Chapter #

MATHEMATICAL MACHINES: FROM HISTORY TO MATHEMATICS CLASSROOM

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Abstract: The aim of this chapter is to present some issues concerning secondary teacher education, drawing on the activity of the Laboratory of Mathematical Machines at the Department of Mathematics of the University of Modena and Reggio Emilia (MMLab: http://www.mmlab.unimore.it). The name comes from the most important collection of the Laboratory, containing more than two hundred working reconstructions (based on the original sources) of mathematical artefacts taken from the history of geometry. In this chapter we intend to discuss, in the setting of teacher education and within a suitable theoretical framework, a single case, i.e., an ellipse drawing device, from different perspectives (historic-epistemological, manipulative and virtual), to develop expertise in selecting and adjusting appropriate tools for the mathematics classroom.

Key words: artefact, history, mathematical laboratory, mathematical machine, semiotic mediation.

INTRODUCTION

Mathematical machines are cultural artefacts, that draw on centuries (and even millennia) of tradition. Briefly, a mathematical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law. They are collected in the Laboratory of Mathematical Machines at the Department of Mathematics of the University of Modena and Reggio Emilia (MMLab: http://www.mmlab.unimore.it). The Laboratory is a well known research centre for the teaching and learning of mathematics by means of artefacts (Maschietto, 2005).

Familiar examples of mathematical machines are the standard compass (that forces a point to go on a circular trajectory, Figure #.1) and the Dürer glass (Figure #.2) used as a perspectograph (that transform a point into its perspective image on a glass from a given point).

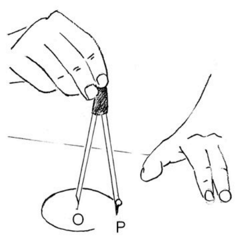


Figure #.1 The compass.

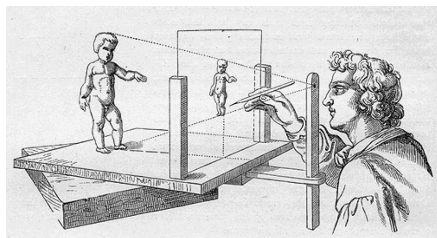


Figure #.2 Dürer glass.

As argued by Bartolini Bussi and Maschietto (2006), they are part of the historical phenomenology of geometry: ruler and compass are at the roots of elementary geometry (e.g., Euclid); curve drawing devices are at the roots of algebraic geometry (e.g., Descartes, van Schooten, Newton), perspectographs are at the roots of projective geometry (e.g., Desargues). They are linked to the cultural development of mankind in a sense that does not consist merely of mathematics but encompasses also art and technology. They are concretely manipulable, in order to produce the intended effect. In a nutshell, they are good candidate to equip the mathematics classroom for meaningful mathematical experiences, where practice (manipulation and real experiments) and theory (elaboration of definitions, production of conjectures and construction of proofs) are strictly interlaced within a historic-cultural perspective, up to the present modelling of concrete machines by means of Dynamic Geometry Environments (DGE).

All the above activities are consistent with the idea of mathematical laboratory: this idea has a long tradition not only in the professional mathematical practice - as we have said above - but also in the history of mathematics education (see for instance Maschietto, & Martignone, 2008 Bartolini Bussi, in press). The laboratory activity is a great challenge for teachers. In this chapter, we discuss some kind of activity concerning a particular mathematical machine as paradigmatic examples of mathematical laboratory activities. They are proposed to prospective teachers:

- to be experienced in a mathematical laboratory session;
- to provide a model that might serve for future class activity;
- to make they think over the relationships between manipulative and theoretical aspects in doing mathematics, on the basis that the only manipulation is not enough to construct mathematical knowledge.

These activities can be transferred to students' classes because of the availability of materials (working sheets[1] and artefacts that can be reconstructed, using plastic or cardboard bars, by students too).

The chapter is composed by four sections. The first section presents some elements concerning the idea of mathematical laboratory connected to teacher education, then the theoretical background developed within a Vygotskian perspective. The other three sections propose three different activities about van Schooten's ellipse drawing device, according to three different dimensions: in particular, the second session focuses on historical sources (historic-epistemological dimension); the third session on the manipulation of the mathematical machine (manipulative dimension) and the fourth session on the construction of a model of the same mathematical machine by a DGE (digital dimension).

SOME THEORETICAL ELEMENTS

Mathematical Laboratory And Teachers Educations

The Italian Mathematical Union has drawn on the ancient idea of the mathematical laboratory, when the new mathematics standards from 5 to 18 years old students were prepared (Anichini et al., 2004). The document reads:

> A mathematics laboratory is not considered a place (e.g., a computer classroom) but rather a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people, structures, ideas. We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together. It is important to bear in mind that a tool is always the

[1] For the Italian version see http://www.mmlab.unimore.it/on-line/Home/VisitealLaboratorio/Materiale.html

> result of a cultural evolution, and that it has been made for specific aims, and insofar, that it embodies ideas. This has a great significance for the teaching practices, because the meaning can not be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used and in the schemes of use of the tool itself. (pp.60)

In this quotation, the last sentences evoke the distinction between artefact and instrument (Rabardel, 1995). The *instrument* (to be distinguished from the artefact) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*. The utilization schemes are progressively elaborated when an artefact is used to accomplish a particular task; thus the instrument is a construction of an individual. It has a psychological character and it is strictly related to the context within which it originates and its development occurs. The elaboration and evolution of the instruments is a long and complex process that Rabardel names *instrumental genesis.* Instrumental genesis can be articulated into two coordinated processes: *instrumentalisation*, concerning the emergence and the evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the emergence and development of the utilization schemes.

According to the Italian governmental regulations issued in 1998, teacher education (including mathematics teachers education) is organized around three main kinds of activities: lectures (for large groups of prospective teachers, up to 100 and more), in-school apprenticeship (individual participation in standard classroom activities, under the supervision of expert teachers) and laboratories (with a number of prospective teachers around 25, i.e., the standard size of a classroom). In these laboratories, prospective secondary mathematics teachers come personally into contact with new methodologies, with new tools that offer innovative models for their future teaching practice: the personal experience is accompanied by a reflection of the possible application in secondary school teaching. The laboratory activity is a great challenge for teachers, as it requires specific professional competences, which cannot be taken for granted. Some authors have discussed the domains of professional knowledge for teachers. For instance, Ball, Thames and Phelps (2008), suggest at least the following domains, as a refinement of Shulman's categories of Subject Matter Knowledge and Pedagogical content knowledge:

- the *common content knowledge*, i.e., the mathematical knowledge at stake in the material to be taught;
- the *knowledge of content and students,* related to the prediction and interpretation of students' processes when a task is given;

- the *knowledge of content and teaching*, related to the teacher's actions aiming at the students' construction of mathematical meaning;
- the *specialised content knowledge*, that is the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work.

Elsewhere Bartolini Bussi and Maschietto (2008) have linked Ball's analysis to the model developed in the Laboratory of Mathematical Machines (MMLab) for teacher education, as both encompass the needed complex and systemic approach. Our aim is to put the prospective teacher in a situation where the artefacts of the Laboratory (either mathematical machines or computers) are used according to an approach based on the Vygotskian perspective of semiotic mediation (details in the quoted paper and in Bartolini Bussi, & Mariotti, 2008). In this way, prospective teachers can experiment with both exploration processes (that could be activated in their students) and a model of didactic management of activities with artefacts used as tools of semiotic mediation (by the teacher educator).

Tools Of Semiotic Mediation

The theoretical construct of semiotic mediation draws on Vygotsky's papers[2] published in the Thirties (for an English translation, see Vygotsky 1978). It has been elaborated and applied to mathematics education by some authors. In this chapter we follow the elaboration of Bartolini Bussi and Mariotti (2008), that is shortly outlined below.

The process of semiotic mediation may be described schematically by means of the following drawing (Figure #.3).

[2] See Goos' chapter in this volume for other elements concerning the socio-cultural perspective and cultural tools.

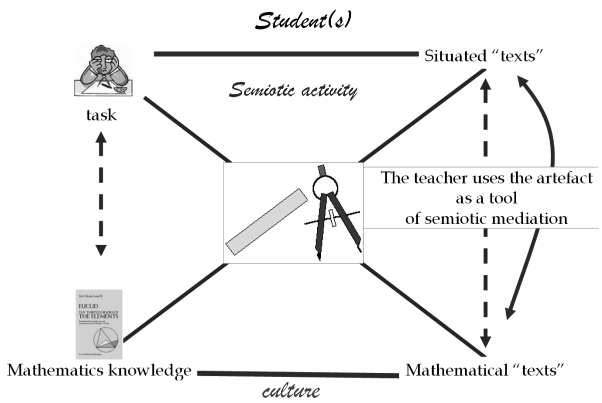


Figure #.3 Semiotic mediation diagram

A learner (either a secondary student or a prospective teacher) is given a task (left-top vertex of the rectangle of the Figure #.3, for an example, see below), to be solved by means of a specific artefact (e.g., the pair straightedge and compass, centre of the rectangle of the Figure #.3). The piece of mathematics knowledge at stake may concern the meaning of circle and of straight line and geometrical properties of some figures.

In the resolution process of the given task, two levels can be distinguished. At the first level, a technical solution of the task may be given using the artefact mechanically, i.e., repeating, in automatized way, a set of instruction, without wondering why the geometrical construction works. At the second level, a solution becomes "meaningful" (in etymological sense) when it is justified and commented with reference to the properties of circles, triangles and so on, as, in this way, the meaning of geometrical construction is approached at and enriched. This meaning is a piece of mathematics knowledge (left-bottom vertex of the rectangle of the Figure #.3).

If the activity stays on the technical plane (task, artefact and situated texts triangle in the Figure #.3), the justification of the correctness may be not at stake. The control by either perception or measuring might be enough, to agree that the solution is correct. The justification belongs to the theoretical plane. The technical description answers the question "how?", whilst the theoretical description is the first step to answer the question "why?". The path towards the justification is neither simple nor fast. For example, from the initial situated expressions which refer to the actual use of the ruler and the compass, the reference to the artefact disappears, remaining embodied or evoked in the straight line and in the circle, i.e., the geometrical objects traced by means of them.

Consider, for instance, the task to bisect a given finite straight line[3] by ruler and compass, and compare the following texts, that accompany similar (yet not identical) drawings (Figure #.4a and Figure #.4b). In the two texts, the artefact is the same (the pair ruler and compass). On the left, there is an evident reference to the physical operations to be performed by means of the concrete available tools, whilst on the right the reference is to geometrical objects, that evoke their geometrical properties. The two sets of instructions are different: the left one evokes a text of technical drawing or engineering[4], whilst the right one evokes Euclid's construction. A novice might be at ease with the left set of instructions, whilst an expert might be annoyed by it. In the left set of instructions, the characteristic properties of the circle are not explicitly evoked, at both linguistic and graphical level. The text only mentions (and the drawing only contains) a "small arc" instead of a "circle". In the right list of instructions, the references to the circle and its properties are explicit. The text on the left is situated, whilst the text on the right is decontextualised (hence, it is a mathematical text). This may be interpreted, after Rabardel (1995, see the section 2.1. above), saying the authors are referring to two different instruments.

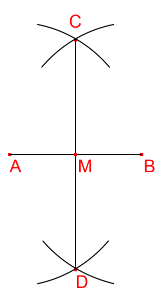


Figure #.4a[5]

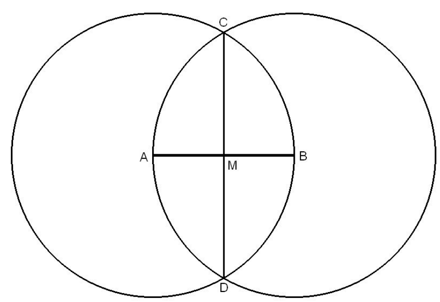


Figure #.4b

Set the needlepoint of the compass on A and the lead point on B and draw a small arc on each side of the line AB.	Draw a circle with centre A and radius AB.
Set the needlepoint of the compass on B and the lead point on A and draw a small arc on each side of the line AB.	Draw a circle with centre B and radius BA.
Mark by means of a pencil the points	Find the intersection C and D of the

[3] This construction problem is taken from the First Book of Euclid's elements (Proposition 10, see Heath, 1956, p. 267). The solution we propose is a bit different from Euclid's one.
[4] http://www.tpub.com/engbas/4.htm. Accessed February 2010.
[5] Java animation: http://www.mathopenref.com/constbisectline.html. Accessed February 2010.

C and D where the arcs intersect each other.	two circles.
Put the edge of the ruler on C and D and draw by means of a pencil a line r.	Draw a straight line r joining C and D.
Mark by means of a pencil the point M where the line r intersects AB.	Find the intersection M of r and AB.

The reader might be interested to write, for the same task, the instructions for another artefact, e.g., a DGE like Cabri or Geometer's Sketchpad. The situated text in this case is different, as the reference is to the commands available on the menus. For instance, in DGE there is no needlepoint and arcs can be drawn only after having drawn the whole circle. Yet the Euclid-style text on the right can serve still as a geometrical reference text.

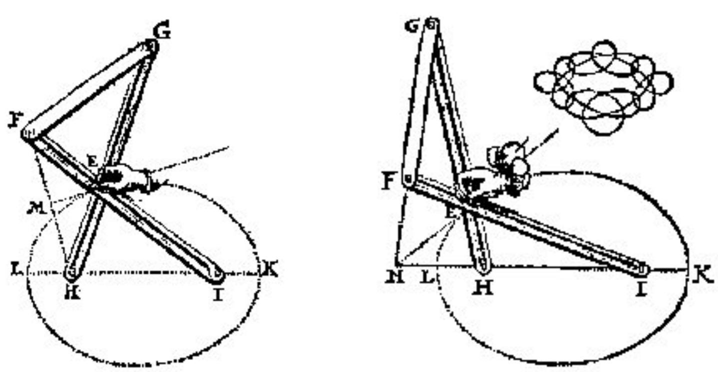
Whichever is the artefact (the concrete pair ruler and compass on the paper, but also the virtual commands on the screen in the case of a DGE), the mathematics teacher's aim is not (only) the technical process, but also the geometrical process that evokes the properties (either definitions or theorems) of geometrical objects. The artefacts allow to perform concrete actions (i.e., they are outward oriented) and, on the other hand, they allow to form the subject's plane of consciousness (i.e., they are inward oriented). In this second case, culturally based psychological processes are created (Vygostky 1978), in the sense that by means of the physical activity (either ruler and compass or the menu commands) the user is constructing the meanings of circles and lines. According to Vygotskian approach, within the social use of artefacts in the accomplishment of a task, shared signs are generated. These signs are related to the accomplishment of the task and to the used artefact, on the one hand, and they may be related to the content that is to be mediated, on the other hand. They can be intentionally used by the teacher to exploit semiotic processes, aiming at guiding the evolution of meanings by the evolution of signs centred on the use of an artefact within the class community. In other words, the teacher acts as mediator using the artefact to mediate mathematical content to the students. In this sense, the teacher uses the artefacts as tools of semiotic mediation (Bartolini Bussi, & Mariotti, 2008, Figure #.3).

The ruler and the compass are the most known drawing devices. In the following sections we study the case of another drawing device, based on the geometrical properties of antiparallelogram, i.e., a quadrilateral in which the pairs of nonadjacent sides are congruent, but in which the pairs of opposite sides intersect (unlike in a parallelogram). The analysis is distinguished into three parts: the historic-epistemological dimension concerning textual descriptions; the manipulative dimension involving material copies and the digital dimension based on simulations by a DGE. For all dimensions, the focus is on tasks for teacher's education.

HISTORIC - EPISTEMOLOGICAL DIMENSION

The Background

The ruler and the compass have been used from the Euclid’s age to solve construction problems in plane geometry. The discussion about acceptable tools to solve construction problems was raised in the classical age (Heath, 1956) and later attacked directly by Descartes, in the XVII century, when he wrote the *Géométrie* (1637), i.e., the appendix to the *Discourse de la Méthode*. His aim was to delineate the frontier between those curves that are acceptable in geometry, which Descartes called “geometric”, and the rest, which he called “mechanical” (Bos, 2001; Dennis and Confrey, 1995; see also Bartolini Bussi, 2001). As said above, in the classical age the “identification” of curves and artefacts (drawing devices) had been realized for straight line (ruler) and circles (compass). Conics were rather considered as solid curves (conic sections), i.e., curves obtained by cutting a cone. Yet conics and other curves could be used to solve construction problems (e.g., the trisection of an angle, see Heath, 1956) that could not be solved using only straight lines and circles. Descartes looked for artefacts able to draw curves by a continuous motion: in this way the perceptual evidence of intersection between curves could be used to state the existence of a rigorous solution of a construction problem (Lebesgue, 1950).



a *b*

Figure #.5 Van Schooten's antiparallelogram (van Schooten, 1657).

Van Schooten followed him in the same direction: he translated Descartes' *Géométrie* into Latin and appended commentaries (*Exercitationes*) about curve drawing devices. The Figure #.5 shows an articulated antiparallelogram used as a curve drawing device from van Schooten (1657). The Figure #.7 shows students using a modern wooden reconstruction of it.

Drawings And Texts As Artefacts

With respects to artefacts, Wartofsky (1979) distinguished primary, secondary and tertiary artefacts:

> What constitutes a distinctively human form of action is the creation and use of artifacts, as tools, in the production of the means of existence and in the reproduction of the species. *Primary artifacts* are those directly used in this production; *secondary artifacts* are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artifacts are therefore representations of such modes of actions (Wartofsky, 1979, p. 200 ff.).

In this chapter, we have examples of primary artefacts (the antiparallelogram of the Figure #.7) and of secondary artefacts (drawings and text from van Schooten's book). There is also another class of artefacts (tertiary artefacts):

> (…) which can come to constitute a relatively autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or 'free' play or game activity. This is particularly true (…) when the relation to direct productive or communicative praxis is so weakened, that the formal structures of the representation are taken in their own right as primary, and are abstracted from their use in productive praxis (Wartofsky, 1979, p. 208 ff.).

Mathematical theories are examples of tertiary artefacts, organizing the models constructed as secondary artefacts. Mathematical theories have the potential of being expanded to create something anew, that maintains links with practical and representative activities.

The two drawings of the Figure #.5 (van Schooten, 1657) show two different positions (like two 'frames' in a modern motion picture) of the articulated antiparallelogram. They seem realistic (bars, pivots, and even the hands), but we discuss this point below. Beside the locus of E also the tangent line in E is drawn.

Van Schooten's text follows (the reference is to the Figure #.5a and the Figure #.5b):

> *Chapter VIII. About the way of tracing ellipses in a plane, when the foci and the vertices are given.*
> There are several ways to trace ellipses: the one when foci and vertices are given is not more complex than others [...]. Given in a plane the foci H and I, a vertex L and the other vertex K, so that LK is the transverse axis, to trace, in the same plane, the drawing of an ellipse, with those vertices and foci. To prepare, in either brass or wood or other hard material, three bars HG, GF and FI, with HG and FI equal to LK, whilst FG is equal to the distance HI between the two foci. Besides, let the bars HG and FI be fissured (along their length) by two runners with the same width of the diameter of the cylindrical stylus, that will be inserted into them to trace the elliptical drawing. Let each of the bars HG and FI be drilled at the ends H and I, to insert the hinges pegged down in the foci H and I; the ends G and F of the same bars will be hinged on the ones of the bars FG, to create the configuration of the figure. That done, if the stylus inserted in both runners (i.e., in the point E where the bars HG and FI intersect each other) is moved, it will drag the bars Hg and FI, which will rotate on the points H and I: moving it from L to K the stylus will trace half (LEK) of the elliptical drawing. In the same way the other half will be traced (van Schooten, 1657, p. 339, translated by the authors).

The Task

Van Schooten's text hints at the process of instrumental genesis for both the coordinated processes of instrumentalisation and instrumentation (see Section Mathematical Laboratory And Teacher Education above). This suggests the following task for prospective teachers, as an example of analysis of a secondary artefact:

Read van Schooten's texts about the ellipse drawing device by antiparallelogram. Find the parts concerning the components of the artefact and the constraints for its points and the parts concerning the utilization schemes[6] of the artefact.

[6] Bèguin and Rabardel (2000) define *instrumentation* as follows:

> "Utilization schemes have both a private and a social dimension. The private dimension is specific to each individual. The social dimension, i.e., the fact that it is shared by many members of a social group, results from the fact that schemes develop during a process involving individuals who are not isolated. Other users as well as the artefact's designers contribute to the elaboration of the scheme" (Bèguin and Rabardel, 2000, p. 182).

The antiparallelogram construction is related yet different from the better known string construction of ellipses (or gardener's string construction, see the Figure #.6, taken from van Schooten, 1657). An additional task may be designed, for prospective teachers, as a comparison between them:

> Compare van Schooten's text about antiparallelogram and the gardener drawing of the artefact pencil-string, with regard to the components of the artefacts and the utilization schemes.

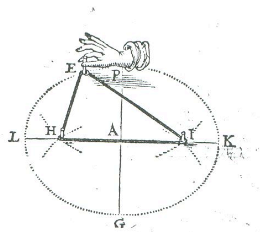


Figure #.6 Drawing of gardener's string (van Schooten, 1657).

In the antiparallelogram there is a linkage whose motion is perfectly determined by the physical constraints, whilst in the gardener's string construction the string has to be taut by the user by means of a pencil during the process. Hence, in the former the motion is controlled by the artefact, whilst in the latter is controlled by the user. The hand in the former has mainly the function to keep the pencil in the right position, although the motion might be given to the artefact pushing other points of the bars (e.g., G, F and others); the hand in the latter has both functions: it holds the pencil and moves it as well, keeping the string taut.

Following Ball, Thames and Phelps' approach (see Section Mathematical Laboratory And Teacher Education above), these tasks are related to the specialised content knowledge. In fact, they concern the mathematical knowledge needed for teaching: for instance, social dimension of the instrumental genesis and different instruments (artefacts + utilisation schemes) related to the same mathematical meanings. In this case, they also contribute to enrich the knowledge of content and teaching.

MANIPULATIVE DIMENSION

The Background

In the MMLab there are more than two hundred working reconstructions (based on the original sources) of mathematical artefacts taken from the history of geometry. Some of them (e.g., van Schooten's antiparallelogram, see Figure #.7) are reproduced in multiple copies to allow small groups (four or five people) use them in the same session (with either secondary school students or prospective teachers). Afterwards, we present the features of a mathematical laboratory session. The structure of a session and the working sheet result from a long process of revision and refinement, based on our analysis of the laboratory sessions realised in the MMLab (for both students and prospective teachers).

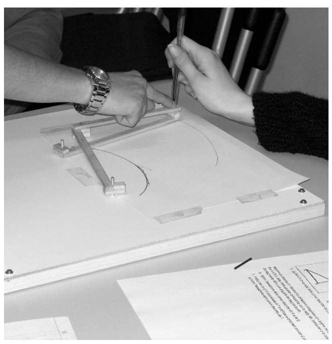


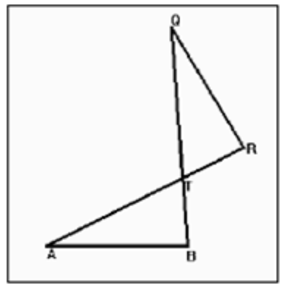
Figure #.7 The concrete artefact.

The Task

The working sessions are usually split into three parts:

- historical introduction for the whole group;
- small group work on the linkage, by means of a working sheet;
- collective work on the solutions for the given tasks.

In the second part, a copy of the van Schooten's antiparallelogram (considered as a primary artefact) with an exploration sheet (Figure #.8), where a schema of the artefact is drawn, is given to each group. Each group is asked to write its answers to the questions. Each working sheet contains several different questions, that support the exploration process of the mathematical machine. They take into account on one hand the process of instrumental genesis (Rabardel, 1995), on the other hand our intention to foster the processes of both production of conjecture and construction of proof, beyond the pure manipulation. In fact, questions concern not only how the artefact is made and works, but also the properties of the drawn curve and the characteristics of the device permitting to draw that curve. The proposed sequence of questions considers the temporal commitment of two hours (at the maximum) for a session, in order to permit a suitable work.



1. How many rigid rods make up the linkage?
2. Measure the lengths of the individual rods.

Which figures do the rods form?

3. Which are the elements of the instrument which are fixed at the plan?

Move the linkage.

4. Which are the segments that do not change in length during the movement?
5. Which are the segments that change their length during the movement?
6. Which variable length segments are equal?

This instrument has three tracer points: Q, R and T. Answer the following questions:

7. Which curves do the points Q and R trace?
8. Put your pencil in T and draw a part of a curve. Which is the property of the curve plotted by the point T?
9. Choose a suitable Cartesian axes system. Write the equations of the curves plotted by the points Q, R and T.

Figure #.8 Working sheet.

Questions 1, 2 and 3 aim at highlighting the physical features of the given artefact (the emergence of the components in the instrumentalisation process). In particular, Question 2 offers elements to justify the functioning of the linkage and the property of the drawn curve. Questions 4, 5 and 6 require the movement of the quadrilateral and aim at highlighting some invariants in its structure during this movement. In this request, the first elements of the instrumentation process are in play, because the users have to choice a pilot point, often in an implicit way. This instrument has three tracer points (Q, R and T), but there is only a hole for pencil in T. So, the trajectories of Q and R could not be really traced, but only supposed. Question 8 concerns the instrumentation process. It also requires to explicit the property of the drawn curve, on the basis of the exploration. The definition of ellipse as a locus of points in a plane such that the sum of the distances to two fixed points (foci A and B) is a constant is expected. In particular, Question 8 prompts a process of conjecture production (*what*) and proof construction (*why*). Question 9 imposes the passage to the analytic geometrical register. The best choice for the Cartesian axes system is as follow: straight line containing the line segment AB as x-axis, the perpendicular bisector of the line segment AB as y-axis. Furthermore, the solver can choice the distance AB as a and the distance AR as b in writing the required equations.

The collective part of the session (third part) concerns the shift from the texts (right-top vertex in the Figure #.3) produced by the prospective teachers towards mathematical texts with definition and properties of ellipse (right-bottom vertex in the Figure #.3). In particular, Questions 2 and 7 are interesting to be developed in a collective discussion, because the former is related to the mathematical meaning of tangent line to ellipse and the latter to a definition of ellipse different from the definition evoked by Question 8. As regards to Question 2, if the quadrilateral ABRQ is recognize as a isosceles trapezoid, its symmetry axis is the tangent line to ellipse at its point T (as it appears in van Schooten's drawings, Figure #.5). Question 7 allows to pay attention to the relationship between the circle with centre on focus A and the point T. In fact, T is a point at the same distance from the focus B and the circle traced by R with centre on focus A (in other term, ellipse as a locus of points in a plane such that the distances to a fixed point and to a circle with centre on another fixed point is equal). The circle with centre A is named "directrix".

In a mathematics laboratory session, the teacher educator uses the artefact as a tool of semiotic mediation. At the same time, prospective teachers are involved and test an example of didactic management of this session.

DIGITAL DIMENSION

The Background

Dynamic Geometry Environments (DGE; e.g., Cabri) are used in MMLab as modelling contexts for dynamic artefacts. Prospective secondary mathematics teachers, after having explored the physical drawing device, are asked to produce a digital model of it. This task represents a challenge for prospective and practising teachers. In fact, the main idea is to use DGE not to explore open problems or as a model for theoretical systems (for a discussion, see Laborde, 2000), but as a modelling environment.

The Task

Prospective teachers are given again a working copy of the drawing device (Figure #.7) and the following task:

> Construct on the Cabri screen a model of the drawing device, that may be piloted in order to work in the same way of the physical one.

In this case, the artefact is DGE (i.e., Cabri) and the prospective teacher has the possibility to use the menus to solve the task. Some different solutions emerge: we illustrate only two solutions[7] (and Figure #.10) and discuss the difference.

First solution.

Line segments are assembled to produce an antiparallelogram.

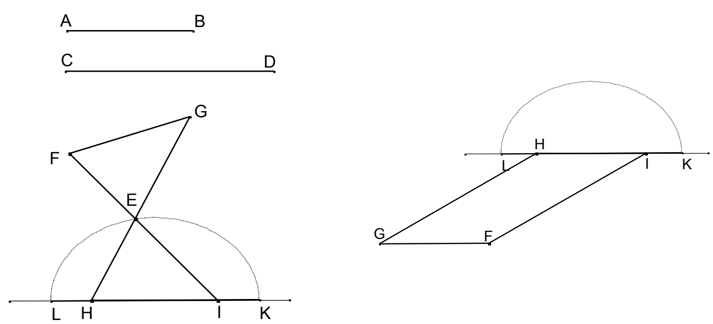


Figure #.9 A technical solution.

Two prototypes of the bar are drawn (AB and CD) (Figure #.9).

1. compass: AB in H

[7] We refer in a short way to the Cabri commands. Legend:

- *compass*: to transport the given segment with a vertex in a given point (the software draws a circle);
- *intersection*: to find the intersection point of two objects on the screen;
- *intersection* (after *compass* command): to intersect the circle with another object on the screen;
- *segment*: to draw a segment joining two points;

The others (*axis, locus, symmetrical point*) hint at geometrical meanings, and are realized by means of the available commands.

2. intersection: I
3. compass: CD in H: select G on the circle
4. compass: CD in I
5. compass AB in G
6. intersection: F
7. segment: IF
8. segment: HG
9. intersection: FI and HG: E
10. intersection: L and K
11. drag G to pilot E.

Locus: the same as the one drawn by the physical device; different from van Schooten drawing (see Figure #.5).

Motion: when the point G is dragged on the circle suddenly the antiparallelogram unknits and becomes a parallelogram.

If one recognizes that the quadrilateral HFGI is an isosceles trapezoid, whose HG and FI are its diagonals, he/she is able to design a digital antiparallelogram, satisfying the two previous conditions. In this case, the symmetry axis of the antiparallelogram is the tangent line to the ellipse in each point, as van Schooten's drawing clearly shows. The second solution is described below.

Second solution.

A geometric property of antiparallelogram is used.

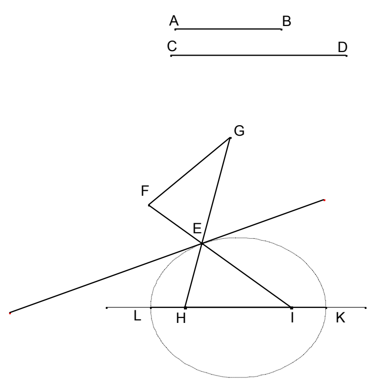


Figure #.10 A geometrical solution.

Two prototypes of the bar are drawn (AB and CD) (Figure #.10)

1. compass: AB in H
2. intersection: I
3. compass: CD in H: select G on the circle
4. axis of GI

5. symmetrical point of H with respect to the axis: F
6. segment: IF
7. segment: HG
8. intersection: FI and HG: E
9. intersection: L and k
10. drag G to pilot E

Locus: the same as van Schooten drawing (see Figure #.5); different from the drawing produced by the physical device.

Motion: when the point G is dragged on the circle the antiparallelogram is maintained.

In the two solutions, the same commands (artefacts) are instrumented in different ways.

In all the cases a difference emerges. The task is impossible if it is taken literally. Actually, it is not possible to design a model that works exactly like the physical one. As we have observed, the physical artefact can be moved pushing many points of the bars, provided that the pencil is firmly inserted into the moving hole E. This cannot be realised with Cabri. Every construction is ordered: the user has to define which is the starting point (G in the above constructions), to be assumed as independent variable, and what follows is strictly dependent on this choice. This is a general property. If one wishes to select the point E as the piloting point, s/he should produce a different set of instruction where E is a piloting point (independent variable) and the others are dependent on E. The choice of the piloting point (a point with one degree of freedom) has to be done explicitly before starting the Cabri construction. This means to look at the antiparallelogram according to the constraints of Cabri (and the same is true for a whichever other DGE). The second construction produces Van Schooten's model, but does not work as the linkage. The first construction (with adjustment) is closer to the linkage but produces only a part of the ellipse.

If one goes back to the schema of the Figure #.3, the first solution may be described by means of a situated text (right-up vertex of the rectangle of the Figure #.3): copies of the prototypes of the bars are assembled as in a meccano setting. The names used are bars rather than straight lines. The observation of prospective teachers at work shows that they try to mime the rotation of the bars GH and FI on the screen with fingers, pointing with thumb in H and I and with forefingers in G and F, and look for a position where FG has the given length. The second solution, instead, hints at a non transparent property of the artefact (the presence of a symmetry axis), that is better acknowledged when a static frame is considered. This is not a spontaneous solution, as the manipulation of the concrete artefact suggests rather the first one. Yet, as soon as the second solution is found, a new exploration of van Schooten's antiparallelogram may be started on the screen, to highlight the tangent line and the relationships between the length of the longest bar and the major axis of the ellipse (as said in van Schooten's text).

CONCLUDING REMARKS

In this paper we have presented and discussed three different ways of introducing a mathematical machine (i.e., a curve drawing device, that produces an elliptical trajectory) into the mathematical laboratory of a secondary teacher education program: the discussion and the interpretation of an artefact given by the pair text and drawings from a XVII century treatise; the manipulative exploration, according to a working sheet, of a material copy of the ancient artefact; the production of a digital simulation of the ancient artefact. Additional tasks may be designed (e.g., building a material copy, drawing on van Schooten's description) and analysed as well. All these activities can be carry out in two hours (at the maximum) sessions. For this reason, they can be easily proposed in both teacher training and students' mathematics course. Nevertheless, a systematic use of mathematical machines for all conic sections needs a careful planning and it represents a methodological choice of the teacher.

In all cases the instrumental genesis (according to Rabardel, 1995) is at work, yet in different ways. In the first case the prospective teacher is invited to recognize in the text hints at the instrumentation and the instrumentalisation process concerning the task of drawing an ellipse: as usual in most ancient treatises, the two processes are intertwined and not easily separable from each other. In the second case the prospective teacher is invited to experience in a personal way the instrumental genesis working with suitable tasks on a material model: the tasks are similar to the ones that he/she might give to his/her students. In the third case the curve drawing device is paired with another artefact (i.e., a DGE), that introduces additional strong constraints which force a new exploration of the material artefact and produce another way for drawing the same curve. The expert geometer might say that what is focused is "the same" artefact, i.e., van Schooten's ellipse drawing device by means of an antiparallelogram. Actually the artefacts are different. According to Wartofsky's classification (1979) in the first case it is a secondary artefact, used in transmission of modes of actions; in the second case it is a primary artefact that is directly used, although the justification required to introduce also secondary and tertiary artefacts; in the third case what is called into play is a tertiary artefact, i.e., the geometrical properties (referred to a mathematical theory) of the figure "antiparallelogram" in the Cabri setting. From a didactical perspective, the instruments (Rabardel, 1995) are different in the three cases because of different utilization schemes and constraints (material or digital) as well.

This experience shows to be paradigmatic for prospective teachers, in order to make them aware that, in spite of some widespread simplifications (see for instance the National Library of Virtual Manipulatives, http://nlvm.usu.edu/) it is quite different to operate on textual descriptions (even with "realistic" drawings), on material copies, on digital simulations. This is obviously true not only for the van Schooten's parallelogram but also for other teaching aids that may exist in either descriptive or material or digital forms. In every case, for every task, a careful analysis of the instrumental genesis and of its relationships with the construction of mathematical meaning is needed for the use in the mathematical laboratory with secondary school students.

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