

# A note about complexity of lens spaces

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Communicated by Andrew Ranicki

**Abstract.** Within crystallization theory, (Matveev's) complexity of a 3-manifold can be estimated by means of the combinatorial notion of *GM-complexity*. In this paper, we prove that the GM-complexity of any lens space  $L(p, q)$ , with  $p \geq 3$ , is bounded by  $S(p, q) - 3$ , where  $S(p, q)$  denotes the sum of all partial quotients in the expansion of  $\frac{q}{p}$  as a regular continued fraction. The above upper bound had been already established with regard to complexity; its sharpness was conjectured by Matveev himself and has recently been proved for some infinite families of lens spaces by Jaco, Rubinstein and Tillmann. As a consequence, infinite classes of 3-manifolds turn out to exist, where complexity and GM-complexity coincide. Moreover, we present and briefly analyze results arising from crystallization catalogues up to order 32, which prompt us to conjecture, for any lens space  $L(p, q)$  with  $p \geq 3$ , the following relation:  $k(L(p, q)) = 5 + 2c(L(p, q))$ , where  $c(M)$  denotes the complexity of a 3-manifold  $M$  and  $k(M) + 1$  is half the minimum order of a crystallization of  $M$ .

**Keywords.** 3-manifold, complexity, crystallization, 2-bridge knot, 4-plat, lens space.

**2010 Mathematics Subject Classification.** 57Q15, 57M27, 57M25, 57M15.

## 1 Introduction

The complexity  $c(M)$  of a 3-manifold  $M$  was originally defined by Matveev as the minimum number of true vertices among all almost simple spines of  $M$  ([24]).<sup>1</sup>

The 3-sphere, the real projective space, the lens space  $L(3, 1)$  and the spherical bundles  $S^1 \times S^2$  and  $S^1 \tilde{\times} S^2$  have complexity zero. Excluding these examples, it is well known that the complexity of a closed prime 3-manifold  $M$  coincides with the number of tetrahedra in a minimal pseudocomplex triangulating  $M$ . Therefore, it is possible to obtain tables of 3-manifolds for increasing values of their complex-

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<sup>1</sup> Recall that a *spine* of  $M$  is a 2-dimensional subpolyhedron  $P$  of  $\text{Int}(M)$ , such that  $M$  (or  $M$  minus an open 3-ball if  $M$  is closed) collapses to  $P$ ; moreover, a spine  $P$  is *almost simple* if the link of each point of  $P$  can be embedded in the 1-skeleton of a 3-simplex. A point whose link is the whole 1-skeleton of the 3-simplex is called a *true vertex* of  $P$ .

This work was performed under the auspices of the G.N.S.A.G.A. of I.N.d.A.M. (Italy) and financially supported by M.I.U.R. of Italy, University of Modena and Reggio Emilia, funds for selected research topics.

ity, simply by generating all triangulations (resp. spines) with a given number of tetrahedra (resp. true vertices) and by identifying the related manifolds. Up-to-date censuses of closed orientable (resp. non-orientable) 3-manifolds are available at the Web page <http://www.matlas.math.csu.ru/> (resp. in [6, Appendix]).<sup>2</sup>

The complexity of a manifold is generally hard to compute from the theoretical point of view, leaving aside the concrete enumeration of its spines.

As far as lens spaces are concerned, a well-known estimation for complexity exists:  $c(L(p, q)) \leq S(p, q) - 3$ , where  $S(p, q)$  denotes the sum of all partial quotients in the expansion of  $\frac{q}{p}$  as a regular continued fraction (see [21, 25]). The sharpness of this estimation was conjectured by Matveev himself ([25, p. 77]); in the last decade Jaco, Rubinstein and Tillmann succeeded to prove that it is sharp for some infinite families of lens spaces, comprehending  $L(2r, 1)$  and  $L(4r, 2r - 1)$ , for all  $r \geq 2$  (see [21, 22]).

In the present paper we approach the problem of computing complexity of lens spaces by making use of *crystallization theory*, i.e. by representing any PL-manifold via suitable edge-coloured graphs, called *crystallizations*: see [1, 2, 19] for a survey on the topic. Within this theory, the invariant *GM-complexity* has been introduced for 3-manifolds: see [8] (resp. [9] and [11]) for the closed non-orientable (resp. orientable) case, and [13] for the extension to 3-manifolds with non-empty boundary.

The GM-complexity of a 3-manifold  $M$  (denoted by  $c_{\text{GM}}(M)$ ) turns out to be an upper bound for the complexity of  $M$ : via GM-complexity, direct estimations of complexity have been obtained for interesting classes of 3-manifolds, both in the closed case (see [9] for 2-fold and 3-fold branched coverings of  $\mathbb{S}^3$ , and for 3-manifolds obtained by Dehn surgery on framed links in  $\mathbb{S}^3$ ) and in the boundary case (see [13] for torus knot complements).

Conjectures about significant upper bounds for infinite families of 3-manifolds have also arisen by the analysis of existing crystallization catalogues, which have been generated up to 30 vertices ([3, 7, 12]).

In Section 3, we present results concerning the extension of the above catalogues to 32 vertices, with regard to the orientable case: as a consequence, the conjectures are proved up to this order. In particular, we remark that

$$c_{\text{GM}}(L(p, q)) = c(L(p, q)) = S(p, q) - 3$$

for any lens space involved in crystallization catalogues.

On the other hand, the main result of the present paper states that  $S(p, q) - 3$  is an upper bound for the GM-complexity of any lens space  $L(p, q)$  with  $p \geq 3$  (Proposition 4.1 (a)).

<sup>2</sup> See the references in [1, Section 1] for subsequent results about the classification of 3-manifolds according to complexity, established by various authors.

The coincidence with the known upper bound for  $c(L(p, q))$  emphasizes the strong relationship between GM-complexity and complexity, which is suggested by the analysis of the crystallization catalogues (see for example [14, Conjecture 7], where equality between the two invariants is conjectured for all closed 3-manifolds).

In particular, lens spaces of type  $L(2r, 1)$  and  $L(4r, 2r - 1)$ , for all  $r \geq 2$ , turn out to be infinite classes of 3-manifolds where complexity and GM-complexity actually coincide (see Corollary 4.2).

Within crystallization theory, a slightly different complexity notion has been introduced and analyzed, too: a closed PL  $n$ -manifold  $M^n$  is said to have *gem-complexity*  $k(M^n) = p - 1$  if  $2p$  is the minimum order of a crystallization representing  $M^n$  (see [7, 10, 23]).

In Section 4 (Proposition 4.1 (b)) we prove that  $k(L(p, q)) \leq 2S(p, q) - 1$  for all  $p \geq 2$ . The sharpness of the above bound for a double infinite family of lens spaces (Corollary 4.4) and for all lens spaces up to complexity 5 suggests the equality to hold in the general case (Conjecture 4.3).

Moreover, we prove that, whenever  $c(L(p, q)) = S(p, q) - 3$ , then

$$k(L(p, q)) \leq 5 + 2c(L(p, q))$$

follows (Proposition 4.8). Actually, in Proposition 4.7, we list some infinite families of lens spaces for which equality holds. Therefore, it is natural to conjecture it for all lens spaces, i.e.  $k(L(p, q)) = 5 + 2c(L(p, q))$  for all  $p \geq 3$  (Conjecture 4.9).

## 2 Crystallizations, gem-complexity and GM-complexity

Edge-coloured graphs are a representation tool for the whole class of piecewise linear (PL) manifolds, without restrictions about dimension, connectedness, orientability or boundary properties. In the present work, however, we will deal only with closed and connected PL-manifolds of dimension  $n = 3$ ; hence, we will briefly review basic notions and results of the theory with respect to this particular case.

A *4-coloured graph (without boundary)* is a pair  $(\Gamma, \gamma)$  with  $\Gamma = (V(\Gamma), E(\Gamma))$  a regular multigraph (i.e. it may include multiple edges, but no loop) of degree four and  $\gamma : E(\Gamma) \rightarrow \Delta_3 = \{0, 1, 2, 3\}$  a proper edge-coloration (i.e. it is injective when restricted to the set of edges incident to any vertex of  $\Gamma$ ).

The elements of the set  $\Delta_3$  are called the *colours* of  $\Gamma$ ; thus, for every  $i \in \Delta_3$ , an  *$i$ -coloured edge* is an element  $e \in E(\Gamma)$  such that  $\gamma(e) = i$ . For every  $i, j \in \Delta_3$  let  $\Gamma_{\hat{i}}$  (resp.  $\Gamma_{ij}$ ) (resp.  $\Gamma_{\hat{i}\hat{j}}$ ) be the subgraph obtained from  $(\Gamma, \gamma)$  by deleting all the edges of colour  $i$  (resp.  $c \in \Delta_3 - \{i, j\}$ ) (resp.  $c \in \{i, j\}$ ). The connected

components of  $\Gamma_{\hat{i}}$  (resp.  $\Gamma_{ij}$ ) (resp.  $\Gamma_{\hat{i}\hat{j}}$ ) are called  $\hat{i}$ -residues (resp.  $\{i, j\}$ -coloured cycles) (resp.  $\{\hat{i}, \hat{j}\}$ -coloured cycles) of  $\Gamma$ , and their number is denoted by  $g_{\hat{i}}$  (resp.  $g_{ij}$ ) (resp.  $g_{\hat{i}\hat{j}}$ ). A 4-coloured graph  $(\Gamma, \gamma)$  is called *contracted* iff, for each  $i \in \Delta_3$ , the subgraph  $\Gamma_{\hat{i}}$  is connected (i.e. iff  $g_{\hat{i}} = 1$  for all  $i \in \Delta_3$ ).

Every 4-coloured graph  $(\Gamma, \gamma)$  may be thought of as the combinatorial visualization of a 3-dimensional labelled pseudocomplex  $K(\Gamma)$ , which is constructed according to the following instructions:

- for each vertex  $v \in V(\Gamma)$ , take a 3-simplex  $\sigma(v)$ , with its vertices labelled 0, 1, 2, 3,
- for each  $j$ -coloured edge between  $v$  and  $w$  ( $v, w \in V(\Gamma)$ ), identify the 2-dimensional faces of  $\sigma(v)$  and  $\sigma(w)$  opposite to the vertex labelled  $j$ , so that equally labelled vertices coincide.

In case  $K(\Gamma)$  triangulates a (closed) PL 3-manifold  $M$ , then  $(\Gamma, \gamma)$  is called a *gem* (gem = graph encoded manifold) *representing*  $M$ .<sup>3</sup>

Finally, a 4-coloured graph representing a (closed) 3-manifold  $M$  is a *crystallization* of  $M$  if it is also a contracted graph; by construction, it is not difficult to check that this is equivalent to requiring that the associated pseudocomplex  $K(\Gamma)$  contains exactly one  $i$ -labelled vertex, for every  $i \in \Delta_3$ . The representation theory of PL-manifolds by edge-coloured graphs is often called *crystallization theory*, since it has been proved that every PL-manifold admits a crystallization: see the Pezzana Theorem and its subsequent improvements ([19] or [2]).

A cellular embedding of a coloured graph into a surface is said to be *regular* if its regions are bounded by the images of bicoloured cycles; interesting results of crystallization theory (mainly related to an  $n$ -dimensional extension of Heegaard genus, called *regular genus* and introduced in [20]) rely on the existence of this type of embeddings for graphs representing manifolds of arbitrary dimension.

As far as the 3-dimensional case is concerned, it is well known that, if  $(\Gamma, \gamma)$  is a crystallization of an orientable (resp. non-orientable) manifold  $M$ , then for every pair  $\alpha, \beta \in \Delta_3$ , there exists a regular embedding  $i_{\alpha, \beta} : \Gamma \rightarrow F_{\alpha\beta}$ , such that  $F_{\alpha\beta}$  is the closed orientable (resp. non-orientable) surface of genus  $g_{\alpha\beta} - 1$ . The minimum genus of  $F_{\alpha\beta}$  taken over all pairs  $\alpha, \beta \in \Delta_3$  is called the *regular genus* of  $(\Gamma, \gamma)$ .

Let now  $\mathcal{D}$  (resp.  $\mathcal{D}'$ ) be an arbitrarily chosen  $\{\alpha, \beta\}$ -coloured (resp.  $\{\hat{\alpha}, \hat{\beta}\}$ -coloured) cycle of a crystallization  $(\Gamma, \gamma)$ ; we denote by  $\mathcal{R}_{\mathcal{D}, \mathcal{D}'}$  the set of regions of  $F_{\alpha\beta} - i_{\alpha, \beta}((\Gamma_{\alpha\beta} - \mathcal{D}) \cup (\Gamma_{\hat{\alpha}\hat{\beta}} - \mathcal{D}'))$ .

<sup>3</sup> The construction of  $K(\Gamma)$  directly ensures that, if  $(\Gamma, \gamma)$  is a gem of  $M$ , then  $M$  is orientable (resp. non-orientable) iff  $\Gamma$  is bipartite (resp. non-bipartite).

**Definition 2.1.** Let  $M$  be a closed 3-manifold, and let  $(\Gamma, \gamma)$  be a crystallization of  $M$ . With the above notations, the *Gem-Matveev complexity* (or *GM-complexity*, for short) of  $\Gamma$  is defined as the non-negative integer

$$c_{\text{GM}}(\Gamma) = \min\{\#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) \mid \mathcal{D} \in \Gamma_{\alpha\beta}, \mathcal{D}' \in \Gamma_{\hat{\alpha}\hat{\beta}}, \\ \Xi \in \mathcal{R}_{\mathcal{D}, \mathcal{D}'}\},$$

while the *Gem-Matveev complexity* (or *GM-complexity*, for short) of  $M$  is defined as

$$c_{\text{GM}}(M) = \min\{c_{\text{GM}}(\Gamma) \mid (\Gamma, \gamma) \text{ crystallization of } M\}.$$

**Remark 2.2.** The notion of GM-complexity can also be extended to non-contracted gems (see [9, Definition 4]); however, it has been proved that, for each closed 3-manifold  $M$ , the minimum value of GM-complexity is always realized by crystallizations ([14, Proposition 6]). For this reason, in this paper we restrict our attention to GM-complexity of crystallizations.

The following key result, due to [8], justifies the choice of terminology:

**Proposition 2.3.** *For every closed 3-manifold  $M$ , GM-complexity gives an upper bound for complexity of  $M$ :*

$$c(M) \leq c_{\text{GM}}(M).$$

Within crystallization theory, another quite natural notion of complexity has been introduced ([7, 23]):

**Definition 2.4.** For each closed PL  $n$ -manifold  $M^n$ , *gem-complexity* is defined as the non-negative integer

$$k(M^n) = \min\left\{\frac{\#V(\Gamma)}{2} - 1 : (\Gamma, \gamma) \text{ crystallization of } M^n\right\}.$$

As far as dimension 3 is concerned, existing 3-manifold censuses via crystallizations (see Section 3) induce to investigate possible relations between complexity  $c(M)$  and gem-complexity  $k(M)$ . In particular, in [11], the following conjecture is stated:

**Conjecture 2.5.** For any closed orientable 3-manifold  $M$ ,

$$k(M) \leq 5 + 2c(M).$$

### 3 Up-to-date results from crystallization catalogues

The totally combinatorial nature of graphs encoding manifolds makes the theory particularly effective for generating catalogues of PL-manifolds for increasing order of the representing graphs (i.e. for increasing values of  $k(M^n)$ ). The main tool for the algorithmic generation of tables of crystallizations is the *code*, a numerical “string” which completely describes the combinatorial structure of a coloured graph, up to colour-isomorphisms (see [15]); afterwards, suitable moves on gems (see [18]) are applied, to yield a classification procedure which allows to detect crystallizations of the same manifold.

In particular, the generation and classification procedures have been successfully developed in dimension 3, both in the orientable and non-orientable case, allowing the complete topological identification of each involved 3-manifold: see [12,23] (resp. [3]) for censuses of orientable (resp. non-orientable) 3-manifolds up to gem-complexity 14.

In the orientable case (to which we restrict our attention in the present paper), the obtained results allow to state:

**Proposition 3.1.** *There are exactly 110 closed prime orientable 3-manifolds, up to gem-complexity 14.*

Details about the quoted catalogues are available at the Web page <http://cdm.unimo.it/home/matematica/casali.mariarita/CATALOGUES.htm>.

In particular, Table 1, Table 2 and Table 3 of the above WEB page contain the JSJ-decomposition of each manifold up to gem-complexity 14 and an analysis of the distribution of these manifolds with respect to complexity and geometry.

By optimizing the program code and by exploiting high-powered computers<sup>4</sup>, we succeeded in extending the above crystallization catalogue to 3-manifolds with  $k(M) = 15$ .<sup>5</sup>

The classification of the involved manifolds has been performed by means of the powerful computer program “3-Manifold Recognizer”.<sup>6</sup>

The analysis of the catalogue yields the following result, concerning the geometries of the involved manifolds and the comparison between  $k(M)$  and  $c(M)$ .

<sup>4</sup> In particular, we made use of CINECA facilities, such as an IBM SP Power6 system for high-performance computing, which are available in virtue of some established Italian Supercomputing Resource Allocation (ISCR) projects.

<sup>5</sup> The same task has independently been achieved by Tarkaev and Fominykh, too: see [25, p. 367].

<sup>6</sup> “3-Manifold Recognizer” has been written by V. Tarkaev as an application of the results about recognition of 3-manifolds obtained by S. Matveev and his research group. It is available on the Web: <http://www.matlas.math.csu.ru/>.

**Proposition 3.2.** *There are exactly 110 closed prime orientable 3-manifolds with gem-complexity 15. They are:*

- *thirty-eight elliptic 3-manifolds (in particular: twenty lens spaces, which are exactly those with complexity five, eighteen other elliptic, of which eleven with complexity six<sup>7</sup> and seven with complexity seven),*
- *four Nil Seifert 3-manifolds (one with complexity seven, one with complexity eight and two with complexity nine),*
- *ten 3-manifolds with Sol geometry (one with complexity seven<sup>8</sup>, three with complexity eight and six with complexity nine),*
- *two 3-manifolds with  $\mathbb{H}^2 \times \mathbb{R}$  geometry (which are exactly those with complexity eight),*
- *thirty-four 3-manifolds with  $\widetilde{\text{SL}}_2(\mathbb{R})$  geometry (six with complexity seven, nineteen with complexity eight and nine with complexity nine),*
- *seventeen non-geometric 3-manifolds (five with complexity eight and twelve with complexity nine),*
- *five hyperbolic 3-manifolds (one with complexity nine, three with complexity ten and one with complexity eleven).<sup>9</sup>*

Apart from  $\mathbb{S}^3$ ,  $L(2, 1)$ ,  $L(3, 1)$  and  $\mathbb{S}^2 \times \mathbb{S}^1$  (i.e. the ones with complexity zero), the distribution of prime 3-manifolds with  $k(M) \leq 15$  with respect to complexity and geometry is summarized in Table 1 (where the symbol  $x/n$  means that  $x$  3-manifolds appear in the catalogue among the  $n$  ones having the corresponding complexity and geometry, and bold character is used to indicate the presence of all manifolds of the considered type).

It is worthwhile to note that this new table emphasizes an idea already suggested by the cases up to gem-complexity 14: for any fixed  $c$ , subsequent crystallization catalogues, for increasing order, appear to cover first the most “complicated” types of complexity  $c$  3-manifolds and then the simplest ones, such as lens spaces (see the above quoted Table 3, containing also details about the cases  $c \leq 4$ ).

On the other hand 3-manifold censuses via crystallizations suggest that “restricted” gem-complexity implies “restricted” complexity, and viceversa: see for instance [7, Section 5], [8, Remark 1] and [11, Propositions 5 and 6]). In particu-

<sup>7</sup> They are exactly the missing ones with that complexity in the crystallization catalogue for  $k \leq 14$ .

<sup>8</sup> It is exactly the missing one with that complexity in the crystallization catalogue for  $k \leq 14$ .

<sup>9</sup> They are all Dehn fillings of the complement of  $6_1^3$  (i.e. the chain link with three components in  $\mathbb{S}^3$ ).

complexity	$\leq 4$	5	6	7	8	9	10	11
lens	<b>21/21</b>	<b>20/20</b>	0/36	0/72	0/136	0/272	0/528	0/1056
other elliptic	<b>6/6</b>	<b>11/11</b>	<b>25/25</b>	7/45	0/78	0/142	0/270	0/526
$\mathbb{E}^3$	-	-	<b>6/6</b>	-	-	-	-	-
Nil	-	-	<b>7/7</b>	4/10	3/14	2/15	0/15	0/15
$\widetilde{\text{SL}}_2(\mathbb{R})$	-	-	-	19/39	24/162	11/513	0/1416	0/3696
Sol	-	-	-	<b>5/5</b>	5/9	6/23	0/39	0/83
non-geometric	-	-	-	<b>4/4</b>	6/35	14/185	0/777	0/2921
$\mathbb{H}^2 \times \mathbb{R}$	-	-	-	-	<b>2/2</b>	-	0/8	0/4
hyperbolic	-	-	-	-	-	3/4	4/25	1/120
TOTAL	<b>27/27</b>	<b>31/31</b>	38/74	39/175	40/436	36/1154	4/3078	1/8421

Table 1. Prime 3-manifolds with complexity  $c \neq 0$  and gem-complexity  $k \leq 15$ .

lar, if we improve the known results in the orientable case with the ones obtained from the analysis of the catalogue of order 32 bipartite crystallizations, we can state:<sup>10</sup>

**Proposition 3.3.** *Let  $M$  be a closed orientable prime 3-manifold.*

- (a) *If  $c(M) \leq 6$  and  $M \neq L(p, q)$ , then  $k(M) < 5 + 2c(M)$ .*
- (b) *If  $M = L(p, q)$  and  $1 \leq c(M) \leq 5$ , then  $k(M) = 5 + 2c(M)$ .*

## 4 Results about lens spaces

We are now able to state our main result:

**Proposition 4.1** (Main Theorem). *Let  $L(p, q)$  be a lens space. Then:*

- (a)  *$c_{\text{GM}}(L(p, q)) \leq S(p, q) - 3$ , for all  $p \geq 3$ ,*
- (b)  *$k(L(p, q)) \leq 2 \cdot S(p, q) - 1$ , for all  $p \geq 2$ .*

*Moreover, if  $c(L(p, q)) \leq 5$ , then*

$$k(L(p, q)) = 2 \cdot S(p, q) - 1 \quad \text{and} \quad c_{\text{GM}}(L(p, q)) = c(L(p, q)) = S(p, q) - 3.$$

*Proof.* Without loss of generality, we may assume that  $p, q \in \mathbb{Z}$  are coprime with  $1 \leq q \leq \frac{p}{2}$ . Let  $b(p, q)$  be the 2-bridge knot or link, with numbers  $p$  and  $q$  in Schubert's normal form, so that  $L(p, q)$  is the 2-fold covering of  $S^3$  branched over  $b(p, q)$  (see [5, Proposition 12.3]).

<sup>10</sup> Note that the results of Proposition 3.3 strengthen Conjecture 2.5.

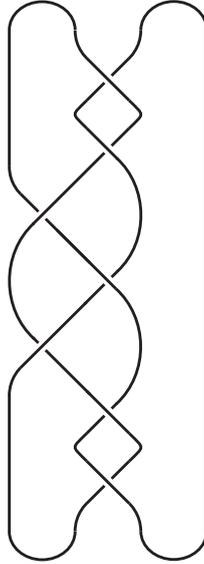


Figure 1. A 4-plat presentation of  $b(21, 8)$ .

According to [5, Proposition 12.13],  $b(p, q)$  admits a presentation  $\bar{P}$  as a 4-plat with a defining braid

$$\bar{3} = \sigma_2^{a_1} \cdot \sigma_1^{-a_2} \cdots \sigma_1^{-a_{m-1}} \cdot \sigma_2^{a_m},$$

with  $a_i > 0$  for all  $i \in \{1, \dots, m\}$  and  $m$  odd, where the  $a_i$  are the quotients of the continued fraction  $[a_1, \dots, a_m] = \frac{q}{p}$ . In Figure 1 the presentation  $\bar{P}$  is shown, for the case  $p = 21, q = 8$ : in fact,  $\frac{8}{21} = [2, 1, 1, 1, 2]$ .

Note that  $\bar{P}$  admits  $\sum_{i=1}^m a_i = S(p, q)$  crossings; for sake of notational simplicity, we set  $S(p, q) = \bar{s}$ .

Let  $(\bar{\Gamma}, \bar{\gamma}) = F(\bar{P})$  be the crystallization obtained by means of Ferri's construction of crystallizations of 2-fold branched coverings (see [17]), applied to  $\bar{P}$  considered as a bridge presentation of  $b(p, q)$ , with a bridge for each crossing point.<sup>11</sup>

The crystallization  $(\bar{\Gamma}, \bar{\gamma})$  may be easily “drawn over”  $\bar{P}$ , as it can be seen in Figure 2 for the case  $p = 21, q = 8$ , starting from the bridge presentation of the 4-plat presentation of  $b(21, 8)$  depicted in Figure 1.

<sup>11</sup> Crystallizations arising from Ferri's construction are said to be 2-symmetric (see [16]) since an involution exists on their edges, which exchanges colour 0 (resp. 2) with colour 1 (resp. 3): this involution can be thought of as the planar symmetry whose axis contains all the bridges of the given knot or link presentation.

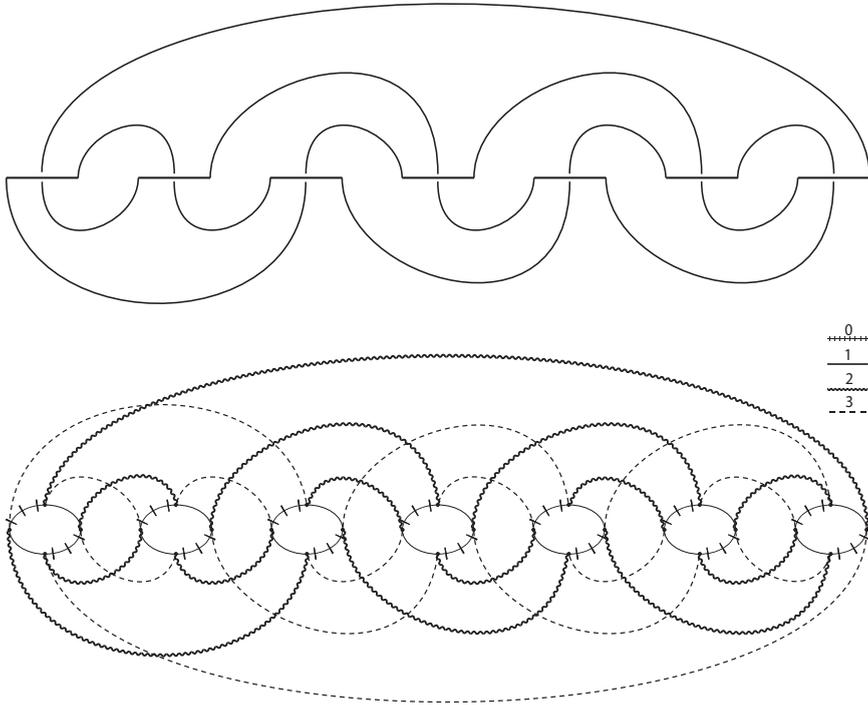


Figure 2. A bridge presentation of  $b(21, 8)$  and the associated crystallization of the lens space  $L(21, 8)$ .

In this way, each  $\{0, 1\}$ -coloured cycle of  $\bar{\Gamma}$  arises from a bridge and has length four; hence,  $(\bar{\Gamma}, \bar{\gamma})$  turns out to have  $4\bar{s}$  vertices, which directly proves part (b) of the statement.

Suppose now  $p \geq 3$ , i.e.  $\bar{s} \geq 3$ . Let us label the crossings of  $\bar{P}$  according to the braid sequence  $\sigma_2^{a_1} \cdot \sigma_1^{-a_2} \dots \sigma_1^{-a_{m-1}} \cdot \sigma_2^{a_m}$  and let  $\pi$  be the “ideal” axis containing all bridges of  $\bar{P}$  with the orientation induced by the sequence  $a_1, \dots, a_m$ . For each crossing  $c_j$  ( $1 \leq j \leq \bar{s}$ ), we label  $v_{j,1}, v_{j,2}, v_{j,3}, v_{j,4}$  the vertices of its associated  $\{0, 1\}$ -coloured cycle, so that  $v_{j,2}$  and  $v_{j,4}$  belong to  $\pi$  and the sequence  $v_{1,2}, v_{1,4}, v_{2,2}, v_{2,4}, \dots, v_{\bar{s},2}, v_{\bar{s},4}$  is consistent with the fixed orientation of  $\pi$ . Moreover,  $v_{j,1}, v_{j,2}$  and  $v_{j,3}, v_{j,4}$  (resp.  $v_{j,2}, v_{j,3}$  and  $v_{j,4}, v_{j,1}$ ) are 0-adjacent (resp. 1-adjacent), for each  $j \in 1, \dots, \bar{s}$ . The 2-coloured edges follow the arcs of  $\bar{P}$ , while 3-coloured edges are obtained from arcs of  $\bar{P}$  via symmetry with respect to the axis  $\pi$  (see again Figure 2).

As a consequence we have that, for each  $j, k \in \{1, \dots, \bar{s}\}$  and each  $i \in \{2, 4\}$ , the vertices  $v_{j,i}$  and  $v_{k,1}$  (resp.  $v_{j,i}$  and  $v_{k,3}$ ) are 3-adjacent iff  $v_{j,i}$  and  $v_{k,3}$  (resp.  $v_{j,i}$  and  $v_{k,1}$ ) are 2-adjacent.

In order to estimate  $c_{GM}(\bar{\Gamma})$  (via Definition 2.1), let us now fix  $\{\alpha, \beta\} = \{0, 2\}$  and  $\{\alpha', \beta'\} = \{1, 3\}$ , and let us consider the  $\{0, 2\}$ -coloured (resp.  $\{1, 3\}$ -coloured) cycle  $\mathcal{D}$  (resp.  $\mathcal{D}'$ ) corresponding to the boundary of the “external” region  $\bar{R}$  of  $\bar{P}$  (resp. obtained via symmetry from the boundary of the “external” region  $\bar{R}$  of  $\bar{P}$ ), and the region  $\Xi \in \mathcal{R}_{\mathcal{D}, \mathcal{D}'}$  induced by the inner region  $\bar{R}'$  of  $\bar{P}$  containing the fourth string of the 4-braid associated to the 4-plat  $\bar{P}$ .

Let us define the following sets of integers:

$$I_1 = \{j \in \{2, \dots, \bar{s} - 1\} \mid c_j \text{ corresponds to a factor } \sigma_1 \text{ in } \mathfrak{z}\} \cup \{\bar{s}\},$$

$$I_2 = \{j \in \{1, \dots, \bar{s} - 1\} \mid c_j \text{ corresponds to a factor } \sigma_2 \text{ in } \mathfrak{z}\}.$$

Since the boundary of  $\bar{R}$  contains the crossing points  $c_j$  with  $j \in I_1 \cup \{1\}$ , it is not difficult to check that  $\mathcal{D} \cup \mathcal{D}'$  consists of the vertices

$$\{v_{j,i} \mid j \in I_1, i = 1, 3, 4\} \cup \{v_{1,1}, v_{1,2}, v_{1,3}\}.$$

On the other hand, the structure of  $\bar{P}$  and the symmetry properties of  $\bar{\Gamma}$ , which arise from Ferri’s construction, allow to check that the region  $\Xi$  can be obtained by the union of the four regions bounded by the following cycles of  $\bar{\Gamma}$ :

- the  $\{1, 2\}$ -coloured cycle corresponding to  $\bar{R}'$ ; its vertex-set is

$$\{v_{j,i} \mid j \in I_2, i = 1, 4\} \cup \{v_{\bar{s},1}, v_{\bar{s},4}\},$$

- the  $\{0, 3\}$ -coloured cycle obtained via symmetry from  $\bar{R}'$ ; its vertex-set is

$$\{v_{j,i} \mid j \in I_2, i = 3, 4\} \cup \{v_{\bar{s},3}, v_{\bar{s},4}\},$$

- the  $\{0, 1\}$ -coloured cycle corresponding to the last crossing  $c_{\bar{s}}$ ,
- the  $\{2, 3\}$ -coloured cycle containing the edge corresponding to the fourth string of the braid  $\mathfrak{z}$  (whose vertices are exactly  $\{v_{1,1}, v_{1,3}, v_{2,2}, v_{\bar{s},4}\}$ , since  $q < \frac{p}{2}$  implies  $a_1 \geq 2$ ).

Hence, the set  $V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)$  contains all vertices  $v_{j,i}$  with  $1 \leq j \leq \bar{s}$ ,  $i \in \{1, 3, 4\}$ , and the vertices  $v_{j,2}$  with  $j \in \{1, 2, \bar{s}\}$ . As a consequence,

$$V(\bar{\Gamma}) - (V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) = \{v_{j,2} \mid 3 \leq j \leq \bar{s} - 1\},$$

which implies  $c_{GM}(\bar{\Gamma}) \leq \bar{s} - 3$ . So, part (a) of the statement directly follows.

Finally, the last part of the statement can be checked by means of the existing crystallization catalogues, up to order 32 (presented in the previous section).  $\square$

Proposition 4.1 (a), together with the results of [21, 22], directly proves the existence of infinite classes of 3-manifolds where complexity and GM-complexity coincide.

**Corollary 4.2.** *The following statements hold:*

- $c_{\text{GM}}(L(2r, 1)) = c(L(2r, 1)) = 2r - 3$ , for each  $r \geq 2$ ,
- $c_{\text{GM}}(L(4r, 2r - 1)) = c(L(4r, 2r - 1)) = r$ , for each  $r \geq 2$ ,
- $c_{\text{GM}}(L((r + 2)(t + 1) + 1, t + 1)) = c(L((r + 2)(t + 1) + 1, t + 1)) = r + t$ , for each  $t > r > 1$ ,  $r$  odd and  $t$  even,
- $c_{\text{GM}}(L((r + 1)(t + 2) + 1, t + 2)) = c(L((r + 1)(t + 2) + 1, t + 2)) = r + t$ , for each  $t > r > 1$ ,  $r$  even and  $t$  odd.  $\square$

With regard to gem-complexity, Proposition 4.1 (b) suggests the following conjecture:

**Conjecture 4.3.** For  $p \geq 2$ ,  $k(L(p, q)) = 2 \cdot S(p, q) - 1$ .

Actually, a recent result by Swartz ([26])<sup>12</sup> proves that, for each  $q, r \geq 1$  odd,  $k(L(qr + 1, q)) = 2(r + q) - 1$ ; since  $S(qr + 1, q) = q + r$  trivially holds for all  $r, q$ , we can state:

**Corollary 4.4.** *Conjecture 4.3 is true for any lens space  $L(qr + 1, q)$ , with  $r, q \geq 1$  odd.*  $\square$

**Remark 4.5.** Note that our Main Theorem yields  $k(L(p, 1)) \leq 2p - 1$ . Actually, in this case, the described crystallization  $(\bar{\Gamma}, \bar{\gamma})$  – which, by [26], realizes gem-complexity of  $L(p, 1)$  in the case of  $p$  even – is the well-known standard crystallization of  $L(p, 1)$ , having  $4p$  vertices and regular genus one. On the contrary, if  $q \neq 1$  is assumed, the standard crystallization of  $L(p, q)$  has regular genus one and  $4p$  vertices, too, but it does not realize minimum order, since it contains *clusters* or, more generally, structures that can be eliminated (see [12, Section 6] or [18, Section 5] for details).

**Remark 4.6.** Proposition 4.1 (b) yields a general result, which admits [4, Theorem 2.6 (v)] and [4, Theorem 2.6 (vi)] as particular cases. Note that, in the cited paper, the authors' approach via fundamental groups allows to rediscover examples of minimal crystallizations already presented in [23] (see <http://cdm.unimo.it/home/matematica/casali.mariarita/Table1.pdf>).

Finally, we can summarize the relation between complexity and gem-complexity of lens spaces as follows.

<sup>12</sup> The preprint [26] was posted in ArXiv (<http://arxiv.org/abs/1310.1991v1>) a few days after we posted the first version of the present paper (<http://arxiv.org/abs/1309.5728v1>). It makes use of part (b) of our Main Theorem in order to prove that  $4(q + r)$  is exactly the minimum order of a crystallization of  $L(qr + 1, q)$ , for all  $r, q \geq 1$  odd.

**Proposition 4.7.** *The following statements hold:*

(a) *Let  $L(p, q)$  be a lens space, with  $p \geq 3$ . Then*

$$k(L(p, q)) = 5 + 2 \cdot c(L(p, q))$$

*whenever:*

(a<sub>1</sub>)  $c(L(p, q)) \leq 5$ ,

(a<sub>2</sub>)  $p$  is even and  $q = 1$ ,

(a<sub>3</sub>)  $p = (r + 2)(t + 1) + 1$  and  $q = t + 1$ , for  $t > r > 1$ ,  $r$  odd and  $t$  even,

(a<sub>4</sub>)  $p = (r + 1)(t + 2) + 1$  and  $q = t + 2$ , for  $t > r > 1$ ,  $r$  even and  $t$  odd.

(b) *For each  $r, q \geq 1$  odd, then*

$$k(L(qr + 1, q)) \geq 5 + 2 \cdot c(L(qr + 1, q)).$$

(c) *For each  $r \geq 2$ , then*

$$k(L(4r, 2r - 1)) \leq 5 + 2 \cdot c(L(4r, 2r - 1)).$$

*Proof.* Part (a<sub>1</sub>), already stated in Proposition 4.1 (a), directly follows from the analysis of crystallization catalogues up to gem-complexity 15.

In order to prove part (a<sub>2</sub>) (resp. (a<sub>3</sub>)) (resp. (a<sub>4</sub>)), the equality

$$k(L(qr + 1, q)) = 2(r + q) - 1$$

(which holds for all  $r, q \geq 1$  odd: see Corollary 4.4) has to be applied to the involved infinite class of lens spaces, together with [21, Theorem 1] (resp. [21, Theorem 2]) (resp. [21, Theorem 3]).

Part (b) follows from the equality  $k(L(qr + 1, q)) = 2(r + q) - 1$ , for all  $r, q \geq 1$  odd, too, by making use of the well-known general estimation

$$c(L(p, q)) \leq S(p, q) - 3.$$

Part (c) is a direct consequence of Proposition 4.1 (b) and [22, Corollary 3].  $\square$

Moreover, we can state

**Proposition 4.8.** *Assuming Matveev's conjecture  $c(L(p, q)) = S(p, q) - 3$  to be true, then*

$$k(L(p, q)) \leq 5 + 2 \cdot c(L(p, q)) \quad \text{for all } p \geq 2.$$

*In particular,  $k(L(p, q)) \leq 5 + 2 \cdot c(L(p, q))$  for each lens space  $L(p, q)$  with  $6 \leq c(L(p, q)) \leq 12$  and equality holds if  $p = qr + 1$ , with  $q, r$  odd.*

*Proof.* If  $c(L(p, q)) = S(p, q) - 3$  is assumed for all  $p \geq 3$ , then

$$k(L(p, q)) \leq 2 \cdot S(p, q) - 1 = 5 + 2 \cdot c(L(p, q))$$

trivially follows from part (b) of our Main Theorem.

In case  $6 \leq c(L(p, q)) \leq 12$ , condition  $c(L(p, q)) = S(p, q) - 3$  is always satisfied (see <http://www.matlas.math.csu.ru/>) and, therefore,

$$k(L(p, q)) \leq 5 + 2 \cdot c(L(p, q)).$$

On the other hand, if, further,  $p = qr + 1$  ( $q, r$  odd), then Proposition 4.7 (b) applies to ensure equality.  $\square$

The above Propositions 4.7 and 4.8 naturally suggest the following:

**Conjecture 4.9.** For all  $p \geq 3$ ,  $k(L(p, q)) = 5 + 2 \cdot c(L(p, q))$ .

**Remark 4.10.** A slightly different notion of GM-complexity for closed 3-manifolds was originally defined by taking into account only the values of  $c_{\text{GM}}(\Gamma)$  when  $(\Gamma, \gamma)$  is a *minimal* crystallization with respect to the number of vertices: see [8, Definition 3]. As regards lens spaces, existing crystallization catalogues prove that this restricted notion actually coincides with the one including *all* crystallizations. Moreover, the arguments of the present paper induce to *conjecture* – according to the above Conjectures 4.3 and 4.9 – *that complexity of any lens space is always realized by GM-complexity of a minimal crystallization* (despite what happens in the general case: see [11, Proposition 7 and Remark 4]).

**Acknowledgments.** We wish to thank E. Fominykh for the helpful discussions which originated this paper. We are also indebted to him, to S. Matveev and V. Tarkaev for the use of “3-Manifold Recognizer”.

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Received November 23, 2013.

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