22 November 2001

PHYSICS LETTERS B

Physics Letters B 521 (2001) 96–104

www.elsevier.com/locate/npe

Gravity on a 3-brane in 6D bulk

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Received 8 August 2001; accepted 1 October 2001
Editor: M. Cvetič

Abstract

We study gravity in codimension-2 brane world scenarios with infinite volume extra dimensions. In particular, we consider the case where the brane has non-zero tension. The extra space then is a two-dimensional "wedge" with a deficit angle. In such backgrounds we can effectively have the Einstein–Hilbert term on the brane at the classical level if we include higher curvature (Gauss–Bonnet) terms in the bulk. Alternatively, such a term would be generated at the quantum level if the brane matter is not conformal. We study (linearized) gravity in the presence of the Einstein–Hilbert term on the brane in such backgrounds. We find that, just as in the original codimension-2 Dvali–Gabadadze model with a tensionless brane, gravity is almost completely localized on the brane with ultra-light modes penetrating into the bulk. © 2001 Published by Elsevier Science B.V.

1. Introduction and summary

In the Brane World scenario the Standard Model gauge and matter fields are assumed to be localized on branes (or an intersection thereof), while gravity lives in a larger dimensional bulk of space–time [1–17]. There is a big difference between the footings on which gauge plus matter fields and gravity come in this picture. [1] Thus, for instance, if gauge and matter fields are localized on D-branes [3], they propagate only in the directions along the D-brane world-volume. Gravity, however, is generically not confined to the branes—even if we have a graviton zero mode localized on the brane as in [14], where the volume of the extra dimension is finite, massive graviton modes are still free to propagate in the bulk.

On the other hand, as was originally proposed in [16], in the cases with infinite volume extra dimensions [18–23], we can have almost completely localized gravity on higher codimension (δ-function-like) branes with the ultra-light modes penetrating into the bulk. [2] As was explained in [16], this dramatic modification of gravity in higher codimension models with infinite volume extra dimensions is due to the Einstein–Hilbert term on the brane, which, as was originally pointed out in [15,16], is induced via loops of non-conformal brane matter.

In the original models of [16] the brane is tensionless, so that the D-dimensional space–time is Minkowski. The purpose of this Letter is to consider such models with non-zero tension brane. In this case

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[1] This, at least in some sense, might not be an unwelcome feature—see, e.g., [4,7,12].

[2] A rather different mechanism was also proposed in [17], which leads to a complete localization of gravity on a codimension-1 brane with no (perturbative) modes propagating in the bulk.

0370-2693/01/$ – see front matter
PII: S0370-2693(01)01189-3

the bulk is no longer flat (but the brane is). In fact, at the origin of the extra space (that is, at the location of the brane) we have curvature singularities in these models. In codimension-3 and higher cases these curvature singularities are difficult to deal with. However, in the codimension-2 case, which we focus on in this Letter, the singularity is δ-function-like. That is, the space away from the brane is locally flat, and all the curvature is concentrated at the location of the brane. In fact, the extra space in this case is a two-dimensional "wedge" with a deficit angle, which depends on the brane tension.

Thus, in this Letter we analyze brane world gravity in such codimension-2 backgrounds. [3] The Einstein–Hilbert term on the brane can effectively be present classically if we include higher curvature (Gauss–Bonnet) terms in the bulk. Alternatively, such a term on the brane is generated at the quantum level if the brane matter is not conformal [15,16]. We study gravity in the presence of the Einstein–Hilbert term on the brane in such backgrounds. We find that, just as in the original codimension-2 Dvali–Gabadadze model with a tensionless brane [16], we still have almost complete localization of gravity on the brane. Thus, in the case of a non-zero tension 3-brane in infinite volume 6-dimensional space we have 4-dimensional gravity on the brane with ultra-light modes penetrating into the bulk.

The remainder of the Letter is organized as follow. In Section 2 we present the model along with the aforementioned background solution. In Section 3 we study small fluctuations around the solution in the presence of brane matter sources.

2. The model

In this section we discuss a brane world model with a codimension-2 brane embedded in a D-dimensional bulk space. (For calculational convenience we will keep the number of space–time dimensions D unspecified, but we are mostly interested in the case $D = 6$, where the brane is a 3-brane.) The action for this model is given by:

$$S = -f \int_\Sigma d^{D-2}x \sqrt{-\widehat{G}} + M_P^{D-2} \int d^D x \sqrt{-G} \times \left[R + \lambda\left(R^2 - 4R_{MN}^2 + R_{MNRS}^2\right)\right]. \tag{1}$$

Here M_P is the (reduced) D-dimensional Planck-mass; Σ is a δ-function-like codimension-2 source brane, which is a hypersurface $x^i = 0$ (x^i, $i = 1, 2$, are the two spatial coordinates transverse to the brane); the tension f of the brane is assumed to be positive; also,

$$\widehat{G}_{\mu\nu} \equiv \delta_\mu{}^M \delta_\nu{}^N G_{MN}\big|_\Sigma, \tag{2}$$

where x^μ are the $(D-2)$ coordinates along the brane (the D-dimensional coordinates are given by $x^M = (x^\mu, x^i)$, and the signature of the D-dimensional metric is $(-, +, \ldots, +)$); finally, the higher curvature terms in the bulk action are chosen in the Gauss–Bonnet combination, and the Gauss–Bonnet coupling λ is *a priori* a free parameter (which, as we will see below, is restricted to be non-negative by unitarity considerations).

The equations of motion following from the action (1) are given by:

$$R_{MN} - \frac{1}{2}G_{MN}\left[R + \lambda\left(R^2 - 4R_{MN}^2 + R_{MNRS}^2\right)\right] + 2\lambda\left(RR_{MN} - 2R_{MS}R_N^S + R_{MRST}R_N^{RST} - 2R^{RS}R_{MRNS}\right) + \frac{1}{2}\frac{\sqrt{-\widehat{G}}}{\sqrt{-G}}\delta_M{}^\mu \delta_N{}^\nu \widehat{G}_{\mu\nu} \tilde{f}\delta^{(2)}\left(x^i\right) = 0, \tag{3}$$

where $\tilde{f} \equiv f/M_P^{D-2}$.

Consider the following ansatz for the metric [4]

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + \exp(2\omega)\,\delta_{ij}\, dx^i\, dx^j, \tag{4}$$

where ω is a function of x^i but is independent of x^μ. With this ansatz we have:

$$R_{\mu\nu} = R_{\mu i} = 0, \qquad R_{ij} = \widetilde{R}_{ij} = \frac{1}{2}\widetilde{G}_{ij}\widetilde{R}, \tag{5}$$

$$\sqrt{\widetilde{G}}\,\widetilde{R} = \tilde{f}\delta^{(2)}\left(x^i\right), \tag{6}$$

[3] Certain codimension-2 solutions were discussed in [24–26].

[4] A similar solution was recently discussed in [26].

where $\widetilde{R}$ and $\widetilde{R}_{ij}$ are, respectively, the 2-dimensional Ricci scalar and Ricci tensor constructed from the 2-dimensional metric

$$\widetilde{G}_{ij} = \exp(2\omega)\,\delta_{ij}. \tag{7}$$

Since this metric is conformally flat, we have $\sqrt{\widetilde{G}}\,\widetilde{R} = -2\partial^i\partial_i\omega$ (where the indices are lowered and raised with δ_{ij} and δ^{ij}, respectively), so we have:

$$\partial^i\partial_i\omega = -\frac{1}{2}\tilde{f}\delta^{(2)}\big(x^i\big). \tag{8}$$

The solution to this equation is given by:

$$\omega\big(x^i\big) = -\frac{1}{8\pi}\tilde{f}\,\ln\!\left(\frac{x^2}{a^2}\right), \tag{9}$$

where $x^2 \equiv x^i x_i$, and a is an integration constant.

Let us go to the polar coordinates (ρ,ϕ): $x^1 = \rho\cos(\phi)$, $x^2 = \rho\sin(\phi)$ (ρ takes values from 0 to ∞, while ϕ takes values from 0 to 2π). In these coordinates the two dimensional metric is given by

$$d\tilde{s}_2^2 = \left(\frac{a^2}{\rho^2}\right)^{\nu}\big[(d\rho)^2 + \rho^2(d\phi)^2\big], \tag{10}$$

where

$$\nu \equiv \frac{1}{4\pi}\tilde{f}. \tag{11}$$

Let us change the coordinates to (r,ϕ), where

$$r \equiv \frac{1}{1-\nu}a^{\nu}\rho^{1-\nu}, \tag{12}$$

where we are assuming that $\nu < 1$. Then we have

$$d\tilde{s}_2^2 = (dr)^2 + \exp(-2\beta)\,r^2(d\phi)^2, \tag{13}$$

where

$$\exp(-\beta) \equiv 1-\nu. \tag{14}$$

Thus, we see that the D-dimensional space–time in this solution is the $(D-2)$-dimensional Minkowski space times a 2-dimensional "wedge" with the deficit angle

$$\theta = 2\pi\big[1-\exp(-\beta)\big] = \frac{\tilde{f}}{2}. \tag{15}$$

That is, the brane is flat for a continuous range of values of the brane tension f. Note that for the critical value f_c of the brane tension, where

$$f_c \equiv 4\pi M_P^{D-2}, \tag{16}$$

the deficit angle is 2π. Thus, we have a flat solution for the brane tension $0 < f < f_c$. Note that the Gauss–Bonnet coupling does not enter in this solution due to the fact that the space–time is factorizable, the curvature comes from the 2-dimensional wedge (in fact, the origin thereof), and the Gauss–Bonnet combination is trivial in two dimensions. However, as we will see in the following, the higher curvature bulk terms in this model do contribute to fluctuations around the background, and, in fact, effectively give rise to the $(D-2)$-dimensional Einstein–Hilbert term on the brane.

3. Brane world gravity

In this section we study gravity in the brane world solution discussed in the previous section. Thus, let us consider small fluctuations around the solution

$$G_{MN} = G^{(0)}_{MN} + h_{MN}, \tag{17}$$

where $G^{(0)}_{MN}$ is the background metric

$$G^{(0)}_{MN} = \begin{bmatrix} \eta_{\mu\nu} & 0 \\ 0 & \exp(2\omega)\delta_{ij} \end{bmatrix}. \tag{18}$$

The $(D-2)$-dimensional graviton $H_{\mu\nu} \equiv h_{\mu\nu}$ couples to the matter localized on the brane via

$$S_{\text{int}} = \frac{1}{2}\int_{\Sigma} d^{D-2}x\, T_{\mu\nu}H^{\mu\nu}, \tag{19}$$

where $T_{\mu\nu}$ is the conserved energy–momentum tensor for the matter localized on the brane:

$$\partial^{\mu}T_{\mu\nu} = 0. \tag{20}$$

The equations of motion read

$$\begin{aligned} &R_{MN} - \frac{1}{2}G_{MN}\big[R + \lambda\big(R^2 - 4R^2_{MN} + R^2_{MNRS}\big)\big] \\ &\quad + 2\lambda\big(RR_{MN} - 2R_{MS}R^S{}_N \\ &\qquad + R_{MRST}R_N{}^{RST} - 2R^{RS}R_{MRNS}\big) \\ &\quad + \frac{1}{2\sqrt{-G}}\delta_M{}^{\mu}\delta_N{}^{\nu}\big[\sqrt{-\widehat{G}}\,\widehat{G}_{\mu\nu}\tilde{f} - M_P^{2-D}T_{\mu\nu}\big] \\ &\quad \times \delta^{(2)}\big(x^i\big) = 0, \end{aligned} \tag{21}$$

where we should keep terms linear in the fluctuations h_{MN}, which are assumed to vanish once we turn off the brane matter source $T_{\mu\nu}$.

In fact, the linearized equations of motion are quite simple. The reason for this simplification is that the background is factorizable, and the Gauss–Bonnet terms give contributions only on the brane. Indeed, outside of the brane the D-dimensional space–time locally is Minkowski, and the linearized Gauss–Bonnet contributions vanish for flat backgrounds. Thus, it is not difficult to show that the linearized equations of motion have the following form:

$$\begin{aligned} & R_{MN} - \frac{1}{2} G_{MN} R \\ & \quad + \frac{1}{2\sqrt{-G}} \delta_M{}^\mu \delta_N{}^\nu \\ & \qquad \times \left[\sqrt{-\widehat{G}} \tilde{f} \left[\widehat{G}_{\mu\nu} + 4\lambda \left(\widehat{R}_{\mu\nu} - \frac{1}{2} \widehat{G}_{\mu\nu} \widehat{R} \right) \right] \right. \\ & \qquad \left. - M_P^{2-D} T_{\mu\nu} \right] \delta^{(2)}(x^i) = 0. \end{aligned} \tag{22}$$

Note that these linearized equations are the same as those following from the action $S_* + S_{\rm int}$, where

$$\begin{aligned} S_* = {} & \widehat{M}_P^{D-4} \int_\Sigma d^{D-2}x \sqrt{-\widehat{G}} \left[\widehat{R} - \widehat{\Lambda} \right] \\ & + M_P^{D-2} \int d^D x \sqrt{-G} R, \end{aligned} \tag{23}$$

while

$$\widehat{M}_P^{D-4} \equiv 2\lambda f, \tag{24}$$

and $\widehat{\Lambda} \equiv 1/2\lambda$. That is, at the linearized level the model (1) coincides with the model (23), albeit they are different beyond the linearized approximation. In particular, the backgrounds corresponding to the ansatz (4) are the same in both models.

Thus, as far as the linearized level is concerned, the contributions from the Gauss–Bonnet term are equivalent to having a tree-level Einstein–Hilbert term on the brane. As was pointed out in [15,16], such a term drastically modifies the behavior of gravity at short vs. long distances. Note that in our case the $(D-2)$-dimensional Planck scale on the brane is given by $\widehat{M}_P$. Positivity of $\widehat{M}_P^{D-4}$ then requires that λ be positive. Finally, note that for

$$\exp(-\beta) = 1/N, \tag{25}$$

where N is a positive integer, the wedge is nothing but the $\mathbf{R}^2/\mathbf{Z}_N$ orbifold with the origin of the wedge identified as the orbifold fixed point.

3.1. Linearized equations of motion

The linearized equations of motion for the fluctuations h_{MN} induced by the brane matter are given by:

$$\begin{aligned} & -\nabla^L \nabla_L h_{MN} + 2\nabla^L \nabla_{(M} h_{N)L} - \nabla_M \nabla_N h \\ & \quad + G^{(0)}_{MN} \left(\nabla^L \nabla_L h - \nabla^L \nabla^K h_{LK} \right) \\ & \quad = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}} \widetilde{T}_{MN} \delta^{(2)}(x^i), \end{aligned} \tag{26}$$

where the components of the “effective” energy–momentum tensor $\widetilde{T}_{MN}$ are given by

$$\begin{aligned} \widetilde{T}_{\mu\nu} = {} & T_{\mu\nu} - \widehat{M}_P^{D-4} \left[-\partial^\lambda \partial_\lambda H_{\mu\nu} + 2\partial^\lambda \partial_{(\mu} H_{\nu)\lambda} \right. \\ & \left. - \partial_\mu \partial_\nu H + \eta_{\mu\nu} \left(\partial^\lambda \partial_\lambda H - \partial^\lambda \partial^\sigma H_{\lambda\sigma} \right) \right], \end{aligned} \tag{27}$$

$$\widetilde{T}_{\mu i} = f h_{\mu i}, \tag{28}$$

$$\widetilde{T}_{ij} = 0, \tag{29}$$

and we have defined $h \equiv h_M^M$ and $H \equiv H_\mu^\mu$. These equations of motion are invariant under certain gauge transformations corresponding to unbroken diffeomorphisms. Since the brane has non-zero tension, some of the diffeomorphisms

$$\delta h_{MN} = \nabla_M \xi_N + \nabla_N \xi_M, \tag{30}$$

corresponding to the D-dimensional reparametrizations

$$x^M \to x^M - \xi^M(x), \tag{31}$$

are actually broken at the origin of the wedge. Thus, it is not difficult to show that the (ij) components of (26) are invariant under the full D-dimensional diffeomorphisms (30), while the invariance of the (μi) and the $(\mu\nu)$ components requires that

$$\begin{aligned} & f \frac{\delta^{(2)}(x^k)}{\sqrt{\widetilde{G}}} \nabla_i \xi_\mu = 0, \\ & f \nabla^i \left[\xi_i \frac{\delta^{(2)}(x^k)}{\sqrt{\widetilde{G}}} \right] = 0, \end{aligned} \tag{32}$$

respectively. Note that these conditions are trivial in the case of a tensionless brane (where the space is flat everywhere, including at the origin). However, for a non-zero tension brane these conditions give non-trivial restrictions on the gauge parameters at the origin of the wedge (away from the origin these conditions, once again, are trivial). Thus, we have:

$$\partial_i \xi_\mu \big|_{\rho=0} = 0, \tag{33}$$

$$\partial^i \left[\xi_i \delta^{(2)}\left(x^k\right)\right] = 0. \tag{34}$$

Because of these conditions, some care is needed in gauge fixing in this model. In particular, there are subtleties with imposing a gauge such as the harmonic gauge (if $f \neq 0$). At any rate, we will solve the above equations of motion without appealing to such gauge fixing.

Since we are looking for solutions to the above linearized equations of motion such that h_{MN} vanish for vanishing $T_{\mu\nu}$, it is clear that the graviphoton components $h_{\mu i}$ must be vanishing everywhere:

$$h_{\mu i} \equiv 0. \tag{35}$$

Indeed, the graviphotons do not couple to the conserved energy–momentum tensor $T_{\mu\nu}$ on the brane. Moreover, the graviscalar components $\chi_{ij} \equiv h_{ij}$ only couple to the trace of $T_{\mu\nu}$, that is, $T \equiv T^\mu_\mu$. This implies that we have

$$\chi_{ij} = \frac{1}{2} G^{(0)}_{ij} \chi, \tag{36}$$

where $\chi \equiv (G^{(0)})^{ij} \chi_{ij}$. The equations of motion for $H_{\mu\nu}$ and χ then simplify as follows (note that $h = H + \chi$):

$$\begin{aligned} &-\left(\partial^\lambda \partial_\lambda + \nabla^i \nabla_i\right) H_{\mu\nu} + 2\partial^\lambda \partial_{(\mu} H_{\nu)\lambda} - \partial_\mu \partial_\nu H \\ &\quad - \partial_\mu \partial_\nu \chi + \eta_{\mu\nu} \bigg(\partial^\lambda \partial_\lambda H + \partial^\lambda \partial_\lambda \chi + \nabla^i \nabla_i H \\ &\qquad + \frac{1}{2} \nabla^i \nabla_i \chi - \partial^\lambda \partial^\sigma H_{\lambda\sigma} \bigg) \\ &\quad = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}} \widetilde{T}_{\mu\nu} \delta^{(2)}\left(x^i\right), \end{aligned} \tag{37}$$

$$\partial_i \left[\partial^\lambda H_{\mu\lambda} - \partial_\mu H - \frac{1}{2} \partial_\mu \chi \right] = 0, \tag{38}$$

$$\begin{aligned} &-\nabla_i \nabla_j H + G^{(0)}_{ij} \bigg[\partial^\lambda \partial_\lambda H + \frac{1}{2} \partial^\lambda \partial_\lambda \chi \\ &\qquad + \nabla^k \nabla_k H - \partial^\lambda \partial^\sigma H_{\lambda\sigma} \bigg] = 0. \end{aligned} \tag{39}$$

From the last equation it follows that

$$\partial^\lambda \partial_\lambda H + \frac{1}{2} \partial^\lambda \partial_\lambda \chi + \frac{1}{2} \nabla^i \nabla_i H - \partial^\lambda \partial^\sigma H_{\lambda\sigma} = 0, \tag{40}$$

$$\nabla_i \nabla_j H = \frac{1}{2} G^{(0)}_{ij} \nabla^k \nabla_k H. \tag{41}$$

Let us begin discussing this system of equations by studying the last equation for H.

This equation can be rewritten as follows:

$$\begin{aligned} &\partial_i \partial_j H - \partial_i \omega \partial_j H - \partial_j \omega \partial_i H \\ &\quad = \delta_{ij} \left[\frac{1}{2} \partial^k \partial_k H - \partial^k \omega \partial_k H \right]. \end{aligned} \tag{42}$$

Consider axially symmetric solutions: $H = H(x^\mu, \rho)$ (recall that $\rho^2 = x^i x_i$). Then we have:

$$H'' + \frac{1}{\rho}(2\nu - 1) H' = 0, \tag{43}$$

where prime stands for derivative w.r.t. ρ. The general solution to this equation is given by

$$H\left(x^\mu, \rho\right) = B\left(x^\mu\right) \left(\frac{\rho^2}{a^2} \right)^{1-\nu} + C\left(x^\mu\right), \tag{44}$$

where B, C *a priori* are arbitrary functions of x^μ. Note, however, that since $0 < \nu < 1$, we must have $B(x^\mu) \equiv 0$. This implies that H is only a function of x^μ. We can then always gauge it away using the $(D-2)$-dimensional diffeomorphisms with the gauge parameters $\xi_\mu(x^\sigma)$ independent of x^i (note that such gauge transformations do not affect the graviphoton or graviscalar components). Thus, we conclude that H can be set to zero everywhere. Note that this is actually correct even for $\nu = 0$, that is, in the case of a tensionless brane.

With $H \equiv 0$ the equations of motion simplify as follows:

$$\begin{aligned} &-\left(\partial^\lambda \partial_\lambda + \nabla^i \nabla_i\right) H_{\mu\nu} + 2\partial^\lambda \partial_{(\mu} H_{\nu)\lambda} - \partial_\mu \partial_\nu \chi \\ &\quad + \eta_{\mu\nu} \left(\partial^\lambda \partial_\lambda \chi + \frac{1}{2} \nabla^i \nabla_i \chi - \partial^\lambda \partial^\sigma H_{\lambda\sigma} \right) \\ &\quad = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}} \widetilde{T}_{\mu\nu} \delta^{(2)}\left(x^i\right), \end{aligned} \tag{45}$$

$$\partial_i \left[\partial^\lambda H_{\mu\lambda} - \frac{1}{2} \partial_\mu \chi \right] = 0, \tag{46}$$

$$\frac{1}{2} \partial^\lambda \partial_\lambda \chi - \partial^\lambda \partial^\sigma H_{\lambda\sigma} = 0. \tag{47}$$

Also, note that

$$\begin{aligned} \widetilde{T}_{\mu\nu} = T_{\mu\nu} - \widehat{M}_P^{D-4} \big[&-\partial^\lambda \partial_\lambda H_{\mu\nu} + 2\partial^\lambda \partial_{(\mu} H_{\nu)\lambda} \\ &- \eta_{\mu\nu} \partial^\lambda \partial^\sigma H_{\lambda\sigma} \big]. \end{aligned} \tag{48}$$

We, therefore, have

$$\widetilde{T} = T + \frac{D-4}{2}\widehat{M}_P^{D-4}\partial^\lambda\partial_\lambda\chi. \tag{49}$$

On the other hand, taking the trace of (45), we have:

$$\left[\partial^\lambda\partial_\lambda + \nabla^i\nabla_i\right]\chi = \frac{2}{D-2}\frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\widetilde{T}\delta^{(2)}\left(x^i\right). \tag{50}$$

Next, note that (46) and (47) imply that

$$\partial^\lambda H_{\mu\lambda} - \frac{1}{2}\partial_\mu\chi = \frac{1}{2}\partial_\mu g, \tag{51}$$

where $g = g(x^\mu)$ is independent of x^i, and satisfies the $(D-2)$-dimensional Klein–Gordon equation

$$\partial^\lambda\partial_\lambda g = 0. \tag{52}$$

It then follows that

$$-\left(\partial^\lambda\partial_\lambda + \nabla^i\nabla_i\right)H_{\mu\nu} + \partial_\mu\partial_\nu g = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\left[\widetilde{T}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\widetilde{T}\right]\delta^{(2)}\left(x^i\right), \tag{53}$$

where

$$\widetilde{T}_{\mu\nu} = T_{\mu\nu} - \widehat{M}_P^{D-4}\left[-\partial^\lambda\partial_\lambda H_{\mu\nu} + \partial_\mu\partial_\nu(\chi+g) - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial_\lambda\chi\right]. \tag{54}$$

We are now ready to solve for $H_{\mu\nu}$ and χ.

To do this, let us Fourier transform the coordinates x^μ on the brane. Let the corresponding momenta be p^μ, and let $p^2 \equiv p^\mu p_\mu$. Then we have

$$-\left(\nabla^i\nabla_i - p^2\right)H_{\mu\nu} - p_\mu p_\nu g = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\left[\widetilde{T}_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}\widetilde{T}(p)\right]\delta^{(2)}\left(x^i\right), \tag{55}$$

$$\left[\nabla^i\nabla_i - p^2\right]\chi = \frac{2}{D-2}\frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\widetilde{T}(p)\delta^{(2)}\left(x^i\right), \tag{56}$$

where

$$\widetilde{T}_{\mu\nu}(p) = T_{\mu\nu}(p) - \widehat{M}_P^{D-4}\left[p^2 H_{\mu\nu} - p_\mu p_\nu(\chi+g) + \frac{p^2}{2}\eta_{\mu\nu}\chi\right]. \tag{57}$$

Note that Eq. (52) now reads

$$p^2 g = 0, \tag{58}$$

so $g \equiv 0$ for $p^2 \neq 0$. On the other hand, for $p^2 = 0$ the equation for $H_{\mu\nu}$ *away* from the brane reads:

$$\nabla^i\nabla_i H_{\mu\nu} = -p_\mu p_\nu g. \tag{59}$$

This gives (for axially symmetric $H_{\mu\nu}$):

$$H''_{\mu\nu} + \frac{1}{\rho}H'_{\mu\nu} = -p_\mu p_\nu\left(\frac{a^2}{\rho^2}\right)^\nu g, \tag{60}$$

where g is independent of ρ. For non-vanishing g we would then have

$$H_{\mu\nu} \sim -\frac{p_\mu p_\nu a^2}{4(1-\nu)^2}\left(\frac{\rho^2}{a^2}\right)^{1-\nu} g \tag{61}$$

for large ρ. This implies that even for $p^2 = 0$ we must set $g = 0$.

Thus, the equations of motion for $H_{\mu\nu}$ and χ read:

$$-\left(\nabla^i\nabla_i - p^2\right)H_{\mu\nu} = \frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\left[\widetilde{T}_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}\widetilde{T}(p)\right]\delta^{(2)}\left(x^i\right), \tag{62}$$

$$\left[\nabla^i\nabla_i - p^2\right]\chi = \frac{2}{D-2}\frac{M_P^{2-D}}{\sqrt{\widetilde{G}}}\widetilde{T}(p)\delta^{(2)}\left(x^i\right), \tag{63}$$

where

$$\widetilde{T}_{\mu\nu}(p) = T_{\mu\nu}(p) - \widehat{M}_P^{D-4}\left[p^2 H_{\mu\nu} - p_\mu p_\nu\chi + \frac{p^2}{2}\eta_{\mu\nu}\chi\right]. \tag{64}$$

To solve these equations, we must distinguish between the cases where $p^2 = 0$ and $p^2 \neq 0$.

Let us start with the $p^2 \neq 0$ case. Then, due to the fact that the two-dimensional propagator is logarithmically divergent at the origin, we have (this is in complete parallel with the discussion in [16])

$$H_{\mu\nu} = \chi = 0, \qquad \rho \neq 0, \tag{65}$$

while on the brane we have

$$H_{\mu\nu}(p_\lambda, \rho=0) = \frac{\widehat{M}^{4-D}}{p^2}\left[T_{\mu\nu}(p) - \frac{1}{D-4}\eta_{\mu\nu}T(p) + \frac{2}{D-4}\frac{p_\mu p_\nu}{p^2}T(p)\right], \tag{66}$$

$$\chi(p_\lambda, \rho=0) = \frac{2}{D-4}\frac{\widehat{M}^{4-D}}{p^2}T(p). \tag{67}$$

Note that, according to these expression, the $p^2 \neq 0$ modes are completely localized on the brane. We will, however, come back to this point after discussing the $p^2 = 0$ case.

Thus, in the $p^2 = 0$ case we have:

$$-\partial^i \partial_i H_{\mu\nu} = M_P^{2-D}\left[T_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}T(p) + \widehat{M}_P^{D-4} p_\mu p_\nu \chi\right]\delta^{(2)}\left(x^i\right), \quad (68)$$

$$\partial^i \partial_i \chi = \frac{2}{D-2} M_P^{2-D} T(p)\delta^{(2)}\left(x^i\right). \quad (69)$$

The solution to the equation for χ is given by:

$$\chi\left(p^2 = 0, \rho\right) = \frac{M_P^{2-D}}{2\pi(D-2)} T(p) \ln\left(\frac{\rho^2}{b^2}\right), \quad (70)$$

where b is an integration constant. Note that unless $T(p) \equiv 0$, the solution for χ is singular at the origin, so that the equation for $H_{\mu\nu}$ is ill-defined due to the term proportional to $p_\mu p_\nu \chi$ as the latter blows up at the origin. [5] Here we would like to emphasize that this term *cannot* be removed by a gauge transformation. Since this singularity is a short-distance singularity, it is expected to be smoothed out by ultra-violet effects which we are neglecting here. [6] This smoothing out can simply be modeled via

$$\chi\left(p^2 = 0, \rho\right) = \frac{M_P^{2-D}}{2\pi(D-2)} T(p) \ln\left(\frac{\rho^2 + \epsilon^2}{b^2}\right), \quad (71)$$

where ϵ is a small parameter with the dimension of length. [7] This amounts to smoothing out the δ-function source via

$$\delta^{(2)}\left(x^i\right) \to \frac{1}{\pi}\frac{\epsilon^2}{(\rho^2+\epsilon^2)^2}. \quad (72)$$

We then have the following solution for the graviton $H_{\mu\nu}$:

$$H_{\mu\nu}\left(p^2 = 0, \rho\right) = -\frac{M_P^{2-D}}{4\pi}\left[T_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}T(p) + \widehat{M}_P^{D-4} p_\mu p_\nu \chi\left(p^2 = 0, \rho = 0\right)\right] \ln\left(\frac{\rho^2+\epsilon^2}{b'^2}\right), \quad (73)$$

where b' is an integration constant.

In fact, smoothing out of the aforementioned singularity also smoothes out a singularity in the $p^2 \neq 0$ case if the brane has non-zero tension. Indeed, from (67) it follows that, if $T(p) \neq 0$, χ is non-vanishing on the brane (but it vanishes in the bulk). The corresponding graviscalar components are given by:

$$\chi_{ij} = \frac{1}{2}\delta_{ij} \exp(2\omega)\chi, \quad (74)$$

which are infinite as $\exp(2\omega)$ diverges on the brane if $0 < \nu < 1$. However, if we smooth out the δ-function via (72), then we have

$$\exp(2\omega) = \left(\frac{a^2}{\rho^2+\epsilon^2}\right)^\nu, \quad (75)$$

which is now non-singular at $\rho = 0$. Note that for a smoothed out brane the $p^2 \neq 0$ modes now also penetrate into the bulk as can be seen from (62). However, as was originally pointed out in [16], for small enough ϵ, only ultra-light modes penetrate into the bulk efficiently (that is, with a substantial-wave function in the bulk).

3.2. The tensionless brane case

The conclusions of the previous subsection are applicable in the case of a tensionless brane. In this case we can arrive at the same conclusions in a somewhat simpler way. Thus, consider the codimension-2 Dvali–Gabadadze model:

$$S = \widehat{M}_P^{D-4} \int_\Sigma d^{D-2}x \sqrt{\widehat{G}}\, \widehat{R} + M_P^{D-2} \int d^D x \sqrt{G}\, R. \quad (76)$$

The background in this model is flat: $G_{MN}^{(0)} = \eta_{MN}$. The linearized equations of motion for the fluctuations

[5] Note that this term does not affect the coupling of the graviton $H_{\mu\nu}$ to the brane matter as $p^\mu T_{\mu\nu}(p) = 0$ for such matter. However, this term can be probed by bulk matter as $p^\mu T_{\mu\nu}^{\text{bulk}}(p)$ need *not* be zero.

[6] For instance, if the brane has small width instead of being δ-function-like, this singularity is absent. Note that, as was pointed out in [16], in this case complete localization of gravity is not expected to be the case either. Instead, it is expected that gravity is $(D-2)$-dimensional below some cross-over distance scale r_c (which depends on the brane width), while it become D-dimensional at distances larger than r_c.

[7] At least in some cases we can expect that $\epsilon \sim 1/\Lambda$, where Λ is an ultra-violet cut-off in the theory.

h_{MN} are given by:

$$\begin{aligned}&\big[-\partial^L\partial_L h_{MN}+2\partial^L\partial_{(M}h_{N)L}-\partial_M\partial_N h\\&\quad+\eta_{MN}\big(\partial^L\partial_L h-\partial^L\partial^K h_{LK}\big)\big]\\&\quad=M_P^{2-D}\widetilde{T}_{MN}\delta^{(2)}\big(x^i\big),\end{aligned}\tag{77}$$

where the components of the "effective" energy–momentum tensor $\widetilde{T}_{MN}$ are given by

$$\begin{aligned}\widetilde{T}_{\mu\nu}={}&T_{\mu\nu}-\widehat{M}_P^{D-4}\big[-\partial^\lambda\partial_\lambda H_{\mu\nu}+2\partial^\lambda\partial_{(\mu}H_{\nu)\lambda}\\&-\partial_\mu\partial_\nu H+\eta_{\mu\nu}\big(\partial^\lambda\partial_\lambda H-\partial^\lambda\partial^\sigma H_{\lambda\sigma}\big)\big],\end{aligned}\tag{78}$$

while $\widetilde{T}_{\mu i}$ and $\widetilde{T}_{ij}$ are zero.

Note that, since the brane is tensionless, the full D-dimensional diffeomorphisms are intact:

$$\delta h_{MN}=\partial_M\xi_N+\partial_N\xi_M.\tag{79}$$

We can, therefore, use the harmonic gauge:

$$\partial^M h_{MN}=\frac{1}{2}\partial_N h.\tag{80}$$

This gives:

$$-\partial^L\partial_L h_{MN}+\frac{1}{2}\eta_{MN}\partial^L\partial_L h=M_P^{2-D}\widetilde{T}_{MN}\delta^{(2)}\big(x^i\big).\tag{81}$$

It then follows that

$$\begin{aligned}-\partial^L\partial_L h_{MN}={}&M_P^{2-D}\left[\widetilde{T}_{MN}-\frac{1}{D-2}\eta_{MN}\widetilde{T}\right]\\&\times\delta^{(2)}\big(x^i\big).\end{aligned}\tag{82}$$

Once again, the graviphoton components vanish ($h_{\mu i}=0$), while for the graviscalar components we have

$$\chi_{ij}=\frac{1}{2}\delta_{ij}\chi.\tag{83}$$

This together with the harmonic gauge (80) implies that

$$\partial^\mu H_{\mu\nu}=\frac{1}{2}\partial_\nu H+\frac{1}{2}\partial_\nu\chi,\tag{84}$$

$$\partial_j H=0.\tag{85}$$

The latter allows to set $H\equiv 0$, and

$$\partial^\mu H_{\mu\nu}=\frac{1}{2}\partial_\nu\chi.\tag{86}$$

In particular, we have

$$\begin{aligned}\widetilde{T}_{\mu\nu}={}&T_{\mu\nu}-\widehat{M}_P^{D-4}\Big[-\partial^\lambda\partial_\lambda H_{\mu\nu}+\partial_\mu\partial_\nu\chi\\&-\frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial_\lambda\chi\Big].\end{aligned}\tag{87}$$

The equations of motion simplify as follows:

$$\begin{aligned}&-\big(\partial^i\partial_i-p^2\big)H_{\mu\nu}\\&\quad=M_P^{2-D}\left[\widetilde{T}_{\mu\nu}(p)-\frac{1}{D-2}\eta_{\mu\nu}\widetilde{T}(p)\right]\delta^{(2)}\big(x^i\big),\end{aligned}\tag{88}$$

$$\big[\partial^i\partial_i-p^2\big]\chi=\frac{2}{D-2}M_P^{2-D}\widetilde{T}(p)\delta^{(2)}\big(x^i\big),\tag{89}$$

where

$$\begin{aligned}\widetilde{T}_{\mu\nu}(p)={}&T_{\mu\nu}(p)-\widehat{M}_P^{D-4}\Big[p^2H_{\mu\nu}-p_\mu p_\nu\chi\\&+\frac{p^2}{2}\eta_{\mu\nu}\chi\Big],\end{aligned}\tag{90}$$

and we have Fourier transformed the coordinates x^μ. Note that these equations are precisely the same as at the end of the previous subsection for the case where the transverse space is flat. Note that, just as in the case of a non-zero tension brane, for the $p^2=0$ modes the $p_\mu p_\nu\chi$ term in $\widetilde{T}_{\mu\nu}$ is still singular on the brane. This singularity is removed once we smooth out the δ-function as in (72). On the other hand, for a strictly δ-function-like brane this singularity in the *linearized* theory would lead to inconsistencies somewhat similar to those discussed in [27], which, in particular, could be probed by bulk matter (see footnote 5). Note, however, that the presence of this term indicates that the linearized theory might be breaking down, which would imply that a more complete non-perturbative analysis (which is outside of the scope of this Letter), say, along the lines suggested in [28] might be required here. [8] If, however, this inconsistency persists non-perturbatively in the case of a (both tensionless as well as non-zero tension) strictly δ-function-like brane, it appears that we would have to appeal to smoothing out via ultra-violet physics. It would be interesting to understand this point better. [9]

[8] This singularity might be analogous to that arising in a linearized theory of a massive graviton as discussed in [28].

[9] Here we note that consistent infinite-volume brane world scenarios with non-conformal brane matter were recently discussed in

Acknowledgements

We would like to thank Gia Dvali and Gregory Gabadadze for valuable discussions. This work was supported in part by the National Science Foundation and an Alfred P. Sloan Fellowship. Parts of this work were completed during Z.K.'s visit at New York University. Z.K. would also like to thank Albert and Ribena Yu for financial support.

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the string theory context in [29,30]. These are generalizations of their conformal counterparts of [31–34]. Their orientifold generalizations can also be discussed, but some caution is needed due to the issues discussed in [35,36].