Contact problem of a Timoshenko beam bonded to a half-plane

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The contact problem of beams, rods, ribs and plates bonded to a half-plane has been widely investigated by many Authors. In particular, the problem of prismatic beams resting on a finite or semi-infinite elastic substrate deserves great interest because its practical applications in many engineering application. As an example, Shield and Kim (1992) investigated the problem of an Eulero-Bernoulli beam resting on an elastic half-plane under symmetric loading conditions, founding the interfacial stresses as well as the SIFs at the edges of the beam. The Authors also studied the effect induced by an elastic-perfectly plastic cohesive interface. Nonetheless, a complete analytical study of the contact problem of a Timoshenko beam bonded to a half-plane cannot be found in literature.
The present study concerns the contact problem of a Timoshenko beam resting on an elastic half-plane under general edge loading. Let N_0, T_0, M_0 external horizontal and vertical forces and bending moment, respectively, acting at the edge of a Timoshenko beam. Reference is made to a Cartesian coordinate system centered at the middle of the beam, as depicted in Figure 1. In absence of further external load distributions acting on the system, the equilibrium equations of the beam read:

$$N' + \tau = 0; \qquad T' + q = 0; \qquad M' - T' + \tau \frac{h}{2} = 0; \tag{1}$$

being N, T and M the axial force, shear force and bending moment, respectively, τ and q denote the interfacial shear and peeling tractions, respectively, h represent the thickness of the beam and prime denotes differentiation with respect to coordinate x.

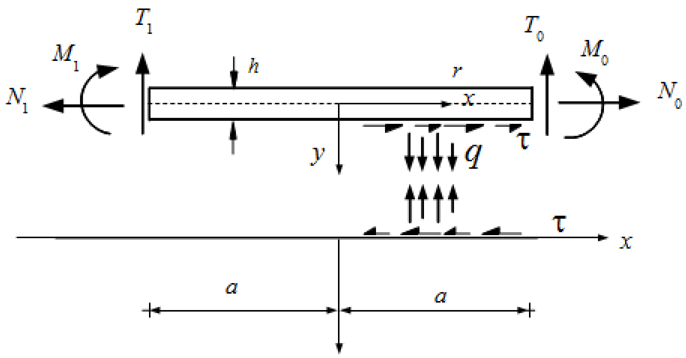

Figure 1. Free-body diagram of a beam bonded to a half-plane subjected to edge loads.

The constitutive laws (the slope of the beam cross section φ is positive if counterclockwise) give the following relationships:

$$\frac{M}{E_b I} = \varphi\,' ; \qquad u\,' = \frac{N}{E_b A} + \varphi\,' \, \frac{h}{2} ; \qquad v\,' = -\varphi + \frac{T}{G_b A_{eq}} ; \tag{2}$$

where E_b denote the Young modulus of the beam, A and I are the area and the moment of inertia of the beam cross section respectively, G_b represents the shear modulo of the beam and A_eq is the shear area of the beam. The equilibrium conditions read:

$$N(x) = N_0 + \int_x^a \tau(s)ds ; \qquad T(x) = T_0 + \int_x^a q(s)ds ; \tag{3}$$

$$M(x) = M_0 + T_0\,(a-x) + \frac{h}{2}\int_x^a \tau(s)ds + \int_x^a q(s)(x-s)ds .$$

By replacing eqs (3) in expressions (2), the horizontal and vertical components $u_\mathrm{b}'(x)$, $v_\mathrm{b}'(x)$ of the beam strain field are found. The normal strains of the surface (i.e. for y = 0) an anisotropic half plane under 2D strain read (Johnson, 1985) for $|x| \le a$ are known from the Green fundamental solution:

$$u_s\,'(x) = -\frac{2(1-\nu_\mathrm{s}^2)}{E_s\,\pi}\int_{-a}^{a}\frac{\tau(\xi)}{\xi - x}\,\mathrm{d}\xi + \frac{(1+\nu_\mathrm{s})(1-2\nu_\mathrm{s})}{E_\mathrm{s}}\,q(x) ;$$

$$v_s\,'(x) = -\frac{2(1-\nu_\mathrm{s}^2)}{E_s\,\pi}\int_{-a}^{a}\frac{q(\xi)}{\xi - x}\,\mathrm{d}\xi - \frac{(1+\nu_\mathrm{s})(1-2\nu_\mathrm{s})}{E_\mathrm{s}}\,\tau(x) . \tag{4}$$

The interfacial stresses within the contact region $|x| \le a$ are suitably expressed as follows:

$$\tau(x) = E_s\,(a+x)^s\,(a-x)^s \sum_n C_n\, P_n^{(s,s)}(x/a) , \qquad \sigma_{yy}(x) = E_s\,(a+x)^s\,(a-x)^s \sum_n D_n\, P_n^{(s,s)}(x/a) . \tag{5}$$

The problem is solved by imposing the compatibility condition $u_\mathrm{b}'(x) = u_s\,'(x)$ and $v_\mathrm{b}'(x) = v_s\,'(x)$ among the strain components of the beam and those of the half-plane in N+1 collocation points $x_k = \cos[\pi k/(2(N+1))]$, beng $k = 1, 2, \ldots , N + 1$. Thus, the system of 2 integral equations leads to an algebraic system of 2N+1 equations in the N+1 unknowns coefficients C_i ($i = 1, 3, 5, \ldots, 2N+1$), and N coefficients D_j ($j = 2, 4, \ldots, 2N$) of the interfacial stresses (5). Once that the stresses (5) are known, the internal forces (3) and the displacement field of the system at the interface can be found.

References

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