big) = f(\zeta \circ z) \qquad \forall \zeta \in \mathbb{R}^{N+1}. \tag{1.6}$$

We will use the following notation in our further assumptions. As usual, we identify the first order linear partial differential operator X_j with the vector-valued function

$$X_j(x) = (b_{j1}(x), \ldots, b_{jN}(x)) \qquad j = 1, \ldots, m.$$

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For any $z_0 \in \mathbb{R}^{N+1}$ and any piecewise constant function $\omega : [0, T_0] \to \mathbb{R}^m$, let γ be a solution of the following initial value problem

$$\gamma'(s) = \sum_{j=1}^m \omega_j(s) X_j(\gamma(s)) + Y(\gamma(s)), \qquad \gamma(0) = z_0. \tag{1.7}$$

We say that the solution γ to (1.7) is an $\mathscr{L}$-*admissible path.*

Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set and let $z_0 \in \Omega$. The *attainable set*

$$\mathscr{A}_{z_0}(\Omega) := \overline{A_{z_0}(\Omega)} \tag{1.8}$$

is defined as the closure in Ω of

$$A_{z_0}(\Omega) := \big\{ z \in \Omega \mid \exists\, \mathscr{L}\text{-admissible path } \gamma : [0, \tau] \to \Omega \text{ s.t. } \gamma(0) = z_0, \gamma(\tau) = z \big\}.$$

When $\Omega = \mathbb{R}^N \times \mathbb{R}_T$ (see (1.3)), we use the simplified notation $\mathscr{A}_{z_0} := \mathscr{A}_{z_0}(\mathbb{R}^N \times \mathbb{R}_T)$.

Our last requirement is concerned with a $\mathscr{L}$-admissible path with a constant $\omega \in \mathbb{R}^m$. As we will see in the sequel, it yields a *restricted uniform Harnack inequality* suitably modeled on the Lie group structure of $\mathbb{G}$ (cf. [38]). For $X = (X_1, \ldots, X_m)$, and $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m$, we denote

$$\omega \cdot X := \omega_1 X_1 + \cdots + \omega_m X_m,$$

$$\exp\left(s\left(\omega \cdot X + Y\right)\right) z_0 := \gamma(s), \qquad \text{where } \gamma \text{ is defined in (1.7)},$$

$$\exp\left(s\left(\omega \cdot X + Y\right)\right) := \exp\left(s\left(\omega \cdot X + Y\right)\right)(0, 0).$$

Note that, by the invariance of the vector fields with respect to $\mathbb{G}$, we have

$$\exp\left(s\left(\omega \cdot X + Y\right)\right) z_0 = z_0 \circ \exp\left(s\left(\omega \cdot X + Y\right)\right). \tag{1.9}$$

Moreover, from (1.2) we see that the *time* component of $\exp\left(s\left(\omega \cdot X + Y\right)\right)(x_0, t_0)$ is always $t_0 - s$. With these notation, our last hypothesis reads as follows

(H2) There exists a bounded open set Ω containing the origin, a vector $\omega \in \mathbb{R}^m$ and a positive s_0 such that

$$\exp\left(s\left(\omega \cdot X + Y\right)\right) \in \operatorname{Int}\left(\mathscr{A}_{(0,0)}(\Omega)\right) \quad \text{for any} \quad s \in\,]0, s_0]. \tag{1.10}$$

Remark 1.1. Some comments on our assumptions (H0), (H1) and (H2) are worth noting.

1. The heat operator $\mathscr{L} = \partial_t - \Delta$ is of the form (1.1). Moreover, it is invariant with respect to the Euclidean translations $(x, t) \circ (\xi, \tau) = (x + \xi, t + \tau)$ and (H2) is satisfied by any $\omega \in \mathbb{R}^N$. In this particular case, if we choose $\omega = 0$, and we recall

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that $X_0 = 0$, we see that $\exp\left(s\left(\omega \cdot X + Y\right)\right) = (0, -s)$. Note that a restricted uniform Harnack inequality $u(x, t - \varepsilon) \le C_\varepsilon u(x, t)$ follows from the classical parabolic Harnack inequality first proved by Hadamard [19] and Pini [42].

2. More generally, hypothesis (H2) is satisfied in the case of an operator of the form $\partial_t + \mathscr{L}_0 u$ in $\mathbb{R}^N \times]-\infty, T[$, where $\mathscr{L}_0$ is a time-independent locally uniformly elliptic operator with bounded coefficients, and also in the case of a manifold M with a cocompact group action G and an operator of the form $\partial_t + \mathscr{L}_0 u$ on $M \times]-\infty, T[$, where $\mathscr{L}_0$ is a (time-independent) G-invariant elliptic operator on M (see [29, 33, 38, 40, 41]). Moreover, for the class of linear degenerate operators satisfying (H0) of the form

$$\mathscr{L}u(x, t) = \partial_t u(x, t) - \sum_{j=1}^{m} X_j^2 u(x, t) \tag{1.11}$$

(so with $X_0 = 0$), we have that (H2) is satisfied for every $\omega \in \mathbb{R}^m$.

3. We further note that there are operators $\mathscr{L}$ of the form (1.1) that satisfy (H0) and (H1), for which (H2) is not satisfied *for all* $\omega \in \mathbb{R}^m$. We refer to Mumford operator (7.1) discussed in Section 7, and to Example 9.2.

4. We eventually recall that Krener's Theorem states that for any open set $\Omega \subseteq \mathbb{R}^{N+1}$ and $z_0 \in \Omega$, the interior of $\mathscr{A}_{z_0}(\Omega)$ is not empty whenever (H0) is satisfied (see [30] or [1, Theorem 8.1, p. 107]). We note here, that for this reason, it is not clear to us whether there exists an operator $\mathscr{L}$ satisfying (H0) and (H1), but not satisfying (H2).

Our assumptions (H0), (H1) and (H2) provide us with some compactness properties that are needed for proving that all points in the convex closed cone $\mathscr{H}_+$ can be represented in terms of its extremal rays. These compactness properties hinge on the following local Harnack inequality which holds true under our assumptions (see the main result of [28]).

(H*) Let $\Omega \subseteq \mathbb{R}^{N+1}$ be a bounded open set and let $z_0 \in \Omega$. For any compact set $K \subset \mathrm{Int}\left(\mathscr{A}_{z_0}(\Omega)\right)$ there exists a positive constant C_K, only depending on Ω, K, z_0 and $\mathscr{L}$, such that

$$\sup_K u \le C_K \, u(z_0), \tag{1.12}$$

for any nonnegative solution u of $\mathscr{L}u = 0$ in Ω.

Note that, from (H*) and from the hypoellipticity of $\mathscr{L}$ we have that $\mathscr{H}$ is a Fréchet space with respect to the topology of uniform convergence on compact sets. Moreover, in this topology, $\mathscr{H}_+$ is clearly a closed convex cone in $\mathscr{H}$. We denote by $\mathrm{exr}\,\mathscr{H}_+$ the set of all extreme rays of $\mathscr{H}_+$.

Properties (H1), (H2) and (H*) yield the following *restricted uniform Harnack inequality* (cf. [38]).

Proposition 1.2 (restricted uniform Harnack inequality)**.** *Let $\mathscr{L}$ be an operator of the form* (1.1), *satisfying* (H0), (H1), *and* (H2)*. Let ω, Ω and s_0 be as in* (H2)*. For any $s > 0$ there exists a positive constant $C_s > 0$ depending only on ω, s and $\mathscr{L}$, such that for any nonnegative solution u of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ we have*

$$u\left(\exp\left(s\left(\omega \cdot X + Y\right)\right) z\right) \leq C_s u(z) \qquad \forall z \in \mathbb{R}^N \times \mathbb{R}_T. \tag{1.13}$$

Moreover, if for $j = 1, \ldots, k$, ω_j are as in (H2), *and s_j are any positive constants, then there exists a positive constant $C_{\mathbf{s}} > 0$ (where $\mathbf{s} = (s_1, \ldots, s_k)$) depending only on $\omega_1, \ldots, \omega_k, \mathbf{s}$ and $\mathscr{L}$, such that for any nonnegative solution u of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ we have*

$$u\left(\exp\left(s_k\left(\omega_k \cdot X + Y\right)\right) \ldots \exp\left(s_1\left(\omega_1 \cdot X + Y\right)\right) z\right) \leq C_{\mathbf{s}} u(z) \qquad \forall z \in \mathbb{R}^N \times \mathbb{R}_T. \tag{1.14}$$

Proof. Let u be a nonnegative solution u of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$. For any $z \in \mathbb{G}$ the function $u^z(y) := u(z \circ y)$ is a nonnegative solution of the equation $\mathscr{L}u = 0$. Therefore, for every $s \in]0, s_0]$, by the local Harnack inequality (H*) and (1.9), we have

$$\begin{aligned} u\left(\exp\left(s\left(\omega \cdot X + Y\right)\right) z\right) &= u\left(z \circ \exp\left(s\left(\omega \cdot X + Y\right)\right)\right) = \\ u^z\left(\exp\left(s\left(\omega \cdot X + Y\right)\right)\right) &\leq C_s u^z(0) = C_s u(z). \end{aligned} \tag{1.15}$$

This proves (1.13) if $s \in]0, s_0]$. If $s > s_0$ we choose $\widetilde{s} \in]0, s_0]$ and $k \in \mathbb{N}$ such that $s = k\widetilde{s}$. By (1.9) and (1.15) we find.

$$\begin{aligned} u\left(\exp\left(k\widetilde{s}\left(\omega \cdot X + Y\right)\right) z\right) &\leq C_{\widetilde{s}} u\left(\exp\left((k-1)\widetilde{s}\left(\omega \cdot X + Y\right)\right) z\right) \leq \\ &\ldots \leq C_{\widetilde{s}}^{k-1} u\left(\exp\left(\widetilde{s}\left(\omega \cdot X + Y\right)\right) z\right) C_{\widetilde{s}}^k u(z). \end{aligned}$$

This concludes the proof of (1.13), with $C_s = C_{\widetilde{s}}^k$.

The proof of (1.14) follows by the same argument. $\qquad \square$

Remark 1.3. In the proof of Proposition 1.2 we have constructed a *Harnack chain* based on the *local* Harnack inequality (H*). For this reason, (1.13) and (1.14) do not require the boundedness assumption of the open set Ω and of the interval $]0, s_0]$ in Condition (H2). Hence, when we apply Proposition 1.2 in the sequel, we do not refer to Ω and s_0.

The following theorem is a version of the *separation principle* (see [38] and [41, Definition 2.2]). We note that the restricted uniform Harnack inequality (Proposition 1.2) is used in the proof of our separation principle to construct *Harnack chains* along the path $\gamma(s) = \exp\left(s\left(\omega \cdot X + Y\right)\right)(x_0, t_0)$.

Theorem 1.4 (Separation principle)**.** *Let $\mathscr{L}$ be an operator of the form* (1.1), *satisfying* (H0), (H1), *and* (H2). *Let ω be as in* (H2), *and suppose that for every $u \in \mathscr{H}_+$, and every positive s*

$$(x,t) \mapsto u\left(\exp(s(\omega \cdot X + Y))(x,t)\right) \qquad \text{is a solution of } \mathscr{L}u = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}_T. \tag{1.16}$$

Then, for every $u \in \mathrm{exr}\, \mathscr{H}_+$, $u \neq 0$, there exists $\beta \in \mathbb{R}$ such that

$$u\left(\exp(s(\omega \cdot X + Y))(x,t)\right) = \mathrm{e}^{-\beta s} u(x,t) \tag{1.17}$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$ and for every $s > 0$. In particular, for every $u \in \mathrm{exr}\, \mathscr{H}_+$ and $z_0 = (x_0, t_0)$ in $\mathbb{R}^N \times \mathbb{R}_T$, if $u(z_0) > 0$, then $u > 0$ in a neighborhood of the integral curve

$$\gamma := \left\{ \exp\left(s\left(\omega \cdot X + Y\right)\right) z_0 \mid s \in \,]t_0 - T, +\infty[\right\}. \tag{1.18}$$

We also have the following result, useful in the study of stratified Lie groups and the Mumford operator. It is weaker than Theorem 1.4 in that the *right-invariance* of solutions is not assumed to hold for every positive s.

Proposition 1.5. *Let $\mathscr{L}$ be an operator of the form* (1.1), *satisfying* (H0), (H1), *and* (H2). *Let ω_j is as in* (H2) *for $j = 1, \ldots, k$, and suppose that there exists $\mathbf{s} = (s_1, \ldots, s_k) \in (\mathbb{R}^+)^k$ such that*

$$(x,t) \mapsto u\left(\exp\left(s_k\left(\omega_k \cdot X + Y\right)\right)\ldots \exp\left(s_1\left(\omega_1 \cdot X + Y\right)\right)(x,t)\right) \tag{1.19}$$

is a solution of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ whenever $u \in \mathscr{H}_+$. Then, for every $u \in \mathrm{exr}\, \mathscr{H}_+$, $u \neq 0$, there exists a positive constant $C = C(\mathbf{s}, \omega_1, \ldots, \omega_k)$ such that

$$u\left(\exp\left(s_k\left(\omega_k \cdot X + Y\right)\right)\ldots \exp\left(s_1\left(\omega_1 \cdot X + Y\right)\right)(x,t)\right) = Cu(x,t) \tag{1.20}$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$.

We prove Theorem 1.4 and Proposition 1.5 in the next subsection devoted to our functional setting.

Remark 1.6. Assumption (1.16) of Theorem 1.4 appears to be quite strong. Indeed, since $\mathscr{L}$ is *left-invariant* with respect to the operation "$\circ$", it follows that $(x,t) \mapsto u((x_0, t_0) \circ (x,t))$ is a solution of $\mathscr{L}u = 0$ for every fixed $(x_0, t_0) \in \mathbb{R}^{N+1}$ and $u \in \mathscr{H}$. On the other hand, (1.9), says that $u\left(\exp(s(\omega \cdot X + Y))(x,t)\right) = u\left((x,t) \circ \exp(s(\omega \cdot X + Y))\right)$, and therefore, we also assume, in fact, a *right-invariance* condition, with respect to the point $\exp(s(\omega \cdot X + Y))$.

However, both conditions are satisfied in the class of linear degenerate operators satisfying (H0) of the form (1.11) (so, $X_0 = 0$). In this case (H2) is satisfied for every $\omega \in \mathbb{R}^m$. In particular, for $\omega = 0$ and $s > 0$,

$$\exp(s(\omega \cdot X + Y))(x,t) = (x, t - s)$$

and $\mathscr{L}u(x, t-s) = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ if $\mathscr{L}u(x,t) = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$.

In Section 3 we discuss some classes of operators of the form (1.11) satisfying (H0), (H1), and (H2). In this case, Theorem 1.4 says that for any nonnegative extremal solution u of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ there exists $\beta \in \mathbb{R}$ such that for any $s > 0$

$$u(x, t-s) = \mathrm{e}^{-\beta s} u(x,t) \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}_T. \tag{1.21}$$

Note that a separation principle also holds when the drift term has the form $X_0 = \sum_{j=1}^{N} b_j \partial_{x_j}$, where $b = (b_1, \ldots, b_N)$ is any constant vector. Indeed, if u is a positive solution of

$$\partial_t u = \sum_{j=1}^{m} X_j^2 u + \sum_{j=1}^{N} b_j \partial_{x_j} u$$

then $v(x,t) := u(x - tb, t)$ is a solution of the analogous equation

$$\partial_t v = \sum_{j=1}^{m} X_j^2 v.$$

Then we can apply Theorem 1.4 to v with $\omega = b$, and finally we obtain

$$u(x + sb, t-s) = \mathrm{e}^{-\beta s} u(x,t) \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}_T.$$

In Section 7, we present a remarkable example of an operator satisfying assumption (1.16) of Theorem 1.4, namely, the well-known Mumford operator:

$$\mathscr{M}u := \partial_t u - \cos(x)\partial_y u - \sin(x)\partial_w u - \partial_x^2 u \qquad (x, y, w, t) \in \mathbb{R}^4,$$

an operator that is discussed in detail in Section 7. Clearly its drift $X_0 = \cos(x)\partial_y + \sin(x)\partial_w$ is nontrivial. It is also worth noting that $\mathscr{M}$ satisfies the assumptions of Proposition 1.5, with $s = 2\pi$, but it does not satisfy the hypotheses of Theorem 1.4. We also note that Section 9.1 contains some remarks on the validity of (1.16) for operators with nontrivial drift.

The outline of the paper is as follows. In Section 2 we introduce representation formulas that play a crucial role in our study, and we give the proof of Theorem 1.4. In sections 3–6 we study operators $\mathscr{L}$ such that the drift term X_0 vanishes identically. In particular, in Section 4 we study stationary solutions, Section 3 deals with solutions of the evolution equation, while Section 5 discusses parabolic Liouville-type theorems, and Section 6 contains a uniqueness result for the positive Cauchy problem. In Section 7 we prove a new uniqueness result for Mumford's operator. In Section 8 we compute the Martin boundary for Kolmogorov-Fokker-Planck operators in $\mathbb{R}^N \times \mathbb{R}_T$. Finally, Section 9 is devoted to some concluding remarks concerning the results of the present paper and to a discussion of some open problems.

2 Functional setting

In the present section we introduce some notation, and recall some known facts about convex cones in vector spaces. The following definition plays a crucial role in our study. It leads to some compactness results that enable us to apply Choquet's theory. We first introduce the following notation. If $z \in \mathbb{R}^{N+1}$ and Ω is a bounded open subset of $\mathbb{R}^{N+1}$, we set

$$\Omega_z = z \circ \Omega = \left\{ z \circ \zeta \mid \zeta \in \Omega \right\}. \tag{2.1}$$

Definition 2.1. Let $\mathscr{L}$ be an operator satisfying (H2). A sequence $\mathscr{R} := (z_k)_{k \in \mathbb{N}} \subset \mathbb{R}^N \times \mathbb{R}_T$ is said to be a *reference set* for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}_T$, if

$$\bigcup_{k=1}^{\infty} \mathrm{Int} \left(\mathscr{A}_{z_k} \left(\Omega_{z_k} \right) \right) = \mathbb{R}^N \times \mathbb{R}_T,$$

where Ω is the bounded open set satisfying (H2).

We next prove that, in our setting, a reference set always exists.

Proposition 2.2. *If $\mathscr{L}$ satisfies assumptions* (H0), (H1) *and* (H2)*, then a* reference set $\mathscr{R}$ *exists.*

Proof. Let $(K_j)_{\in \mathbb{N}}$ be a sequence of compact sets such that

$$\bigcup_{j=1}^{\infty} K_j = \mathbb{R}^N \times \mathbb{R}_T.$$

We claim that for every $j \in \mathbb{N}$ there exist $k_j \in \mathbb{N}$ and $z_{j_1}, \dots, z_{j_{k_j}} \in \mathbb{R}^N \times \mathbb{R}_T$ such that

$$K_j \subset \bigcup_{i=1}^{k_j} \mathrm{Int} \left(\mathscr{A}_{z_{j_i}} \left(\Omega_{z_{j_i}} \right) \right). \tag{2.2}$$

In order to prove (2.2) we consider $\omega \in \mathbb{R}^m, s_0 > 0$ and Ω satisfying (H2). For every $(\xi, \tau) \in K_j$ we choose $s \in]0, s_0]$ such that $s + \tau < T$, and we set

$$(x, t) = \exp \left(-s \left(\omega \cdot X + Y \right) \right) (\xi, \tau).$$

Since $t = s + \tau < T$, it follows that $(x, t) \in \mathbb{R}^N \times \mathbb{R}_T$. Moreover

$$\exp \left(s \left(\omega \cdot X + Y \right) \right) (x, t) = (\xi, \tau),$$

then, by (H1) and (H2) we have that $(\xi, \tau) \in \mathrm{Int} \mathscr{A}_{(x,t)} \left(\Omega_{(x,t)} \right)$. Hence, (2.2) follows from the compactness of K_j. Therefore, a reference set for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}_T$ is given by

$$\mathscr{R} := \bigcup_{j=1}^{\infty} \left\{ z_{j_1}, \dots, z_{j_{k_j}} \right\}.$$

$\square$

We equip $\mathscr{H}$ with the compact open topology, that is, the topology of uniform convergence on compact sets.

Let $\mathscr{R} := (z_k)_{k \in \mathbb{N}}$ be a reference set for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}_T$, and let $a = (a_k)_{k \in \mathbb{N}}$ be a strictly positive sequence. We set

$$\mathscr{H}_a := \left\{ u \in \mathscr{H}_+ \mid \sum_{k=1}^{\infty} a_k u(z_k) \leq 1 \right\}, \tag{2.3}$$

$$\mathscr{H}_a^1 := \left\{ u \in \mathscr{H}_+ \mid \sum_{k=1}^{\infty} a_k u(z_k) = 1 \right\}. \tag{2.4}$$

Lemma 2.3. *For any positive sequence $a = (a_k)_{k \in \mathbb{N}}$, the convex set $\mathscr{H}_a$ is compact in $\mathscr{H}_+$.*

Proof. By the hypoellipticity of $\mathscr{L}$, it is sufficient to show that $\mathscr{H}_a$ is locally bounded on $\mathbb{R}^N \times \mathbb{R}_T$. With this aim, we consider any compact set $K \subset \mathbb{R}^N \times \mathbb{R}_T$. By (H2), and Proposition 2.2, there exist $w_1, \ldots, w_k$ in $\mathbb{R}^N \times \mathbb{R}_T$ such that

$$K \subset \text{Int} \left(\mathscr{A}_{w_1} \left(\Omega_{w_1} \right) \right) \cup \cdots \cup \text{Int} \left(\mathscr{A}_{w_k} \left(\Omega_{w_k} \right) \right).$$

We claim that there exist $z_{n_1}, \ldots, z_{n_k}$ in $\mathscr{R}$ and k compact sets $K_1, \ldots, K_k$ such that

$$K = K_1 \cup \cdots \cup K_k, \quad \text{and} \quad K_j \subset \text{Int} \left(\mathscr{A}_{z_{n_j}} \left(\widetilde{\Omega}_j \right) \right), \tag{2.5}$$

for $j = 1, \ldots, k$, where every $\widetilde{\Omega}_j$ is a bounded open set containing z_{n_j}.

Indeed, let $K_j := K \cap \left(\overline{\text{Int} \left(\mathscr{A}_{w_j} \left(\Omega_{w_j} \right) \right)} \right)$ for $j = 1, \ldots, k$. As in the proof of Proposition 2.2, we take $z_{n_j} \in \mathscr{R}$ such that $w_j \in \text{Int} \left(\mathscr{A}_{z_{n_j}} \left(\Omega_{z_{n_j}} \right) \right)$. We then choose a bounded open set $\widetilde{\Omega}_j$ containing $\Omega_{z_{n_j}} \cup K_j$, and we have that $K_j \subset \text{Int} \left(\mathscr{A}_{z_{n_j}} \left(\widetilde{\Omega}_j \right) \right)$ for $j = 1, \ldots, k$. This proves (2.5).

As a consequence of (2.5), the restricted uniform Harnack inequality (1.13) yields

$$\sup_K u \leq C_K \max_{j=1,\ldots,k} u \left(z_{n_j} \right),$$

for some positive constant C_K depending only on $\mathscr{L}$ and K. On the other hand, from the definition of $\mathscr{H}_a$ it follows that for any $u \in \mathscr{H}_a$, we clearly have that $u(z_j) \leq \frac{1}{a_j}$. Consequently,

$$\sup_K u \leq C_K \max_{j=1,\ldots,k} \left\{ \frac{1}{a_{n_j}} \right\}.$$

$\square$

Note that $\mathscr{H}_+$ is the union of the caps $\mathscr{H}_a$. Indeed, for every $u \in \mathscr{H}_+$, we easily see that $u \in \mathscr{H}_a$ where the sequence $a = (a_k)_{k \in \mathbb{N}}$ is defined as follows $a_k := \frac{b_k}{u(z_k)+1}$ and $(b_k)_{k \in \mathbb{N}}$ is any nonnegative sequence such that $\sum b_k \leq 1$.

Thus, $\mathscr{H}_a$ is a metrizable *cap* in $\mathscr{H}_+$ (*i.e.* $\mathscr{H}_a$ is a compact convex set and $\mathscr{H}_+ \backslash \mathscr{H}_a$ is convex) and $\mathscr{H}$ is *well-capped* (*i.e.* $\mathscr{H}_+$ is the union of the caps $\mathscr{H}_a$). Furthermore, since $\mathscr{H}_+$ is a harmonic space in the sense of Bauer, it follows that $\mathscr{H}_a$ is a *simplex* (see [4, 11]).

Let $\mathscr{C}$ be a convex cone, we denote by $\mathrm{exr}\,\mathscr{C}$ the set of all extreme rays of $\mathscr{C}$. Analogously, if K is a convex set, we denote by $\mathrm{ex}\,K$ the set of the extreme points of K.

Since $\mathscr{H}_+$ is a proper cone (*i.e.* it contains no one-dimensional subspaces), we have

$$\mathrm{ex}\,\mathscr{H}_a = \{0\} \cup \{\mathrm{exr}\,\mathscr{H}_+ \cap \mathscr{H}_a^1\}. \tag{2.6}$$

We next prove Theorem 1.4 and Proposition 1.5. The argument of the proof is standard, we give here the details for reader's convenience.

Proof of Theorem 1.4. Clearly, $\mathscr{H}_+ \neq \{0\}$ since $\mathbf{1} \in \mathscr{H}_+$. By the Krein-Milman theorem and (2.6), it follows that $\mathrm{exr}\,\mathscr{H}_+$ contains a nontrivial ray. Consider any function $u \in \mathrm{exr}\,\mathscr{H}_+$ such that $u \neq 0$, and let $\omega \in \mathbb{R}^m$ be as in Proposition 1.2. We claim that, for every positive s, there exists a positive constant α_s such that

$$u\left(\exp\left(s\left(\omega \cdot X + Y\right)(x,t)\right)\right) = \alpha_s u(x,t). \tag{2.7}$$

Indeed, let

$$v_s(x,t) := C_s^{-1} u\left(\exp\left(s\left(\omega \cdot X + Y\right)(x,t)\right)\right),$$

and recall that by our hypothesis (1.16), v_s is a nonnegative solution of the equation $\mathscr{L}v_s = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$. Moreover, the restricted uniform Harnack inequality (Proposition 1.2) implies that $v_s \leq u$. Since $u \in \mathrm{exr}\,\mathscr{H}_+$, it follows that $v_s(z) = \nu_s u(z)$ for all $z \in \mathbb{R}^N \times \mathbb{R}_T$, where $\nu_s \geq 0$. If $\nu_s > 0$, then we obviously have (2.7). Suppose that $\nu_s = 0$, then by applying the exponential map forward, it follows that $u(x,t) = 0$ for all $(x,t) \in \Omega_{T-s}$. This completes the proof if $T = \infty$. If $T < \infty$ we repeat the argument for a vanishing sequence $(s_j)_{j \in \mathbb{N}}$ of positive numbers. This contradicts our assumption that $u \neq 0$. Hence (2.7) is proved.

In order to conclude the proof of (1.17), we note that for every $\omega \in \mathbb{R}^m$ satisfying the assumption of Proposition 1.2, $z \in \mathbb{R}^N \times \mathbb{R}_T, s > 0$, and any $k \in \mathbb{N}$ we have

$$\exp\left(ks\left(\omega \cdot X + Y\right)\right) z = \underbrace{\exp\left(s(\omega \cdot X + Y)\right) \circ \ldots \circ \exp\left(s(\omega \cdot X + Y)\right)}_{k \text{ times}} z.$$

By iterating (2.7), we then find

$$\alpha_{ks} u(x,t) = u\left(\exp\left(ks\left(\omega \cdot X + Y\right)\right)(x,t)\right) = \alpha_s^k u(x,t).$$

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Hence, $\alpha_k = \alpha_1^k$, and $\alpha_{1/k} = \alpha_1^{1/k}$, for every $k \in \mathbb{N}$. Therefore, $\alpha_r = \alpha_1^r$ for every $r \in \mathbb{Q}$. The conclusion of the proof thus follows from the continuity of u, by setting $\beta := \log(\alpha_1)$.

For the proof of the last assertion of the theorem, take z_0 such that $u(z_0) > 0$. Then by (1.17) $u > 0$ on the integral curve γ given by (1.18). $\qquad\square$

Proof of Proposition 1.5. It is analogous to the proof of (2.7), which is based only on the Harnack inequality and on the assumption concerning the (restricted) right-invariance of the solutions in $\mathscr{H}_+$. We omit the details. $\qquad\square$

Remark 2.4. When considering the classical heat equation in $\mathbb{R}^N \times \mathbb{R}_T$, or more generally when $X_0 = 0$, the separation principle reads as follows (see [29, 38, 40] for the corresponding result in the nondegenerate case):

For any $u \in \mathrm{exr}\, \mathscr{H}_+$ there exists $\lambda \le \lambda_0$ such that

$$u(x,t) = \mathrm{e}^{-\lambda t} u(x,0) \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}_T, \tag{2.8}$$

where λ_0 is the *generalized principal eigenvalue* of the operator $\mathscr{L}_0 := -\sum_{j=1}^m X_j^2$ defined by

$$\lambda_0 := \sup\left\{\lambda \in \mathbb{R} \mid \exists u_\lambda \gneq 0 \text{ s.t. } \left(-\sum_{j=1}^m X_j^2 - \lambda\right) u_\lambda = 0 \text{ in } \mathbb{R}^N\right\}. \tag{2.9}$$

Moreover, using Choquet's theorem and the argument in the proof of [40, Theorem 2.1], (2.8) implies that u is a nontrivial extremal solution of the equation $\mathscr{L}w = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ if and only if it is of the form $u(x,t) := e^{-\lambda t} u_\lambda(x)$, where $\lambda \le \lambda_0$ and u_λ is a nonzero extremal solution of the equation $\mathscr{L}_\lambda \phi = (-\sum_{j=1}^m X_j^2 - \lambda)\phi = 0$ in $\mathbb{R}^N$. In particular, it follows that any nontrivial solution in $\mathscr{H}_+$ is strictly positive.

In fact, for the heat equation it is known (see for example [17]) that any nonnegative extremal caloric function $u \ne 0$ in $\mathbb{R}^{N+1}$ or in $\mathbb{R}^N \times \mathbb{R}_-$ is of the form

$$u(x,t) = \exp\left(\langle x, v\rangle + t\|v\|^2\right),$$

where $v \in \mathbb{R}^N$ is a fixed vector.

When a drift term X_0 appears in the operator $\mathscr{L}$, (2.8) does not holds necessarily, even for nondegenerate parabolic equations. Consider, for instance, the nondegenerate Ornstein-Uhlenbeck operator

$$\mathscr{L}u := \partial_t u - \Delta u - \langle x, \nabla u\rangle = 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}_T. \tag{2.10}$$

Clearly, $\mathscr{L}$ is of the form (1.1) with $X_j = \partial_{x_j}, j = 1, \ldots, N$, and $X_0 = \langle x, \nabla\rangle \simeq x$. Moreover, $\mathscr{L}$ is invariant with respect to the following change of variable. Fix any $(y, s) \in \mathbb{R}^{N+1}$, and set $v(x,t) := u(x + e^{-t}y, t + s)$. We have that $\mathscr{L}v = 0$ in $\mathbb{R}^{N+1}$, if

and only if $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$. Thus, the Ornstein-Uhlenbeck operator satisfies (H0), (H1) and (H2). Note that in this case, the restricted Harnack inequality reads as

$$u\left(e^s x, t - s\right) \leq C_s u(x,t) \qquad \forall (x,t) \in \mathbb{R}^{N+1},$$

and that (1.16) does not hold for $y \neq 0$. On the other hand, the expression of a *minimal* solution of the equation in one space variable, given in [14, 41], is

$$u_\lambda(x,t) = \exp\left(\lambda^2 e^{2t} - \sqrt{2}\lambda x e^t\right),$$

where $\lambda \in \mathbb{R}$. Clearly, (2.8) does not hold for u_λ.

3 Degenerate equations without drift

We first derive from (H*) a Harnack inequality for the operator $\mathscr{L} - \lambda$, where $\mathscr{L}$ is of the form (1.1) and λ is a real constant. After that, we focus on operators $\mathscr{L}$ such that the drift term X_0 does not appear. In particular, we prove a representation theorem for the extremal nonnegative solutions of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$, when $X_0 = 0$ and the Lie group on $\mathbb{R}^N$ is nilpotent and stratified.

Proposition 3.1. *Let $\mathscr{L}$ be an operator of the form (1.1) that satisfies (H0), (H1), and (H2). Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set and let $z_0 = (x_0, t_0) \in \Omega$. For any compact set $K \subset \mathrm{Int}\left(\mathscr{A}_{z_0}(\Omega)\right)$ and for every $\lambda \in \mathbb{R}$ there exists a positive constant $C_{K,\lambda}$, only depending on Ω, K, z_0, λ and $\mathscr{L}$, such that*

$$\sup_K u \leq C_{K,\lambda}\, u(z_0),$$

for any nonnegative solution u of the equation $\mathscr{L}w - \lambda w = 0$ in Ω.

Proof. If u is a nonnegative solution of $\mathscr{L}w - \lambda w = 0$ in Ω, then $u_\lambda(x,t) := e^{-\lambda t}u(x,t)$ is a nonnegative solution of $\mathscr{L}w = 0$ in Ω. The claim then follows from (H*) with $C_{K,\lambda} := C_K \max_{(x,t) \in K} e^{\lambda(t_0 - t)}$. $\qquad \square$

We next consider operators $\mathscr{L}$ such that the drift term X_0 does not appear. We will use the following notation

$$\mathscr{L}_0 := -\sum_{j=1}^m X_j^2, \qquad \mathscr{L}_\lambda := \mathscr{L}_0 - \lambda. \tag{3.1}$$

We consider the degenerate *elliptic* equation $\mathscr{L}_\lambda u = 0$ in $\mathbb{R}^N$ and its parabolic counterpart $\mathscr{L} = \partial_t u + \mathscr{L}_\lambda u = 0$ in $\mathbb{R}^N \times]0, T[$. In this case Hörmander's condition (H0) is equivalent to:

(H0')   rank Lie$\{X_1, \ldots, X_m\}(x) = N$ for every $x \in \mathbb{R}^N$.

Moreover, (H1) is equivalent to:

(H1') there exists a Lie group $\mathbb{G}_0 = (\mathbb{R}^N, \cdot)$ such that the vector fields $X_1, \ldots, X_m$ are invariant with respect to the left translation of $\mathbb{G}_0$.

Indeed, as (H1') is satisfied, then a group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ satisfying (H1) is defined by $\mathbb{G} := \mathbb{G}_0 \times \mathbb{R}$, with the operation

$$(x, t) \circ (y, s) := (x \cdot y, t + s) \qquad (x, t), (y, s) \in \mathbb{R}^{N+1}. \tag{3.2}$$

Finally, the Chow-Rashevskii theorem (see for example [36]) implies that for any open cylinder $\Omega = O \times I$, with $O \subseteq \mathbb{R}^N$ an open connected set, and an interval $I \subset \mathbb{R}$, we have for every $(x_0, t_0) \in \Omega$ that

$$\mathscr{A}_{(x_0, t_0)}(\Omega) = \Omega \cap \{(x, t) \mid t \leq t_0\}, \tag{3.3}$$

whenever (H0') holds. Thus condition (H2) is satisfied with any $\omega \in \mathbb{R}^m$. In the sequel of the present section we will always consider $\omega = 0$.

Based on Proposition 3.1, we next prove a Harnack inequality for the operators $\mathscr{L}_\lambda$. We refer to the monograph [6] and to the reference therein for an exhaustive bibliography on Harnack inequalities for operators of the form $\mathscr{L}_0$.

Proposition 3.2. *Let $\mathscr{L}_0$ be an operator of the form* (3.1), *satisfying* (H0'), *and* (H1'), *and let λ be a given constant. Let $O \subseteq \mathbb{R}^N$ be an open connected set and let $x_0 \in O$. For any compact set $H \subset O$ there exists a positive constant $C_{H,\lambda}$, only depending on O, H, x_0, λ and $\mathscr{L}_0$, such that*

$$\sup_H u \leq C_{H,\lambda} \, u(x_0),$$

for any nonnegative solution u of $\mathscr{L}_\lambda u = 0$ in O.

Proof. If u is a nonnegative solution of $\mathscr{L}_\lambda w = 0$ in O, then the function $v(x, t) := u(x)$ is a nonnegative solution of $\partial_t w + \mathscr{L}_\lambda w = 0$ in $\Omega := O \times I$, where I is any open interval of $\mathbb{R}$. We choose $I =]-2, 1[$, $z_0 := (x_0, 0)$ and $K := H \times \{-1\}$. The Chow-Rashevskii theorem implies $\mathscr{A}_{z_0}(\Omega) = \Omega \cap \{t \leq 0\}$, thus $K \subset \mathrm{Int}\,(\mathscr{A}_{z_0}(\Omega))$. We then apply Proposition 3.1 to v, and we obtain the Harnack estimate for u. $\square$

We consider now operators of the form $\mathscr{L}_0$, satisfying (H0'), and (H1') with the further property that they are invariant with respect to a family of dilations. Specifically, we suppose that $\mathbb{R}^N$ can be split as follows

$$\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n}, \quad \text{and denote } x = (x^{(m)}, x^{(m_2)} \ldots, x^{(m_n)}) \in \mathbb{R}^N,$$

where $n \geq 2$. We assume that there exists a group of dilations $D_r : \mathbb{R}^N \to \mathbb{R}^N$, defined for every $r > 0$ as follows

$$D_r(x) = D_r\left(x^{(m)}, x^{(m_2)}, \ldots, x^{(m_n)}\right) := \left(rx^{(m)}, r^2 x^{(m_2)}, \ldots, r^n x^{(m_n)}\right),$$

which are automorphisms of $(\mathbb{R}^N, \cdot)$. In this case we say that $\mathbb{G}_C = \left(\mathbb{R}^N, \cdot, (D_r)_{r>0}\right)$ is a *homogeneous Lie group*. It is well-known that $\mathbb{G}_C$ is nilpotent (see for example [6, Proposition 1.3.12]) and compactly generating. Moreover, the following two properties follow from the homogeneous structure of Carnot groups (see for example [6, Theorem 1.3.15]).

$$(x \cdot y)^{(m)} = x^{(m)} + y^{(m)} \qquad \text{for every} \quad x, y \in \mathbb{R}^N. \tag{3.4}$$

$$x \cdot y = y \cdot x = x + y \qquad \text{whenever} \quad x = (0^{(m)}, 0^{(m_2)} \ldots, 0^{(m_n)}, x^{(m_n)}). \tag{3.5}$$

We point out that (3.5) means, in particular, that right and left multiplications by a point x belonging to the last layer of the group agree.

When the vector fields $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to the dilation $(D_r)_{r>0}$, we say that $\mathbb{G}_C := \left(\mathbb{R}^N, \cdot, (D_r)_{r>0}\right)$ is a *Carnot group* and $X_1, \ldots, X_m$ are called *generators* of $\mathbb{G}_C$. In this case, it is always possible to choose the X_j's such that $X_j = \partial_{x_j} + \sum_{k=m+1}^{N} b_{jk}(x)\partial_{x_k}$, for $j = 0, \ldots, m$, and the coefficients $b_{jk}(x)$ are polynomials. Moreover all commutators $[X_j, X_k]$ only acts on $(x^{(m_2)}, \ldots, x^{(m_n)})$, third order commutators $[X_i, [X_j, X_k]]$ only acts on $(x^{(m_3)}, \ldots, x^{(m_n)})$, n-th order commutators only act on $x^{(m_n)}$.

The corresponding *sub-Laplacian* $\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2$ agrees with $-\mathscr{L}_0$, and is always self-adjoint, that is $\Delta_{\mathbb{G}}^* = \Delta_{\mathbb{G}}$ (see [6]).

Example 3.3. HEISENBERG GROUP. $\mathbb{H} := (\mathbb{R}^3, \cdot)$, is defined by the multiplication

$$(\xi, \eta, \zeta) \cdot (x, y, z) := (\xi + x, \eta + y, \zeta + z + (\eta x - \xi y)) \quad (\xi, \eta, \zeta), (x, y, z) \in \mathbb{R}^3.$$

The vector fields X_1 and X_2

$$X_1 := \partial_x - \tfrac{1}{2}y\partial_z, \quad X_2 := \partial_y + \tfrac{1}{2}x\partial_z,$$

are invariant with respect to the left translation of $\mathbb{H} = (\mathbb{R}^3, \cdot)$, and with respect to the following dilation in $\mathbb{R}^3$

$$D_r(x, y, z) := \left(rx, ry, r^2 z\right) \qquad (x, y, z) \in \mathbb{R}^3, r > 0.$$

Note that we have $[X_1, X_2] = \partial_z$, and any other commutator is zero.

The sub-Laplacian on the Heisenberg group acts on a function $u = u(x, y, z)$ as follows

$$\Delta_{\mathbb{H}} u := \left(\partial_x - \tfrac{1}{2}y\partial_z\right)^2 u(x, y, z) + \left(\partial_y + \tfrac{1}{2}x\partial_z\right)^2 u(x, y, z). \tag{3.6}$$

If $-\mathscr{L}_0$ is a sub-Laplacian in a Carnot group $\mathbb{G}_C := \big(\mathbb{R}^N, \cdot, (D_r)_{r>0}\big)$, we define a homogeneous group $\mathbb{G} = \big(\mathbb{R}^{N+1}, \circ, (\delta_r)_{r>0}\big)$

$$(x, t) \circ (\xi, \tau) := (x \cdot \xi, t + \tau), \qquad \delta_r(x, t) := \big(D_r x, r^2 t\big)$$

for every $(x, t)(\xi, \tau) \in \mathbb{R}^{N+1}$ and for any $r > 0$. For $\omega = 0$ we have $\exp\big(\tau(\omega \cdot X + Y)\big) = (0, -\tau)$. Thanks to the invariance with respect to translations and dilations, the restricted uniform Harnack inequality of Proposition 1.2 for such an operator $\mathscr{L}$ reads as

$$u(x, t - \tau) \le C_\tau u(x, t) \qquad \text{for every } (x, t) \in \mathbb{R}^N \times \mathbb{R}_T, \tau > 0, \tag{3.7}$$

and for any nonnegative solution of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$.

The main result of this Section is the following version of the separation principle.

Theorem 3.4. *Let* $\mathbb{G}_C = \big(\mathbb{R}^N, \cdot, (D_r)_{r>0}\big)$ *be a Carnot group, let* $\Delta_{\mathbb{G}}$ *be its sub-Laplacian, and assume that* $\mathscr{L}_0 = -\sum_{j=1}^m X_j^2$ *agrees with* $-\Delta_{\mathbb{G}}$. *If* u *is an extremal nonzero nonnegative solution of* $\mathscr{L}u = \partial_t u - \Delta_{\mathbb{G}} u = 0$ *in* $\mathbb{R}^N \times \mathbb{R}_T$, *then*

$$u(x, t) = C \exp\big(\langle x, \alpha\rangle + |\alpha|^2 t\big)$$

for some vector $\alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$ *and* $C > 0$. *Moreover, any nontrivial solution* $v \in \mathscr{H}_+$ *does not depend on the 'degenerate' variables* $x_{m+1}, \ldots, x_N$, *and* v *is strictly positive.*

Proof. We first give the proof in the simplest (nontrivial) case of the Heisenberg group $\mathbb{H}$, in order to show the main idea of the proof. Let c be any real constant, and let (x, y, z, t) be a given point of $\mathbb{R}^4$. A direct computation shows that

$$\exp\big(s(-cX_2 - \partial_t)\big)\exp\big(s(-cX_1 - \partial_t)\big)\exp\big(s(cX_2 - \partial_t)\big)\exp\big(s(cX_1 - \partial_t)\big)(x, y, z, t) =$$

$$(x, y, z + c^2 s^2, t - 4s), \quad (3.8)$$

for every positive s. Note that for any $u \in \mathscr{H}_+$, we have that

$$v(x, y, z, t) := u(x, y, z + c^2 s^2, t - 4s) \in \mathscr{H}_+.$$

Since hypothesis (H2) holds true, Proposition 1.5 implies that for any extremal solution $u \in \mathscr{H}_+$ there exists a positive constant C_s, that may depend on c, such that

$$u(x, y, z + c^2 s^2, t - 4s) = C_s u(x, y, z, t) \qquad \forall(x, y, z, t) \in \mathbb{R}^N \times \mathbb{R}_T,$$

and for every positive s. The standard argument used in the last part of the proof of Theorem 1.4 implies that

$$u(x, y, z + c^2 s^2, t - 4s) = e^{\beta c s} u(x, y, z, t) \qquad \forall(x, y, z, t) \in \mathbb{R}^N \times \mathbb{R}_T, \tag{3.9}$$

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and for every positive s. Note that for $c = 0$, the above identity restores (1.21)

$$u(x, y, z, t - s) = e^{\widetilde{\beta}_0 s} u(x, y, z, t).$$

Combining it with (3.9) we find

$$u(x, y, z + c^2 s^2, t) = e^{\widetilde{\beta}_c s} u(x, y, z, t) \qquad \forall (x, y, z, t) \in \mathbb{R}^N \times \mathbb{R}_T,$$

for some real constant $\widetilde{\beta}_c$. The above identity can be written equivalently as

$$u(x, y, z', t) = e^{\widetilde{\beta}_c \sqrt{|z' - z|}} u(x, y, z, t) \qquad \forall (x, y, z, t), (x, y, z', t) \in \mathbb{R}^N \times \mathbb{R}_T. \qquad (3.10)$$

We finally note that (3.10) contradicts the regularity of u unless $\widetilde{\beta}_c = 0$. Since u is smooth by Hörmander's condition (H0), we have necessarily $\widetilde{\beta}_c = 0$. Hence $u = u(x, y, t)$ is a nonnegative extremal solution of $\partial_t u = \Delta u$, and the conclusion of the proof, in the case of the Heisenberg group $\mathbb{H}$, follows from the classical representation theorem for the heat equation [40].

Before considering any Carnot Group $\mathbb{G}_C$, we point out that the above proof only relies on the fact that ∂_z is the highest order commutators of a nilpotent Lie group. In particular, the operator $\mathscr{L}$ is translation invariant with respect to z and that ∂_z has been obtained by (3.8). Then Proposition 1.5 gives (3.10), that in turns contradicts the smoothness of u.

Let $\mathscr{L} = \partial_t - \Delta_{\mathbb{G}}$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian on a Carnot group $\mathbb{G}_C$. We recall the Baker-Campbell-Hausdorff formula. If X_j, X_k are the vector fields belonging to the first layer of $\mathbb{G}_C$, then

$$\exp\left(s\left(X_j - \partial_t\right)\right) \exp\left(s\left(X_k - \partial_t\right)\right)(x, t) =$$
$$\exp\left(s\left(\left(X_j + X_k\right) - 2\partial_t\right) + \tfrac{s^2}{2}\left[X_k, X_j\right] + R_{jk}(s)\right)(x, t)$$

for any $s \in \mathbb{R}$, where R_{jk} denotes a polynomial function of the form

$$R_{jk}(s) = \sum_{i=3}^{m} c_{i,jk} s^i$$

whose coefficients $c_{i,jk}$'s are sums of commutators of $X_1, \ldots, X_m$ of order i. In particular, we have

$$\exp\left(s(-X_j - \partial_t)\right) \exp\left(s(-X_k - \partial_t)\right) \exp\left(s(X_j - \partial_t)\right) \exp\left(s(X_k - \partial_t)\right)(x, t) =$$
$$\exp\left(-4s\partial_t + \tfrac{s^2}{2}\left[X_k, X_j\right] + R_{jk}(s)\right)(x, t).$$

We can express the variable $x^{(m_n)}$ of the last layer of $\mathbb{G}_{\mathrm{C}}$ in terms of commutators of order n with zero remainder. In particular, by repeating the use of the Baker-Campbell-Hausdorff formula, we can express every vector $x_j^{(m_n)}$ of a basis of the last layer of $\mathbb{G}_{\mathrm{C}}$ as

$$x_j^{(m_n)} = \exp\left(-X_{j_k}\right)\ldots\exp\left(-X_{j_1}\right),$$

for a suitable choice of $X_{j_1},\ldots,X_{j_k}$ in the first layer of $\mathbb{G}_{\mathrm{C}}$. In particular, we have that

$$u\left(x + s^n x_j^{(m_n)}, t - ks\right) = u\left(\exp\left(-s(X_{j_k} - \partial_t)\right)\ldots\exp\left(-s(X_{j_1} - \partial_t)\right)(x,t)\right),$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$ and every positive s. On the other hand, by (3.5) $x + sx_j^{(m_n)}$ is at once a *right* and *left* translation on the group $\mathbb{G}_{\mathrm{C}}$. Then, in particular, $(x,t) \mapsto u\left(x + sx_j^{(m_n)}, t\right)$ is a solution of $\mathscr{L}_0 u = 0$ for every $s \in \mathbb{R}$. Thus, if u is an extremal solution of $\mathscr{L}_0 u = 0$, Proposition 1.5, combined with (1.21), yields

$$u\left(x + cs^n x_j^{(m_n)}, t\right) = e^{\beta s} u(x,t),$$

for every $x \in \mathbb{R}^N$ and $s \geq 0$. Here c is a real constant that may depend on $x_j^{(m_n)}$. As in the case of the Heisenberg group, this identity contradicts the smoothness of u, unless u does not depend on $x^{(m_n)}$. Thus, $u = u\left(x^{(m)},\ldots,x^{(m_{n-1})}\right)$ is an extremal solution of $\mathscr{L}' u = 0$, where $\mathscr{L}' = \partial_t - \Delta_{\mathbb{G}'}$, and $\Delta_{\mathbb{G}'}$ is a sub-Laplacian on a Carnot group $\mathbb{G}'$ on $\mathbb{R}^{N-m_n}$ defined as the restriction of $\mathbb{G}$ to the first $N - m_n$ variables of $\mathbb{R}^N$. The conclusion of the proof follows by a backward iteration of the above argument. $\qquad\square$

4 Stationary equations

In the present section we consider stationary equations, and we prove a result analogous to Theorem 3.4. We first introduce some notation. Fix any $\lambda \in \mathbb{R}$, and consider an operator $\mathscr{L}_\lambda$ of the form (3.1) on $\mathbb{R}^N$, satisfying (H0'), and (H1'). We set

$$\mathscr{H}_\lambda := \left\{u \in C^\infty(\mathbb{R}^N) \mid \mathscr{L}_\lambda u = 0 \quad \text{in } \mathbb{R}^N\right\}, \tag{4.1}$$

$$\mathscr{H}_\lambda^+ := \left\{u \in \mathscr{H}_\lambda \mid u \geq 0, u(0) = 1\right\}. \tag{4.2}$$

Note that in light of Proposition 3.2, the generalized principal eigenvalue λ_0 defined in (2.9) can be characterized as

$$\lambda_0 := \sup\left\{\lambda \in \mathbb{R} \mid \mathscr{H}_\lambda^+ \neq \emptyset\right\}.$$

Moreover, by the strong minimum principle (or Proposition 3.2), any function $u \in \mathscr{H}_\lambda^+$ never vanishes. The results proved in Section 2 for $\mathscr{H}$ and $\mathscr{H}_+$, and $\mathscr{H}_a$ plainly extend to $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^+$. In particular, it follows that $\mathscr{H}_\lambda^+$ is a convex compact set (for a reference set for $\mathscr{L}_\lambda$ in $\mathbb{R}^N$ one can choose any singleton). Hence any function in $\mathscr{H}_\lambda^+$ can be represented by the set of all extreme points of $\mathscr{H}_\lambda^+$.

Theorem 4.1. *Let* $\mathbb{G}_C = \left(\mathbb{R}^N, \cdot, (D_r)_{r>0}\right)$ *be a Carnot group, let* $\Delta_{\mathbb{G}}$ *be its sub-Laplacian, and assume that* $\mathscr{L}_0 = -\sum_{j=1}^m X_j^2$ *agrees with* $-\Delta_{\mathbb{G}}$. *Then* $\lambda_0 = 0$, *and for any* $\lambda \le 0$, $u \in \mathscr{H}_\lambda^+$ *is an extremal solution if and only if*

$$u(x) = u_\alpha(x) := \exp\left(\langle x, \alpha\rangle\right)$$

for some vector $\alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0)$ *such that* $\|\alpha\|^2 = -\lambda$. *Moreover,* $u \in \mathscr{H}_\lambda^+$ *if and only if there exists a unique probability measure* μ *on* $\mathbb{S}^{m-1}$ *such that*

$$u(x) = \int_{\xi \in \mathbb{S}^{m-1}} \exp\left(\sqrt{-\lambda}\langle x, \xi\rangle\right)\, \mathrm{d}\mu(\xi).$$

Proof. It is a direct consequence of Theorem 3.4 and Choquet's theorem. Recall that as in [40, Theorem 2.1], if the separation principle of the form (2.8) holds true, then u_λ is an extremal solution of $\mathscr{L}_\lambda v = 0$ in $\mathbb{R}^N$ if and only if the function $u(x, t) := e^{\lambda t}u_\lambda(x)$ is a nonzero extremal solution of $\mathscr{L}w = 0$ in $\mathbb{R}^{N+1}$ (see also, Remark (2.4)). The conclusion immediately follows from Theorem 3.4. $\qquad\square$

As a result we obtain the following nonnegative Liouville theorem.

Corollary 4.2. *If* $u \in \mathscr{H}_0^+$ *and* $-\mathscr{L}_0$ *is a sub-Laplacian* $\Delta_{\mathbb{G}}$ *on a Carnot group* $\mathbb{G}$, *then* $u = \mathbf{1}$, *where* $\mathbf{1}$ *is the constant function taking the value* 1 *in* $\mathbb{R}^N$.

Remark 4.3. If $\mathscr{L}_0 = -\sum_{jk}^m a_{jk}X_j X_k$ for some symmetric positive definite constant matrix $A = \left(a_{jk}\right)_{j,k=1,m}$, then the result of Theorem 4.1 clearly applies with

$$u(x) = u_{A;\alpha}(x) = \exp\left(\langle A^{-1}x, \alpha\rangle\right),$$

with $\alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0) \in \mathbb{R}^N$ such that $\langle A^{-1}\alpha, \alpha\rangle = -\lambda$.

5 Parabolic Liouville theorems

In the present section we assume that $\mathscr{L}$ is a hypoelliptic operator of the form

$$\mathscr{L} := \partial_t + \mathscr{L}_0, \qquad \mathscr{L}_0 := -\sum_{j=1}^m X_j^2 \tag{5.1}$$

satisfying (H0') and (H1'). In particular, $\mathscr{L}$ is of the form (1.1) with $X_0 = 0$.

We say that $\mathscr{L}_0$ satisfies the *nonnegative Liouville property* if any nonnegative solution of $\mathscr{L}_0 u = 0$ in $\mathbb{R}^N$ is equal to a constant. Recall that

$$\lambda_0 := \sup\left\{\lambda \in \mathbb{R} \mid \exists u_\lambda \gneqq 0 \ \text{s.t.} \ (\mathscr{L}_0 - \lambda)\,u_\lambda = 0 \text{ in } \mathbb{R}^N\right\}$$

denotes the generalized principal eigenvalue of the operator $\mathscr{L}_0$.

We assume that

a) $\mathscr{L}_0$ satisfies the nonnegative Liouville property,

b) $\lambda_0 = 0$.

We note that the nonnegative Liouville property clearly implies the Liouville property for *bounded* solutions: any bounded solution of $\mathscr{L}_0 u = 0$ in $\mathbb{R}^N$ is equal to a constant.

Properties *(a)-(b)* hold whenever $\mathbb{G}$ is nilpotent and $\mathscr{L}_0 = \mathscr{L}_0^*$ (see [33] for a similar statement), and in particular, under the assumptions of Theorem 4.1 (see the aforementioned theorem and Corollary 4.2, see also [26]). Property *(a)* also holds when all the X_j's are homogeneous of degree 1 with respect to a dilation group. It is also true for a wide class of operators including Grushin-type operators

$$\mathscr{L}_0 = -\partial_x^2 - x^{2\alpha}\partial_y^2\,,$$

where α is any positive constant (see [27]). Property *(b)* is well studied in the nondegenerate case, and our Theorem 4.1 is a first result for degenerate operators. We aim to study this property under more general assumptions in a forthcoming work.

Since $X_0 = 0$, Theorem 1.4 implies that a nonzero $u \in \exp \mathscr{H}^+$ if and only if it satisfies the separation principle, namely,

$$u(x,t) = e^{-\lambda t}\varphi_\lambda(x),$$

where φ_λ is an extreme positive solution of the equation $\left(\sum_{j=1}^m X_j^2 + \lambda\right)u = 0$ in $\mathbb{R}^N$, and $\lambda \leq \lambda_0 = 0$. Consequently, the following nonnegative Liouville theorem holds for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}$.

Theorem 5.1. *Assume that $\mathscr{L}_0$ satisfies the nonnegative Liouville property and that $\lambda_0 = 0$. Let $u \geq 0$ be a solution of the equation*

$$\left(\partial_t + \mathscr{L}_0\right)u = \partial_t u - \sum_{j=1}^m X_j^2 u = 0 \qquad in \quad \mathbb{R}^N \times \mathbb{R}$$

such that
$$u(0,t) = O(e^{\varepsilon t}) \qquad as \; t \to \infty,$$

for any $\varepsilon > 0$. Then $u = $ constant.

This result should be compared with the Liouville theorems proved by Kogoj and Lanconelli in [23, 24, 26], where it was assumed that the operator $\mathscr{L}$ is of the form (1.1), $\mathscr{L}$ is not necessarily translation invariant, but it is invariant with respect to a dilation group $(\delta_r)_{r>0}$, and satisfies an *oriented connectivity condition* that is, (using our notation)

$$\mathscr{A}_{(x_0,t_0)} = \mathbb{R}^N \times\,]-\infty, t_0[, \qquad \text{for every} \quad (x_0,t_0) \in \mathbb{R}^N \times \mathbb{R}_T. \tag{5.2}$$

In this case, a (stronger) sufficient growth condition for the validity of the above Liouville theorem is

$$u(0, t) = O(t^m) \qquad \text{as } t \to \infty, \text{ for some } m > 0.$$

In particular, in this case, the nonnegative Liouville theorem holds true for the stationary equation (without any growth condition, see [23, Corollary 1.2]).

6 Positive Cauchy Problem

In the present section we consider the positive Cauchy problem for $\mathscr{L}$ in $S_T :=$ $\mathbb{R}^N \times]0, T[$ with $0 < T \leq +\infty$, where $\mathscr{L}$ is of the form (5.1). Our aim is to prove the following uniqueness result for the positive Cauchy problem under the assumption that $X_0 = 0$.

Theorem 6.1. *Let $\mathscr{L}$ be an operator of the form* (5.1), *satisfying* (H0') *and* (H1'), *and let $u_0 \geq 0$ be a continuous function in $\mathbb{R}^N$. Then the positive Cauchy problem*

$$\begin{cases} \partial_t u = \sum_{j=1}^m X_j^2 u & (x, t) \in S_T, \\ u(x, 0) = u_0(x) \geq 0 & x \in \mathbb{R}^N, \\ u(x, t) \geq 0 & (x, t) \in S_T, \end{cases} \tag{6.1}$$

admits at most one solution.

We note that the first uniqueness result for the positive Cauchy problem was established by Widder for the classical heat equation in the Euclidean space [45].

The proof of Theorem 6.1 relies on Theorem 1.4 which, under the additional assumption $X_0 = 0$, asserts that every nonnegative extremal solution u of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ satisfies

$$u(x, t) = \mathrm{e}^{-\lambda t} u_0(x) \qquad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}_T,$$

where $\lambda \leq \lambda_0$, and λ_0 is the generalized principal eigenvalue (see Remark 2.4).

Before giving the proof of Theorem 6.1, we should compare it with a result of Chiara Cinti [12] who considered a class of left translation invariant hypoelliptic operators with nontrivial drift X_0 under the additional hypothesis that the operator is *homogeneous* with respect to a group of dilations on the underlying Lie group. The method used in [12] relies on some accurate upper and lower bounds of the fundamental solution of $\mathscr{L}$. We note that the lower bounds for the fundamental solution are usually obtained by constructing suitable *Harnack chains*, as the ones used in the proof of Theorem 1.4. On the other hand, in order to apply the method used in [12], the upper and lower bounds need to agree asymptotically. Hence, the Harnack chains need to be chosen

in some optimal way. An advantage of our method is that it does not require such an optimization step. Actually, a priori bounds of the fundamental solution, and even its existence are not needed. We also note that the bibliography of [12] contains an extensive discussion of known results on the uniqueness of the Cauchy problem. We also recall a recent result by Bumsik Kim [21] for the heat equation associated with subelliptic diffusion operators. In his work, Kim proves uniqueness results for the heat equation under curvature bounds through the generalized curvature-dimension criterion developed by Baudoin and Garofalo and thus without the Lie group assumption.

We start the proof of Theorem 6.1 with some preliminary results that do not require the assumption $X_0 = 0$.

Consider the positive Cauchy problem

$$\begin{cases} \mathscr{L}u(x,t) = 0 & (x,t) \in S_T, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \\ u(x,t) \geq 0 & (x,t) \in S_T, \end{cases} \tag{6.2}$$

with $u_0 \geq 0$ continuous function in $\mathbb{R}^N$.

We first recall some basic results on hypoelliptic operators of the form (1.1). Usually, hypoelliptic operators have been studied under the further assumption that they are *non-totally degenerate*, namely, there exists a vector $\nu \in \mathbb{R}^N$ and $j \in \{1, \ldots, m\}$ such that

$$\langle X_j(x), \nu \rangle \neq 0 \qquad \text{for all } x \in \mathbb{R}^N. \tag{6.3}$$

This condition was introduced by Bony in [7] and is not very restrictive. We also refer to [5] for a weaker version of this condition.

We observe that (6.3) can be always satisfied by a simple *lifting* procedure. Indeed, let $\mathscr{L}$ be of the form (1.1), and consider the operator $\widetilde{\mathscr{L}}$ acting on $(x_0, x, t) \in \mathbb{R}^{N+2}$ and defined by

$$\widetilde{\mathscr{L}}u := -\partial_{x_0}^2 u + \mathscr{L}u = \partial_t u - \partial_{x_0}^2 u - \sum_{j=1}^m X_j^2 u + X_0 u.$$

Clearly, $\widetilde{\mathscr{L}}$ is non-totally degenerate with respect to $\nu = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$. Moreover, $\widetilde{\mathscr{L}}$ is hypoelliptic and satisfies (H1) and (H2) if $\mathscr{L}$ is hypoelliptic and satisfies (H1) and (H2). Our uniqueness result for $\mathscr{L}$ readily follows from the uniqueness for $\widetilde{\mathscr{L}}$. Therefore, in the sequel we assume that $\mathscr{L}$ satisfies (6.3).

We recall Bony's *strong maximum principle* [7, Théorème 3.2] for hypoelliptic operators $\mathscr{L}$ of the form (1.1) that satisfy (6.3). With our notation, it reads as follows. *Let Ω be any open subset of $\mathbb{R}^{N+1}$ and let $u \in C^2(\Omega)$ be such that $\mathscr{L}u \leq 0$ in Ω. Let $z_0 \in \Omega$ be such that $u(z_0) = \max_\Omega u$. If $\gamma : [0, T_0] \to \Omega$ is an $\mathscr{L}$–admissible path such that $\gamma(0) = z_0$, then $u(\gamma(s)) = u(z_0)$ for every $s \in [0, T_0]$.*

The following *weak maximum principle* can be obtained as a consequence of Bony's strong maximum principle. *Let Ω be any bounded open set of $\mathbb{R}^{N+1}$ and let $u \in C^2(\Omega)$ be such that $\mathscr{L}u \leq 0$ in Ω. If $\limsup_{\substack{z \to w \\ z \in \Omega}} u(z) \leq 0$ for every $w \in \partial\Omega$, then $u \leq 0$ in Ω.*

Let Ω be any bounded open set of $\mathbb{R}^{N+1}$, and let $\varphi \in C(\partial\Omega)$. The axiomatic potential theory provides us with the Perron solution u_φ of the boundary value problem $\mathscr{L}u = 0$ in Ω, $u = \varphi$ in $\partial\Omega$. It is known that u_φ might attain the prescribed boundary data only in a subset of $\partial\Omega$. We say that $w \in \partial\Omega$ is *regular* for $\mathscr{L}$ if $\lim_{\substack{z \to w \\ z \in \Omega}} u_\varphi(z) \to \varphi(w)$ for every $\varphi \in C(\partial\Omega)$. we denote by $\partial_r(\Omega)$ the set of the regular points of $\partial\Omega$

$$\partial_r(\Omega) := \Big\{ w \in \partial\Omega \mid \lim_{\substack{z \to w \\ z \in \Omega}} u_\varphi(z) \to \varphi(w) \ \text{ for every } \varphi \in C(\partial\Omega) \Big\}.$$

Under assumption (6.3) it is possible to construct a family of *regular cylinders* of $\mathbb{R}^{N+1}$, that is cylinders such that their regular boundary agree with their *parabolic boundary* [31]. Specifically, we denote by $B(x, r)$ the Euclidean ball centered at $x \in \mathbb{R}^N$ with radius r. Let ν be a vector satisfying (6.3), and assume, as it is not restrictive, that $|\nu| = 1$. For every $x \in \mathbb{R}^N$ and $k \in \mathbb{N}$ we set

$$B_k(x) := B(x + k\nu, 2k) \cap B(x - k\nu, 2k).$$

It turns out that for every $x \in \mathbb{R}^N$, $k \in \mathbb{N}$, and $0 < T_0 < \infty$, the cylinder $Q_{k,T_0}(x) := B_k(x) \times \,]0, T_0[$ is regular, see [31] for a detailed proof of this statement.

We note that the sequence of regular cylinders $(Q_{k,T_0}(0))_{\substack{0 < T_0 < T \\ k \in \mathbb{N}}}$ exhausts the set S_T. This property will be used in the sequel.

Consider a regular cylinder $Q := B \times \,]0, T_0[$ and a function $f \in C(\overline{Q})$. In [31, Theorem 2.5] it is proved that for a hypoelliptic operator $\mathscr{L}$ of the form (1.1) satisfying (6.3) there exists a unique solution $u \in C^\infty(Q) \cap C(Q \cup \partial_r(Q))$ to the following initial-boundary value problem

$$\begin{cases} \mathscr{L}u = f & \text{in } Q, \\ u = 0 & \text{in } \partial_r Q. \end{cases} \tag{6.4}$$

We next show that the same result holds when a continuous compactly supported initial condition is prescribed on the bottom of Q.

Lemma 6.2. *Let $\mathscr{L}$ be a hypoelliptic operator of the form (1.1) satisfying (6.3), and let $Q := B \times \,]0, T_0[$ be a regular cylinder. Let $\varphi \in C(B)$ be such that $\operatorname{supp}(\varphi) \subset B$. Then there exists a unique solution $u \in C^\infty(Q) \cap C(Q \cup \partial_r Q)$ to the following initial-boundary value problem*

$$\begin{cases} \mathscr{L}u = 0 & \text{in } Q, \\ u(x, t) = 0 & \text{in } \partial B \times [0, T_0], \\ u(x, 0) = \varphi(x) & \text{in } B \times \{0\}. \end{cases} \tag{6.5}$$

Proof. We use a standard argument. Consider, for any positive ε, a function $w_\varepsilon \in C^\infty\left(\overline{Q}\right)$ such that $w_\varepsilon(\cdot, 0) \to \varphi$, uniformly as $\varepsilon \to 0$, and takes the zero boundary condition at the lateral boundary of Q. Denote by $f_\varepsilon := \mathscr{L}w_\varepsilon$, and note that f_ε is continuous on $\overline{Q}$. We recall that we can solve uniquely the initial-boundary value problem of the form (6.4). So, let v_ε be the unique solution of the following problem

$$\begin{cases} \mathscr{L}v_\varepsilon = f_\varepsilon & \text{in } Q, \\ v_\varepsilon = 0 & \text{in } \partial_r Q. \end{cases}$$

The function $u_\varepsilon := w_\varepsilon - v_\varepsilon$ is clearly the unique solution of

$$\begin{cases} \mathscr{L}u_\varepsilon = 0 & \text{in } Q, \\ u_\varepsilon(x, t) = 0 & \text{in } \partial B \times [0, T_0], \\ u_\varepsilon(x, 0) = w_\varepsilon(x, 0) & \text{in } B \times \{0\}. \end{cases}$$

By the maximum principle, u_ε uniformly converges to a continuous function u that is a classical solution of (6.5). The uniqueness follows from the weak maximum principle. $\square$

Next, we apply the well-known argument (introduced by Donnelly for nondegenerate parabolic equations [16]) to show that the uniqueness for the positive Cauchy problem is equivalent to the uniqueness of the positive Cauchy problem with the *zero* initial condition. For this sake, we prove the following proposition, which clearly implies the above equivalence.

Proposition 6.3. *Let $\mathscr{L}$ be a hypoelliptic operator of the form* (1.1), *satisfying* (6.3). *If $u \in C(\overline{S_T}) \cap C^\infty(S_T)$ is a solution of the positive Cauchy problem* (6.2), *then there exists a minimal nonnegative solution $\widetilde{u}$ of* (6.2). *Namely, $0 \leq \widetilde{u} \leq v$ in S_T for any solution v of* (6.2).

Proof. We use a standard exhaustion argument. Consider a sequence of continuous functions $\psi_k : \mathbb{R}^N \to \mathbb{R}$ such that $0 \leq \psi_k(x) \leq 1$, for any $k \in \mathbb{N}$, and that $\psi_k(x) = 1$ whenever $|x| \leq k$, and $\psi_k(x) = 0$ if $|x| \geq k + 1$. Consider a sequence $Q_k := \Omega_k \times]0, T_k[$ of regular cylinders, such that $\text{supp}(\psi_k) \subset \Omega_k$, and $T_k \nearrow T$. Let $\widetilde{u}_k$ be the solution to

$$\begin{cases} \mathscr{L}\widetilde{u}_k = 0 & \text{in } Q_k, \\ \widetilde{u}_k(x, t) = 0 & \text{in } \partial\Omega_k \times [0, T_k], \\ \widetilde{u}_k(x, 0) = \psi_k(x)u_0(x) & \text{in } \Omega_k \times \{0\}, \end{cases}$$

whose existence is given by Lemma 6.2. By the comparison principle, $(\widetilde{u}_k)_{k \in \mathbb{N}}$ is a nondecreasing sequence of nonnegative solutions of the equation $\mathscr{L}\widetilde{u}_k = 0$, such that $\widetilde{u}_k(x, t) \leq u(x, t)$. Then, the function

$$\widetilde{u}(x, t) := \lim_{k \to \infty} \widetilde{u}_k(x, t)$$

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is a distributional solution of $\mathscr{L}\widetilde{u} = 0$ in S_T such that $0 \leq \widetilde{u} \leq u$ in S_T. By the hypoellipticity of $\mathscr{L}$, $\widetilde{u}$ is a smooth classical solution of the equation $\mathscr{L}\widetilde{u} = 0$ in S_T. In order to prove that $\widetilde{u}$ takes the initial condition, we fix any $x_0 \in \mathbb{R}^N$, and we choose $k_0 > |x_0|$. We have

$$\widetilde{u}_{k_0}(x, t) \leq \widetilde{u}(x, t) \leq u(x, t),$$

for every $(x, t) \in Q_k$ with $k \in \mathbb{N}$. Consequently, $\widetilde{u}(x, t) \to u_0(x_0)$ as $(x, t) \to (x_0, 0)$, and this concludes the proof. $\square$

Corollary 6.4. *The positive Cauchy problem has a unique solution if and only if any nonnegative solution of the positive Cauchy problem with $u_0 = 0$ is the trivial solution $u = 0$.*

In the following proof of Theorem 6.1, which relies on Choquet's integral representation theorem and the separation principle (1.21), we resume the assumption $X_0 = 0$.

Proof of Theorem 6.1. By Corollary 6.4, we may assume that $u_0 = 0$.

So, let $S_T = \mathbb{R}^N \times]0, T[$ with $0 < T \leq +\infty$, and let $u : S_T \to \mathbb{R}$ be a solution of the positive Cauchy problem

$$\begin{cases} \mathscr{L}u(x, t) = 0 & (x, t) \in S_T, \\ u(x, 0) = 0 & x \in \mathbb{R}^N, \\ u(x, t) \geq 0 & (x, t) \in S_T. \end{cases} \tag{6.6}$$

We need to prove that $u = 0$.

As in [29], we extend the solution u of the Cauchy problem (6.6) to the whole domain $\mathbb{R}^N \times \mathbb{R}_T$ by setting

$$\widetilde{u}(x, t) := \begin{cases} u(x, t) & t \in [0, T[, \\ 0 & t < 0. \end{cases}$$

It is easy to see that $\widetilde{u}$ is a distributional solution of $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$. Hence, the hypoellipticity of $\mathscr{L}$ yields that $\widetilde{u}$ is a nonnegative smooth classical solution of the equation

$$\mathscr{L}w = 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}_T, \tag{6.7}$$

and $\widetilde{u} = 0$ in $\mathbb{R}^N \times \mathbb{R}^-$. We need to prove that $\widetilde{u} = 0$ in S_T.

Suppose that $u \neq 0$, and let $a \in C(]-\infty, T[)$ be a nonnegative function such that $\widetilde{u} \in \mathscr{H}_a^1$. By Choquet's integral representation theorem and (2.6), it follows that $\widetilde{u}$ can be represented as

$$\widetilde{u}(x, t) = \int_{\mathscr{H}_+} v(x, t) d\mu(v) \tag{6.8}$$

for some probability measure μ supported on $\{0\} \cup \{\text{exr } \mathscr{H}_+ \cap \mathscr{H}_a^1\}$. Recall that $\widetilde{u}(x, t) = 0$ for $t \leq 0$. On the other hand, by (1.18) with $\omega = 0$, any nonnegative

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solution $v \in \{\operatorname{exr} \mathscr{H}_+ \cap \mathscr{H}_a^1\}$ is strictly positive in a neighborhood of an integral curve of the form

$$\gamma := \{\exp(sY)\, z_0 \mid s \in \,]t_0 - T, +\infty[\} = \{x_0\} \times \,] - \infty, t_0[,$$

where $Y = -\partial_t$, and $z_0 = (x_0, t_0)$ might depend on v. Bony's strong maximum principle implies that v is strictly positive in a neighborhood of the line $\{x_0\} \times \mathbb{R}^-$. Therefore, Proposition 2.2 concerning the existence of a reference set, implies that all such v are strictly positive in $\mathbb{R}^N \times \mathbb{R}^-$. Therefore, (6.8) implies that

$$\mu\{\operatorname{exr} \mathscr{H}_+ \cap \mathscr{H}_a^1\} = 0.$$

Hence, $\widetilde{u} = 0.$ $\qquad\square$

7 Mumford operator

The Mumford operator $\mathscr{M}$ is defined as

$$\mathscr{M}u := \partial_t u - \cos(x)\partial_y u - \sin(x)\partial_w u - \partial_x^2 u \qquad (x, y, w, t) \in \mathbb{R}^4. \tag{7.1}$$

It models the relative likelihood of different edges disappearing in some scene to be matched up by some hidden edges, and explains the role of *elastica* in computer vision [37]. In the present section we prove the uniqueness of the positive Cauchy problem for $\mathscr{M}$, and we establish some properties of the minimal positive solutions of $\mathscr{M}u = 0$. The following proposition allows us to apply our results to $\mathscr{M}$.

Proposition 7.1. *The Mumford operator $\mathscr{M}$ satisfies conditions* (H0) *with the group operation*

$$\begin{aligned}(x_0, y_0, w_0, t_0) \circ (x, y, w, t) := \\ \big(x_0 + x, y_0 + y\cos(x_0) - w\sin(x_0), \\ w_0 + y\sin(x_0) + w\cos(x_0), t_0 + t\big)\end{aligned} \tag{7.2}$$

for every $(x_0, y_0, w_0, t_0), (x, y, w, t) \in \mathbb{R}^4$. Moreover, $\mathscr{M}$ satisfies (H2) *with $\omega \neq 0$.*

Proof. Condition (H0) is verified by a direct computation. Moreover, it is known that $\mathscr{M}$ is invariant with respect to the left translations of the group $\mathbb{G} := (\mathbb{R}^3 \times \mathbb{R}, \circ)$ on $\mathbb{R}^4$ whose operation is defined by (7.2), (see [5, Formula (61)]). $\mathbb{G}$ is called in the literature the *roto-translation group*.

In order to check (H2), we note that

$$\exp(sY)(x, y, w, t) = (x, y + s\cos(x), w + s\sin(x), t - s), \tag{7.3}$$

where $Y = \cos(x)\partial_y + \sin(x)\partial_w - \partial_t$ (see (1.2)), while

$$\exp\left(s(\omega X + Y)\right)(x, y, w, t) =$$
$$\left(x + s\omega, y + \frac{\sin(x + s\omega) - \sin(x)}{\omega}, w - \frac{\cos(x + s\omega) - \cos(x)}{\omega}, t - s\right), \quad (7.4)$$

for every $(x, y, w, t) \in \mathbb{R}^4$, and $s, \omega \in \mathbb{R}$, with $\omega \neq 0$.

We first show that

$$\mathscr{A}_{z_0} = \left\{(x, y, w, t) \in \mathbb{R}^4 \mid \sqrt{(y - y_0)^2 + (w - w_0)^2} \leq t_0 - t\right\}, \quad (7.5)$$

for every $z_0 = (x_0, y_0, w_0, t_0) \in \mathbb{R}^4$. The inclusion $\mathscr{A}_{z_0}$ in the right hand side of (7.5) follows directly from the definition of attainable set, and from the fact that the norm of the drift term $X_0 = \cos(x)\partial_y + \sin(x)\partial_w \simeq (0, \cos(x), \sin(x), 0)$ equals 1.

We next prove the inclusion of the right hand side of (7.5) in $\mathscr{A}_{z_0}$. We first note that, by the invariance with respect to the Lie operation (7.2), it is not restrictive to assume that $(x, y, w, t) = 0$. We also assume that $(y_0, w_0) \neq (0, 0)$ since $\mathscr{A}_{z_0}$ is the closure of the set of the reachable points. We introduce polar coordinates; $\widetilde{x} = -\arg(y_0, w_0)$, and $\widetilde{t} = \sqrt{y_0^2 + w_0^2}$, and we note that

$$(y_0, w_0) = -\widetilde{t}(\cos(\widetilde{x}), \sin(\widetilde{x})) \qquad 0 < \widetilde{t} \leq t_0. \quad (7.6)$$

We define the sequence of paths $(\gamma_k)_{k \in \mathbb{N}}$ in the interval $[0, \widetilde{t}]$ by choosing

$$x_k(0) = x_0, \qquad x_k(\widetilde{t}) = 0, \qquad x_k(s) = \widetilde{x}, \quad \text{for} \quad \frac{\widetilde{t}}{4k} \leq s \leq \left(1 - \frac{1}{4k}\right)\widetilde{t},$$

and x_k linear in $\left[0, \frac{\widetilde{t}}{4k}\right]$ and in $\left[\left(1 - \frac{1}{4k}\right)\widetilde{t}, \widetilde{t}\right]$. If $\widetilde{t} < t_0$, we set $x_k(s) = 2\pi(s - t_0 + \widetilde{t})/\widetilde{t}$, for every $s \in [t_0 - \widetilde{t}, t_0]$. Moreover,

$$y_k(s) = y_0 + \int_0^s \cos(x_k(\tau))d\tau, \quad w_k(s) = w_0 + \int_0^s \sin(x_k(\tau))d\tau, \quad t_k(s) = t_0 - s.$$

We clearly have that $x_k(t_0) = 0, t_k(t_0) = 0$. Moreover, a simple computation based on (7.6) gives $|y_k(t_0)| = |y_k(\widetilde{t})| \leq \frac{1}{2k}(|y_0| + \widetilde{t}) \leq \frac{1}{2k}(|y_0| + t_0)$ and, analogously, $|w_k(t_0)| \leq \frac{1}{2k}(|w_0| + t_0)$. This proves that $\gamma_k(t_0) \to 0$ as $k \to +\infty$. In particular $0 \in \mathscr{A}_{z_0}$, and the proof of (7.5) is completed.

The above argument also applies to any bounded open box Ω which is sufficiently wide in the x-direction. More precisely, if $\Omega =]x_0 - R_x, x_0 + R_x[\times]y_0 - R_y, y_0 + R_y[\times]w_0 - R_w, w_0 + R_w[\times]t_0 - R_t, t_0 + R_t[$ with $R_x > \pi$, then

$$\mathscr{A}_{z_0}(\Omega) = \left\{(x, y, w, t) \in \Omega \mid \sqrt{(y - y_0)^2 + (w - w_0)^2} \leq t_0 - t\right\}.$$

Note that, by (7.3) and (7.4), we have that $\exp\left(s(\omega X + Y)\right)(z_0)$ belongs to the interior of $\mathscr{A}(z_0)(\Omega)$ if, and only if, $\omega \neq 0$. This proves (H2). $\qquad \square$

We next prove a separation principle for the extremal solutions of the equation $\mathscr{M}u = 0$. We have

Proposition 7.2. *For every $u \in \mathrm{exr}\, \mathscr{H}_+$ there exist two constants $\beta \in \mathbb{R}$ and $C_0 > 0$ such that*

$$u(x + 2k\pi, y, w, t) = C_0^k e^{\beta t} u(x, y, w, 0) \qquad \text{for every} \quad (x, y, w, t) \in \mathbb{R}^N \times \mathbb{R}_T, \ k \in \mathbb{Z}.$$

In particular for $k = 0$, we have

$$u(x, y, w, t) = e^{\beta t} u(x, y, w, 0) \qquad \text{for every} \quad (x, y, w, t) \in \mathbb{R}^N \times \mathbb{R}_T.$$

Proof. We first prove that

$$u(x, y, w, t - s) = e^{-\beta s} u(x, y, w, t) \quad \text{for every} \quad (x, y, w, t) \in \mathbb{R}^N \times \mathbb{R}_T, \ s > 0. \quad (7.7)$$

Fix any positive s, and choose $\omega = 2\pi/s$. Recall (7.4), and note that

$$\exp\left(s(-\omega X + Y)\right)\left(\exp\left(s(\omega X + Y)\right)(x, y, w, t)\right) = (x, y, w, t - 2s),$$

and that the change of variable $(x, y, w, t) \mapsto (x, y, w, t - 2s)$ preserves the equation $\mathscr{M}u = 0$. Then the hypotheses of Proposition 1.5 are satisfied with $\omega_1 = -\omega_2 := \omega$ and $s_1 = s_2 := s$. Hence we have

$$u(x, y, w, t - 2s) = Cu(x, y, w, t)$$

for some positive constant $C = C(s)$. Hence, (7.7) followed as in the last part of the proof of Theorem 1.4.

In order to conclude the proof, we consider again a positive s, we set $\omega = 2\pi/s$, and we note that

$$\exp\left(s(\omega X + Y)\right)(x, y, w, t) = (x + 2\pi, y, w, t - s).$$

Also in this case the assumptions of Proposition 1.5 are satisfied with $\omega_1 := \omega$ and $s_1 := s$, thus there exists a positive constant C such that

$$u(x + 2\pi, y, w, t - s) = Cu(x, y, w, t) \quad \text{for every} \quad (x, y, w, t) \in \mathbb{R}^N \times \mathbb{R}_T.$$

The conclusion of the proof then follows by combining the above identity with (7.7). $\quad\square$

The following result is a corollary of Proposition 7.2.

Theorem 7.3. *Let $\mathscr{M}$ be the Mumford operator (7.1), and let $u_0 \geq 0$ be a continuous function in $\mathbb{R}^3$. Then the positive Cauchy problem*

$$\begin{cases} \mathscr{M}u(x, y, w, t) = 0 & (x, y, w, t) \in S_T, \\ u(x, y, w, 0) = u_0(x, y, w) & (x, y, w) \in \mathbb{R}^3, \\ u(x, y, w, t) \geq 0 & (x, t) \in S_T, \end{cases}$$

admits at most one solution.

Proof. The proof is exactly as in the proof of Theorem 6.1, once the separation principle (7.7) has been established. We omit the details. $\quad\square$

8 Kolmogorov-Fokker-Planck operators

Consider the Kolmogorov operator

$$\mathscr{L}u(x,y,t) := \partial_t u(x,y,t) - \sum_{j=1}^m \partial_{x_j}^2 u(x,y,t) - \sum_{j=1}^m x_j \partial_{y_j} u(x,y,t), \qquad (8.1)$$

with $(x,y,t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. As usual, we denote $\mathbb{R}^N \times \mathbb{R}_T = \mathbb{R}^{2m} \times]-\infty, T[$. The operator $\mathscr{L}$ can be written in the form (1.1) by setting $X_j := \partial_{x_j}$ for $j = 1, \ldots, m$, and $X_0 := \sum_{j=1}^m x_j \partial_{y_j}$. It follows that $\mathscr{L}$ satisfies Hörmander's condition (H0). The vector fields X_j's and $Y := X_0 - \partial_t$ are invariant with respect to the left translations and the dilation defined by

$$(\xi, \eta, \tau) \circ (x,y,t) := (x + \xi, y + \eta - t\xi, t + \tau), \qquad \delta_r(x,y,t) := (rx, r^3 y, r^2 t), \quad (8.2)$$

respectively. An invariant Harnack inequality for Kolmogorov equations was first proved by Garofalo and Lanconelli in [18]. It can be written in its restricted form as in Proposition 1.2 with $\omega = 0$. It reads as

$$u(x, y + \tau x, t - \tau) \le C_\tau\, u(x, y, t) \qquad \text{for every } (x,y,t) \in \mathbb{R}^{2m+1} \text{ and } \tau > 0. \quad (8.3)$$

We stress that due to the drift term $X_0 - \partial_t$, the Harnack inequality for Kolmogorov equations is different from (3.7). The above discussion applies to the following more general class of operators of the above type, first studied by Lanconelli and Polidoro in [32]. We also refer to the book by Lorenzi and Bertoldi [34] and to the bibliography therein for results on Kolmogorov equations obtained by semigroup theory.

We summarize the properties of $\mathscr{L}$ that are needed for its study in our functional setting. Condition (H0) can be verified by a direct computation, while the group operation required to satisfy (H1) is defined in (8.2). Condition (H2) holds for every $\omega \in \mathbb{R}^m$. In the sequel we choose $\omega = 0$.

We use the explicit expression of the (nonnegative) fundamental solution Γ of $\mathscr{L}$ to compute the Martin functions for $\mathscr{L}$ in $\mathbb{R}^{2m} \times]-\infty, T[$. We recall that this method has been used in [14] (see (1.2) therein) to compute the complete parabolic and elliptic Martin boundary for nondegenerate Ornstein-Uhlenbeck processes in dimension two (see also [17] for other explicit examples of computing parabolic Martin boundaries).

We recall the definition of Martin functions for our case. Assume for simplicity that $T < \infty$. We say that a sequence $\{(\xi_k, \eta_k, \tau_k)\}_{k \in \mathbb{N}}$ is a *fundamental sequence* if $\|(\xi_k, \eta_k, \tau_k)\| \to +\infty$ as $k \to \infty$ and the corresponding sequence of *Martin quotients* $\{u_k\}$ given by

$$u_k(x,y,t) := \frac{\Gamma(x,y,t,\xi_k,\eta_k,\tau_k)}{\Gamma(0,0,T,\xi_k,\eta_k,\tau_k)} \qquad (8.4)$$

converges to a nonnegative solution $u(x,y,t) := \lim_{k \to \infty} u_k(x,y,t)$ in $\mathscr{H}_+$. Such a u is called a *Martin function u* for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}_T$. It is a is a nonnegative solution of

$\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ which is defined by some fundamental sequence $(\xi_k, \eta_k, \tau_k)_{k \in \mathbb{N}}$. Note that $\Gamma(0, 0, T, \xi_k, \eta_k, \tau_k) = 0$ if and only if $T \le \tau_k$, hence we need to assume $T > \tau_k$ for every $k \in \mathbb{N}$.

The explicit form of the fundamental solution Γ of Kolmogorov operator is known and is given by

$$\Gamma(x, y, t, \xi, \eta, \tau) = \left(\tfrac{3}{2\pi}\right)^{m/2} \frac{1}{(t-\tau)^{2m}} \exp\left(-\frac{\|x-\xi\|^2}{4(t-\tau)} - 3\frac{\|y - \eta + \frac{t-\tau}{2}(x+\xi)\|^2}{(t-\tau)^3}\right) \quad (8.5)$$

if $t > \tau$, while $\Gamma(x, y, t, \xi, \eta, \tau) = 0$ if $t \le \tau$.

We have

Proposition 8.1. *Let $\mathscr{L}$ be the Kolmogorov operator (8.1), and let u be a Martin function for $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$. Then either $u = 0$, or there exists $v \in \mathbb{R}^m$ such that*

$$u(x, y, t) = \exp\left(\langle x, v\rangle + t\|v\|^2\right) \quad \text{for all } (x, y, t) \in \mathbb{R}^N \times \mathbb{R}_T. \quad (8.6)$$

Since in any Bauer harmonic space all the extremal solutions are Martin kernels (see, Proposition 4.1 and Theorem 5.1 in [35]), we have

Corollary 8.2. *Any nonnegative solution $u = u(x, y, t)$ of the Kolmogorov equation $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ does not depend on the variable y, and u is a nonnegative solution of the heat equation $\partial_t w(x, t) = \Delta w(x, t)$ in $\mathbb{R}^m \times \mathbb{R}_T$.*

In particular, any nonzero nonnegative solution of the equation $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ is strictly positive, and the uniqueness of the positive Cauchy problem in S_T holds true.

The uniqueness of the positive Cauchy problem in S_T for the Kolmogorov equation was first proved in [43] by a different method.

Proof of Proposition 8.1. Assume, as it is not restrictive, that $T = 0$, let u be a Martin function for $\mathscr{L}$ in $\mathbb{R}^N \times \mathbb{R}_T$, and let $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}_T$. In order to prove our claim, we preliminarily note that

$$-\frac{1}{4}\left(\frac{\|x-\xi_k\|^2}{t-\tau_k} - \frac{\|\xi_k\|^2}{-\tau_k}\right) = -\frac{1}{4}\left(\frac{\|x\|^2}{t-\tau_k} - 2\frac{\langle x, \xi_k\rangle}{t-\tau_k} - t\frac{\|\xi_k\|^2}{(t-\tau_k)(-\tau_k)}\right), \quad (8.7)$$

and that

$$-3\left(\frac{\|y - \eta_k + \frac{t-\tau_k}{2}(x+\xi_k)\|^2}{(t-\tau_k)^3} - \frac{\|\eta_k + \frac{\tau_k}{2}\xi_k\|^2}{(-\tau_k)^3}\right) =$$
$$-3\frac{\|y + \frac{t}{2}x\|^2}{(t-\tau_k)^3} - \frac{3}{4}\frac{\|t\xi_k - \tau_k x\|^2}{(t-\tau_k)^3} + 6\frac{\langle y + \frac{t}{2}x, \eta_k + \frac{\tau_k}{2}\xi_k\rangle}{(t-\tau_k)^3}$$
$$-3\frac{\langle y + \frac{t}{2}x, t\xi_k - \tau_k x\rangle}{(t-\tau_k)^3} + 3\frac{\langle \eta_k + \frac{\tau_k}{2}\xi_k, t\xi_k - \tau_k x\rangle}{(t-\tau_k)^3}$$
$$+ \left(9t\tau_k^2 - 9t^2\tau_k + 3t^3\right)\frac{\|\eta_k + \frac{\tau_k}{2}\xi_k\|^2}{(t-\tau_k)^3(-\tau_k)^3}. \quad (8.8)$$

We next choose a fundamental sequence $\left((\xi_k, \eta_k, \tau_k)\right)_{k \in \mathbb{N}}$ such that $u(x, y, t) = 0$ for every $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}_T$. We fix any vector $w \in \mathbb{R}^m$ such that $w \neq 0$, and we set $(\xi_k, \eta_k, \tau_k) = (kw, 0, -1)$. Since $\Gamma(x, y, t, \xi, \eta, \tau) = 0$ if $t \leq \tau$, we have $u_k(x, y, t) = 0$ whenever $t < -1$. A direct computation based on (8.7) and (8.8) shows that $u_k(x, y, t) \to 0$ also if $-1 < t < 0$. We then conclude that $u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$.

Note that, we find the trivial solution whenever a bounded subsequence of $\left(\tau_k\right)_{k \in \mathbb{N}}$ exists. Indeed, let $\left(\tau_{k_j}\right)_{j \in \mathbb{N}}$ be a convergent subsequence of $\left(\tau_k\right)_{k \in \mathbb{N}}$, and denote by $\widetilde{\tau} \in] - \infty, T]$ its limit. Let $(x, y, t) \in \mathbb{R}^{2m+1}$ be fixed, with $t < \widetilde{\tau}$. Then there exists a $J \in \mathbb{N}$ such that $\tau_{k_j} > t$, so that $u_{k_j}(x, y, t) = 0$ for every $j > J$. Thus $u(x, y, t) = 0$ for every (x, y, t) such that $t < \widetilde{\tau}$. This proves the claim if $\widetilde{\tau} = T$. If $\widetilde{\tau} > T$ the uniqueness of the positive Cauchy problem for Kolmogorov equations (see Theorem 3.2 in [43]) implies that $u(x, y, t) = 0$ also when $\widetilde{\tau} < t < T$. For this reason, in the sequel we will always assume that $\tau_k \to -\infty$ as $k \to +\infty$.

We next show that *nontrivial* Martin functions for $\mathscr{L}$ have the form (8.6). We fix $w_1, w_2 \in \mathbb{R}^m$ and we set $(\xi_k, \eta_k, \tau_k) = (2kw_1, k^2 w_2, -k)$. A direct computation based on (8.7) shows that

$$-\frac{1}{4}\left(\frac{\|x - \xi_k\|^2}{t - \tau_k} - \frac{\|\xi_k\|^2}{-\tau_k}\right) \to \langle x, w_1 \rangle + t\|w_1\|^2 \qquad \text{as} \quad k \to \infty. \tag{8.9}$$

A similar argument based on (8.8) applies to the last term in the exponent of (8.5). We have

$$\eta_k + \tfrac{\tau_k}{2}\xi_k = k^2\left(w_2 - w_1\right), \qquad t\xi_k - \tau_k x = k\left(2tw_1 - x\right),$$

then

$$y - \eta_k + \tfrac{t-\tau_k}{2}(x + \xi_k) = -k^2\left(w_2 - w_1\right) + k\left(tw_1 - \tfrac{1}{2}x\right) + y + \tfrac{t}{2}x.$$

Consequently, we find that

$$-3\left(\frac{\|y - \eta_k + \frac{t-\tau_k}{2}(x + \xi_k)\|^2}{(t - \tau_k)^3} - \frac{\|\eta_k + \frac{\tau_k}{2}\xi_k\|^2}{(-\tau_k)^3}\right) =$$
$$-3\frac{-k^6\langle w_2 - w_1, 2tw_1 + x \rangle - 3tk^6\|w_1 - w_2\|^2}{k^3(t + k)^3} + \omega(k),$$

for some function ω such that $\omega(k) \to 0$ as $k \to \infty$. Hence,

$$-3\left(\frac{\|y - \eta_k + \frac{t-\tau_k}{2}(x + \xi_k)\|^2}{(t - \tau_k)^3} - \frac{\|\eta_k + \frac{\tau_k}{2}\xi_k\|^2}{(-\tau_k)^3}\right) \to 3\langle w_2 - w_1, 2tw_1 + x \rangle + 9t\|w_1 - w_2\|^2, \tag{8.10}$$

as $k \to \infty$. Note that the variable y does not appear in last limit. Thus, also using the obvious fact $\left(\frac{-\tau_k}{t-\tau_k}\right)^{2m} \to 1$ as $k \to \infty$, we find

$$u(x, y, t) = \exp\left(\langle x, 3w_2 - 2w_1 \rangle + t\|3w_2 - 2w_1\|^2\right),$$

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and we conclude that u has the form (8.6) if we choose $v = 3w_2 - 2w_1$.

We next show that either u is zero, or has the form (8.6), for every fundamental sequence. With this aim, we consider any sequence $(\xi_k, \eta_k, \tau_k)_{k\in\mathbb{N}}$, with $\tau_k < 0$ for every $k \in \mathbb{N}$, and such that $\tau_k \to -\infty$ as $k \to +\infty$, since we know that, otherwise, u is the trivial solution. We also assume that the function u in (8.4) is well defined.

We set
$$\widetilde{\xi}_k := \tfrac{1}{-\tau_k}\xi_k, \qquad \widetilde{\eta}_k := \tfrac{1}{(-\tau_k)^2}\eta_k, \quad k \in \mathbb{N}. \tag{8.11}$$
and, after some elementary, but lengthly computations, we find that

$$u_k(x,y,t) = \exp\left(\left(\langle x, 3\widetilde{\eta}_k - \widetilde{\xi}_k\rangle + t\|3\widetilde{\eta}_k - \widetilde{\xi}_k\|^2\right)(1 + R_k)\right), \tag{8.12}$$

where $R_k \to 0$ denotes a vanishing sequence. Thus, either $\left\|3\widetilde{\eta}_k - \widetilde{\xi}_k\right\| \to +\infty$ as $k \to +\infty$, or the sequence $\left(3\widetilde{\eta}_k - \widetilde{\xi}_k\right)_{k\in\mathbb{N}}$ has a bounded subsequence.

In the first case we plainly find $u(x,y,t) = 0$ for every $(x,y,t) \in \mathbb{R}^{m+1}$ with $t < 0$.

In the second case there exists a subsequence $\left(3\widetilde{\eta}_{k_j} - \widetilde{\xi}_{k_j}\right)_{j\in\mathbb{N}}$ converging to some point $w \in \mathbb{R}^m$. From (8.12) we have that

$$u(x,y,t) = \exp\left(\langle x, w\rangle + t\|w\|^2\right),$$

and hence, u has the form (8.6). This concludes the proof. $\qquad\square$

9 Concluding remarks and further developments

As was stressed in Remark 2.4, our separation principle (Theorem 1.4) gives valuable information concerning nonnegative solutions for operators $\mathscr{L}$ of the form

$$\mathscr{L}u = \partial_t u - \sum_{j=1}^{m} X_j^2 u,$$

and for Mumford's operator $\mathscr{M}$

$$\mathscr{M}u := \partial_t u - \cos(x)\partial_y u - \sin(x)\partial_w u - \partial_x^2 u.$$

On the other hand, in recent years, operators of the form (1.1) with $X_0 \neq 0$ that satisfy (H0), (H1) and (H2) have received considerable attention. It would be interesting to study their positivity properties using our functional analytic approach. We give here two examples of such operators.

Example 9.1. LINKED OPERATORS. Let $(\partial_x + y\partial_s)^2 + (\partial_y - x\partial_s)^2$ be the sub-Laplacian on the Heisenberg group given by (3.6), and let $x\partial_w - \partial_t$ be the first order term of the simplest Kolmogorov operator (8.1), that is

$$\mathscr{L} := \partial_t - x\partial_w - \partial_x^2 \qquad (x, w, t) \in \mathbb{R}^3.$$

Define

$$\mathscr{L} := \partial_t - x\partial_w - (\partial_x + y\partial_s)^2 - (\partial_y - x\partial_s)^2 \qquad (x, y, s, w, t) \in \mathbb{R}^5. \qquad (9.1)$$

Note that the operator $\mathscr{L}$ acts on the variables (x, y, s, t) as the heat equation on the Heisenberg group, and on the variables (x, y, w, t) as a Kolmogorov operator in $\mathbb{R}^3 \times \mathbb{R}$. It is easy to see that $\mathscr{L}$ satisfies the Hörmander condition. Moreover, it can be shown that there exists a homogeneous Lie group on $\mathbb{R}^5$ that *links* the Heisenberg group on $\mathbb{R}^4$ and the Kolmogorov group in $\mathbb{R}^3$, and such that $\mathscr{L}$ is invariant with respect to this new Lie group.

The notion of a *link of homogeneous groups* has been introduced by Kogoj and Lanconelli in [22, 25]. It gives a general procedure for the construction of sequences of homogeneous groups of arbitrarily large dimension and step.

Example 9.2. Consider the following operator studied by Cinti, Menozzi and Polidoro [13]

$$\mathscr{L}u = \partial_t u - x\partial_w u - x^2\partial_y u - \partial_x^2 u \qquad (x, y, w, t) \in \mathbb{R}^4. \qquad (9.2)$$

It is invariant with respect to the following Lie group operations

$$(x, y, w, t) \circ (\xi, \eta, \omega, \tau) := (x + \xi, y + \eta + 2x\omega - \tau x^2, w + \omega - \tau x, t + \tau), \qquad (9.3)$$

and verifies Hörmander hypoellipticity condition, so, (H0) and (H1) are satisfied. Note that, in this case, the drift term $X_0 := x^2\partial_y + x\partial_w$ is essential for the validity of (H0). $\mathscr{L}$ is also invariant with respect to the following dilation

$$\delta_r(x, y, w, t) := \left(rx, r^4y, r^3w, r^2t\right). \qquad (9.4)$$

We next show that the attainable set of the point $z_0 = (x_0, y_0, w_0, t_0)$ in $\mathbb{R}^4$ is

$$\mathscr{A}_{z_0} = \left\{ (x, y, w, t) \in \mathbb{R}^4 \mid t \le t_0, y_0 \le y, (w - w_0)^2 \le (y - y_0)(t_0 - t) \right\}. \qquad (9.5)$$

To prove (9.5), we recall that in [13, Lemma 5.11] it has been shown that, if $z_0 = 0 \in \mathbb{R}^4$, and $\Omega = \left(]-1, 1[\right)^4$ is the open unit cube in $\mathbb{R}^4$, then

$$\mathscr{A}_0(\Omega) = \left\{ (x, y, w, t) \in \Omega \mid 0 \le y \le -t, w^2 \le -ty \right\}.$$

In accordance with (9.4), we consider the r dilation of Ω

$$\delta_r\Omega =]-r, r[\times]-r^4, r^4[\times]-r^3, r^3[\times]-r^2, r^2[.$$

By the dilation invariance of $\mathscr{L}$, we then have

$$\mathscr{A}_0 = \bigcup_{r>0} \mathscr{A}_0(\delta_r\Omega) = \bigcup_{r>0} \left\{ (x, y, w, t) \in \delta_r\Omega \mid 0 \le y \le -r^2t, w^2 \le -ty \right\},$$

and we get (9.5) for $z_0 = 0$. Eventually, (9.5) for any $z_0 \in \mathbb{R}^4$ follows from the invariance of $\mathscr{L}$ with respect to the translations defined in (9.3).

Note that the point $\exp(sY) z_0 \notin \mathrm{Int}(\mathscr{A}_{z_0})$, where $Y = x^2 \partial_y + x \partial_w - \partial_t$ is defined by (1.2). Since $\mathscr{A}_{z_0}(\Omega) \subset \mathscr{A}_{z_0}$, for every bounded set $\Omega \subset \mathbb{R}^4$, we conclude that (H2) is not satisfied if we choose $\omega = 0$. Nevertheless, $\mathscr{L}$ defined in (9.2) satisfies assumption (H2), for any $\omega \neq 0$ provided that we choose Ω big enough.

We note that the operator $\mathscr{L}$ in (9.2) is an approximation of the Mumford operator (7.1). Indeed, the Taylor expansion at $x = 0$ of the drift term $X_0 = \cos(x)\partial_y + \sin(x)\partial_w$, leads us to approximate $\mathscr{M}$ with

$$\widetilde{\mathscr{M}} = \partial_t - \left(1 - \tfrac{x^2}{2}\right)\partial_y - x\partial_w - \partial_x^2.$$

Moreover, it can be easily checked that u is a solution of the equation $\mathscr{L}u = 0$ (where $\mathscr{L}$ is the operator defined by (9.2)) if and only if the function $v(x, y, w, t) := u\left(x, -\tfrac{y}{2} - t, w, t\right)$ is a solution of the equation $\widetilde{\mathscr{M}}v = 0$, and the claim is verified.

9.1 On the separation principle

We discuss here the main assumption (1.16) of Theorem 1.4. We recall that it is satisfied whenever $X_0 = 0$, and therefore, it is natural to study operators with $X_0 \neq 0$ and a non-abelian $\mathbb{G}$ that still satisfy (1.16). In order to discuss this question, we focus on the consequence of (1.16), that is

$$u\left(\exp(s(\omega \cdot X + Y))(x, t)\right) = \mathrm{e}^{-\beta s} u(x, t) \quad \forall(x, t) \in \mathbb{R}^N \times \mathbb{R}_T \text{ and } \forall s > 0, \qquad (9.6)$$

where u is a nonnegative extremal solution. The following result answers this question

Proposition 9.3. *Let $\mathscr{L}$ be an operator of the form* (1.1), *satisfying* (H0), (H1) *and* (H2). *Let $u : \mathbb{R}^N \times \mathbb{R}_T \to \mathbb{R}$ be a nonnegative smooth function satisfying* (9.6). *Then*

$$[X_j, X_k]u(x, t) = 0, \qquad \forall j, k = 0, 1, \ldots, m, \text{ and } \forall(x, t) \in \mathbb{R}^N \times \mathbb{R}_T. \qquad (9.7)$$

The same result holds for all higher-order commutators.

Moreover, if any nonnegative extremal solution in $\mathscr{H}_+$ satisfies (9.6), *then the conclusion* (9.7) *holds for any $u \in \mathscr{H}_+$.*

As an application, we apply the above result to the degenerate Kolmogorov equations in two space variables $\mathscr{K} := \partial_t - x\partial_y - \partial_x^2$, and let $\mathscr{H}_+$ denote the corresponding cone of nonnegative solutions in $\mathbb{R}^2 \times]-\infty, T[$. In this case $X_1 = \partial_x$, $X_0 = x\partial_y$, and Proposition 9.3 says that, if u is a nontrivial nonnegative extremal solution in $\mathscr{H}_+$ that satisfies (9.6), then

$$[X_1, X_0]u(x, y, t) = [\partial_x, x\partial_y]u(x, y, t) = \partial_y u(x, y, t) = 0.$$

Hence, u does not depend on y. Therefore, u is a nontrivial nonnegative solution of the heat equation $\partial_t u = \partial_x^2 u$ in $\mathbb{R} \times\,] - \infty, T[$, and in particular u is strictly positive.

In conclusion, all nontrivial nonnegative extremal solutions in $\mathcal{H}_+$ satisfying (9.6), do not depend on the 'degenerate' variable y. Recall that in fact, by Corollary 8.2, all solutions in $\mathcal{H}_+$ do not depend on y.

Next, we present the proof of Proposition 9.3. It relies on the following Lemma, whose proof is analogous to that of Theorem 3.4.

Lemma 9.4. *Let $u : \mathbb{R}^N \times \mathbb{R}_T \to \mathbb{R}$ be a nonnegative smooth function, and ω_1, ω_2 be two vectors of $\mathbb{R}^m$ such that (9.6) holds. Then, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$, we have*

$$\big[\omega_1 \cdot X + Y, \omega_2 \cdot X + Y\big] u(x,t) = 0.$$

Proof. Let $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$, and consider the function $v := \log(u)$. Using (9.6) with $s > t - T$, we obtain

$$v\Big(\exp(-s(\omega_2 \cdot X + Y))\exp(-s(\omega_1 \cdot X + Y))\exp(s(\omega_2 \cdot X + Y))\exp(s(\omega_1 \cdot X + Y))(x,t)\Big) =$$
$$s\beta_{\omega_1} + s\beta_{\omega_2} - s\beta_{\omega_1} - s\beta_{\omega_2} + v(x,t) = v(x,t).$$

We recall the Baker-Campbell-Hausdorff formula

$$\exp\big(s\widetilde{Y}\big)\exp\big(s\widetilde{X}\big)(x,t) = \exp\Big(s\big(\widetilde{Y} + \widetilde{X}\big) + \tfrac{s^2}{2}\big[\widetilde{X}, \widetilde{Y}\big] + o(s^2)\Big)$$

where $o(s^2)$ denotes a function such that $o(s^2)/s^2 \to 0$ as $s \to 0$, and we apply it twice. The first time we choose $\widetilde{X} = \omega_1 \cdot X + Y$ and $\widetilde{Y} = \omega_2 \cdot X + Y$, the second time we set $\widetilde{X} = -\big(\omega_1 \cdot X + Y\big)$ and $\widetilde{Y} = -\big(\omega_2 \cdot X + Y\big)$, and we find

$$\frac{v\Big(\exp\big(s^2[\omega_1 \cdot X + Y, \omega_2 \cdot X + Y] + o(s^2)\big)(x,t)\Big) - v(x,t)}{s^2} = 0,$$

for every $s > t - T$. Then from the differentiability of the functions v and $\exp$, by letting $s \to 0$ we obtain

$$\big[\omega_1 \cdot X + Y, \omega_2 \cdot X + Y\big] v(x,t) = 0.$$

The proof of the claim then follows from the fact that $u(x,t) = \exp\big(v(x,t)\big)$. $\quad\square$

Proof of Proposition 9.3. Let $u : \mathbb{R}^N \times \mathbb{R}_T \to \mathbb{R}$ be a nonnegative smooth function, and let $\omega \in \mathbb{R}^m$ be any vector satisfying (H2). We claim that

$$\big[X_k, \omega \cdot X + Y\big] u(x,t) = 0 \qquad k = 1, \ldots, m, \tag{9.8}$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$.

In order to prove (9.8) we note that, since $\exp\big(s(\omega \cdot X + Y)\big)(x,t) \in \text{Int}\big(\mathscr{A}_{(x,t)}\big)(\Omega)$ for any $s \in]0, s_0[$, there exists $r > 0$ such that $\exp\big(s\big(\omega \cdot X + rX_k + Y\big)\big)(x,t) \in \text{Int}\big(\mathscr{A}_{(x,t)}(\Omega)\big)$ for $k = 1, \ldots, m$. We denote by e_k the k-th vector of the canonical basis of $\mathbb{R}^m$, and we apply Lemma 9.4 with $\omega_1 := \omega + re_k$ and $\omega_2 := \omega$. We find

$$r\big[X_k, \omega \cdot X + Y\big]u(x,t) = \big[\omega \cdot X + rX_k + Y, \omega \cdot X + Y\big]u(x,t) = 0,$$

for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_T$. This proves (9.8).

We apply again Lemma 9.4 with $\omega_1 := \omega + re_k$ and $\omega_2 := \omega + re_j$, for $j, k = 1, \ldots, m$, and (9.8) to obtain

$$r^2\big[X_j, X_k\big]u(x,t) = \big[rX_j + \omega \cdot X + Y, rX_k + \omega \cdot X + Y\big]u(x,t)$$
$$- r\big[X_j, \omega \cdot X + Y\big]u(x,t) + r\big[X_k, \omega \cdot X + Y\big]u(x,t) = 0.$$

This proves

$$\big[X_j, X_k\big]u(x,t) = 0 \qquad j, k = 1, \ldots, m. \tag{9.9}$$

From (9.9) and from the fact that $\big[X_k, \partial_t\big] = 0$, we eventually obtain

$$\big[X_k, X_0\big]u(x,t) = \big[X_k, \omega \cdot X + Y\big]u(x,t) - \sum_{j=1}^m \omega_j\big[X_k, X_j\big]u(x,t) = 0,$$

for $k = 1, \ldots, m$. This concludes the proof of (9.7). A plain application of the Baker-Campbell-Hausdorff formula gives the result for all higher-order commutator.

The result for any nonnegative solution then clearly follows from the representation formula (6.8). $\qquad\square$

9.2   Liftable operators

Our approach applies also to operators that are not invariant with respect to any Lie group structure, but that can be *lifted* to a suitable operator $\widetilde{\mathscr{L}}$ that satisfies assumption (H1). Consider, for instance, the following Grushin-type evolution operator

$$\mathscr{L}u = \partial_t u - \partial_x^2 u - x^2 \partial_y^2 u \qquad (x, y, t) \in \mathbb{R}^3. \tag{9.10}$$

Since it is degenerate at $\{x = 0\}$ and nondegenerate in the set $\{x \neq 0\}$, a change of variables that preserves the operator cannot exist. If we lift the operator by adding a new variable w and introducing the vector fields $\widetilde{X}_1 := X_1$ and $\widetilde{X}_2 := X_2 + \partial_w$, then we get the lifted operator

$$\widetilde{\mathscr{L}}u := \partial_t u - \partial_x^2 u - (\partial_w + x\partial_y)^2 u \qquad (x, y, w, t) \in \mathbb{R}^4, \tag{9.11}$$

that belongs to the class considered in Section 3. The uniqueness result proved for (9.11) directly extends to (9.10).

Analogously, the operator

$$\mathscr{L}u = \partial_t u - \partial_x^2 u - x^2 \partial_y u \qquad (x, y, t) \in \mathbb{R}^3 \tag{9.12}$$

studied in [13], is not invariant with respect to any Lie group structure. However, it can be lifted to the operator defined in (9.2) and, also in this case, the uniqueness result for (9.2) extends to (9.12). We note that (9.12) appears in stochastic theory (see the references in [13] for a bibliography on this subject).

Clearly, the lifting method can be applied to a wide class of operators.

9.3 Open problems

In this subsection we list several open problems related to the results of the present paper.

1. Our first problem concerns the strict positivity of nonzero nonnegative solutions of the equation $\mathscr{L}u = 0$ in $\mathbb{R}^N \times \mathbb{R}_T$ (cf. Theorem 1.4).

2. We would like to extend our main results to operators with nontrivial zero-order term, namely to operators of the form

$$\mathscr{L}_c u := \partial_t u - \sum_{j=1}^m X_j^2 u - X_0 u - c(x)u.$$

3. We would like to weaken the left-invariance condition, as well.

4. We aim to study property *(b)* of Section 5 for degenerate operators. More precisely, we would like to find conditions under which the generalized principal eigenvalue λ_0 of $\mathscr{L}_0$ is equal 0. Moreover, we would like to understand whether $\mathscr{L}_0$ is critical in $\mathbb{R}^N$.

5. In another direction, we would like to extend the nonnegative Liouville-type theorem in $\mathbb{R}^{N+1}$ (Theorem 5.1) to the case of operators with a nontrivial drift term.

6. Finally, it is natural to extend our work to the case where $\mathscr{L}$ of the form (1.1) is defined on a noncompact Lie group, and even to the more general setting of a noncompact manifold M with a cocompact group action (cf. [33]). We expect that the acting group should be nilpotent.

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