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Dimensional regularization of the path integral in curved space on an infinite time interval

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Abstract

We use dimensional regularization to evaluate quantum mechanical path integrals in arbitrary curved spaces on an infinite time interval. We perform 3-loop calculations in Riemann normal coordinates, and 2-loop calculations in general coordinates. It is shown that one only needs a covariant two-loop counterterm ($V_{\mathrm{DR}} = \frac{\hbar^2}{8}R$) to obtain the same results as obtained earlier in other regularization schemes. It is also shown that the mass term needed in order to avoid infrared divergences explicitly breaks general covariance in the final result.

1. Introduction

The path integral formulation of quantum mechanics [1] is quite subtle when applied to particles moving in a curved space [2]. It can be used to evaluate anomalies in quantum field theories, but only when the corresponding quantum mechanical models are defined on a finite interval of the worldline. When viewed as one dimensional QFTs on the worldline (but with higher-dimensional target spaces), one is dealing with nonlinear sigma models with double-derivative interactions. Such theories are super-renormalizable: though certain one- and two-loop Feynman diagrams are superficially divergent and regularization is necessary. However, there is no need to renormalize infinities away because the infinities of different graphs cancel each other and quantum mechanics is finite. Different regularization prescriptions give in general different finite answers for the same Feynman diagram. This situation is rather familiar in QFT: it simply means that there are free parameters entering in the theory (which are equivalent to the ordering ambiguities of canonical quantization) that can only be fixed by requiring further constraints (finite renormalization conditions). The latter simply parametrize different physical phenomena which can be described by the quantum mechanical model under consideration.

It is sometimes claimed that one does not need any 'artificial' counterterm at all because the theory

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has no divergences. As the results of this article show, in all regularization schemes studied so far one always needs finite local counterterms. In fact, finite local counterterms are in general to be expected because they just amount to finite additive renormalizations needed to implement the renormalization conditions.

In the recent past, two different regularization schemes for nonlinear sigma models on a finite time interval have been discussed carefully: mode regularization [3–5] and time discretization [5–7]. A detailed comparison carried out to three loops shows that both schemes produce the same physics [8]. However, they both break manifest general coordinate invariance at intermediate stages and require noncovariant counterterms to restore that symmetry in the final result. It is important to stress that these counterterms are unambiguously determined in each scheme. Nevertheless, lack of manifest covariance is annoying and constitutes a technical limitation: at higher loops one must expand the non-covariant counterterms to get the corresponding vertices but one cannot employ covariant techniques to simplify that computation.

Recently, dimensional regularization has been employed to define a new regulated version of the path integral for an infinite time interval [9]. By evaluating the partition function of a particular massive nonlinear sigma model with a one-dimensional flat target space, it was found that no noncovariant counterterms were needed to obtain the correct result. Since target space in [9] was only one-dimensional, covariant counterterms could not be detected since these are proportional to the scalar curvature R. It is the purpose of this letter to extend the proposal of Ref. [9] to a higher dimensional target space and to demonstrate that a covariant counterterm is needed. This counterterm turns out to be $V_{\mathrm{DR}} = \frac{\hbar^2}{8}R$.

Let us present first a discussion on the limits of dimensional regularization applied to quantum mechanics as used in [9]. The main problem is that it seems to require an infinite propagation time. In fact, one obtains a continuum momentum space (the energy in one dimension) only upon Fourier transforming the infinite time dimension. Integrals in momentum space are regulated dimensionally afterwards [10]. Instead, it would be desirable to regulate and compute the path integral for a finite propagation time. The latter could be interpreted as a proper time, thus making it useful for relativistic applications in the world line approach to QFT [11]. A related problem is that the infinite propagation time introduces infrared divergences in massless models, and requires a harmonic term as infrared regulator. In Ref. [9] only a massive model was considered. The harmonic term ruins general coordinate invariance: a potential of the form $V \sim \omega^2 g_{ij}(x) x^i x^j$ is not a scalar since the coordinates x^i do not transform as the components of a vector. Invariance in the final result could be recovered in the limit $\omega \to 0$ if the propagation time would be kept finite, but that limit is not possible in the dimensional regularization described above which requires an infinite propagation time. Given that general coordinate invariance is necessarily softly broken, one may use as well a potential $V \sim \omega^2 g_{ij}(0) x^i x^j$ as infrared regulator. The latter is quadratic even far away from the origin of the chosen coordinate system and will not modify the interaction vertices. This soft breaking of general coordinate invariance is not expected to modify the counterterm V_{CT} since such a counterterm is sensitive only to the ambiguities due to ultraviolet divergences.

We now proceed to test the proposal of Ref. [9] in a class of sufficiently general models and relate it to the other regularization methods mentioned above. The calculation in [9] is enough to indicate that possible counterterms will be covariant, but since it involves a single coordinate it misses terms proportional to the curvature. Our strategy will be to compute terms in the effective action using both mode regularization (MR) and dimensional regularization (DR). Equating the results fixes the counterterm needed in dimensional regularization to be $V_{\mathrm{DR}} = \frac{1}{8}\hbar^2 R$.

First, let us briefly review some known facts. Quantization of a free particle on a curved space produces in the quantum Hamiltonian $\hat{H}$ an undetermined term proportional to the scalar curvature, $\hat{H} = -\frac{1}{2}\hbar^2\Delta + \alpha\hbar^2 R$. This is easily seen using canonical (operatorial) methods: ordering ambiguities are encountered in the construction of the quantum Hamiltonian from the classical one and give rise to terms with at most two derivatives on the metric. Then, requiring general coordinate invariance leaves only a term proportional to the scalar curvature. Using path integrals this arbitrary coupling will appear as a correction to the effective action proportional to the scalar curvature

$$\langle 0|\mathrm{e}^{-\beta\hat{H}/\hbar}|0\rangle = \int \mathscr{D}x\,\mathrm{e}^{-S[x]/\hbar} = \mathrm{e}^{-\Gamma/\hbar} = \exp\left\{-\frac{1}{\hbar}\int_0^\beta dt\left[\cdots + \left(\alpha+\tfrac{1}{12}\right)\hbar^2 R + \cdots\right]\right\} \quad (1)$$

where the first equality reminds us of the equivalence of canonical and path integral quantization ($|0\rangle$ and $\langle 0|$ are eigenstates of the position operator $\hat{x}$ with eigenvalue zero) and in the second equality we have the definition of the effective action Γ. The term $\frac{1}{12}\hbar^2 R$ is partially due to the counterterm and partially due to two-loop diagrams, see Eq. (36). Henceforth we set $\hbar = 1$.

We are going to compute the corrections to the effective action Γ as function of the various couplings using both mode and dimensional regularization. In the former we can be general and allow for a finite propagation time β. Then we take the limit $\beta \to \infty$, which is safe in the presence of an infrared regulator, and compare the result with dimensional regularization. It is known that the former requires the counterterm

$$V_{\mathrm{MR}} = \tfrac{1}{8}R - \tfrac{1}{24}g^{ij}g^{kl}g_{mn}\Gamma^m_{ik}\Gamma^n_{jl} \quad (2)$$

to produce a general coordinate invariant result with $\alpha = 0$ [5]. We will see that dimensional regularization will match the result when using a counterterm

$$V_{\mathrm{DR}} = \tfrac{1}{8}R \quad (3)$$

which is manifestly covariant. For comparison we mention that the counterterm for time-slicing, needed to obtain the same result as mode regularization, is different, see Eq. (40).

2. The 3-loop calculation with Riemann normal coordinates

The model we analyze is given by

$$S[x^i] = \int_{t_i}^{t_f} dt\left[\tfrac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j + \tfrac{1}{2}\omega^2 g_{ij}(0)x^i x^j + V_{\mathrm{CT}}\right] \quad (4)$$

where ω is a frequency needed as infrared regulator and V_{CT} is the counterterm for the regularization scheme chosen. Using Riemann normal coordinates, we will need to compute up to three loops since the noncovariant part of the counterterm (2), when expanded around the origin of the coordinates, only gives contributions from 3 loops onwards. We want to make sure that noncovariant counterterms are not required when using dimensional regularization, as noticed in Ref. [9]. In the next section we shall repeat the calculation below for general coordinates but with only two-loop graphs. Since in general coordinates the first derivatives of the metric do not vanish, we get nonvanishing contributions from the noncovariant parts of (2) already at the two-loop level. This gives an additional nontrivial check on the covariance of the counterterm of dimensional regularization on the infinite time interval.

The counterterm is effectively of order $\hbar^2$ since it first appears at two loops, but for notational convenience we are using units where $\hbar = 1$. As already mentioned, the harmonic potential breaks general coordinate invariance since it selects those coordinates in which the potential is quadratic. We have chosen them to be Riemann normal coordinates as a definition of our model, so that the metric has the expansion

$$g_{ij}(x) = \delta_{ij} + \tfrac{1}{3}R_{kijl}(0)x^k x^l + \tfrac{1}{6}\nabla_m R_{kijl}(0)x^k x^l x^m + \left(\frac{1}{20}\nabla_m\nabla_n R_{kijl}(0) + \frac{2}{45}R_{kipl}R_{mj}{}^p{}_n(0)\right) \times x^k x^l x^m x^n + O(x^5). \quad (5)$$

We find it convenient to use a rescaled time parameter τ with $t=\beta\tau+t_f$ and $\beta=t_f-t_i$, so that $-1\le\tau\le 0$. An infinite propagation time will be recovered in the limit $\beta\to\infty$, while for finite β this setting allows us to compare easily with the results for $\omega=0$ which were reported in [8] using similar notations[1].

With this rescaling, and introducing the ghost a^i, b^i, c^i for a correct treatment of the measure [3,4], we aim to compute the following path integral with two different regularization schemes, mode regularization (MR) and dimensional regularization (DR),

$$\int \mathscr{D}x\mathscr{D}a\mathscr{D}b\mathscr{D}c\, \mathrm{e}^{-\frac{1}{\beta}S} \tag{6}$$

with

$$\begin{aligned} S &\equiv S[x,a,b,c] \\ &= \int_{-1}^{0} d\tau \Big(\tfrac{1}{2} g_{ij}(x)(\dot x^i \dot x^j + a^i a^j + b^i c^j) \\ &\quad + \tfrac{1}{2}(\beta\omega)^2 g_{ij}(0)\, x^i x^j + \beta^2 V_{\mathrm{CT}}(x)\Big) \end{aligned} \tag{7}$$

and with the boundary conditions that all fields vanish at $t=t_i, t_f$, (i.e. at $\tau=-1,0$).

For the perturbative evaluation (in the coupling constants contained in the metric $g_{ij}(x)$) it is convenient to split the action into a quadratic part S_2 and an interacting part $S_{int}=S_3+S_4+S_5+S_6+\cdots$

$$\begin{aligned} S_2 = \int_{-1}^{0} d\tau \Big[& \tfrac{1}{2}\delta_{ij}(\dot x^i \dot x^j + a^i a^j + b^i c^j) \\ & + \tfrac{1}{2}\delta_{ij}(\beta\omega)^2 x^i x^j \Big] \end{aligned} \tag{8}$$

$$S_3 = 0 \tag{9}$$

$$\begin{aligned} S_4 = \int_{-1}^{0} d\tau \Big[& \tfrac{1}{6} R_{kijl}\, x^k x^l (\dot x^i \dot x^j + a^i a^j + b^i c^j) \\ & + \beta^2 V_{\mathrm{CT}} \Big] \end{aligned} \tag{10}$$

$$\begin{aligned} S_5 = \int_{-1}^{0} d\tau \Big[& \tfrac{1}{12} \nabla_m R_{kijl}\, x^k x^l x^m (\dot x^i \dot x^j + a^i a^j + b^i c^j) \\ & + \beta^2 x^i \partial_i V_{\mathrm{CT}} \Big] \end{aligned} \tag{11}$$

$$\begin{aligned} S_6 = \int_{-1}^{0} d\tau \Bigg[& \left(\frac{1}{40} \nabla_m \nabla_n R_{kijl} + \frac{1}{45} R_{kipl} R_{mj}{}^{p}{}_{n} \right) \\ & \times x^k x^l x^m x^n (\dot x^i \dot x^j + a^i a^j + b^i c^j) \\ & + \frac{\beta^2}{2} x^i x^j \partial_i \partial_j V_{\mathrm{CT}} \Bigg]. \end{aligned} \tag{12}$$

Note that all structures like R_{ijkl}, V_{CT} and derivatives thereof are evaluated at the origin of the Riemann coordinate system, but for notational simplicity we do not indicate so explicitly from now on.

From S_2 one recognizes the propagators

$$\begin{aligned} \langle x^i(\tau)\, x^j(\sigma)\rangle &= -\beta\,\delta^{ij}\,\Delta(\tau,\sigma) \\ \langle a^i(\tau)\, a^j(\sigma)\rangle &= \beta\,\delta^{ij}\,\Delta_{\mathrm{gh}}(\tau,\sigma) \\ \langle b^i(\tau)\, c^j(\sigma)\rangle &= -2\beta\,\delta^{ij}\,\Delta_{\mathrm{gh}}(\tau,\sigma) \end{aligned} \tag{13}$$

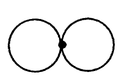
where the functions $\Delta(\tau,\sigma)$, $\Delta_{\mathrm{gh}}(\tau,\sigma)$ are to be defined shortly in each regularization scheme. Then, the transition element, Eq. (1), at three loops is given by

$$\begin{aligned} \mathscr{Z} &= \mathrm{e}^{-\Gamma} \\ &= A \exp\left\{ \left\langle -\frac{1}{\beta}(S_4+S_5+S_6) \right\rangle \right. \\ &\quad \left. + \left\langle \frac{1}{2\beta^2} S_4^2 \right\rangle_{\mathrm{con}} + \cdots \right\} \end{aligned} \tag{14}$$

where the subscript 'con' refers to connected diagrams only. The constant A is the normalization of the exact path integral for S_2 which describes a harmonic oscillator in D dimensions [1,2]

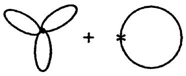
$$A = \left(\frac{\omega}{2\pi \sinh(\beta\omega)} \right)^{\frac{D}{2}}. \tag{15}$$

For $\omega=0$ this term becomes the familiar Feynman measure for a free particle $(2\pi\beta)^{-D/2}$. The perturbative contributions are obtained by computing the various Wick contractions. We record the results in terms of Δ and Δ_{gh} through three loops; the symbol * denotes counterterms. The nonzero contributions are

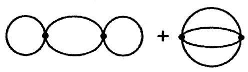
$$\left\langle -\frac{1}{\beta} S_4 \right\rangle =$$  $$+ \ * \ = -I_2 \frac{\beta}{6} R - \beta V_{CT}$$

(16)

[1] Our conventions follow from $[\nabla_i, \nabla_j] V^k = R_{ij}{}^{k}{}_{l} V^l$, $R_{ij} = R_{ik}{}^{k}{}_{j}$. Thus, the scalar curvature $R = R_i{}^i$ of a sphere is negative.

$$\left\langle -\frac{1}{\beta} S_6 \right\rangle =$$ 

$$= I_8\, \beta^2 \left(\frac{1}{20} \nabla^2 R + \frac{1}{45} R_{ij}^2 + \frac{1}{30} R_{ijmn}^2 \right) + I_9\, \frac{\beta^2}{2} \partial^i \partial_i V_{CT} \qquad (17)$$

$$\left\langle \frac{1}{2\beta^2} S_4^2 \right\rangle_{con} =$$ 

$$= I_{14}\, \frac{\beta^2}{36} R_{ij}^2 + I_{15}\, \frac{\beta^2}{24} R_{ijmn}^2 \qquad (18)$$

being $\langle S_5 \rangle$ proportional to at least one classical field that is zero since $x_i = x_f = 0$. The integrals I_n are given by

$$I_2 = \int_{-1}^{0} d\tau \Big(\Delta \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right) - {}^\bullet\Delta^2 \Big)\Big|_\tau \qquad (19)$$

$$I_8 = \int_{-1}^{0} d\tau \Big(\Delta^2 \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right) - {}^\bullet\Delta^2\, \Delta \Big)\Big|_\tau \qquad (20)$$

$$I_9 = \int_{-1}^{0} d\tau\, \Delta|_\tau \qquad (21)$$

$$\begin{aligned} I_{14} = \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \Big(& \Delta|_\tau \left({}^\bullet\Delta^{\bullet\,2} - \Delta_{gh}^2 \right) \Delta|_\sigma \\ & - 4\, \Delta|_\tau\, {}^\bullet\Delta^\bullet\, {}^\bullet\Delta\, \Delta^\bullet|_\sigma \\ & + 2\, \Delta|_\tau\, {}^\bullet\Delta^2 \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right)|_\sigma \\ & + 2\, \Delta^\bullet|_\tau\, \Delta\, {}^\bullet\Delta^\bullet\, \Delta^\bullet|_\sigma + 2\, \Delta^\bullet|_\tau\, {}^\bullet\Delta\, \Delta^\bullet\, \Delta^\bullet|_\sigma \\ & - 4\, \Delta^\bullet|_\tau\, \Delta\, {}^\bullet\Delta \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right)|_\sigma \\ & + \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right)|_\tau\, \Delta^2 \left({}^\bullet\Delta^\bullet + \Delta_{gh} \right)|_\sigma \Big) \end{aligned} \qquad (22)$$

$$\begin{aligned} I_{15} = \int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \Big(& \Delta^2 \left({}^\bullet\Delta^{\bullet\,2} - \Delta_{gh}^2 \right) \\ & + {}^\bullet\Delta^2\, \Delta^{\bullet\,2} - 2\, \Delta\, {}^\bullet\Delta\, \Delta^\bullet\, {}^\bullet\Delta^\bullet \Big). \end{aligned} \qquad (23)$$

We have kept the same names and notations for the integrals I_n as in [8] to facilitate comparison for the limit $\omega \to 0$ possible in mode regularization when β is kept finite. We recall that $\Delta|_\tau \equiv \Delta(\tau,\tau)$ and ${}^\bullet\Delta \equiv \frac{\partial}{\partial\tau}\Delta(\tau,\sigma)$ while $\Delta^\bullet \equiv \frac{\partial}{\partial\sigma}\Delta(\tau,\sigma)$.

Let us first consider mode regularization. Here one expands all fields in a Fourier sine series and keeps all modes up to a large mode number M. The limit $M \to \infty$ is taken after having computed all integrals. In practice, one manipulates the integrals by partial integration to put them into a form which can be computed directly and without ambiguities in the continuum. One partially integrates such that all double derivatives of Δ, namely ${}^\bullet\Delta^\bullet$ and $\Delta_{gh} \equiv {}^{\bullet\bullet}\Delta_0$ are removed. If this is not possible, one casts the expressions in a form such that the integrands vanish at the end-points. In the latter case, singularities like $\delta(\tau)$ and $\delta(\tau+1)$ are neutralized. With this prescription one recognizes that the function $\Delta(\tau,\sigma)$ appearing in the propagator is given by

$$\Delta(\tau,\sigma) = \sum_{m=1}^{M} \left[\frac{-2}{(\pi m)^2 + (\beta\omega)^2} \times \sin(\pi m \tau) \sin(\pi m \sigma) \right] \qquad (24)$$

while as anticipated one can represent $\Delta_{gh}(\tau,\sigma) = {}^{\bullet\bullet}\Delta_0(\tau,\sigma)$ with

$$\Delta_0(\tau,\sigma) = \sum_{m=1}^{M} \left[\frac{-2}{(\pi m)^2} \sin(\pi m \tau) \sin(\pi m \sigma) \right]. \qquad (25)$$

Their continuum limit ($M \to \infty$) is given by

$$\begin{aligned} \Delta(\tau,\sigma) = \frac{1}{\beta\omega \sinh(\beta\omega)} \big[& \theta(\tau-\sigma) \sinh(\beta\omega\tau) \\ & \times \sinh(\beta\omega(\sigma+1)) \\ & + \theta(\sigma-\tau) \sinh(\beta\omega\sigma) \\ & \times \sinh(\beta\omega(\tau+1)) \big] \end{aligned} \qquad (26)$$

$$\Delta_{gh}(\tau,\sigma) = \delta(\tau-\sigma). \qquad (27)$$

Table 1
Results in mode regularization at finite β

I_2	I_8	I_9	I_{14}	I_{15}
$-\frac{1}{4}$	$-\frac{1}{8}I(\beta\omega)$	$\frac{1}{2}I(\beta\omega)$	$\frac{1}{4}I(\beta\omega)$	$\frac{1}{6}I(\beta\omega)$

It is easy to check that $\left[\partial_\tau^2 - (\beta\omega)^2\right]\Delta(\tau,\sigma) = \delta(\tau - \sigma)$ and $\Delta(0,\sigma) = \Delta(-1,\sigma) = \Delta(\tau,0) = \Delta(\tau,-1) = 0$.

Now, we can compute the various I_n and obtain the results summarized in Table 1, where we have found it convenient to define the function

$$I(a) = \frac{1 - a\coth(a)}{a^2}. \tag{28}$$

As an example how these results are obtained, consider the 'clover leaf' graph in (17) corresponding to I_8. Using that in mode regularization

$$\left({}^\bullet\Delta^\bullet + \Delta_{gh}\right)|_\tau = \partial_\tau\left(({}^\bullet\Delta)|_\tau\right) - (\beta\omega)^2\,\Delta|_\tau \tag{29}$$

the first term in I_8 yields

$$\int_{-1}^0 d\tau\left(-4\,{}^\bullet\Delta^2\Delta - (\beta\omega)^2\,\Delta^3\right)|_\tau. \tag{30}$$

Hence

$$I_8 = \int_{-1}^0 d\tau\left(-5\,{}^\bullet\Delta^2\Delta - (\beta\omega)^2\,\Delta^3\right)|_\tau. \tag{31}$$

Then from Eq. (26) we obtain

$$\Delta|_\tau = \frac{\sinh(\beta\omega\tau)\sinh(\beta\omega(\tau+1))}{\beta\omega\sinh(\beta\omega)} \tag{32}$$

$$({}^\bullet\Delta)|_\tau = \frac{\sinh(\beta\omega(2\tau+1))}{2\sinh(\beta\omega)} \tag{33}$$

and substitution into (31) yields the result for I_8 as given in Table 1.

In this regularization scheme the counterterm to be used is V_{MR} as given in Eq. (2). When evaluated at the origin of the Riemann normal coordinates it produces

$$V_{MR} = \tfrac{1}{8}R \tag{34}$$

$$\partial^i\partial_i V_{MR} = \tfrac{1}{8}\nabla^2 R - \tfrac{1}{36}R_{ijkl}R^{ijkl}. \tag{35}$$

As an aside, we can check the correctness of the $\omega \to 0$ limit. Since $I(\beta\omega) \to -\frac{1}{3}$ for $\omega \to 0$, one can verify that the results in [8] are reproduced

$$\mathscr{Z} = A\exp\Big\{-\Big[\beta\tfrac{1}{12}R + \beta^2\left(\tfrac{1}{120}\nabla^2 R + \tfrac{1}{720}R_{ij}^2 - \tfrac{1}{720}R_{ijkl}^2\right) + \cdots\Big]\Big\}. \tag{36}$$

This result is expected to be covariant [5,8] and the use of Riemann normal coordinates shows immediately which is the covariant form of the effective action.

On the other hand, for $\omega \neq 0$ and $\beta \to \infty$ one gets $\beta I(\beta\omega) \to -\frac{1}{\omega}$, and thus

$$\mathscr{Z} = A\exp\left\{-\beta\left[\tfrac{1}{12}R + \frac{1}{\omega}\left(\tfrac{1}{40}\nabla^2 R + \tfrac{1}{240}R_{ij}^2 - \tfrac{1}{240}R_{ijkl}^2\right) + \cdots\right]\right\}. \tag{37}$$

Now, this result in not expected to be covariant because of the presence of the mass term ω. The apparent covariance of (37) is just a coordinate artifact of the Riemann normal coordinates (this point will be self-evident in the calculations of the next section). The result (37) is what one should obtain in dimensional regularization as well.

Thus, let us turn to dimensional regularization. The propagators are represented as in (13) with

$$\Delta(\tau,\sigma) = -\frac{1}{\beta}\int\frac{dk}{2\pi}\,\frac{e^{-ik\beta(\tau-\sigma)}}{k^2+\omega^2} \tag{38}$$

$$\Delta_{gh}(\tau,\sigma) = -\frac{1}{\beta}\int\frac{dk}{2\pi}\,e^{-ik\beta(\tau-\sigma)}. \tag{39}$$

Note that, strictly speaking, one should use an infinite β, which anyway cancels in (13), and a finite $t \equiv \beta\tau$ and $s \equiv \beta\sigma$. Now one can use dimensional regularization to compute the various integrals (with momenta contracted as suggested by the kinetic term continued to D dimensions) and then take the limit $D \to 1$. Using the formulas given in [9] (and also in [12] where dimensional regularization is used in configuration space), one recognizes that the ghosts are effectively regulated to give a vanishing contribution (this is due to the fact that $\delta^{(n)}(0)$ is zero in

Table 2
Results in dimensional regularization at $\beta=\infty$

I_2	I_8	I_9	I_{14}	I_{15}
$-\frac{1}{4}$	$\frac{1}{8\beta\omega}$	$-\frac{1}{2\beta\omega}$	$-\frac{1}{4\beta\omega}$	0

dimensional regularization), while the remaining integrals give the results summarized in Table 2.

It is immediate to verify that the result (37) is reproduced once one uses the counterterm $V_{\rm DR}=\frac{1}{8}R$ (of course, in the limit of infinite β this result is unaffected by the infrared divergence related to the infinite time integral and remains finite). Thus, we conclude that $V_{\rm DR}=\frac{1}{8}R$ is the counterterm needed in dimensional regularization to have $\alpha=0$ in Eq. (1).

Of course, we could have compared as well dimensional regularization with time slicing regularization [6] and obtain the same result. In that case, one should remember that time slicing (TS) requires different rules to compute the integrals in Eqs. (19)–(23) but also a different counterterm [6,13]

$$V_{\rm TS}=\tfrac{1}{8}R+\tfrac{1}{8}g^{ij}\Gamma^{l}_{ik}\Gamma^{k}_{jl}. \tag{40}$$

As an extra check, in what follows we also verify the necessity of the counterterm $V_{\rm DR}$ at two loops but using arbitrarily chosen coordinates.

3. The two-loop calculation with general coordinates

In this section we repeat the calculation for the amplitude (1) using general coordinates going as far as two loops. Again we perform the calculation using mode regularization along with the counterterm (2) and dimensional regularization with the counterterm (3) applied to the model (4) where x^i are *now* general coordinates. Writing (2) explicitly in terms of the metric tensor

$$\begin{aligned}V_{\rm MR}&=\tfrac{1}{8}R-\tfrac{1}{24}g^{ij}g^{kl}g_{mn}\Gamma^{m}_{ik}\Gamma^{n}_{jl}\\&=\tfrac{1}{8}R-\tfrac{1}{32}\left(\partial_i g_{jk}\right)^2+\tfrac{1}{48}\left(\partial_i g_{jk}\right)\left(\partial_j g_{ik}\right)\end{aligned} \tag{41}$$

makes it clear that one will get nonzero contribution from the noncovariant parts of the counterterms already at the two-loop level. Indeed the derivatives of the metric do not vanish at the origin of an arbitrary system of coordinates contrarily to what happens in Riemann normal coordinates where they do vanish. The expansion of the metric $g_{ij}(x)$ around the origin gives the same quadratic action of the previous section and thus the propagators are the same as well. The interacting part is $S_{int}=S_3+S_4+\cdots$, being

$$S_3=\int_{-1}^{0}d\tau\,\tfrac{1}{2}\partial_k g_{ij}x^k(\dot x^i\dot x^j+a^ia^j+b^ic^j) \tag{42}$$

$$\begin{aligned}S_4=\int_{-1}^{0}d\tau\Big[&\tfrac{1}{4}\partial_k\partial_l g_{ij}x^kx^l(\dot x^i\dot x^j+a^ia^j+b^ic^j)\\&+\beta^2V_{\rm CT}\Big]\end{aligned} \tag{43}$$

where metric and derivative thereof and $V_{\rm CT}$ are evaluated at the origin of the system of coordinates. The transition element at two-loop is given by

$$\begin{aligned}\mathcal{Z}=A\exp\bigg\{&\left\langle-\frac{1}{\beta}(S_3+S_4)\right\rangle\\&+\left\langle\frac{1}{2\beta^2}S_3^2\right\rangle_{\rm con}+\cdots\bigg\}\end{aligned} \tag{44}$$

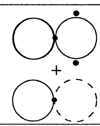
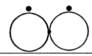
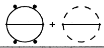
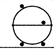
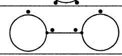
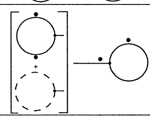
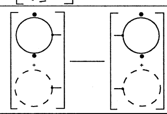
where $\langle S_3\rangle$ vanishes because it contains an odd number of quantum fields while

$$\left\langle-\frac{1}{\beta}S_4\right\rangle=-\frac{\beta}{4}\left[A_1\partial^2g+2A_2\partial^jg_j\right]-\beta V_{\rm CT} \tag{45}$$

$$\begin{aligned}\left\langle\frac{1}{2\beta^2}S_4\right\rangle_{\rm con}=-\frac{\beta}{8}\Big[&B_1(\partial_i g)^2+4B_2(\partial_j g)g^j\\&+2B_3(\partial_i g_{jk})^2+4B_4(\partial_i g_{jk})\partial_j g_{ik}\\&+4B_5g_j^2\Big].\end{aligned} \tag{46}$$

We have used the shorthand notation: $\partial^2g\equiv g^{ij}g^{kl}\partial_k\partial_l g_{ij}$, $\partial_k g\equiv g^{ij}\partial_k g_{ij}$, $g_k\equiv g^{ij}\partial_i g_{jk}$, $\partial^jg_j\equiv g^{ik}g^{jl}\partial_k\partial_l g_{ij}$. The results obtained from this calculation are summarized in Table 3, where the column ‘Result’ refers to the computations done using dimensional regularization (DR) and mode regularization (MR) of the integrals shown in the column aside. In the same line we also report a pictorial representation and the ‘tensor structure’ associated to

Table 3
2-loop results with dimensional and mode regularization

Integral	*Result* DR[MR]	*Diagram*	*Tensor structure*
$A_1 \equiv \int_{-1}^{0} \Delta\|_\tau ({}^\bullet\Delta^\bullet + \Delta_{gh})\|_\tau$	$-\frac{1}{4}$ $\left[-\frac{1}{4}\right]$		$\partial^2 g$
$A_2 \equiv \int_{-1}^{0} ({}^\bullet\Delta\|_\tau)^2$	0 [0]		$\partial^j g_j$
$B_3 \equiv \int_{-1}^{0}\int_{-1}^{0} \Delta({}^\bullet\Delta^{\bullet 2} - \Delta_{gh}^2)$	$\frac{7}{24}$ $\left[\frac{5}{12}\right]$		$(\partial_i g_{jk})^2$
$B_4 \equiv \int_{-1}^{0}\int_{-1}^{0} ({}^\bullet\Delta^\bullet)\, \Delta^\bullet\, ({}^\bullet\Delta)$	$\frac{1}{24}$ [0]		$(\partial_i g_{jk})\partial_j g_{ik}$
$B_5 \equiv \int_{-1}^{0}\int_{-1}^{0} {}^\bullet\Delta\|_\tau\, ({}^\bullet\Delta^\bullet)\, \Delta^\bullet\|_\sigma$	0 [0]		g_j^2
$B_2 \equiv \int_{-1}^{0}\int_{-1}^{0} ({}^\bullet\Delta^\bullet + \Delta_{gh})\|_\tau \Delta^\bullet\, (\Delta^\bullet\|_\sigma)$	0 [0]		$(\partial_j g) g^j$
$B_1 \equiv \int_{-1}^{0}\int_{-1}^{0} ({}^\bullet\Delta^\bullet + \Delta_{gh})\|_\tau \Delta ({}^\bullet\Delta^\bullet + \Delta_{gh})\|_\sigma$	$-\frac{1}{4}$ $\left[-\frac{1}{4}\right]$		$(\partial_j g)^2$

each diagram. Recalling that the scalar curvature is given by

$$R = \partial^2 g - \partial^j g_j - \tfrac{3}{4}\left(\partial_k g_{ij}\right)^2 + \tfrac{1}{2}\left(\partial_i g_{jk}\right)\partial_j g_{ik} + \tfrac{1}{4}\left(\partial_j g\right)^2 - \left(\partial_j g\right) g^j + g_j^2 \qquad (47)$$

and using the results from Table 3, the amplitude $\mathscr{Z}$ reads

$$\mathscr{Z} = A \exp\left\{ -\frac{\beta}{16}\partial^2 g + \frac{\beta}{8}\partial^j g_j + \frac{\beta}{48}\left(\partial_i g_{jk}\right)^2 - \frac{\beta}{12}\left(\partial_i g_{jk}\right)\partial_j g_{ik} + \frac{\beta}{8}\left(\partial_j g\right) g^j - \frac{\beta}{8} g_j^2 \right\} \qquad (48)$$

for both regularization schemes. Therefore, also in this case, dimensional regularization yields the same transition amplitude as mode regularization only requiring the covariant counterterm $\frac{1}{8}R$.

Note that in Riemann normal coordinates $\partial^2 g = \frac{2}{3}R$ and $\partial^j g_j = -\frac{1}{3}R$ at the origin; substituting these identities, (48) reduces to the two-loop part of (36). Obviously the result is not covariant as the covariance of the model in Eq. (4) is explicitly broken by the mass term and cannot be recovered even in the limit $\omega \to 0$ since (48) is ω independent. Therefore dimensional regularization of the path integral on an infinite time interval does not preserve target space general covariance, contrarily to what stated in [9].

4. Conclusions

In this letter we considered quantum mechanical path integrals in curved space with an infinite propagation time. We computed transition amplitudes both using dimensional regularization (DR) and other (in this context more established) regularization schemes. We showed that DR does not need noncovariant counterterms in order to reproduce the correct answer, as already noticed in a simpler model in [9], but it does need a covariant two-loop counterterm, namely $V_{\mathrm{DR}} = \frac{\hbar^2}{8}R$. We took an infinite propagation time in order to have a continuous momentum spectrum and to be able to use DR in the usual way. This forced us to add an infrared regulator: a mass term. The unpleasant feature of this term is that it breaks manifest general covariance. Furthermore, for applications to quantum field theories such as computations of anomalies, one needs path integrals on a finite time interval. We are at present working on an approach to use dimensional regularization at finite β. The crucial question is whether again only covariant counterterms are needed.

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