DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **5**, Number **3**, June **2012**

doi:10.3934/dcdss.2012.5.xx

pp. **X–XX**

ASYMPTOTIC BEHAVIOR OF THE CAGINALP PHASE-FIELD SYSTEM WITH COUPLED DYNAMIC BOUNDARY CONDITIONS

Monica Conti

Dipartimento di Matematica "F.Brioschi", Politecnico di Milano
Via Bonardi 9, I-20133 Milano, Italy

Stefania Gatti

Dipartimento di Matematica, Università di Modena e Reggio Emilia
Via Campi 213/B, I-41125 Modena, Italy

Alain Miranville

Laboratoire de Mathématiques et Applications, Université de Poitiers
UMR CNRS 6086 - SP2MI, Boulevard Marie et Pierre Curie - Téléport 2
F-86962 Chasseneuil Futuroscope Cedex, France

Abstract. This paper deals with the longtime behavior of the Caginalp phase-field system with coupled dynamic boundary conditions on both state variables. We prove that the system generates a dissipative semigroup in a suitable phase-space and possesses the finite-dimensional smooth global attractor and an exponential attractor.

1. **Introduction.** The Caginalp system is a well-known model in phase transition, introduced in [1] to describe, in particular, melting-solidification phenomena in certain classes of materials: the state variables are the order parameter u and the relative temperature ϑ. If the system undergoing the phase-change is confined in a container, it is natural to take into account the interactions with the walls: this gives rise to the so-called dynamic boundary conditions (introduced for the first time in the context of the Cahn-Hilliard system, see [3, 4, 5]), that is, to evolution equations on the boundary of the vessel, resulting from suitable free energy/enthalpy balances. In most papers dealing with the Caginalp model, such boundary conditions only concern the order parameter (see, e.g., [6, 8]), but, in the abscence of specific physical justifications (e.g., a thermally isolated system, [6]), it is reasonable to impose the same type of conditions on the temperature, or, actually, on the enthalpy $H(t) = \vartheta(t) + u(t)$: a derivation can be obtained along the lines of [7].

2000 *Mathematics Subject Classification.* Primary: 35K55, 35J60; Secondary: 80A22.

Key words and phrases. Caginalp system, phase transition, dynamic boundary conditions, global attractor, exponential attractor.

This work was partially supported by the Italian MIUR-PRIN Research Project 2008 "Transizioni di fase, isteresi e scale multiple" and by the GNAMPA Project 2010 "Proprietà asintotiche per problemi differenziali con memoria".

More precisely, given a bounded and smooth domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$, we will study the following system:

$$\begin{cases} \partial_t u - \Delta u + f(u) = \vartheta \quad \text{in} \quad \Omega \\ \partial_t \psi - \Delta_\Gamma \psi + g(\psi) + \partial_{\boldsymbol{n}} u = \zeta \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta + u) - \Delta\vartheta = 0 \quad \text{in} \quad \Omega \\ \partial_t(\zeta + \psi) - \Delta_\Gamma \zeta + \partial_{\boldsymbol{n}} \vartheta = 0 \quad \text{on} \quad \Gamma \\ u|_\Gamma = \psi, \quad \vartheta|_\Gamma = \zeta \\ u(0) = u_0, \quad \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega \\ \psi(0) = \psi_0, \quad \zeta(0) = \zeta_0 \quad \text{on} \quad \Gamma, \end{cases} \tag{1.1}$$

where Δ_Γ is the Laplace-Beltrami operator, $\partial_{\boldsymbol{n}}$ is the outward normal derivative and all physical constants have been set equal to one. Here, the nonlinear functions f and g represent the derivative of a typically nonconvex configuration potential and of a boundary potential, respectively.

Such models have been studied in [7, 9], for smooth nonlinearities in [7] and for both smooth and singular nonlinearities in [9]. Note however that, in [7], one assumes that there is no diffusion for the temperature on the boundary, i.e., the Laplace-Beltrami operator does not appear in the (dynamic) boundary condition for the temperature, whereas, in [9], there is an additional dissipativity term in this boundary condition, so that the enthalpy is not conserved (see below).

In this paper, we consider smooth nonlinearities f, g of class C^2. We are interested in the existence of strong (i.e., H^2) solutions. Our first task is thus to study the well-posedness, namely, the existence of a unique strong solution and a suitable Lipschitz continuous dependence on the initial data. Due to the nontrivial coupling between the interior and the boundary, the existence is proved in several steps, based on the Leray-Schauder principle. Note that, in [7], the existence of solutions is proved via a standard (contracting) fixed point theorem. Here, the presence of the diffusion on both dynamic boundary conditions allow to interpret these as perturbations of the classical heat equation on the boundary, which makes the use of the Leray-Schauder principle natural. This also has an interest on its own, as it allows to prove additional regularity on the solutions.

It follows from the well-posedness result that the system generates a strongly continuous semigroup in a suitable phase-space and we can address the existence of a global attractor with finite dimension. Indeed, such a result, which entails that the essential asymptotic dynamics can be described by a finite number of parameters, is obtained as a byproduct of the existence of an exponential attractor whose basin of attraction can be extended to the whole phase-space, thanks to the transitivity of the exponential attraction (see [2]).

1.1. **Assumptions.** We make the following assumptions on the nonlinear functions f and g. As far as the bulk nonlinearity f is concerned, we impose the standard dissipativity hypothesis,

$$f \in C^2(\mathbb{R}) \quad \text{with} \quad \liminf_{|u|\to\infty} f'(u) > 0, \tag{1.2}$$

whereas, for physical reasons (cf. [13]), we assume that the surface nonlinearity g satisfies

$$g(u) = u + g_0(u), \quad \text{where} \quad g_0 \in C^2(\mathbb{R}) \quad \text{with} \quad \|g_0\|_{C^2(\mathbb{R})} = c_0 < \infty. \tag{1.3}$$

Notice that we do not impose any growth condition on f. In light of (1.2), it follows that there exist constants $c_f, \mu > 0$ and $\mu' \geq 0$ such that

$$f'(s) \geq -c_f, \quad \text{and} \quad f(s)s \geq \mu s^2 - \mu', \quad \forall s \in \mathbb{R}, \tag{1.4}$$

and, setting $F(s) := \int_0^s f(\xi)\, d\xi$, there exists $C_f > 0$ such that

$$\langle F(u) - f(u)u, 1\rangle_\Omega \leq C_f \|u\|_\Omega^2, \quad \forall u \in L^2(\Omega), \tag{1.5}$$

where $\|\cdot\|_\Omega$ and $\langle \cdot,\cdot\rangle_\Omega$ denote either the $L^2(\Omega)$ or the $[L^2(\Omega)]^3$-norm/scalar product, depending on the context. Besides, it is not difficult to prove that

$$2F(s) \geq \mu s^2 - \mu', \quad \forall s \in \mathbb{R}. \tag{1.6}$$

Preliminaries. Having equipped $L^2(\Gamma)$ and $[L^2(\Gamma)]^3$ with their usual scalar products and norms, both denoted by $\langle \cdot,\cdot\rangle_\Gamma$ and $\|\cdot\|_\Gamma$, we introduce the space $L^2(\overline{\Omega}) := L^2(\Omega) \times L^2(\Gamma)$ endowed with the scalar product

$$(U, W) := \langle u, w\rangle_\Omega + \langle \psi, v\rangle_\Gamma, \quad \forall U = (u,\psi), W = (w, v) \in L^2(\overline{\Omega}),$$

and the corresponding norm

$$\|U\|^2 := \|u\|_\Omega^2 + \|\psi\|_\Gamma^2.$$

For $U = (u,\psi) \in L^2(\overline{\Omega})$, we also set

$$m(U) := \frac{1}{|\Omega| + |\Gamma|}\Big(\int_\Omega u dx + \int_\Gamma \psi d\Sigma\Big) \quad \text{and} \quad \langle U\rangle := (m(U), m(U)).$$

It is easy to check that the following inequalities hold:

$$0 \leq \|V - \langle V\rangle\|^2 = \|V\|^2 - (|\Omega| + |\Gamma|)m(V)^2, \quad \forall V \in L^2(\overline{\Omega}). \tag{1.7}$$

For further convenience, given two normed function spaces X in Ω and Y on Γ, we set, whenever this makes sense,

$$X \otimes Y := \{u \in X : \quad u|_\Gamma \in Y\}$$

and we endow this space with the norm

$$\|u\|_{X\otimes Y}^2 = \|u\|_X^2 + \|u|_\Gamma\|_Y^2.$$

In particular, we set

$$H^k(\overline{\Omega}) := H^k(\Omega) \otimes H^k(\Gamma), \quad k = 1, 2, 3.$$

Then, introducing the dual space $[H^k(\overline{\Omega})]^*$ of $H^k(\overline{\Omega})$, we denote by $\langle\cdot,\cdot\rangle_{H^k(\overline{\Omega})^*, H^k(\overline{\Omega})}$ the corresponding duality pairings. This allows to define the spaces

$$\mathcal{V}^o := \{U \in H^1(\overline{\Omega}) : \quad m(U) = 0\} \subset H^1(\overline{\Omega})$$
$$\mathcal{V}^{o\prime} := \{\varphi \in H^1(\overline{\Omega})^* : \quad \langle \varphi, 1\rangle_{H^1(\overline{\Omega})^*, H^1(\overline{\Omega})} = 0\} \subset H^1(\overline{\Omega})^*$$

and the operator $\mathbf{A} : \mathcal{V}^o \to \mathcal{V}^{o\prime}$, defined as

$$\langle \mathbf{A}U, W\rangle_{H^1(\overline{\Omega})^*, H^1(\overline{\Omega})} := \langle \nabla u, \nabla w\rangle_\Omega + \langle \nabla_\Gamma \psi, \nabla_\Gamma v\rangle_\Gamma,$$

for any $U = (u,\psi)$, $W = (w, v) \in \mathcal{V}^o$, is invertible. Notice that the third and fourth equation of (1.1) can be rewritten as

$$\mathbf{A}^{-1}\partial_t(\Theta + U) + \Theta - \langle\Theta\rangle = 0 \quad \text{in} \quad \mathcal{V}^o. \tag{1.8}$$

Beside, the bilinear form

$$((U, W)) := \langle \nabla u, \nabla w\rangle_\Omega + \langle \nabla_\Gamma \psi, \nabla_\Gamma v\rangle_\Gamma, \quad \forall U = (u,\psi), W = (w, v) \in H^1(\overline{\Omega})$$

is a scalar product on $\mathcal{V}^o$ and we set

$$|U|^2 := ((U,U)), \quad \forall U \in H^1(\overline{\Omega}).$$

We will repeatedly exploit the equivalence between the standard $H^1(\overline{\Omega})$-norm and another norm appearing naturally in the estimates. More precisely, there exists a constant $\gamma > 1$ such that

$$\frac{1}{\gamma}\|U\|^2_{H^1(\overline{\Omega})} \leq |U|^2 + \|\psi\|^2_\Gamma \leq \gamma\|U\|^2_{H^1(\overline{\Omega})}, \quad \forall U = (u,\psi) \in H^1(\overline{\Omega}). \tag{1.9}$$

Finally, we set

$$\mathcal{H} = [L^2(\overline{\Omega})]^2 \quad \text{and} \quad \mathcal{H}^k = [H^k(\overline{\Omega})]^2, \quad \text{for} \quad k = 1,2,3,$$

and, given any $M > 0$, we introduce the phase space

$$\mathcal{H}^2_M = \{(U,\Theta) \in \mathcal{H}^2 : \quad |m(\Theta + U)| \leq M\}$$

endowed with the $\mathcal{H}^2-$topology. Notice that

$$\mathcal{H}^3_M =: \mathcal{H}^2_M \cap \mathcal{H}^3 \Subset \mathcal{H}^2_M.$$

Enthalpy conservation. Denoting any solution to (1.1) by $(U(t),\Theta(t))$, where $U(t) = (u(t),\psi(t))$ and $\Theta(t) = (\vartheta(t),\zeta(t))$, the problem is characterized by the conservation of the *enthalpy*, defined as $H(t) = \Theta(t) + U(t)$. Indeed, the third and fourth equations in (1.1) immediately provide

$$\frac{\mathrm{d}}{\mathrm{dt}}\langle H(t)\rangle = 0, \tag{1.10}$$

that is,

$$m(H(t)) = m(\Theta(t) + U(t)) = m(\Theta_0 + U_0), \tag{1.11}$$

for all $t \geq 0$.

Notation. Throughout the paper, $c > 0$ stands for a constant allowed to vary within a same line and only influenced by the structural data of the problem; further dependencies will be specified on occurrence.

2. **Existence and uniqueness.** This section is devoted to the proof of existence and uniqueness of global solutions to problem (1.1).

Definition 2.1. For any fixed $M > 0$ and $T > 0$, given an arbitrary initial datum $z_0 = (U_0,\Theta_0) = (u_0,\psi_0,\vartheta_0,\zeta_0) \in \mathcal{H}^2_M$, a solution to our system is a quadruplet $z(t) = (U(t),\Theta(t)) = (u(t),\psi(t),\vartheta(t),\zeta(t)) \in C([0,T];\mathcal{H}^2_M)$ satisfying (1.1) in the sense of distributions.

Our first result is the following:

Theorem 2.2. *For any $T > 0$, system (1.1) admits a unique solution $z(t) = (U(t),\Theta(t))$ defined on the whole time interval $[0,T]$ departing from an arbitrary initial datum in $\mathcal{H}^2$.*

To prove this theorem, we argue as in [12, Theorem 2.1]. The idea is to interpret problem (1.1) as a *nonlinear compact perturbation* of its linearized version (see (2.8) below) and relies on the application of the Leray-Schauder principle. To accomplish this program, some L^p-regularity estimates are needed.

L^p-regularity estimates. We will use the anisotropic Sobolev spaces $W^{1,2}_p(\Omega_T)$ and $W^{1,2}_p(\partial\Omega_T)$, where $\Omega_T = [0,T]\times\Omega$ and $\partial\Omega_T = [0,T]\times\partial\Omega$, consisting of functions which, together with their first time derivative and first and second space derivatives, belong to $L^p(\Omega_T)$ and $L^p(\partial\Omega_T)$, respectively (see, e.g., [10]).

In what follows, we will need the embeddings $W^{1,2}_p(\Omega_T) \Subset C(\Omega_T)$ and $H^2(\Omega)\subset W^{2-2/p}_p(\Omega)$. The former compact inclusion follows from the classical Aubin-Simon theorem, provided that $W^2_p(\Omega)\Subset C(\overline{\Omega})$, that is, when $2-3/p>0$. The second embedding is satisfied when $2\le p\le 10/3$. This leads us to confine p to the interval $[2, 10/3]$.

We first consider the linear nonhomogeneous problem with homogeneous Dirichlet boundary conditions,

$$\begin{cases} \partial_t u - \Delta u - \vartheta = h_1 \quad \text{in} \quad \Omega \\ u|_\Gamma = 0 \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta+u) - \Delta\vartheta = h_2 \quad \text{in} \quad \Omega \\ \vartheta|_\Gamma = 0 \quad \text{on} \quad \Gamma \\ u(0)=u_0, \quad \vartheta(0)=\vartheta_0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

Lemma 2.3. *If $u_0,\vartheta_0\in W^{2(1-\frac{1}{p})}_p(\Omega)$ and $h_1,h_2\in L^p(\Omega_T)$, then there exists $C=C(T)>0$ and a unique solution $(u,\vartheta)\in W^{1,2}_p(\Omega_T)\times W^{1,2}_p(\Omega_T)$ to (2.1) such that*

$$\begin{aligned} &\|u\|_{W^{1,2}_p(\Omega_T)} + \|\vartheta\|_{W^{1,2}_p(\Omega_T)} \\ &\le C\big(\|u_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} + \|\vartheta_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}\big). \end{aligned}$$

Proof. We first obtain an estimate needed in the subsequent argument. Multiplying the first equation by $\partial_t u$ and the third one by ϑ in $L^2(\Omega)$, we obtain, on account of the Poincaré inequality,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}[\|\vartheta\|^2_\Omega + \|\nabla u\|^2_\Omega] &= -\|\nabla\vartheta\|^2_\Omega - \|\partial_t u\|^2_\Omega + \langle h_1,\partial_t u\rangle_\Omega + \langle h_2,\vartheta\rangle_\Omega \\ &\le -\frac{1}{2}\|\nabla\vartheta\|^2_\Omega - \frac{1}{2}\|\partial_t u\|^2_\Omega + c(\|h_1\|^2_\Omega + \|h_2\|^2_\Omega), \end{aligned}$$

leading to

$$\frac{\mathrm{d}}{\mathrm{dt}}[\|\vartheta\|^2_\Omega + \|\nabla u\|^2_\Omega] + \|\nabla\vartheta\|^2_\Omega + \|\partial_t u\|^2_\Omega \le c(\|h_1\|^2_\Omega + \|h_2\|^2_\Omega)$$

and, integrating over $[0,T]$, to

$$\begin{aligned} &\|\vartheta\|_{L^\infty(0,T;L^2(\Omega))} + \|\vartheta\|_{L^2(0,T;H^1(\Omega))} \\ &\le c(\|\vartheta_0\|_\Omega + \|u_0\|_{H^1(\Omega)} + \|h_1\|_{L^2(\Omega_T)} + \|h_2\|_{L^2(\Omega_T)}) \\ &\le c(\|\vartheta_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} + \|u_0\|_{W^{2(1-\frac{1}{p})}_p(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}), \end{aligned} \tag{2.2}$$

where here and below in this proof, $c=c(T)$. We can now apply the classical L^p-theory to the system

$$\begin{cases} \partial_t u - \Delta u = h_1 + \vartheta \quad \text{in} \quad \Omega \\ u|_\Gamma = 0 \quad \text{on} \quad \Gamma \\ u(0) = u_0 \quad \text{in} \quad \Omega \end{cases}$$

to conclude that

$$\|u\|_{W_p^{1,2}(\Omega_T)} \le c(\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|\vartheta\|_{L^p(\Omega_T)}).$$

Here, the last term can be controlled by proper interpolation inequalities (see, e.g., [10, Chapter II, (3.2)] with $q = r = p$) and by (2.2), that is,

$$\begin{aligned}\|\vartheta\|_{L^p(\Omega_T)} &\le c\|\vartheta\|_{L^\infty(0,T;L^2(\Omega))}^{1-\frac{2}{p}}\|\vartheta\|_{L^2(0,T;H^1(\Omega))}^{\frac{2}{p}}\\ &\le c(\|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}),\end{aligned}$$

giving the bound

$$\|u\|_{W_p^{1,2}(\Omega_T)} \le c(\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}). \quad (2.3)$$

We apply the same theory to the system for ϑ, where $\partial_t u$ is read from the above equation, namely,

$$\begin{cases}\partial_t\vartheta - \Delta\vartheta + \vartheta = h_2 - h_1 - \Delta u \quad \text{in} \quad \Omega\\ \vartheta|_\Gamma = 0 \quad \text{on} \quad \Gamma\\ \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega,\end{cases}$$

obtaining, by (2.3),

$$\begin{aligned}\|\vartheta\|_{W_p^{1,2}(\Omega_T)} &\le c(\|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)} + \|u\|_{W_p^{1,2}(\Omega_T)}) \quad (2.4)\\ &\le c(\|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}),\end{aligned}$$

which finishes the proof. ∎

As a second step, we consider the linear homogeneous problem with nonhomogeneous Dirichlet boundary conditions and null initial data,

$$\begin{cases}\partial_t u - \Delta u - \vartheta = 0 \quad \text{in} \quad \Omega\\ u|_\Gamma = \psi \quad \text{on} \quad \Gamma\\ \partial_t(\vartheta + u) - \Delta\vartheta = 0 \quad \text{in} \quad \Omega\\ \vartheta|_\Gamma = \zeta \quad \text{on} \quad \Gamma\\ u(0) = 0, \quad \vartheta(0) = 0 \quad \text{in} \quad \Omega.\end{cases} \quad (2.5)$$

Lemma 2.4. *If $\psi, \zeta \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T)$, then there exists a unique solution $(u, \vartheta) \in W_p^{1,2}(\Omega_T) \times W_p^{1,2}(\Omega_T)$ to (2.5) such that*

$$\|u\|_{W_p^{1,2}(\Omega_T)} + \|\vartheta\|_{W_p^{1,2}(\Omega_T)} \le C(\|\psi\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T)}), \quad (2.6)$$

for some positive constant C depending on T, but independent of ψ and ζ. Besides,

$$\int_0^t (\langle\partial_{\boldsymbol{n}} u(s), \partial_t\psi(s)\rangle_\Gamma + \langle\partial_{\boldsymbol{n}}\vartheta(s), \zeta(s)\rangle_\Gamma)ds \ge 0. \quad (2.7)$$

Proof. Following [12, Corollary 2.1], we consider the linear and continuous extension operator

$$T_p : W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T) \to W_p^{1,2}(\Omega_T) \quad \text{defined as} \quad (T_p\psi)|_{\partial\Omega_T} = \psi.$$

Performing the change of variables $w = u - T_p\psi$ and $\xi = \vartheta - T_p\zeta$, we obtain by straightforward computations that (w, ξ) solves (2.1) with

$$h_1 := -[\partial_t(T_p\psi) - \Delta(T_p\psi) - T_p\zeta],$$
$$h_2 := -[\partial_t(T_p\zeta) - \Delta(T_p\zeta) + \partial_t(T_p\psi)]$$

and null initial data. By Lemma 2.3, we have the existence and uniqueness of the solution, together with estimate (2.6). In order to prove (2.7), it suffices to sum the first equation times $\partial_t u$ and the equation for ϑ times ϑ: an integration in space and time in light of the null initial conditions provides the required inequality. □

We are now ready to study the linearized version of our problem,

$$\begin{cases} \partial_t u - \Delta u - \vartheta = h_1 \quad \text{in} \quad \Omega \\ \partial_t \psi - \Delta_\Gamma \psi + \psi + \partial_{\boldsymbol{n}} u - \zeta = h_2 \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta + u) - \Delta\vartheta = h_3 \quad \text{in} \quad \Omega \\ \partial_t(\zeta + \psi) - \Delta_\Gamma \zeta + \partial_{\boldsymbol{n}}\vartheta = h_4 \quad \text{on} \quad \Gamma \\ u|_\Gamma = \psi, \quad \vartheta|_\Gamma = \zeta \\ u(0) = u_0, \quad \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega \\ \psi(0) = \psi_0, \quad \zeta(0) = \zeta_0 \quad \text{on} \quad \Gamma. \end{cases} \tag{2.8}$$

Lemma 2.5. *If $h_1, h_3 \in L^p(\Omega_T)$, $h_2, h_4 \in L^p(\partial\Omega_T)$, $u_0, \vartheta_0 \in W_p^{2(1-\frac{1}{p})}(\Omega)$ and $\psi_0, \zeta_0 \in W_p^{2(1-\frac{1}{p})}(\partial\Omega)$, then (2.8) possesses a unique solution $(u(t), \psi(t), \vartheta(t), \zeta(t))$ such that*

$$\begin{aligned} &\|u\|_{W_p^{1,2}(\Omega_T)} + \|\vartheta\|_{W_p^{1,2}(\Omega_T)} + \|\psi\|_{W_p^{1,2}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1,2}(\partial\Omega_T)} \\ &\le c(\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\psi_0\|_{W_p^{2(1-\frac{1}{p})}(\partial\Omega)} + \|\zeta_0\|_{W_p^{2(1-\frac{1}{p})}(\partial\Omega)} \\ &\qquad + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\partial\Omega_T)} + \|h_3\|_{L^p(\Omega_T)} + \|h_4\|_{L^p(\partial\Omega_T)}), \end{aligned}$$

for some constant $c > 0$ depending on T, but independent of the solution $(u, \psi, \vartheta, \zeta)$ and the data (h_1, h_2, h_3, h_4).

Proof. The proof is similar to that of [12, Lemma 2.2], but we report it for the reader's convenience: the constant c is allowed to depend on T.
Let $\mathbb{T} : [W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\partial\Omega_T)]^2 \to [W_p^{1,2}(\Omega_T)]^2$, defined as

$$\mathbb{T}(\psi, \zeta) = (\mathbb{T}_1(\psi, \zeta), \mathbb{T}_2(\psi, \zeta)) = (\bar{u}, \bar{\vartheta}),$$

be the solution operator to the following problem:

$$\begin{cases} \partial_t \bar{u} - \Delta\bar{u} - \bar{\vartheta} = 0 \quad \text{in} \quad \Omega \\ \bar{u}|_\Gamma = \psi \quad \text{on} \quad \Gamma \\ \partial_t(\bar{\vartheta} + \bar{u}) - \Delta\bar{\vartheta} = 0 \quad \text{in} \quad \Omega \\ \bar{\vartheta}|_\Gamma = \zeta \quad \text{on} \quad \Gamma \\ \bar{u}(0) = 0, \quad \bar{\vartheta}(0) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Such an operator is well defined thanks to Lemma 2.4. Besides, a suitable trace theorem, together with interpolation (cf. [12, (67) and (69)]), provides

$$\begin{aligned} \|\partial_{\boldsymbol{n}}(\mathbb{T}_i(\psi, \zeta))\|_{L^p(\partial\Omega_T)} &\le c(\|\psi\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\partial\Omega_T)}) \\ &\le \nu(\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1,2}(\partial\Omega_T)}) + c_\nu(\|\psi\|_{L^2(\partial\Omega_T)} + \|\zeta\|_{L^2(\partial\Omega_T)}) \end{aligned} \tag{2.10}$$

for $i = 1, 2$. Setting $\upsilon(t) = u(t) - \bar{u}(t)$ and $\theta(t) = \vartheta(t) - \bar{\vartheta}(t)$, we obtain, in view of (2.8),

$$\begin{cases} \partial_t \upsilon - \Delta \upsilon - \theta = h_1 \quad \text{in} \quad \Omega \\ \upsilon|_\Gamma = 0 \quad \text{on} \quad \Gamma \\ \partial_t(\theta + \upsilon) - \Delta\theta = h_3 \quad \text{in} \quad \Omega \\ \theta|_\Gamma = 0 \quad \text{on} \quad \Gamma \\ \upsilon(0) = u_0, \quad \theta(0) = \vartheta_0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

Since Lemma 2.3 applies to (2.11), we have

$$\begin{aligned} &\|\upsilon\|_{W_p^{1,2}(\Omega_T)} + \|\theta\|_{W_p^{1,2}(\Omega_T)} \\ &\leq c\big[\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_3\|_{L^p(\Omega_T)}\big]. \end{aligned} \tag{2.12}$$

Next, (ψ, ζ) solves

$$\begin{cases} \partial_t\psi - \Delta_\Gamma\psi + \psi + \partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta)) - \zeta = h_2 - \partial_{\boldsymbol{n}}\upsilon \quad \text{on} \quad \Gamma \\ \partial_t(\zeta + \psi) - \Delta_\Gamma\zeta + \partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta)) = h_4 - \partial_{\boldsymbol{n}}\theta \quad \text{on} \quad \Gamma \\ \psi|_{t=0} = \psi_0, \quad \zeta(0) = \zeta_0, \end{cases} \tag{2.13}$$

where, due to (2.12), $h_2 - \partial_{\boldsymbol{n}}\upsilon, h_4 - \partial_{\boldsymbol{n}}\theta \in L^p(\partial\Omega_T)$. Hence, (2.13) is a compact perturbation of the heat equation on the boundary and the existence and uniqueness of solutions can be verified in a standard way. To prove the required L^p-estimates, we first apply the classical L^p-theory to the system

$$\begin{cases} \partial_t\psi - \Delta_\Gamma\psi + \psi = \tilde{h}_2 + \zeta \quad \text{on} \quad \Gamma \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{2.14}$$

where

$$\tilde{h}_2 := h_2 - \partial_{\boldsymbol{n}}\upsilon - \partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta)),$$

obtaining

$$\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} \leq c(\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\zeta\|_{L^p(\partial\Omega_T)} + \|\tilde{h}_2\|_{L^p(\partial\Omega_T)}), \tag{2.15}$$

where

$$\|\tilde{h}_2\|_{L^p(\partial\Omega_T)} \leq c(\|h_2\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}\upsilon\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta))\|_{L^p(\partial\Omega_T)}). \tag{2.16}$$

We now turn our attention to the ζ-system, where $\partial_t\psi$ has been replaced by the corresponding terms read from (2.14),

$$\begin{cases} \partial_t\zeta - \Delta_\Gamma\zeta + \zeta = \tilde{h}_4 \quad \text{on} \quad \Gamma \\ \zeta(0) = \zeta_0, \end{cases} \tag{2.17}$$

with

$$\tilde{h}_4 := h_4 - \partial_{\boldsymbol{n}}\theta - \partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta)) - \Delta_\Gamma\psi + \psi - \tilde{h}_2.$$

Standard L^p-estimates give

$$\begin{aligned} \|\zeta\|_{W_p^{1,2}(\partial\Omega_T)} &\leq c(\|\zeta_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\tilde{h}_4\|_{L^p(\partial\Omega_T)}) \\ &\leq c(\|\zeta_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|h_4\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}\theta\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta))\|_{L^p(\partial\Omega_T)} \\ &\quad + \|\psi\|_{W_p^{1,2}(\partial\Omega_T)} + \|\psi\|_{L^p(\partial\Omega_T)} + \|\tilde{h}_2\|_{L^p(\partial\Omega_T)}), \end{aligned}$$

which, by (2.15) and (2.16), leads to

$$
\begin{aligned}
&\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1,2}(\partial\Omega_T)} \\
&\le c(\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\zeta_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|h_2\|_{L^p(\partial\Omega_T)} + \|h_4\|_{L^p(\partial\Omega_T)} \\
&\quad + \|\psi\|_{L^p(\partial\Omega_T)} + \|\zeta\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}\upsilon\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}\theta\|_{L^p(\partial\Omega_T)} \\
&\quad + \|\partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta))\|_{L^p(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta))\|_{L^p(\partial\Omega_T)}).
\end{aligned}
$$

In light of (2.10) and (2.12), fixing ν small enough, we have

$$
\begin{aligned}
&\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} + \|\zeta\|_{W_p^{1,2}(\partial\Omega_T)} \\
&\le c[\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\psi_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\vartheta_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\zeta_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} \\
&\quad + \|h_1\|_{L^p(\Omega_T)} + \|h_3\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\partial\Omega_T)} + \|h_4\|_{L^p(\partial\Omega_T)} \\
&\quad + \|\psi\|_{L^p(\partial\Omega_T)} + \|\zeta\|_{L^p(\partial\Omega_T)} + \|\psi\|_{L^2(\partial\Omega_T)} + \|\zeta\|_{L^2(\partial\Omega_T)}].
\end{aligned} \tag{2.18}
$$

Since the first two terms in the last line can be handled by the inequality

$$
\|\cdot\|_{L^p(\partial\Omega_T)} \le c\|\cdot\|^{1-\frac{2}{p}}_{L^\infty(0,T;L^2(\Gamma))}\|\cdot\|^{\frac{2}{p}}_{L^2(0,T;H^1(\Gamma))}, \tag{2.19}
$$

we are left to control the $L^2(\Gamma)$-norms of ψ and ζ. To this aim, we multiply the first equation in (2.13) by $\partial_t\psi$,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\psi\|^2_{H^1(\Gamma)} + \|\partial_t\psi\|^2_\Gamma = \langle h_2 - \partial_{\boldsymbol{n}}\upsilon, \partial_t\psi\rangle_\Gamma - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta)), \partial_t\psi\rangle_\Gamma + \langle\zeta, \partial_t\psi\rangle_\Gamma,
$$

and the second one by ζ,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\zeta\|^2_\Gamma + \|\nabla_\Gamma\zeta\|^2_\Gamma + \langle\zeta, \partial_t\psi\rangle_\Gamma = \langle h_4 - \partial_{\boldsymbol{n}}\theta, \zeta\rangle_\Gamma - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta)), \zeta\rangle_\Gamma.
$$

Summing up, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\|\psi\|^2_{H^1(\Gamma)} + \|\zeta\|^2_\Gamma) + \|\partial_t\psi\|^2_\Gamma + \|\nabla_\Gamma\zeta\|^2_\Gamma = \langle h_2 - \partial_{\boldsymbol{n}}\upsilon, \partial_t\psi\rangle_\Gamma \\
&\quad + \langle h_4 - \partial_{\boldsymbol{n}}\theta, \zeta\rangle_\Gamma - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta)), \zeta\rangle_\Gamma - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta)), \partial_t\psi\rangle_\Gamma,
\end{aligned}
$$

which leads to the differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\|\psi\|^2_{H^1(\Gamma)} + \|\zeta\|^2_\Gamma) + \frac{1}{2}\|\partial_t\psi\|^2_\Gamma + \|\zeta\|^2_{H^1(\Gamma)} \\
&\le \|h_2\|^2_\Gamma + \|h_4\|^2_\Gamma + \|\partial_{\boldsymbol{n}}\upsilon\|^2_\Gamma + \|\partial_{\boldsymbol{n}}\theta\|^2_\Gamma + 2\|\zeta\|^2_\Gamma \\
&\quad - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_2(\psi,\zeta)), \zeta\rangle_\Gamma - \langle\partial_{\boldsymbol{n}}(\mathbb{T}_1(\psi,\zeta)), \partial_t\psi\rangle_\Gamma,
\end{aligned}
$$

having added $\|\zeta\|^2_\Gamma$ to both sides.

An application of the Gronwall's lemma, taking into account (2.7) and (2.12), provides

$$\begin{aligned}
&\|\psi(t)\|^2_{H^1(\Gamma)} + \|\zeta(t)\|^2_\Gamma + \int_0^t [\|\partial_t\psi(s)\|^2_\Gamma + \|\zeta(s)\|^2_{H^1(\Gamma)}]ds \\
&\le e^{4T}(\|\psi_0\|^2_{H^1(\Gamma)} + \|\zeta_0\|^2_\Gamma) \\
&\quad + 2\int_0^t e^{4(t-s)}[\|h_2(s)\|^2_\Gamma + \|h_4(s)\|^2_\Gamma + \|\partial_{\boldsymbol{n}}\upsilon(s)\|^2_\Gamma + \|\partial_{\boldsymbol{n}}\theta(s)\|^2_\Gamma]ds \\
&\le c[\|\psi_0\|^2_{H^1(\Gamma)} + \|\zeta_0\|^2_\Gamma + \|h_2\|^2_{L^2(\partial\Omega_T)} + \|h_4\|^2_{L^2(\partial\Omega_T)} \\
&\quad + \|\partial_{\boldsymbol{n}}\upsilon\|^2_{L^2(\partial\Omega_T)} + \|\partial_{\boldsymbol{n}}\theta\|^2_{L^2(\partial\Omega_T)}] \\
&\le c[\|u_0\|^2_{H^1(\Omega)} + \|\psi_0\|^2_{H^1(\Gamma)} + \|\vartheta_0\|^2_{H^1(\Omega)} + \|\zeta_0\|^2_\Gamma \\
&\quad + \|h_1\|^2_{L^2(\Omega_T)} + \|h_3\|^2_{L^2(\Omega_T)} + \|h_2\|^2_{L^2(\partial\Omega_T)} + \|h_4\|^2_{L^2(\partial\Omega_T)}],
\end{aligned}$$

that is,

$$\begin{aligned}
&\|\psi\|_{L^\infty(0,T;H^1(\Gamma))} + \|\zeta\|_{L^\infty(0,T;L^2(\Gamma))} + \|\zeta\|_{L^2(0,T;H^1(\Gamma))} + \|\partial_t\psi\|_{L^2(\partial\Omega_T)} \\
&\le c(\|z_0\|_{\mathcal{H}^1} + \|h_1\|_{L^2(\Omega_T)} + \|h_3\|_{L^2(\Omega_T)} + \|h_2\|_{L^2(\partial\Omega_T)} + \|h_4\|_{L^2(\partial\Omega_T)}),
\end{aligned} \tag{2.20}$$

where c depends exponentially on T. Injecting this relation into (2.18), we have, in light of (2.19), the required control of the norms $\|\psi\|_{W^{1,2}_p(\partial\Omega_T)}$ and $\|\zeta\|_{W^{1,2}_p(\partial\Omega_T)}$, which concludes the proof. □

Proof of existence. Given $z_0 = (u_0, \psi_0, \vartheta_0, \zeta_0) \in \mathcal{H}^2_M$, we consider the following homotopy of problem (1.1):

$$\begin{cases}
\partial_t u - \Delta u - \vartheta = -sf(u) \quad \text{in} \quad \Omega \\
\partial_t\psi - \Delta_\Gamma\psi + \psi + \partial_{\boldsymbol{n}} u - \zeta = -sg_0(\psi) \quad \text{on} \quad \Gamma \\
\partial_t\vartheta - \Delta\vartheta + \partial_t u = 0 \quad \text{in} \quad \Omega \\
\partial_t\zeta - \Delta_\Gamma\zeta + \partial_{\boldsymbol{n}}\vartheta + \partial_t\psi = 0 \quad \text{on} \quad \Gamma \\
u|_\Gamma = \psi, \quad \vartheta|_\Gamma = \zeta \\
u(0) = u_0, \quad \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega \\
\psi(0) = \psi_0, \quad \zeta(0) = \zeta_0 \quad \text{on} \quad \Gamma.
\end{cases}$$

For any $s \in [0,1]$, this problem is equivalent to the following:

$$\begin{pmatrix} u \\ \psi \\ \vartheta \\ \zeta \end{pmatrix} = \mathbb{M}_0 \begin{pmatrix} u_0 \\ \psi_0 \\ \vartheta_0 \\ \zeta_0 \end{pmatrix} + s\mathbb{M}_h \begin{pmatrix} -f(u) \\ -g_0(\psi) \\ 0 \\ 0 \end{pmatrix}$$

where $\mathbb{M}_0 : z_0 = (u_0, \psi_0, \vartheta_0, \zeta_0) \mapsto z(t) = (u, \psi, \vartheta, \zeta)$ is the solving operator to (2.8) with $h = \underline{0}_4$ and $\mathbb{M}_h : h \mapsto z$ is the solving operator to (2.8) with null initial data. We now introduce the space

$$\Phi := [W^{1,2}_p(\Omega_T) \times W^{1,2}_p(\partial\Omega_T))]^2$$

which is compactly embedded into $[C(\Omega_T) \times C(\partial\Omega_T)]^2$. In light of Lemma 2.5, the operator $(u, \psi, \vartheta, \zeta) \mapsto \mathbb{M}_h\big(-f(u), -g_0(\psi), 0, 0\big)$ is (continuous and) compact in Φ.

As in [12, Theorem 2.1], it is possible to verify that each solution $(u_s, \psi_s, \vartheta_s, \zeta_s)$ to the s-problem satisfies a priori estimates in the space Φ which are *uniform* with respect to $s \in [0,1]$. Indeed, this directly follows from an application of Lemma 2.5

to (2.8) with external forcing $(-sf(u_s), -sg_0(\psi_s), 0, 0)$ for which we have uniform L^∞-estimates in light of Lemma 3.3 below.

Therefore we can apply the Leray-Schauder principle which ensures that the homotopy system has a solution for every $s \in [0,1]$. Letting $s = 1$, we obtain the desired existence result for (1.1).

Proof of uniqueness. Uniqueness is an immediate consequence of the following continuous dependence estimate with respect to the initial data.

Lemma 2.6. *For any pair of initial data $z_1, z_2 \in \mathcal{H}^2_M$, there exist two positive constants c and L, possibly depending on $\|z_i\|_{\mathcal{H}^2}$, such that, denoting by $z^i(t)$ a solution originating from z_i, there holds*

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}^2} + \|\partial_t z^1(t) - \partial_t z^2(t)\|_{\mathcal{H}} \le ce^{Lt}\|z_1 - z_2\|_{\mathcal{H}^2}, \quad t \ge 0.$$

Proof. Having denoted by $z^i(t) = (u^i(t), \psi^i(t), \vartheta^i(t), \zeta^i(t))$, $i = 1, 2$, the solution departing from z_i, we also set, for further convenience,

$$\begin{cases} \ell_1(t) = \displaystyle\int_0^1 f'(su^1(t) + (1-s)u^2(t))ds \\ \ell_2(t) = \displaystyle\int_0^1 g_0'(s\psi^1(t) + (1-s)\psi^2(t))ds, \end{cases} \tag{2.21}$$

noticing that, by the energy estimates in Lemma 3.3 below, we have

$$\|\ell_1(t)\|_{H^2(\Omega)} + \|\partial_t \ell_1(t)\|_\Omega + \|\ell_2(t)\|_{H^2(\Gamma)} + \|\partial_t \ell_2(t)\|_\Gamma \le c, \qquad \forall t \ge 0, \tag{2.22}$$

where the constant c depends on the norms of the initial data. Now, we see that $z(t) = z^1(t) - z^2(t) = (U(t), \Theta(t)) = (u(t), \psi(t), \vartheta(t), \zeta(t))$ satisfies the problem

$$\begin{cases} \partial_t u - \Delta u + \ell_1 u = \vartheta \quad \text{in} \quad \Omega \\ \partial_t \psi - \Delta_\Gamma \psi + \psi + \ell_2 \psi + \partial_{\boldsymbol{n}} u = \zeta \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta + u) - \Delta \vartheta = 0 \quad \text{in} \quad \Omega \\ \partial_t(\zeta + \psi) - \Delta_\Gamma \zeta + \partial_{\boldsymbol{n}} \vartheta = 0 \quad \text{on} \quad \Gamma \\ u|_\Gamma = \psi, \qquad \vartheta|_\Gamma = \zeta \quad \text{on} \quad \Gamma \\ U(0) = U_0, \quad \Theta(0) = \Theta_0 \quad \text{in} \quad \overline{\Omega}, \end{cases} \tag{2.23}$$

where $z_0 = z_1 - z_2 = (U_0, \Theta_0)$.

Multiplying the first equations by $\partial_t U$ in $L^2(\overline{\Omega})$ and the third and fourth ones by Θ in $L^2(\overline{\Omega})$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(|U|^2 + \|\psi\|_\Gamma^2 + \|\Theta\|^2) + |\Theta|^2 + \|\partial_t U\|^2 = -\langle \ell_1 u, \partial_t u\rangle_\Omega - \langle \ell_2 \psi, \partial_t \psi\rangle_\Gamma. \tag{2.24}$$

We next consider the problem formally obtained by differentiating (2.23) with respect to time,

$$\begin{cases} \partial_{tt} u - \Delta \partial_t u + \ell_1 \partial_t u + (\partial_t \ell_1) u = \partial_t \vartheta \quad \text{in} \quad \Omega \\ \partial_{tt} \psi - \Delta_\Gamma \partial_t \psi + \partial_t \psi + \ell_2 \partial_t \psi + (\partial_t \ell_2)\psi + \partial_{\boldsymbol{n}} \partial_t u = \partial_t \zeta \quad \text{on} \quad \Gamma \\ \partial_{tt}(\vartheta + u) - \Delta \partial_t \vartheta = 0 \quad \text{in} \quad \Omega \\ \partial_{tt}(\zeta + \psi) - \Delta_\Gamma \partial_t \zeta + \partial_{\boldsymbol{n}} \partial_t \vartheta = 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{2.25}$$

and we multiply the first two equations by $\partial_t U$ in $L^2(\overline{\Omega})$ and sum the resulting equality to the product of the third and fourth equations of (2.23) by $\partial_t \Theta$ in $L^2(\overline{\Omega})$,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(|\Theta|^2 + \|\partial_t U\|^2) + \|\partial_t \Theta\|^2 + |\partial_t U|^2 + \|\partial_t \psi\|^2_\Gamma \\
&= -\langle \ell_1 \partial_t u, \partial_t u\rangle_\Omega - \langle \ell_2 \partial_t \psi, \partial_t \psi\rangle_\Gamma - \langle (\partial_t \ell_1) u, \partial_t u\rangle_\Omega - \langle (\partial_t \ell_2)\psi, \partial_t \psi\rangle_\Gamma.
\end{aligned} \tag{2.26}$$

Finally, we rewrite the last two equations in (2.25) in the form

$$\partial_{tt} H + \mathbf{A}(\partial_t H) = \mathbf{A}(\partial_t U - \langle \partial_t U\rangle)$$

which we multiply by $\partial_t H$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t H\|^2 + |\partial_t H|^2 = -((\partial_t H, \partial_t U)). \tag{2.27}$$

Summing (2.24), (2.26) and (2.27), we see that the energy functional

$$E(t) = |U(t)|^2 + \|\psi(t)\|^2_\Gamma + \|\Theta(t)\|^2_{H^1(\overline{\Omega})} + \|\partial_t U(t)\|^2 + \|\partial_t H(t)\|^2$$

satisfies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}E + |\Theta|^2 + \|\partial_t U\|^2 + \|\partial_t \Theta\|^2 + |\partial_t U|^2 + \|\partial_t \psi\|^2_\Gamma + |\partial_t H|^2 = \\
&= -\langle \ell_1 \partial_t u, \partial_t u\rangle_\Omega - \langle \ell_2 \partial_t \psi, \partial_t \psi\rangle_\Gamma - \langle \ell_1 u, \partial_t u\rangle_\Omega - \langle \ell_2 \psi, \partial_t \psi\rangle_\Gamma \\
&\quad - \langle (\partial_t \ell_1) u, \partial_t u\rangle_\Omega - \langle (\partial_t \ell_2)\psi, \partial_t \psi\rangle_\Gamma - ((\partial_t H, \partial_t U)).
\end{aligned}$$

We then easily see that

$$-((\partial_t H, \partial_t U)) \leq \frac{1}{2}|\partial_t H|^2 + \frac{1}{2}|\partial_t U|^2.$$

Besides, on account of (2.22) and recalling that, by (1.3) and (1.4), we have $-\ell_1 \leq c_f$, whereas $\|\ell_2\|_{L^\infty(\Gamma)} \leq c_0$, the following inequalities hold:

$$\begin{aligned}
&- \langle \ell_1 \partial_t u, u + \partial_t u\rangle_\Omega - \langle \ell_2 \partial_t \psi, \psi + \partial_t \psi\rangle_\Gamma - \langle (\partial_t \ell_1) u, \partial_t u\rangle_\Omega - \langle (\partial_t \ell_2)\psi, \partial_t \psi\rangle_\Gamma \\
&\leq (\|\ell_1\|_{H^2(\Omega)} + c_0)\|U\|\|\partial_t U\| + (c_f + c_0)\|\partial_t U\|^2 \\
&\quad + (\|\partial_t \ell_1\|_\Omega + \|\partial_t \ell_2\|_\Gamma)\|U\|_{L^4(\overline{\Omega})}\|\partial_t U\|_{L^4(\overline{\Omega})} \\
&\leq c(\|U\|^2_{H^1(\overline{\Omega})} + \|\partial_t U\|^2) + \frac{1}{2}\|\partial_t U\|^2_{H^1(\overline{\Omega})},
\end{aligned}$$

where we have also exploited the continuous embedding $H^1(\overline{\Omega}) \subset L^4(\overline{\Omega})$. Collecting the last two estimates, we deduce, in light of (1.9), that

$$\frac{\mathrm{d}}{\mathrm{dt}}E \leq cE,$$

which gives the desired estimate, since

$$\frac{1}{\gamma+2}(\|z(t)\|^2_{\mathcal{H}^2} + \|\partial_t z(t)\|^2_{\mathcal{H}}) \leq E(t) \leq (2+\gamma)(\|z(t)\|^2_{\mathcal{H}^2} + \|\partial_t z(t)\|^2_{\mathcal{H}}).$$

$\square$

3. **Dissipative semigroup.** As a consequence of our existence result, we can state the

Theorem 3.1. *For any $M > 0$, the solution operator to (1.1) defined as*

$$S(t)z := z(t) = (u(t), \psi(t), \vartheta(t), \zeta(t)) = (U(t), \Theta(t)), \qquad t \geq 0,$$

is a strongly continuous semigroup $(S(t), \mathcal{H}^2_M)$.

The main result of this section is that this semigroup is dissipative, namely, there exists a bounded set which eventually captures all the trajectories $S(t)z$, uniformly with respect to the norm of the initial data z in $\mathcal{H}^2_M$.

Theorem 3.2. *For any $M > 0$, there exists a bounded set $\mathcal{B}_M \subset \mathcal{H}^2_M$ such that*

$$S(t)B \subset \mathcal{B}_M, \qquad t \geq t_B \geq 0$$

for any $B \subset \mathcal{H}^2_M$ bounded.

The dissipativity of the semigroup is an immediate consequence of the following uniform estimate:

Lemma 3.3. *Given any initial datum $z \in \mathcal{H}^2_M$, the solution $S(t)z = z(t) = (U(t), \Theta(t))$ to (1.1) satisfies*

$$\|z(t)\|_{\mathcal{H}^2} + \|\partial_t z(t)\|_{\mathcal{H}} \leq Q(\|z\|_{\mathcal{H}^2})e^{-\delta t/2} + c_M, \quad \forall t \geq 0,$$

where Q is an increasing nonnegative function, $\delta > 0$ and $c_M > 0$ is a constant depending on M and on the structural parameters of the problem. Moreover,

$$\int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \leq Q(\|z\|_{\mathcal{H}^2}), \quad \forall t \geq 0. \tag{3.1}$$

Proof. We set $I_0 = m(\Theta_0 + U_0)$, knowing that $|I_0| \leq M$ and that $m(H(t)) = m(\Theta(t) + U(t)) = I_0$ for all times, in light of (1.11).

First, we multiply the first and second equations in (1.1) by U in $L^2(\overline{\Omega})$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\|U\|^2 + 2|U|^2 + 2\|\psi\|^2_\Gamma + 2\langle f(u), u\rangle_\Omega = 2\langle \Theta, U\rangle - 2\langle g_0(\psi), \psi\rangle_\Gamma.$$

Multiplying then (1.8) by $H - \langle H\rangle$ gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\|H - \langle H\rangle\|^2_{\mathcal{V}^{\circ\prime}} + 2\|\Theta - \langle\Theta\rangle\|^2 = -2\langle U - \langle U\rangle, \Theta - \langle\Theta\rangle\rangle.$$

Here, on account of (1.11), the right-hand side reads

$$\begin{aligned} 2\langle U - \langle U\rangle, \Theta - \langle\Theta\rangle\rangle &= 2\langle U, \Theta\rangle - 2(|\Omega| + |\Gamma|)m(U)m(\Theta) \\ &= 2\langle U, \Theta\rangle - (|\Omega| + |\Gamma|)[I_0^2 - m(\Theta)^2 - m(U)^2]. \end{aligned}$$

Summing the three above equalities, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}(\|H - \langle H\rangle\|^2_{\mathcal{V}^{\circ\prime}} + \|U\|^2) \\ &\quad + 2|U|^2 + 2\|\psi\|^2_\Gamma + \|\Theta\|^2 + \|\Theta - \langle\Theta\rangle\|^2 + (|\Omega| + |\Gamma|)m(U)^2 + 2\langle f(u), u\rangle_\Omega \\ &= (|\Omega| + |\Gamma|)I_0^2 - 2\langle g_0(\psi), \psi\rangle_\Gamma. \end{aligned} \tag{3.2}$$

We now take the product in $L^2(\overline{\Omega})$ of the first and second equations in (1.1) by $\partial_t U$,

$$\frac{\mathrm{d}}{\mathrm{dt}}(|U|^2 + \|\psi\|^2_\Gamma + 2\langle F(u), 1\rangle_\Omega) + 2\|\partial_t U\|^2 = 2\langle \Theta, \partial_t U\rangle - 2\langle g_0(\psi), \partial_t \psi\rangle_\Gamma. \tag{3.3}$$

Then, the product of the third and fourth equations in $L^2(\overline{\Omega})$ by Θ gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\Theta\|^2 + 2|\Theta|^2 = -2\langle\Theta, \partial_t U\rangle,$$

which, added to (3.3), leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}(\|\Theta\|^2+|U|^2+\|\psi\|^2_\Gamma+2\langle F(u),1\rangle_\Omega)+2|\Theta|^2+2\|\partial_t U\|^2=-2\langle g_0(\psi),\partial_t\psi\rangle_\Gamma. \quad (3.4)$$

Finally, we compute the product of the third and fourth equations by $\partial_t\Theta$,

$$\frac{\mathrm{d}}{\mathrm{dt}}|\Theta|^2 + 2\|\partial_t\Theta\|^2 = -2\langle\partial_t\Theta, \partial_t U\rangle. \quad (3.5)$$

We now differentiate the whole system with respect to time,

$$\begin{cases} \partial_{tt}u - \Delta\partial_t u + f'(u)\partial_t u = \partial_t\vartheta \quad \text{in} \quad \Omega \\ \partial_{tt}\psi - \Delta_\Gamma\partial_t\psi + \partial_t\psi + g_0'(\psi)\partial_t\psi + \partial_{\boldsymbol{n}}\partial_t u = \partial_t\zeta \quad \text{on} \quad \Gamma \\ \partial_{tt}(\vartheta+u) - \Delta\partial_t\vartheta = 0 \quad \text{in} \quad \Omega \\ \partial_{tt}(\zeta+\psi) - \Delta_\Gamma\partial_t\zeta + \partial_{\boldsymbol{n}}\partial_t\vartheta = 0 \quad \text{on} \quad \Gamma \end{cases} \quad (3.6)$$

supplemented with the boundary conditions read from the system,

$$\begin{aligned} \partial_t U(0) &= (\Delta u_0 - f(u_0) + \vartheta_0, \Delta_\Gamma\psi_0 - \psi_0 - g_0(\psi_0) - \partial_{\boldsymbol{n}} u_0 + \zeta_0) \\ \partial_t(\Theta+U)(0) &= (\Delta\vartheta_0, \Delta_\Gamma\zeta_0 - \partial_{\boldsymbol{n}}\vartheta_0). \end{aligned}$$

We multiply the first and second equations in (3.6) by $\partial_t U$ in $L^2(\overline{\Omega})$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t U\|^2 + 2|\partial_t U|^2 + 2\|\partial_t\psi\|^2_\Gamma &= -2\langle f'(u)\partial_t u, \partial_t u\rangle_\Omega - 2\langle g_0'(\psi)\partial_t\psi, \partial_t\psi\rangle_\Gamma \\ &\quad + 2\langle\partial_t\Theta, \partial_t U\rangle \end{aligned}$$

and the third and fourth ones by $\partial_t H$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t H\|^2 + 2|\partial_t H|^2 = 2((\partial_t H, \partial_t U))$$

so that, adding the results to (3.5), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}[\|\partial_t U\|^2 + \|\partial_t H\|^2 + |\Theta|^2] + 2\|\partial_t\Theta\|^2 + 2|\partial_t U|^2 + 2\|\partial_t\psi\|^2_\Gamma + 2|\partial_t H|^2 \\ &\quad = -2\langle f'(u)\partial_t u, \partial_t u\rangle_\Omega - 2\langle g_0'(\psi)\partial_t\psi, \partial_t\psi\rangle_\Gamma + 2((\partial_t H, \partial_t U)). \end{aligned}$$

Introducing the functional

$$\begin{aligned} E(t) &= \|\Theta(t)\|^2 + \nu|\Theta(t)|^2 + \|U(t)\|^2_{H^1(\overline{\Omega})} + \|\psi(t)\|^2_\Gamma + \|H(t) - \langle H(t)\rangle\|^2_{\mathcal{V}^{o\prime}} \\ &\quad + 2\langle F(u(t)), 1\rangle_\Omega + \nu\|\partial_t U(t)\|^2 + \nu\|\partial_t H(t)\|^2, \end{aligned}$$

the product of the above equation by $\nu \in (0,1)$, added to (3.2) and (3.4), leads to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}E + \|\Theta\|^2 + \|\Theta - \langle\Theta\rangle\|^2 + 2|\Theta|^2 \\ &\quad + 2|U|^2 + 2\|\psi\|^2_\Gamma + 2\|\partial_t U\|^2 + (|\Omega| + |\Gamma|)m(U)^2 + 2\nu\|\partial_t\Theta\|^2 \\ &\quad + 2\nu|\partial_t U|^2 + 2\nu\|\partial_t\psi\|^2_\Gamma + 2\nu|\partial_t H|^2 + 2\langle f(u), u\rangle_\Omega \\ &= h, \end{aligned} \quad (3.7)$$

where

$$h := -2\langle g_0(\psi), \partial_t\psi\rangle_\Gamma - 2\langle g_0(\psi), \psi\rangle_\Gamma - 2\nu\langle f'(u)\partial_t u, \partial_t u\rangle_\Omega - 2\nu\langle g_0'(\psi)\partial_t\psi, \partial_t\psi\rangle_\Gamma$$
$$+ 2\nu((\partial_t H, \partial_t U)) + (|\Omega| + |\Gamma|)I_0^2$$

satisfies, in light of (1.3) and (1.4),

$$h \le c\|\partial_t\psi\|_\Gamma + c\|\psi\|_\Gamma + \nu c\|\partial_t U\|^2 + \nu|\partial_t U|^2 + \nu|\partial_t H|^2 + (|\Omega| + |\Gamma|)I_0^2.$$

After straightforward computations and provided that ν is small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}E + \|\Theta\|^2 + \|\Theta - \langle\Theta\rangle\|^2 + 2|\Theta|^2 + 2|U|^2 + \|\psi\|_\Gamma^2 + 2\langle f(u), u\rangle_\Omega$$
$$+ (|\Omega| + |\Gamma|)m(U)^2 + \|\partial_t U\|^2 + \nu\|\partial_t\Theta\|^2 + \nu|\partial_t U|^2 + \nu\|\partial_t\psi\|_\Gamma^2 + \nu|\partial_t H|^2 \le c.$$

By (1.4) and (1.5), choosing $0 < \delta < \mu/(2\mu + 2C_f)$, we can now write

$$\begin{aligned} 2\langle f(u), u\rangle_\Omega &= 2\delta\langle f(u), u\rangle_\Omega + 2(1-\delta)\langle f(u), u\rangle_\Omega \\ &\ge 2\delta\langle F(u), 1\rangle_\Omega - 2\delta C_f\|u\|_\Omega^2 + 2(1-\delta)\mu\|u\|_\Omega^2 - 2(1-\delta)\mu'|\Omega| \\ &\ge 2\delta\langle F(u), 1\rangle_\Omega + \mu\|u\|_\Omega^2 - 2(1-\delta)\mu'|\Omega|, \end{aligned}$$

leading to

$$\frac{\mathrm{d}}{\mathrm{dt}}E + \|\Theta\|^2 + \|\Theta - \langle\Theta\rangle\|^2 + 2|\Theta|^2 + 2|U|^2 + \|\psi\|_\Gamma^2 + 2\delta\langle F(u), 1\rangle_\Omega + \mu\|u\|_\Omega^2$$
$$+ (|\Omega| + |\Gamma|)m(U)^2 + \|\partial_t U\|^2 + \nu\|\partial_t\Theta\|^2 + \nu|\partial_t U|^2 + \nu\|\partial_t\psi\|_\Gamma^2 + \nu|\partial_t H|^2 \le c.$$

Here, recovering the lacking norm $\|H - \langle H\rangle\|_{\mathcal{V}^{o\prime}}^2$ by the continuous embedding $L^2(\overline{\Omega}) \subset \mathcal{V}^{o\prime}$ and (1.7),

$$0 \le \|H - \langle H\rangle\|_{\mathcal{V}^{o\prime}}^2 \le c\|H - \langle H\rangle\|^2 \le c(\|\Theta\|^2 + \|U\|^2), \tag{3.8}$$

we obtain, thanks also to (1.9),

$$\frac{\mathrm{d}}{\mathrm{dt}}E + \delta E + \delta\|\partial_t z\|_{\mathcal{H}^1}^2 \le c, \tag{3.9}$$

for some $\delta > 0$. Next, exploiting (1.6), (1.9) and (3.8), we deduce that

$$\frac{\nu}{\gamma+3}[\|z(t)\|_{\mathcal{H}^1}^2 + \|\partial_t z(t)\|_{\mathcal{H}}^2] - \mu' \le E(t) \le c[\|z(t)\|_{\mathcal{H}^1}^2 + \|\partial_t z(t)\|_{\mathcal{H}}^2] + 2\langle F(u(t)), 1\rangle_\Omega.$$

Now, an application of the Gronwall lemma entails

$$\begin{aligned} &\frac{\nu}{\gamma+3}[\|z(t)\|_{\mathcal{H}^1}^2 + \|\partial_t z(t)\|_{\mathcal{H}}^2] - \mu' \le E(t) \le e^{-\delta t}E(0) + c \\ &\le [c\|z(0)\|_{\mathcal{H}^1}^2 + c\|\partial_t z(0)\|_{\mathcal{H}}^2 + 2\langle F(u_0), 1\rangle_\Omega]e^{-\delta t} + c, \end{aligned}$$

so that, on account of the initial conditions in (1.1) and (3.6), we finally have

$$\|z(t)\|_{\mathcal{H}^1}^2 + \|\partial_t z(t)\|_{\mathcal{H}}^2 \le Q(\|z\|_{\mathcal{H}^2}^2)e^{-\delta t} + c, \tag{3.10}$$

for some nonnegative increasing monotone function Q. To prove the integral inequality (3.1), it is now sufficient to integrate (3.9) over the interval $[t, t+1]$.

In order to complete the proof, we are left to control the H^2-norm of the solution. This follows by an application of [12, Appendix, Lemma A.2] to the nonlinear elliptic problem

$$\begin{cases} -\Delta u + f(u) = h_1 := \vartheta - \partial_t u & \text{in} \quad \Omega \\ -\Delta_\Gamma\psi + \psi + \partial_{\boldsymbol{n}} u = h_2 := \zeta - \partial_t\psi - g_0(\psi) & \text{on} \quad \Gamma, \end{cases}$$

which, together with (3.10), furnishes the bound

$$\|U(t)\|_{L^\infty(\overline{\Omega})} \le c(1 + \|h_1(t)\|_\Omega + \|h_2(t)\|_\Gamma) \le Q(\|z\|_{\mathcal{H}^2})e^{-\delta t/2} + c.$$

Notice that this, in turn, yields L^∞-estimates for $f(u)$. Hence, interpreting the nonlinearities as external sources in the elliptic system, we can apply [12, Appendix, Lemma A.1] to derive

$$\|U(t)\|_{H^2(\overline{\Omega})} \le Q(\|z\|_{\mathcal{H}^2})e^{-\delta t/2} + c.$$

Now, we exploit [12, Appendix, Lemma A.1] again for the elliptic problem

$$\begin{cases} -\Delta\vartheta = h_3 := -\partial_t(\vartheta + u) \quad \text{in} \quad \Omega \\ -\Delta_\Gamma\zeta + \zeta + \partial_{\boldsymbol{n}}\vartheta = h_4 := -\partial_t(\zeta + \psi) + \zeta \quad \text{on} \quad \Gamma, \end{cases}$$

which provides the same bound for Θ and concludes the proof of the required energy estimate. $\square$

4. **Global attractors.**

Theorem 4.1. *For any $M > 0$, the semigroup $(S(t), \mathcal{H}^2_M)$ possesses the global attractor $\mathcal{A}_M \subset \mathcal{H}^3_M$.*

Proof. We prove the existence of the global attractor by relying on a standard technique, consisting of the decomposition of the solution operator as the sum of a "contracting map" and a "smoothing" map (see, e.g., [14]), namely, we write $z(t) = z^d(t) + z^c(t)$, where $z^d(t) = (U_d(t), \Theta_d(t)) = (u_d(t), \psi_d(t), \vartheta_d(t), \zeta_d(t))$ solves

$$\begin{cases} \partial_t u_d - \Delta u_d = \vartheta_d \quad \text{in} \quad \Omega \\ \partial_t\psi_d - \Delta_\Gamma\psi_d + \psi_d + \partial_{\boldsymbol{n}} u_d = \zeta_d \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta_d + u_d) - \Delta\vartheta_d = 0 \quad \text{in} \quad \Omega \\ \partial_t(\zeta_d + \psi_d) - \Delta_\Gamma\zeta_d + \partial_{\boldsymbol{n}}\vartheta_d = 0 \quad \text{on} \quad \Gamma \\ U_d(0) = U_0 - \langle U_0\rangle, \quad \Theta_d(0) = \Theta_0 - \langle\Theta_0\rangle \quad \text{in} \quad \overline{\Omega} \end{cases} \tag{4.1}$$

and $z^c(t) = (U_c(t), \Theta_c(t)) = (u_c(t), \psi_c(t), \vartheta_c(t), \zeta_c(t))$ is the solution to

$$\begin{cases} \partial_t u_c - \Delta u_c + f(u) = \vartheta_c \quad \text{in} \quad \Omega \\ \partial_t\psi_c - \Delta_\Gamma\psi_c + \psi_c + g_0(\psi) + \partial_{\boldsymbol{n}} u_c = \zeta_c \quad \text{on} \quad \Gamma \\ \partial_t(\vartheta_c + u_c) - \Delta\vartheta_c = 0 \quad \text{in} \quad \Omega \\ \partial_t(\zeta_c + \psi_c) - \Delta_\Gamma\zeta_c + \partial_{\boldsymbol{n}}\vartheta_c = 0 \quad \text{on} \quad \Gamma \\ U_c(0) = \langle U_0\rangle, \quad \Theta_c(0) = \langle\Theta_0\rangle \quad \text{in} \quad \overline{\Omega}. \end{cases} \tag{4.2}$$

Here and below, the initial datum (U_0, Θ_0) belongs to the absorbing set $\mathcal{B}_M$ provided by Theorem 3.2 and c may depend on the size of $\mathcal{B}_M$.

Step I. We first prove that, for any $t \ge 0$, there holds

$$\|z^d(t)\|^2_{\mathcal{H}^2} + \|\partial_t z^d(t)\|^2_{\mathcal{H}} \le ce^{-\nu t} \tag{4.3}$$

for some $\nu > 0$ and

$$\int_t^{t+1} \|\partial_t z^d(s)\|^2_{\mathcal{H}^1} ds \le c. \tag{4.4}$$

We argue exactly as in the proof of Lemma 3.3 in order to get the differential equality (3.7) for (U_d, Θ_d) in place of (U, Θ). Taking into account that now $H_d(t) :=$

$\Theta_d(t) + U_d(t)$ satisfies $m(H_d(t)) = 0$ for any $t \geq 0$, so that $m(U_d(t)) = -m(\Theta_d(t))$, which, by (1.7), entails

$$\begin{aligned}&\|\Theta_d - \langle\Theta_d\rangle\|^2 + (|\Omega| + |\Gamma|)m^2(U_d) = \|\Theta_d\|^2 - (|\Omega| + |\Gamma|)m^2(\Theta_d)\\&+ (|\Omega| + |\Gamma|)m^2(U_d) = \|\Theta_d\|^2,\end{aligned}$$

we see that the differential equation (3.7) (with $\nu = 1$ and canceling the nonlinearities as in (4.1)) now reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}E_d + 2|U_d|^2 + 2\|\psi_d\|_\Gamma^2 + 2\|\Theta_d\|^2_{H^1(\overline{\Omega})} + 2\|\partial_t U_d\|^2_{H^1(\overline{\Omega})} + 2\|\partial_t\Theta_d\|^2 + 2\|\partial_t\psi_d\|^2_\Gamma\\&+ 2|\partial_t H_d|^2 = 2((\partial_t H_d, \partial_t U_d)) \leq |\partial_t U_d|^2 + |\partial_t H_d|^2,\end{aligned}$$

where

$$\begin{aligned}E_d(t) = {}&\|U_d(t)\|^2_{H^1(\overline{\Omega})} + \|\Theta_d(t)\|^2_{H^1(\overline{\Omega})} + \|\psi_d(t)\|^2_\Gamma + \|H_d(t)\|^2_{\mathcal{V}^{\circ\prime}}\\&+ \|\partial_t U_d(t)\|^2 + \|\partial_t H_d(t)\|^2.\end{aligned}$$

Owing to (1.9), it is straightforward to see that E_d satisfies

$$\frac{1}{2}(\|z^d(t)\|^2_{\mathcal{H}^1} + \|\partial_t z^d(t)\|^2_{\mathcal{H}}) \leq E_d(t) \leq c(\|z^d(t)\|^2_{\mathcal{H}^1} + \|\partial_t z^d(t)\|^2_{\mathcal{H}}),$$

together with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{dt}}E_d + \delta\|z^d\|^2_{\mathcal{H}^1} + \delta\|\partial_t z^d\|^2_{\mathcal{H}^1} \leq 0$$

for some $\delta > 0$. Hence, the Gronwall lemma gives the decay

$$\|z^d(t)\|^2_{\mathcal{H}^1} + \|\partial_t z^d(t)\|^2_{\mathcal{H}} \leq c\|z_0\|^2_{\mathcal{H}^2}e^{-\delta t}$$

and a subsequent integration over $(t, t+1)$ proves (4.4). The final decay estimate in $\mathcal{H}^2$ is read from the elliptic problem included in the system, arguing as in the final part of Lemma 3.3.

We now turn our attention to the smoothing part z^c. We preliminarily observe that, by Lemma 3.3 and Step I, we have, in particular,

$$\|z^c(t)\|^2_{\mathcal{H}^1} + \|\partial_t z^c(t)\|^2_{\mathcal{H}} + \int_t^{t+1}\|\partial_t z^c(s)\|^2_{\mathcal{H}^1}ds \leq c. \tag{4.5}$$

Step II. We are going to prove that

$$\|z^c(t)\|_{\mathcal{H}^3} \leq c.$$

To this aim, due to (4.5), it is enough to prove uniform estimates for $|\Delta u_c|$ and $|\Delta\Theta_c|$, which will follow by controlling the $\mathcal{H}^1$-norm of $\partial_t z^c$.

We differentiate (4.2) with respect to time,

$$\begin{cases}\partial_{tt}u_c - \Delta\partial_t u_c + f'(u)\partial_t u = \partial_t\vartheta_c \quad \text{in} \quad \Omega\\ \partial_{tt}\psi_c - \Delta_\Gamma\partial_t\psi_c + \partial_t\psi_c + g_0'(\psi)\partial_t\psi + \partial_{\boldsymbol{n}}\partial_t u_c = \partial_t\zeta_c \quad \text{on} \quad \Gamma\\ \partial_{tt}(\vartheta_c + u_c) - \Delta\partial_t\vartheta_c = 0 \quad \text{in} \quad \Omega\\ \partial_{tt}(\zeta_c + \psi_c) - \Delta_\Gamma\partial_t\zeta_c + \partial_{\boldsymbol{n}}\partial_t\vartheta_c = 0 \quad \text{on} \quad \Gamma.\end{cases} \tag{4.6}$$

A multiplication of the first two equations by $\partial_{tt}U_c$ and the third and fourth ones by $\partial_t\Theta_c + \partial_{tt}\Theta_c$ leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}[|\partial_t U_c|^2 + \|\partial_t\psi_c\|^2_\Gamma + \|\partial_t\Theta_c\|^2_{H^1(\overline{\Omega})}] + 2\|\partial_{tt}U_c\|^2 + 2\|\partial_{tt}\Theta_c\|^2 + 2|\partial_t\Theta_c|^2 \\
&= -2\langle f'(u)\partial_t u, \partial_{tt}u_c\rangle_\Omega - 2\langle g_0'(\psi)\partial_t\psi, \partial_{tt}\psi_c\rangle_\Gamma - 2\langle\partial_{tt}U_c, \partial_{tt}\Theta_c\rangle \\
&\leq 2(\|\partial_{tt}U_c\|^2 + \|\partial_{tt}\Theta_c\|^2) + \|f'(u)\partial_t u\|^2_\Omega + \|g_0'(\psi)\partial_t\psi\|^2_\Gamma \\
&\leq 2(\|\partial_{tt}U_c\|^2 + \|\partial_{tt}\Theta_c\|^2) + c,
\end{aligned}$$

thanks to the uniform estimate (3.10) provided by Lemma 3.3. Notice that, by (4.5), we can complete the norms of $(\partial_t U_c, \partial_t\Theta_c)$ in order to have

$$\frac{\mathrm{d}}{\mathrm{dt}}[|\partial_t U_c|^2 + \|\partial_t\psi_c\|^2_\Gamma + \|\partial_t\Theta_c\|^2_{H^1(\overline{\Omega})}] + \|\partial_t z^c\|^2_{\mathcal{H}^1} \leq h,$$

where $h(t) := c + \|\partial_t U_c(t)\|^2_{H^1(\overline{\Omega})} + \|\partial_t\Theta_c(t)\|^2$ satisfies $\int_t^{t+1} h(y)dy \leq c$. Besides, there hold

$$\partial_t U_c(0) = -\partial_t\Theta_c(0) = (-f(u_0) + m(\Theta_0), -m(U_0) - g_0(\psi_0) + m(\Theta_0))$$

and, abusing the notation,

$$\nabla\partial_t U_c(0) = -\nabla\partial_t\Theta_c(0) = (-f'(u_0)\nabla u_0, -g_0'(\psi_0)\nabla_\Gamma\psi_0).$$

Hence, $\|\partial_t z^c(0)\|^2_{\mathcal{H}^1} \leq c$, which, together with (1.9) and the uniform Gronwall lemma (see, e.g., [14]), entails

$$\|\partial_t z^c(t)\|^2_{\mathcal{H}^1} \leq c, \qquad t \geq 0.$$

This concludes the proof. □

5. **Exponential attractors.** We have the following exponential attractor's existence result (see [2]).

Theorem 5.1. *For any $M > 0$, let $\mathcal{B}_M$ be the absorbing ball given by Theorem 3.2 and let $t^\star > 0$ be such that $S(t)\mathcal{B}_M \subset \mathcal{B}_M$, for any $t \geq t^\star$. Assume that the following conditions hold.*

(H1) Setting $S(t^\star) = S$, the map S satisfies, for every $z_1, z_2 \in \mathcal{B}_M$,

$$Sz_1 - Sz_2 = L(z_1, z_2) + K(z_1, z_2),$$

where

$$\|L(z_1, z_2)\|_{\mathcal{H}^2} \leq \kappa\|z_1 - z_2\|_{\mathcal{H}^2}, \quad \text{and} \quad \|K(z_1, z_2)\|_{\mathcal{H}^3} \leq \Lambda\|z_1 - z_2\|_{\mathcal{H}^2},$$

for some $\kappa \in (0, 1/2)$ and some $\Lambda > 0$.

(H2) The map

$$z \mapsto S(t)z : \mathcal{B}_M \to \mathcal{B}_M$$

is Lipschitz continuous on $\mathcal{B}_M$, with a Lipschitz constant independent of $t \in [t^\star, 2t^\star]$. Besides, the map

$$(t, z) \mapsto S(t)z : [t^\star, 2t^\star] \times \mathcal{B}_M \to \mathcal{B}_M$$

is Hölder continuous.

Then, there exists an exponential attractor $\mathcal{E}$ on $\widetilde{\mathcal{B}}_M = \overline{\mathcal{B}}_M^{\mathcal{H}^3}$ which attracts $\widetilde{\mathcal{B}}_M$ exponentially fast.

Our aim is to prove that, suitably fixing $t^\star > 0$, the difference of solutions, departing from any pair of initial data $z_0^i \in \mathcal{B}_M$, can be seen as a sum of a contraction and a smoothing map. For this purpose, we set

$$z_0 = z_0^1 - z_0^2 = (U_0, \Theta_0),$$

and we denote by

$$z(t) = z^1(t) - z^2(t) = (U(t), \Theta(t)) = (u(t), \psi(t), \theta(t), \zeta(t))$$

the difference of the corresponding solutions: this can be decomposed as $z(t) = \widehat{z}^d(t) + \widehat{z}^c(t)$, where $\widehat{z}^d(t)$ solves (4.1) and $\widehat{z}^c(t)$ satisfies (4.2) with $f(u)$ and $g_0(\psi)$ replaced, respectively, by $\ell_1 u$ and $\ell_2 \psi$, the ℓ_i being given by (2.21). Arguing as in Lemma 3.3, we see that

$$\|\widehat{z}^d(t)\|_{\mathcal{H}^2} \leq c e^{-\gamma t} \|z_0\|_{\mathcal{H}^2}.$$

We thus accomplish our purpose if we show that

$$\|\widehat{z}^c(t)\|_{\mathcal{H}^3} \leq c \|z_0\|_{\mathcal{H}^2}, \quad \forall t \in [t^\star, 2t^\star].$$

This can be seen as in Step II of Theorem 4.1, by using Lemma 2.6 instead of Lemma 3.3. Finally, taking $t = t^\star$ large enough, the maps $L(z_1, z_2) = \widehat{z}^d(t^\star)$ and $K(z_1, z_2) = \widehat{z}^c(t^\star)$ satisfy (H1).

Verification of (H2). Notice that, thanks to Lemma 2.6, the Lipschitz continuity with respect to the initial data at any fixed time in $[t^\star, 2t^\star]$ (actually, on any bounded time interval) is known. Thus, we are left to show the Hölder continuity with respect to time in the $\mathcal{H}^2$-norm. Indeed, Lemma 3.3 only implies

$$\|z(t_2) - z(t_1)\|_{\mathcal{H}}^2 \leq (t_2 - t_1) \int_0^{2t^\star} \|\partial_t z(s)\|_{\mathcal{H}}^2 ds \leq c(t_2 - t_1),$$

but, by interpolation, we have,

$$\|z(t_2) - z(t_1)\|_{\mathcal{H}^2} \leq \|z(t_2) - z(t_1)\|_{\mathcal{H}^3}^{2/3} \|z(t_2) - z(t_1)\|_{\mathcal{H}}^{1/3}, \quad 2t^\star \geq t_2 > t_1 \geq t^\star.$$

Therefore, $S(\cdot)z$ is Hölder continuous with exponent $1/3$, provided that

$$\sup_{z \in \mathcal{B}_M} \|S(t)z\|_{\mathcal{H}^3} \leq c, \qquad \forall t \in [t^\star, 2t^\star],$$

which will follow by controlling the $\mathcal{H}^1$-norm of $\partial_t z$.

This can be proved by multiplying the first two equations in (3.6) by $\partial_{tt} U$ and the last two ones by $\partial_{tt} \Theta$ and by summing the resulting equalities,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(|\partial_t U|^2 + \|\partial_t \psi\|_\Gamma^2 + |\partial_t \Theta|^2) + \|\partial_{tt} U\|^2 + \|\partial_{tt}\Theta\|^2 \\
&= \langle \partial_t \Theta, \partial_{tt} U\rangle - \langle \partial_{tt}\Theta, \partial_{tt} U\rangle - \langle f'(u)\partial_t u, \partial_{tt} u\rangle_\Omega - \langle g_0'(\psi)\partial_t \psi, \partial_{tt}\psi\rangle_\Gamma.
\end{aligned}$$

Multiplying the above equation by t, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}[t(|\partial_t U|^2 + \|\partial_t \psi\|_\Gamma^2 + |\partial_t \Theta|^2)] + t\|\partial_{tt} U\|^2 + t\|\partial_{tt}\Theta\|^2 \\
&= t\langle \partial_t \Theta, \partial_{tt} U\rangle - t\langle \partial_{tt}\Theta, \partial_{tt} U\rangle - t\langle f'(u)\partial_t u, \partial_{tt} u\rangle_\Omega - t\langle g_0'(\psi)\partial_t \psi, \partial_{tt}\psi\rangle_\Gamma \\
&\quad + \frac{1}{2}(|\partial_t U|^2 + \|\partial_t \psi\|_\Gamma^2 + |\partial_t \Theta|^2) \\
&\leq t\|\partial_{tt} U\|^2 + t\|\partial_{tt}\Theta\|^2 + \frac{1}{2}(|\partial_t U|^2 + \|\partial_t \psi\|_\Gamma^2 + |\partial_t \Theta|^2) + ct(\|\partial_t \Theta\|^2 + \|\partial_t U\|^2),
\end{aligned}$$

that is,
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}[t(|\partial_t U|^2 + \|\partial_t\psi\|_\Gamma^2 + |\partial_t\Theta|^2)] \le c(1+t)\|\partial_t z(s)\|_{\mathcal{H}^1}^2.$$
Integrating over $(0,t)$ for any $t \in [t^\star, 2t^\star]$, it follows, in view of (1.9) and of Lemma 3.3, that
$$t^\star\|\partial_t z(t)\|_{\mathcal{H}^1}^2 \le c(1+t^\star)\int_0^{2t^\star}\|\partial_t z(s)\|_{\mathcal{H}^1}^2 ds \le ct^{\star 2}\sup_{t\in[0,2t^\star]}\int_t^{t+1}\|\partial_t z(s)\|_{\mathcal{H}^1}^2 ds \le c,$$
which concludes the proof.

Remark 1. In particular, $\mathcal{B}_M$ can be made positively invariant, closed in $\mathcal{H}^3$ and absorbing in $\mathcal{H}^2_M$, by taking the set
$$\overline{\cup_{t\ge t^\star} S(t)\mathcal{B}_M}^{\mathcal{H}^3},$$
which we still call $\mathcal{B}_M$. Thus, the transitivity of the exponential attraction [2], thanks to Lemma 2.6 and the exponential attraction exerted by $\mathcal{E}_M$ and $\mathcal{B}_M$, allows to extend the basin of attraction of $\mathcal{E}_M$ to the whole space $\mathcal{H}^2_M$. Therefore, since the global attractor is the minimal (for the inclusion) compact attracting set, we obtain $\mathcal{A}_M \subset \mathcal{E}_M$, which ensures the boundedness of the fractal dimension of the global attractor.

REFERENCES

[1] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal., **92** (1986), 205–245.

[2] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, *Uniform exponential attractors for a singularly perturbed damped wave equation. Partial differential equations and applications*, Discrete Contin. Dynam. Systems, **10** (2004), 211–238.

[3] H. P. Fischer, P. Maass and W. Dieterich, *Novel surface modes in spinodal decomposition*, Phys. Rev. Letters, **79** (1997), 893–896.

[4] H. P. Fischer, P. Maass and W. Dieterich, *Diverging time and length scales of spinodal decomposition modes in thin flows*, Europhys. Lett., **42** (1998), 49–54.

[5] H. P. Fischer, J. Reinhard, W. Dieterich, J.-F. Gouyet, P. Maass, A. Majhofer and D. Reinel, *Time-dependent density functional theory and the kinetics of lattice gas systems in contact with a wall*, J. Chem. Phys., **108** (1998), 3028–3037.

[6] C. G. Gal, M. Grasselli and A. Miranville, *Robust exponential attractors for singularly perturbed phase-field equations with dynamic boundary conditions*, NoDEA Nonlinear Differential Equations Appl., **15** (2008), 535-556.

[7] C. G. Gal, M. Grasselli and A. Miranville, *Nonisothermal Allen-Cahn equations with coupled dynamic boundary conditions*, in "Nonlinear Phenomena with Energy Dissipation," GAKUTO Internat. Ser. Math. Sci. Appl., **29**, Gakkotōsho, Tokyo, 2008, 117-139.

[8] S. Gatti and A. Miranville, *Asymptotic behavior of a phase-field system with dynamic boundary conditions*, in "Differential Equations: Inverse and Direct Problems," Lecture Notes in Pure and Applied Mathematics, **251**, Chapman & Hall/CRC, Boca Raton, FL, (2006), 149–170.

[9] M. Grasselli, A. Miranville and G. Schimperna, *The Caginalp phase-field system with coupled dynamic boundary conditions and singular potentials*, Discrete Contin. Dynam. Systems, **28** (2010), 67–98.

[10] O. A. Ladyžhenskaja, V. A. Solonnikov and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, AMS, Providence, RI, 1967.

[11] A. Miranville and S. Zelik, *Robust exponential attractors for singularly perturbed phase-field type equations*, Electron. J. Differential Equations, **2002**, 28 pp.

[12] A. Miranville and S. Zelik, *Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions*, Math. Meth. Appl. Sci., **28** (2005), 709–735.

[13] A. Miranville and S. Zelik, *The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions*, Discrete Contin. Dynam. Systems, **28** (2010), 275–310.

[14] R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," 2^{nd} edition, Applied Mathematical Sciences, **68**, Springer-Verlag, New York, 1997.

[15] E. Zeidler, "Nonlinear Functional Analysis and its Applications. Part I. Fixed-Point Theorems," Springer-Verlag, New York, 1986.

Received May 2011; revised July 2011.

E-mail address: monica.conti@polimi.it
E-mail address: stefania.gatti@unimore.it
E-mail address: Alain.Miranville@math.univ-poitiers.fr