Stability of Reeb graphs of closed curves

Barbara Di Fabio[1]

ARCES
University of Bologna
Bologna, Italy

Claudia Landi[2]

DISMI
University of Modena and Reggio Emilia
Reggio Emilia, Italy

Abstract

Reeb graphs are very popular shape descriptors in computational frameworks. Indeed, they capture both geometrical properties of the shape, and its topological features. Some different comparison methodologies have been proposed in literature to compare Reeb graphs for estimating the similarity of the shapes they are describing. In this context, one of the most important open questions is whether Reeb graphs are robust against function perturbations. In fact, it is clear that any data acquisition is subject to perturbations, noise and approximation errors and, if Reeb graphs were not stable, then distinct computational investigations of the same object could produce completely different results. In this paper we present an initial contribution to establishing stability properties for Reeb graphs. More precisely, focusing our attention on 1-dimensional manifolds, we define an editing distance between Reeb graphs, in terms of the cost necessary to transform one graph into another. Our main result is that changes in Morse functions imply smaller changes in the editing distance between Reeb graphs.

Keywords: Shape comparison, editing distance, Morse function.

1 Introduction

Let $\mathcal{M}$ be a closed compact smooth manifold endowed with a simple Morse function $f : \mathcal{M} \to \mathbb{R}$. The quotient space obtained from $\mathcal{M}$ identifying points that belong to the same connected component of each level set of f is the body of a finite simplicial complex of dimension 1, called a *Reeb graph*.

[1] Email: difabio@dm.unibo.it
[2] Email: clandi@unimore.it

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Originally defined in 1946 by the mathematician George Reeb [5], Reeb graphs have been introduced in Computer Graphics in 1991 [6], becoming, in a few years, very popular shape descriptors in computational frameworks, especially in applications such as 3D shape matching, shape coding and comparison. Indeed, they capture both geometrical properties of the shape, according to the behavior of the function over the space, and its topological features, described by the connectivity of the graph.

Some different comparison methodologies have been proposed in literature to compare Reeb graphs for estimating the similarity of the shapes they are describing [4,1]. In this context, one of the most important open questions is whether Reeb graphs are robust against function perturbations. In fact, it is clear that any data acquisition is subject to perturbations, noise and approximation errors and, if Reeb graphs were not stable, then distinct computational investigations of the same object could produce completely different results.

In this paper we illustrate the recent results proved in [3] on the stability problem of Reeb graphs under function perturbations, when the considered manifolds are 1-dimensional. To be more precise, we define an editing distance for comparing Reeb graphs, in terms of the cost necessary to transform one graph into another through a finite sequence of allowed editing operations on their vertices (Definitions 3.1 and 3.2). Then we prove that perturbations in the functions imply smaller changes in the editing distance between Reeb graphs, and therefore the stability.

This is only an initial contribution to establishing stability properties for Reeb graphs. Indeed, our purpose is to extend such results to the more interesting case of surfaces. In this sense, we want to underline that, even if the editing operations would need to be appropriately modified, the general technique we have used to prove our main result can be easily generalized.

2 Preliminaries on Reeb graphs

This section provides basic notions on Reeb graphs and the notations we will use in the rest of the paper.

Let $\mathcal{M}$ be a smooth (i.e. differentiable of class C^∞) compact n-manifold without boundary, and let $f : \mathcal{M} \to \mathbb{R}$ be a simple Morse function on $\mathcal{M}$, i.e., a smooth function with finitely many non-degenerate critical points, each of them belonging to a different critical level.

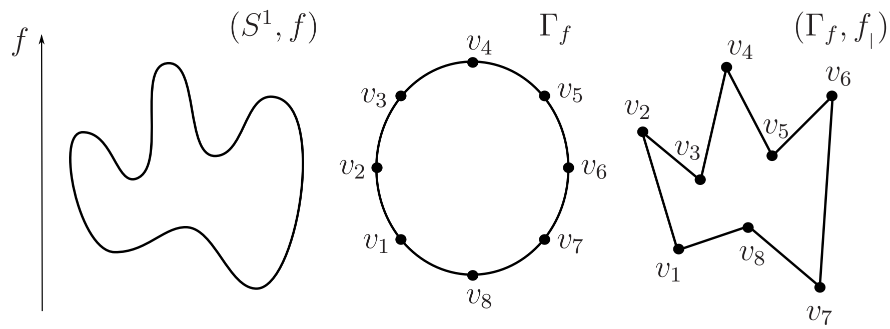
Following [5], let us recall what a Reeb graph is.

Theorem 2.1 (Reeb, 1946) *The quotient space of $\mathcal{M}$ under the equivalence relation "p and q belong to the same connected component of the same level set of f" is a finite and connected simplicial complex of dimension 1.*

This simplicial complex, denoted by Γ_f, is called the *Reeb graph* associated with the pair $(\mathcal{M}, f)$. Its vertex set will be denoted by $V(\Gamma_f)$, and its edge set by $E(\Gamma_f)$. Since the vertices of the Reeb graph correspond in a one to one manner to the critical points of f, we will often identify each $v \in V(\Gamma_f)$ with the corresponding critical point p of f. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, i.e., connected by an edge, we will write $e(v_1, v_2) \in E(\Gamma_f)$.

2.1 Labelled Reeb graphs of closed curves

Let us focus on $\mathcal{M} = S^1$. The Reeb graph Γ_f associated with (S^1, f) is a cycle graph on an even number of vertices, corresponding, alternatively, to the minima and maxima of f on S^1. Furthermore, let us label the vertices of Γ_f, by equipping each of them with the value of f at the corresponding critical point. We denote such a labelled Reeb graph by $(\Gamma_f, f_|)$, where $f_| : V(\Gamma_f) \to \mathbb{R}$ is the restriction of $f : S^1 \to \mathbb{R}$ to its set of critical points. To facilitate the reader, in all the figures of this paper, we consider f as the height function, so that $f_|(v_i) < f_|(v_j)$ if and only if v_i is lower than v_j in the picture.



Moreover, we will also identify two labelled Reeb graphs $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$, and write $(\Gamma_f, f_|) = (\Gamma_g, g_|)$, if there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that $f_|(v) = g_|(\Phi(v))$ for every $v \in V(\Gamma_f)$.

3 Editing distance between labelled Reeb graphs

Now, we define all the editing operations admissible to transform a labelled Reeb graph into another, their costs, and the editing distance in terms of these costs.

3.1 Edit operations on labelled Reeb graphs

Definition 3.1 Let $(\Gamma_f, f_|)$ be a labelled Reeb graph with $2n$ vertices, $n \geq 1$. An *elementary deformation* T of $(\Gamma_f, f_|)$ is any of the following transformations:

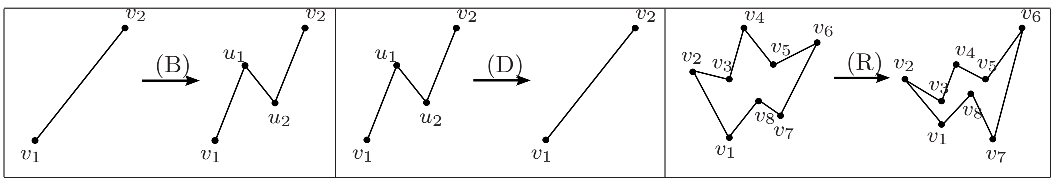
(B) (Birth): If $e(v_1, v_2) \in E(\Gamma_f)$ with $f_|(v_1) < f_|(v_2)$, then $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$ such that

· $V(\Gamma_g) = V(\Gamma_f) \cup \{u_1, u_2\}$;
· $e(v_1, u_1), e(u_1, u_2), e(u_2, v_2) \in E(\Gamma_g)$;
· $g_{|V(\Gamma_f)} = f_|$;
· $g_|(v_1) < g_|(u_2) < g_|(u_1) < g_|(v_2)$.

(D) (Death): If $e(v_1, u_1), e(u_1, u_2), e(u_2, v_2) \in E(\Gamma_f)$ with $f_|(v_1) < f_|(u_2) < f_|(u_1) < f_|(v_2)$, then $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$ such that
· $V(\Gamma_g) = V(\Gamma_f) - \{u_1, u_2\}$;
· $e(v_1, v_2) \in E(\Gamma_g)$;
· $g_| = f_{|V(\Gamma_f)-\{u_1,u_2\}}$.

(R) (Relabelling): $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$ such that
· $\Gamma_g = \Gamma_f$;
· if $e(v_1, v_2), e(v_2, v_3) \in E(\Gamma_f)$, then $f_|(v_2) < f_|(v_1), f_|(v_3)$ if and only if $g_|(v_2) < g_|(v_1), g_|(v_3)$.

The table below schematically illustrates the elementary deformations described in Definition 3.1.



As proved in [3, Prop. 3.2], each elementary deformation transforms a labelled Reeb graph into another one, simply by changing the labels attached to its vertices, or by adding, or deleting a pair of adjacent vertices. So, we can apply elementary deformations iteratively to transform any labelled Reeb graph into any other.

Definition 3.2 We shall call a *deformation* of $(\Gamma_f, f_|)$ any finite ordered sequence $T = (T_1, T_2, \dots, T_r)$ of elementary deformations such that T_1 is an elementary deformation of $(\Gamma_f, f_|)$, T_2 is an elementary deformation of $T_1(\Gamma_f, f_|)$, ..., T_r is an elementary deformation of $T_{r-1}T_{r-2}\cdots T_1(\Gamma_f, f_|)$. We shall denote by $T(\Gamma_f, f_|)$ the result of the deformation T applied to $(\Gamma_f, f_|)$.

Now, we associate a cost with each type of deformation.

Definition 3.3 Let T be an elementary deformation transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$.

- If T is of type (B) inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, then we define the associated cost as
$$c(T) = \frac{|g_|(u_1) - g_|(u_2)|}{2};$$

- If T is of type (D) deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, then we define the associated cost as
$$c(T) = \frac{|f_|(u_1) - f_|(u_2)|}{2};$$
- If T is of type (R) relabelling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, then we define the associated cost as
$$c(T) = \max_{v \in V(\Gamma_f)} |f_|(v) - g_|(v)|.$$

Moreover, if $T = (T_1, \ldots, T_r)$ is a deformation such that $T_r \cdots T_1(\Gamma_f, f_|) = (\Gamma_g, g_|)$, we define the associated cost as $c(T) = \sum_{i=1}^{r} c(T_i)$.

Let us introduce the concept of inverse deformation.

Definition 3.4 Let T be a deformation such that $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$. Then we denote by T^{-1}, and call it the *inverse* of T, the deformation such that $T^{-1}(\Gamma_g, g_|) = (\Gamma_f, f_|)$ defined as follows:

- If T is elementary of type (B) inserting two vertices, then T^{-1} is of type (D) deleting the same vertices;
- If T is elementary of type (D) deleting two vertices, then T^{-1} is of type (B) inserting the same vertices, with the same labels;
- If T is elementary of type (R) relabelling vertices of $V(\Gamma_f)$, then T^{-1} is again of type (R) relabelling these vertices in the inverse way;
- If $T = (T_1, \ldots, T_r)$, then $T^{-1} = (T_r^{-1}, \ldots, T_1^{-1})$.

Obviously, from Definition 3.4, it follows that, if $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$, then $c(T^{-1}) = c(T)$.

3.2 Editing distance

Let $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ be two labelled Reeb graphs. Then the set of all the deformations T such that $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$ is non-empty [3, Prop. 3.8]. Actually, this set, that from now on will be denoted by $\mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$, contains infinitely many possible deformations transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$, each of them with a different cost. We define the distance between such two graphs as the infimum cost we have to pay to transform the first graph into the second.

Theorem 3.5 *For every two labelled Reeb graphs $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$, we set*
$$d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \inf_{T \in \mathcal{T}((\Gamma_f, f_|), (\Gamma_g, g_|))} c(T).$$
Then d is a distance.

Proof (Sketch) The properties of symmetry and triangular inequality can be easily verified. The positive definiteness of d can be proved by observing that, if $(\Gamma_f, f_{|})$ and $(\Gamma_g, g_{|})$ are labelled Reeb graphs associated with (S^1, f) and (S^1, g), respectively, then $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \geq \inf_{\tau} \|f - g \circ \tau\|_{C^0}$ with τ varying in the set of all homeomorphisms on S^1 (see [3, Cor. 4.2]). $\square$

4 Local and global stability

Let $\mathcal{F}(S^1, \mathbb{R})$ be the set of smooth real valued functions on S^1, endowed with the C^∞ topology, and let us stratify such a space, as done by Cerf in [2] (see also [3, Sec. 1.3]). Let us denote by $\mathcal{F}^0$ the submanifold of $\mathcal{F}(S^1, \mathbb{R})$ of co-dimension 0 that contains all the simple Morse functions $f : S^1 \to \mathbb{R}$. Let, then, $\mathcal{F}^1 = \mathcal{F}^1_\alpha \bigcup \mathcal{F}^1_\beta$ be the submanifold of $\mathcal{F}(S^1, \mathbb{R})$ of co-dimension 1, where: $\mathcal{F}^1_\alpha$ represents the set of functions whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one; $\mathcal{F}^1_\beta$ the set of Morse functions whose critical levels contain at most one critical point, except for one level containing exactly two critical points.

Since $\mathcal{F}^0$ is dense in the space $\mathcal{F}(S^1, \mathbb{R})$ endowed with the C^r topology, $2 \leq r \leq \infty$ ([3, Sec. 1.3]), given two arbitrary functions $f, g \in \mathcal{F}^0$,

the main idea we use to prove our stability result consists in

Theorem 4.1 (Global stability) *Let $f, g \in \mathcal{F}^0$. Then*

$$d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \|f - g\|_{C^2},$$

with $\|f\|_{C^2} = \max\left\{\max|f|, \max\left|\frac{df}{dx}\right|, \max\left|\frac{d^2 f}{dx^2}\right|\right\}$.

Proof (Sketch) The proof is divided into three steps.

- (Local stability) Let $f \in \mathcal{F}^0$. Then there exists a positive real number $\delta(f)$ such that, for every δ, $0 \leq \delta \leq \delta(f)$, and for every $g \in \mathcal{F}^0$ such that $\|f - g\|_{C^2} \leq \delta$, it holds that $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \delta$.
- Let $f, g \in \mathcal{F}^0$ and let us consider the path $h : [0,1] \to \mathcal{F}(S^1, \mathbb{R})$ defined by $h(\lambda) = (1-\lambda)f + \lambda g$. If $h(\lambda) \in \mathcal{F}^0$ for every $\lambda \in [0,1]$, then $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \|f - g\|_{C^2}$.
- Let $f, g \in \mathcal{F}^0$ and let us consider the path $h : [0,1] \to \mathcal{F}(S^1, \mathbb{R})$ defined by $h(\lambda) = (1-\lambda)f + \lambda g$. If $h(\lambda) \in \mathcal{F}^0$ for every $\lambda \in [0,1] \setminus \{\overline{\lambda}\}$, with $0 < \overline{\lambda} < 1$, and h transversely intersects $\mathcal{F}^1$ at $\overline{\lambda}$, then $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \|f - g\|_{C^2}$.

$\square$

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