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13 November 1997

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PHYSICS LETTERS B

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Physics Letters B 413 (1997) 349–354

# 't Hooft anomaly matching conditions for generalized symmetries in 2D

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Received 9 July 1997

Editor: R. Gatto

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## Abstract

The 't Hooft anomaly matching conditions are a standard tool to study and test non-perturbative issues in quantum field theory. We give a new, simple proof of the anomaly matching conditions in 2D Poincaré invariant theories. We consider the case of invariance under a large class of generalized symmetries, which include abelian and non-abelian internal symmetries, space-time symmetries generated by the stress tensor, and  $W$ -type of symmetries generated by higher spin currents. © 1997 Elsevier Science B.V.

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## 1. Introduction

The importance of the anomaly matching conditions has been recognized by 't Hooft in a classic paper [1] as a non-perturbative tool to put restrictions on the quantum numbers of composite massless particles. Since then the anomaly matching conditions have been used as a major tool to search for preonic models [2], and to analyze strongly coupled  $N = 1$  supersymmetric models [3]. Their power is related to the fact that they constitute an exact, fully non-perturbative result obtained in the framework of quantum field theory. As it is familiar to field theory practitioners, there are not so many exact results known for interacting quantum field theories.

The arguments given by 't Hooft for the anomaly

matching have been analyzed and extended in several papers [4], and by now they stand on a rather solid footing. Nevertheless, it may be useful to study the matching conditions from different perspectives. It is the purpose of this letter to give a new, simple proof of such an important result in the simple context of two dimensional theories. We show that the coefficients related to the anomalies are invariant under the flow generated by the renormalization group (RG). We will employ a method of proof similar to that originally introduced by Zamolodchikov in his study of the  $C$ -theorem [5].

An extension of the  $C$ -theorem to chiral theories was already studied in [6], where it was noted that a certain function was left invariant by the action of the renormalization group, and related to the gravitational anomalies [7]. In this paper we use a similar philosophy, and search for functions that are constant along the RG group trajectories. Then, we observe

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that these functions are related to chiral anomalies. The values of these chiral anomalies are easily identified as a combination of the left and right central charges of the corresponding conformal symmetry algebra.

The physical interpretation is quite clear. The anomaly coefficients are seen to be invariant along the trajectories generated by the beta functions of the theory: the values of the coefficients generated by the microscopic degrees of freedom, visible in the ultraviolet region, must be reproduced by the macroscopic degrees of freedom which describe the infrared physics.

We will proceed as follows. In Section 2 we prove the 't Hooft anomaly matching conditions for the case of internal abelian and non-abelian symmetries. In Section 3 we extend the proof to the case of generalized symmetries generated by spin- $n$  currents. For  $n = 2$  this reproduce the case of the stress tensor. In Section 4 we consider mixed anomalies. Finally, in Section 5 we present our conclusions.

## 2. The case of global internal symmetries

Let us consider the case of a two dimensional quantum field theory invariant under an abelian  $U(1)$  global symmetry. The internal symmetry is generated by a conserved current  $J_\mu$ :  $\partial^\mu J_\mu = 0$ . Using complex coordinates<sup>3</sup>, we denote the independent components of the symmetry current by  $J = J_z$ ,  $\bar{J} = J_{\bar{z}}$ , so that the conservation equation reads

$$\bar{\partial}J + \partial\bar{J} = 0. \tag{2.1}$$

In a Poincaré invariant quantum field theory, the two-point functions of the symmetry current must have the general form

$$\begin{aligned} \langle J(z, \bar{z}) J(0, 0) \rangle &= \frac{K(z\bar{z})}{z^2}, \\ \langle J(z, \bar{z}) \bar{J}(0, 0) \rangle &= \frac{H(z\bar{z})}{z\bar{z}}, \\ \langle \bar{J}(z, \bar{z}) \bar{J}(0, 0) \rangle &= \frac{L(z\bar{z})}{\bar{z}^2}, \end{aligned} \tag{2.2}$$

<sup>3</sup>We work in the euclidean version of the theory by performing a Wick rotation. Nevertheless, we use a language appropriate to the minkowskian theory.

where  $K, H, L$  are undetermined scalar function of the product  $z\bar{z}$ . In fact, translation invariance implies that the two-point correlation functions depend only on the relative distance between the points, and one can use this invariance to fix the coordinate of the second point at the origin of the coordinate system. Lorentz covariance is then used to extract the expected Lorentz transformation properties. Moreover, since conserved currents do not develop anomalous dimensions, the scalar functions  $K, H$  and  $L$  must be dimensionless, since the expected dimensions are already carried by the  $z$  and  $\bar{z}$  dependence factored out. Imposing the conservation Eq. (1) onto (2), one obtains the following relations:

$$\bar{\partial}\left(\frac{K}{z^2}\right) + \partial\left(\frac{H}{z\bar{z}}\right) = 0, \quad \partial\left(\frac{L}{\bar{z}^2}\right) + \bar{\partial}\left(\frac{H}{z\bar{z}}\right) = 0, \tag{2.3}$$

which can be rewritten as

$$r^2 \frac{\partial}{\partial r^2} (K + H) = H, \quad r^2 \frac{\partial}{\partial r^2} (L + H) = H, \tag{2.4}$$

where  $r^2 = z\bar{z}$ . Note that we consider only non-vanishing finite distances,  $r \neq 0$ , so that it is consistent to drop possible contact terms from Eqs. (2) and (3) (i.e. terms proportional to delta functions and derivatives thereof: indeed such terms are generically present in non-trivial quantum field theories). From Eq. (4) it is immediate to recognize that the quantity

$$A \equiv K - L \tag{2.5}$$

satisfies

$$\frac{\partial A}{\partial r^2} = 0. \tag{2.6}$$

Thus, we see that the ‘‘anomaly coefficient’’  $A$  is independent of the distance scale  $r$ . This is the essence of the 't Hooft anomaly matching conditions: the anomaly coefficient  $A$  is constant over the various length scales. To complete the argument, one can relate this constancy to a constancy along the trajectory generated by the renormalization group flow. We use dimensional analysis

$$\left(\mu \frac{\partial}{\partial \mu} - 2r^2 \frac{\partial}{\partial r^2}\right) A = 0, \tag{2.7}$$

and the renormalization group equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial g^i} \right) A = 0, \quad (2.8)$$

where  $\mu$  is the mass scale parametrizing the choice of the renormalization conditions, and  $\beta^i = \beta^i(g^j)$  are the beta functions of the theory. As usual, the beta functions can be integrated to obtain the running coupling constants  $g^i(t)$

$$\frac{d}{dt} g^i(t) = -\beta^i(g^j). \quad (2.9)$$

Now, we can compute the variation of  $A$  along the trajectory generated by the renormalization group flow

$$\frac{d}{dt} A \equiv -\beta^i \frac{\partial}{\partial g^i} A = \mu \frac{\partial}{\partial \mu} A = 2r^2 \frac{\partial}{\partial r^2} A = 0. \quad (2.10)$$

This proves the 't Hooft anomaly matching conditions. We see that the function  $A$  is constant along the trajectories generated by the renormalization group flow: its value at the ultraviolet fixed point must be reproduced by whatever macroscopic degrees of freedom describe the low-energy physics.

Now, let us relate the function  $A$  to the properties of the conformal field theories describing the ultraviolet (UV) and infrared (IR) fixed points which characterize the ending points of the RG trajectory. At these fixed points conformal invariance takes over, and the Eq. (1) split into two independent pieces generating the left and right moving current algebras

$$\begin{aligned} \bar{\partial} J = 0, \quad J(z)J(0) &= \frac{k}{z^2}, \\ \partial \bar{J} = 0, \quad \bar{J}(\bar{z})\bar{J}(0) &= \frac{\bar{k}}{\bar{z}^2}, \end{aligned} \quad (2.11)$$

where  $k$  and  $\bar{k}$  are the central charges for the left and right moving  $U(1)$  currents. From Eq. (5), and taking into account Eq. (10), we can immediately read off the following equalities:

$$A = k_{UV} - \bar{k}_{UV} = k_{IR} - \bar{k}_{IR}, \quad (2.12)$$

where  $(k_{UV}, \bar{k}_{UV})$  and  $(k_{IR}, \bar{k}_{IR})$  denote the left and right central charges of the UV and IR conformal current algebras, respectively. This result justify the

name ‘‘anomaly coefficient’’ for  $A$ . In fact, the conserved current  $J_\mu$  can be coupled to a gauge field  $A_\mu$  in a gauge invariant way only if  $A = 0$ .

The previous analysis is easily extended to the case of an internal simple symmetry group  $G$  generated by conserved currents  $J_\mu^a$ :  $\partial^\mu J_\mu^a = 0$ ,  $a = 1, \dots, \dim G$ . One can use the additional invariance under the group  $G$  to constrain the two-point functions of the symmetry currents

$$\begin{aligned} \langle J^a(z, \bar{z}) J^b(0, 0) \rangle &= \frac{\gamma^{ab} K(z\bar{z})}{z^2}, \\ \langle J^a(z, \bar{z}) \bar{J}^b(0, 0) \rangle &= \frac{\gamma^{ab} H(z\bar{z})}{z\bar{z}}, \\ \langle \bar{J}^a(z, \bar{z}) \bar{J}^b(0, 0) \rangle &= \frac{\gamma^{ab} L(z\bar{z})}{\bar{z}^2}, \end{aligned} \quad (2.13)$$

where  $\gamma^{ab}$  is the Killing metric of the group  $G$ . One can check that the same function defined in (5) is invariant along the RG trajectory. It is related to the difference of the central charges (the ‘‘levels’’) appearing in the Kac-Moody algebras which characterize the UV and IR fixed points. For completeness, we recall that a Kac-Moody algebra is described by the following operator product expansion

$$J^a(z) J^b(0) = \frac{k\gamma^{ab}}{z^2} + \frac{if_c^{ab} J^c(0)}{z}, \quad (2.14)$$

where the central charge  $k$  is called the level, and where  $\gamma^{ab}$  and  $f_c^{ab}$  denote the Killing metric and the structure constants of a simple Lie group  $G$ , respectively.

### 3. The case of generalized symmetries

We consider now a theory which is invariant under symmetries generated by a spin- $n$  conserved current

$$\partial^{\mu_1} W_{\mu_1 \mu_2 \dots \mu_n} = 0, \quad (3.1)$$

where  $W_{\mu_1 \mu_2 \dots \mu_n}$  is a completely symmetric tensor. A class of 2D theories invariant under such generalized space-time symmetries have been first identified by Zamolodchikov [8] as particular examples of conformal field theories with higher spin symmetry currents, generating the so-called  $W$ -algebras. In general, the current in Eq. (15) will generate off-critical space-time symmetries.

We use again complex coordinates, and denote by  $W_p$  the component of (15) with  $p$  holomorphic indices, since then the number of antiholomorphic indices is uniquely fixed to be  $n - p$  (e.g. for  $n = 3$  we denote  $W_3 \equiv W_{zzz}$ ,  $W_2 \equiv W_{zz\bar{z}}$ ,  $W_1 \equiv W_{z\bar{z}\bar{z}}$ ,  $W_0 \equiv W_{\bar{z}\bar{z}\bar{z}}$ ). In this notation the conservation equations read

$$\bar{\partial}W_p + \partial W_{p-1} = 0, \quad p = 1, \dots, n. \tag{3.2}$$

By using Poincaré covariance, the two-point functions of the current are seen to have the following general form:

$$\langle W_p(z, \bar{z})W_q(0,0) \rangle = \frac{F_{p,q}(z\bar{z})}{z^{p+q}\bar{z}^{2n-p-q}}, \quad 0 \leq p, q \leq n. \tag{3.3}$$

The scalar functions  $F_{p,q}$  form a symmetric matrix with  $\frac{(n+1)(n+2)}{2}$  components, but not all of its components are independent, as we shall see later on. At a critical point only the  $F_{n,n}$  and  $F_{0,0}$  components are non-vanishing, and they will be related to the central charges of the corresponding  $W$ -like conformal algebras.

Imposing the conservation equation on the two-point functions, one deduces the following equations:

$$\begin{aligned} r^2 \frac{\partial}{\partial r^2} (F_{p,q} + F_{p-1,q}) \\ = (2n - p - q)F_{p,q} + (p + q - 1)F_{p-1,q}, \end{aligned} \tag{3.4}$$

where we continue to denote  $r^2 = z\bar{z}$ . To search for a constant function, one can start from

$$r^2 \frac{\partial}{\partial r^2} (F_{n,n} + F_{n-1,n}) = (2n - 1)F_{n-1,n} \tag{3.5}$$

and subtract from it a similar equation to eliminate the  $F_{n-1,n}$  dependence on the right hand side. This is achieved by subtracting a term proportional to

$$\begin{aligned} r^2 \frac{\partial}{\partial r^2} (F_{n-1,n} + F_{n-1,n-1}) \\ = F_{n-1,n} + (2n - 2)F_{n-1,n-1}. \end{aligned} \tag{3.6}$$

Then, in a stepwise fashion one tries to reach  $F_{0,0}$  which satisfies

$$r^2 \frac{\partial}{\partial r^2} (F_{0,0} + F_{1,0}) = (2n - 1)F_{1,0}. \tag{3.7}$$

Proceeding this way, one is led to consider the function

$$\begin{aligned} A = \sum_{k=1}^n \left[ \binom{2n-1}{2k-2} (F_{n-k+1,n-k+1} + F_{n-k,n-k+1}) \right. \\ \left. - \binom{2n-1}{2k-1} (F_{n-k,n-k} + F_{n-k+1,n-k}) \right], \end{aligned} \tag{3.8}$$

which indeed satisfies  $\frac{\partial}{\partial r^2} A = 0$ . The anomaly  $A$  can also be written in the following somewhat more compact form

$$\begin{aligned} A = F_{n,n} - F_{0,0} + \sum_{k=1}^{n-1} \frac{n-2k}{n-k} \binom{2n-1}{2k} F_{n-k,n-k} \\ - \sum_{k=1}^n \frac{2n-4k+2}{2n-2k+1} \binom{2n-1}{2k-1} F_{n-k,n-k+1}. \end{aligned} \tag{3.9}$$

At a critical point the only non-vanishing functions will be  $F_{n,n}$  and  $F_{0,0}$ , so that the anomaly reduces to  $A = F_{n,n} - F_{0,0}$ , where  $F_{n,n}$  and  $F_{0,0}$  are related to the left and right central charges appearing in the critical  $W$ -like algebras. For  $n = 2$ , this reproduces the case of the stress tensor  $T_{\mu\nu}$ , and the anomaly matching  $2A = c_{UV} - \bar{c}_{UV} = c_{IR} - \bar{c}_{IR}$  gives the matching of the gravitational anomalies ( $c$  and  $\bar{c}$  denote the left and right Virasoro central charges, respectively).

We must note that for the special case of higher spin symmetry currents, when the fixed point algebras reduces precisely to those discovered by Zamolodchikov and Fateev-Lukyanov, the anomaly matching does not give a new independent condition. In fact, the central charges  $F_{n,n}$  and  $F_{0,0}$  are linearly related by the Jacobi identities to the left and right central charges appearing in the sub-algebra generated by the stress tensor, the Virasoro algebra. The latter is always present since we consider Poincaré invariant theories.

In more general cases, when the current  $W_{\mu_1\mu_2\dots\mu_n}$  reduces at the fixed points to currents generating conformal algebras with independent central charges, or at least with non-linear relations between the various central charges, the constancy of  $A$  gives new independent anomaly matching conditions.

Finally, we note that there are many equivalent ways of writing the anomaly coefficient  $A$  in (23).

By repeated use of the conservation Eq. (16), one derives the identities  $F_{p,q-1} = F_{p-1,q}$ , valid for  $p + q \neq n + 1$ . These identities may be used to cast the anomaly  $A$  in different looking forms.

#### 4. Anomaly matching for mixed anomalies

Whenever there is a central charge appearing in the conformal algebra describing a fixed point of the renormalization group, one can derive a matching conditions for those massive theories connected to this critical point by a RG group trajectory. What one needs to prove is the invariance along the RG trajectory, starting or ending at the given fixed point, of a combination of the left and right central charges. This may be called ‘‘anomaly matching for mixed anomalies’’.

To be concrete, let us consider a particular example of a conformal algebra describing a fixed point theory. We take for the left moving sector

$$\begin{aligned} T(z)T(0) &= \frac{c}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \\ T(z)J(0) &= \frac{c'}{z^3} + \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}, \\ J(z)J(0) &= \frac{k}{z^2}, \end{aligned} \tag{4.1}$$

and a copy of a similar algebra characterized by the central charges  $(\bar{c}, \bar{c}', \bar{k})$  for the right moving sector. This algebra may be considered as the conformal algebra describing the (UV or IR) fixed point of certain massive 2D theories with a conserved stress tensor and a conserved  $U(1)$  current. We have already seen how to derive an anomaly matching for  $c - \bar{c}$  and  $k - \bar{k}$ . Using the same type of procedure described in the previous sections, it is quite easy to derive a matching condition for the mixed anomaly  $c' + \bar{c}'$ . We introduce the general two-point functions

$$\begin{aligned} \langle J(z, \bar{z})T(0,0) \rangle &= \frac{F_1(z\bar{z})}{z^3}, \\ \langle \bar{J}(z, \bar{z})T(0,0) \rangle &= \frac{F_2(z\bar{z})}{z^2\bar{z}}, \end{aligned}$$

$$\langle \bar{J}(z, \bar{z})\Theta(0,0) \rangle = \frac{F_3(z\bar{z})}{z\bar{z}^2},$$

$$\langle J(z, \bar{z})\Theta(0,0) \rangle = \frac{F_4(z\bar{z})}{z^2\bar{z}},$$

$$\langle J(z, \bar{z})\bar{T}(0,0) \rangle = \frac{F_5(z\bar{z})}{z\bar{z}^2},$$

$$\langle \bar{J}(z, \bar{z})\bar{T}(0,0) \rangle = \frac{F_6(z\bar{z})}{\bar{z}^3}, \tag{4.2}$$

where  $\Theta \equiv T_{z\bar{z}}$ . Imposing the conservation equations

$$\bar{\partial}J + \partial\bar{J} = 0, \quad \bar{\partial}T + \partial\Theta = 0, \quad \partial\bar{T} + \bar{\partial}\Theta = 0, \tag{4.3}$$

one can derive

$$\frac{d}{dr^2}A = 0, \quad A \equiv F_1 - F_2 - F_5 + F_6. \tag{4.4}$$

(Note that one can also prove the relations  $F_2 = F_4$  and  $F_3 = F_5$ ). Then, by using the RG equations, one obtains the required anomaly matching conditions

$$\begin{aligned} \frac{d}{dt}A &\equiv -\beta^i \frac{\partial}{\partial g^i}A = 0 \\ \Rightarrow A &= c'_{UV} + \bar{c}'_{UV} = c'_{IR} + \bar{c}'_{IR}. \end{aligned} \tag{4.5}$$

The general case is treated in a similar way: if a central charge  $c$  appears in the operator product expansion of a spin- $m$  and a spin- $n$  symmetry currents, then one can derive an anomaly matching condition for  $A = c - (-1)^{m+n}\bar{c}$ .

#### 5. Conclusions

We have described a simple method for proving the 't Hooft anomaly matching conditions in 2D quantum field theories, and considered theories which may be invariant under a large class of generalized symmetries. We have employed a method of proof similar to that used by Zamolodchikov for obtaining the  $C$ -theorem. We may note that in our case unitarity was not required as it was in the proof of the  $C$ -theorem. Therefore, we see that the anomaly

matching is still valid in non-unitary Poincaré invariant theories. This fact may be of relevance for applications in condensed matter systems (e.g. as in [9]).

Both the  $C$ -theorem and the 't Hooft anomaly matching conditions are quite powerful non-perturbative results obtained in 2D quantum field theories. The extension of the  $C$ -theorem to four dimensions has not been achieved, yet, even though many proposals have been analyzed [10]. Actually, certain tests indicate that such an extension may be valid, at least in supersymmetric theories [11]. On the other hand, the 't Hooft anomaly matching conditions are certainly valid in 4D. Therefore, it may be possible to prove them in a way similar to that described here.

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