

# Effective action for higher spin fields on (A)dS backgrounds

Fiorenzo Bastianelli,<sup>a</sup> Roberto Bonezzi,<sup>a</sup> Olindo Corradini<sup>b,c</sup> and Emanuele Latini<sup>d,e</sup>

<sup>a</sup>*Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy*

<sup>b</sup>*Centro de Estudios en Física y Matemáticas Básicas y Aplicadas, Universidad Autónoma de Chiapas, Tuxtla Gutiérrez 29050, Mexico*

<sup>c</sup>*Dipartimento di Fisica, Università di Modena e Reggio Emilia, Via Campi 213/A, I-41125 Modena, Italy*

<sup>d</sup>*Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

<sup>e</sup>*INFN, Laboratori Nazionali di Frascati, CP 13, I-00044 Frascati, Italy*

*E-mail:* [bastianelli@bo.infn.it](mailto:bastianelli@bo.infn.it), [bonezzi@bo.infn.it](mailto:bonezzi@bo.infn.it),  
[olindo.corradini@unach.mx](mailto:olindo.corradini@unach.mx), [emanuele.latini@math.uzh.ch](mailto:emanuele.latini@math.uzh.ch)

**ABSTRACT:** We study the one loop effective action for a class of higher spin fields by using a first-quantized description. The latter is obtained by considering spinning particles, characterized by an extended local supersymmetry on the worldline, that can propagate consistently on conformally flat spaces. The gauge fixing procedure for calculating the worldline path integral on a loop is delicate, as the gauge algebra contains nontrivial structure functions. Restricting the analysis on (A)dS backgrounds simplifies the gauge fixing procedure, and allows us to produce a useful representation of the one loop effective action. In particular, we extract the first few heat kernel coefficients for arbitrary even spacetime dimension  $D$  and for spin  $S$  identified by a curvature tensor with the symmetries of a rectangular Young tableau of  $D/2$  rows and  $[S]$  columns.

**KEYWORDS:** Extended Supersymmetry, Field Theories in Higher Dimensions, Sigma Models

ARXIV EPRINT: [1210.4649](https://arxiv.org/abs/1210.4649)

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**1 Introduction**

Higher spin field theory is a topic that enters several aspects of modern theoretical physics. In this paper we quantize higher spin fields on (A)dS spaces using a worldline approach and study their one loop effective action, extending the analysis of [1] that was restricted to flat spacetimes.

The worldline approach to quantum field theory (see [2] for a review), has been known to be an alternative tool to compute Feynman diagrams through the quantization of relativistic point particles. More specifically, one loop effective actions in the presence of external fields find an efficient approach in terms of point particle path integrals computed on the circle, whereas field theory propagators are linked to particle path integrals on the line. In particular, for relativistic higher spin fields (see [3–8] for reviews) the particle approach might be particularly useful to extract information beyond the classical level. It is the main objective of the present manuscript to use a particle approach to compute the one loop effective action for higher spin fields in curved space. Indeed, extensions of the worldline approach to field theories with background gravity are feasible, as discussed for example in [9–14].

The class of higher spin particles that we wish to treat here are those described by the  $O(N)$  spinning particles actions [15–18], that contain a fully-gauged extended supersymmetry on the worldline. These models describe in first quantization higher spin fields that enjoy conformal invariance in flat spacetimes [19–21]. They form the complete set

in  $D = 4$ , and for spin  $S > 1$  they live only in even space-time dimensions. In [22] the conformal invariance was proven by showing that these particle models have classical background reparametrization and Weyl invariance, thus leaving the conformal Killing vectors as generators of true symmetries. This result also implies that these models are consistent on generic conformally flat spaces. The particular coupling to (A)dS spaces was previously known from the work of [23]. The class of higher spin fields treated here can be described by higher spin curvature tensors that obey the symmetries of a Young tableau of  $D/2$  rows and  $[S]$  columns (see [24] for a discussion of the curvature tensors for half-integer spin). More general types of higher spin fields could perhaps be described by using the detour worldline methods of [25–27].

The gauge structure of our particle models on generic conformally flat spaces is quite complex, as it contains non-trivial structure functions [22]. We find it simpler, for the moment being, to investigate the one loop effective action on maximally symmetric spaces, i.e. (A)dS spaces, which allow for an algebraically simpler gauge fixing procedure. Weyl anomalies are generically present in quantum field theories, so that we expect to find a nontrivial effective action, as indeed we do.

One may also approach the problem directly in quantum field theory, as suitable actions are known, see for example [28–36]. However we wish to suggest here the point of view that many results are more efficiently obtained using first quantized methods.<sup>1</sup> Recently the heat kernel for some higher spin fields in (mostly) odd-dimensional maximally symmetric spaces were computed using a group-theoretical approach [42–44]. Our approach deals with a different set of multiplets on even-dimensional spaces. It would be useful to eventually compare the two approaches. Also, a different type of effective action with higher spin backgrounds was studied in [45].

In subsequent sections we first present the gauge fixing of the models under study, then briefly review the regularization techniques needed to compute worldline path integrals in curved spaces. Finally we present the derivation of the worldline representation of the effective action. It is generically difficult to compute it in a closed form, so we aim here to calculate explicitly only the first few heat kernel coefficients for (A)dS backgrounds. For  $D > 2$  these correspond to diverging terms that must be subtracted to renormalize the effective action. We perform the path integral computation with an arbitrary metric, as intermediate calculations might be useful for a larger class of backgrounds. Indeed, as mentioned above, these spinning particles are certainly consistent on conformally flat spaces. However, in that case the gauge fixing procedure is much more laborious and will not be attempted here.

The present analysis could be repeated step by step to carry out similar calculations for the  $U(N)$  spinning particle [46], which gives rise to higher spin fields living on complex spaces [47] (treated already for the particular cases of  $N=1,2$  on arbitrary Kahler manifolds in [48, 49]).

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<sup>1</sup>A worldline approach to quantum massive higher spins in (A)dS [37–41] can be treated along similar lines by dimensionally reducing the  $O(N)$  spinning particle used here.

To conclude, the main results derived here are a worldline representation of the one loop effective action for a class of higher spin fields on (A)dS spaces, see eq. (2.27), and the calculations of the first few heat kernel coefficients, see eqs. (3.4), (3.5) for integer spin and eqs. (3.11), (3.12) for half-integer spin.

## 2 Spinning particle on conformally flat spaces

The model we study here is the (fully) gauged counterpart of the mechanical model with action

$$S = \int_0^1 dt \left( p_\mu \dot{x}^\mu + \frac{i}{2} \psi_i^a \dot{\psi}_{ia} - \frac{1}{2} p_\mu p^\mu \right), \quad i = 1, \dots, N \quad (2.1)$$

where a set of global worldline symmetries (time translation,  $N$  supersymmetries,  $O(N)$   $R$ -symmetry) are rendered local to guarantee unitarity. Here  $x^\mu$  and  $p_\mu$  are spacetime coordinates and momenta of the particle, whereas  $\psi_i^a$  are Majorana fermions, with  $a$  a flat Lorentz index. The resulting phase-space action identifies the so-called  $O(N)$  spinning particle model and, when considering a curved target space, reads

$$S[x, p, \psi, E; g] = \int_0^1 dt \left[ p_\mu \dot{x}^\mu + \frac{i}{2} \psi_i^a \dot{\psi}_{ia} - eH - i\chi_i \underbrace{\pi_\mu e_a^\mu \psi_i^a}_{Q_i} - \frac{1}{2} a_{ij} \underbrace{i\psi_i \cdot \psi_j}_{J_{ij}} \right] \quad (2.2)$$

with  $H = H_0 - \frac{1}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d$  and  $H_0 = \frac{1}{2} g^{\mu\nu} \pi_\mu \pi_\nu$  being the kinetic hamiltonian written in terms of the covariant momenta  $\pi_\mu = p_\mu - \frac{i}{2} \omega_{\mu ab} \psi_i^a \psi_i^b$ . From (2.2) one recognizes the supercharges  $Q_i$  and the  $O(N)$  symmetry generators  $J_{ij}$ .  $E$  collectively denotes the worldline gauge fields  $E = (e, \chi_i, a_{ij})$ , i.e. einbein, gravitini and  $O(N)$  gauge fields respectively. This model describes the first quantization of a particular mixed-symmetry higher spin particle in  $D = 2d$  even-dimensional curved space, that generically (for  $N > 2$ ) must be conformally flat. The spectrum of the model for  $N > 2$  is empty in odd dimensions [18]. For even  $N = 2n$  the model describes equations of motion (the Dirac constraints) for a bosonic field strength characterized by a rectangular Young tableau with  $n$  columns and  $d$  rows. For odd  $N = 2n + 1$  the model describes equations of motion for a fermionic field strength, a spinor-tensor with a tensor structure characterized by the same  $n \times d$  Young tableau. For  $D = 4$  this involves all possible massless representations of the Poincaré group, that at the level of gauge potentials are given by totally symmetric (spinor-) tensors, whereas for  $D > 4$  it corresponds to conformal multiplets only [19–21]. The euclidean configuration space action, that one obtains after integrating out the momenta  $p_\mu$  and Wick rotating, reads

$$S[y, E; g] = \int_0^1 d\tau \left[ \frac{1}{2e} g_{\mu\nu} \left( \dot{x}^\mu - \chi_i \psi_i^\mu \right) \left( \dot{x}^\nu - \chi_j \psi_j^\nu \right) \right. \\ \left. + \frac{1}{2} \psi_i^a \left( \delta_{ij} \delta_{ab} \partial_\tau + \dot{x}^\mu \omega_{\mu ab} \delta_{ij} - a_{ij} \delta_{ab} \right) \psi_j^b - \frac{e}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d \right] \quad (2.3)$$

with  $y = (x^\mu, \psi_i^a)$  being the “matter” fields. For arbitrary  $N$  and generic curved backgrounds the gauge symmetry generators  $(H, Q_i, J_{ij})$  do not form a first class algebra. However in [22] it was found that, if the background is conformally flat, they form a (nonlinear) first-class constraint algebra and the previous action is gauge-invariant under the transformations induced by the gauge symmetry generator  $G = \xi H + i\epsilon_i Q_i + \frac{1}{2}\alpha_{ij} J_{ij} := \Xi^A G_A$ .<sup>2</sup>

At the quantum level the constraint algebra on conformally flat spaces closes as well, provided one adds to the hamiltonian an improvement term proportional to the scalar of curvature, namely

$$H = H_0 - \frac{1}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d - \frac{(N-2)(D+N-2)}{16(D-1)} R \quad (2.4)$$

with the kinetic operator given by

$$\begin{aligned} H_0 &= \frac{1}{2} \left( \pi^a - i\omega_b{}^{ba} \right) \pi_a \\ \pi_a &= e_a^\mu \pi_\mu, \quad \pi_\mu = g^{1/4} p_\mu g^{-1/4} - \frac{i}{2} \omega_{\mu ab} \psi_i^a \psi_i^b. \end{aligned} \quad (2.5)$$

Here we use a path integral formalism and find it more convenient to use the (euclidean) configuration space action

$$\begin{aligned} S[y, E; g] &= \int_0^1 d\tau \left[ \frac{1}{2e} g_{\mu\nu} \left( \dot{x}^\mu - \chi_i \psi_i^\mu \right) \left( \dot{x}^\nu - \chi_j \psi_j^\nu \right) \right. \\ &\quad \left. + \frac{1}{2} \psi_i^a \left( \delta_{ij} \delta_{ab} \partial_\tau + \dot{x}^\mu \omega_{\mu ab} \delta_{ij} - a_{ij} \delta_{ab} \right) \psi_j^b - \frac{e}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d \right. \\ &\quad \left. - e \frac{(N-2)(D+N-2)}{16(D-1)} R \right] \end{aligned} \quad (2.6)$$

that is (2.3) with the addition of the improvement term. The associated path integral evaluated on the circle  $S^1$

$$\Gamma[g] = \int_{S^1} \frac{DE Dy}{\text{Vol}(\text{Gauge})} e^{-S[y, E; g]} \quad (2.7)$$

gives a representation of the one loop effective action for the aforementioned higher spin field coupled to external gravity. It is defined by taking bosonic fields with periodic boundary conditions and fermionic fields with antiperiodic boundary conditions.

In order to be able to perform computations two preliminary issues have to be taken care of:

- i) Firstly, the worldline action must be suitably gauge-fixed; i.e. the gauge fields  $E$  must be fixed to some specific configuration that will depend upon a set of modular parameters that must be integrated over. In the present case the gauge symmetry algebra, associated to the above generators, is nonlinear, i.e. commutators of pairs of generators involve structure *functions* and not structure *constants*. Therefore one must use more powerful hamiltonian BRST methods to gauge fix the action in its hamiltonian form.

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<sup>2</sup>For  $N \leq 2$  the  $R$ -symmetry group is either trivial or abelian, and the algebra closes on an arbitrary background.

ii) The resulting gauge-fixed action depends only upon “matter” fields and modular parameters. However, in curved space, it still is a nonlinear sigma model, so that for perturbative computations one usually Taylor expands the metric about a fixed point of the circle. This results in an infinite set of vertices. In addition some Feynman diagrams present ambiguities and need to be regularized. This is a well-known fact, and several regularization schemes have been used in the past to compute such path integrals; see [60] for an overall review. Up to recently only quantum-mechanical path integrals in curved space with  $N \leq 2$  had been used: these path integrals, in the worldline formalism, correspond to the first quantization of spin  $S \leq 1$  fields in curved space. More recently in [61] the regularization of nonlinear sigma models with arbitrary  $N$  was considered, having in mind applications to the  $O(N)$  spinning particles. What studied in [61] are the globally supersymmetric counterparts of the models studied here. That is enough for the present purposes as the gauging does not introduce additional ambiguities.

### 2.1 Gauge-fixing in (A)dS

In this section we describe the gauge-fixing of the  $O(N)$  spinning particle propagating on (A)dS spaces. For such backgrounds the Riemann curvature can be written as

$$R_{abcd} = b(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \tag{2.8}$$

where  $\Lambda = (D-1)(D-2)b$  is the cosmological constant. Let us start considering the action in hamiltonian form. At the classical level (cfr. (2.2)), in (A)dS spaces the hamiltonian constraint reduces to  $H = H_0 - \frac{b}{4}J_{ij}J_{ij}$  and the first-class algebra reduces to a quadratic algebra (curly brackets here are graded Poisson brackets)

$$\begin{aligned} \{Q_i, Q_j\} &= -2i\delta_{ij}H + ib \left( J_{ik}J_{jk} - \frac{1}{2}\delta_{ij}J_{kl}J_{kl} \right) \\ \{J_{ij}, J_{kl}\} &= \delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk} \\ \{J_{ij}, Q_k\} &= \delta_{jk}Q_i - \delta_{ik}Q_j, \qquad \{H, Q_i\} = \{H, J_{ij}\} = 0 \end{aligned} \tag{2.9}$$

that can be used to obtain the corresponding transformations of the gauge fields.

Upon canonical quantization the latter quadratic algebra turns into the following (anti-)commutation relations

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}H - \frac{b}{2}(J_{ik}J_{jk} + J_{jk}J_{ik} - \delta_{ij}J_{kl}J_{kl}) \\ [J_{ij}, J_{kl}] &= i\delta_{jk}J_{il} - i\delta_{ik}J_{jl} - i\delta_{jl}J_{ik} + i\delta_{il}J_{jk} \\ [J_{ij}, Q_k] &= i\delta_{jk}Q_i - i\delta_{ik}Q_j, \qquad [H, Q_i] = [H, J_{ij}] = 0 \end{aligned} \tag{2.10}$$

with the hamiltonian constraint given by (2.4), that in (A)dS reduces to

$$H = H_0 - \frac{b}{4}J_{ij}J_{ij} - bA(D) \tag{2.11}$$

with  $A(D) = -\frac{D}{8}(D + N - 2)$ .

In order to gauge fix the locally symmetric  $O(N)$  spinning particle action (with quantum gauge algebra given in (2.10)) we use the hamiltonian BRST method reviewed in appendix A. Basically, we define ghost fields  $\mathcal{C}^A = (\mathcal{C}, \mathcal{C}_i, \mathcal{C}_{ij})$  and ghost momenta  $\mathcal{P}_A = (\mathcal{P}, \mathcal{P}_i, \mathcal{P}_{ij})$  for all constraint generators  $G_A = (H, Q_i, J_{ij})$ , such that  $[\mathcal{P}_A, \mathcal{C}^B] = -i\delta_A^B$  and write the quantum BRST operator as a graded sum  $\Omega = \sum_{p \geq 0} \overset{(p)}{\Omega}$ . Starting from

$$\overset{(0)}{\Omega} = \mathcal{C}^A G_A = \mathcal{C}H + \mathcal{C}_i Q_i + \mathcal{C}_{ij} J_{ij} \quad (2.12)$$

and imposing the nilpotency of the BRST charge, we can recursively obtain higher antighost-number operators. Setting

$$[G_A, G_B] = F_{AB}^C(z) G_C \quad (2.13)$$

with  $z^\alpha = (p_\mu, x^\mu, \psi_i^a)$  and  $F_{AB}^C(z)$  structure functions, for the algebra (2.10) we get

$$\begin{aligned} \overset{(1)}{\Omega} &= \frac{i}{2} (-)^{\varepsilon_A} \mathcal{C}^A \mathcal{C}^B F_{BA}^C \mathcal{P}_C \\ &= -i \mathcal{C}_i \mathcal{C}_i \mathcal{P} - 2 \mathcal{C}_k \mathcal{C}_{ki} \mathcal{P}_i + 2 \mathcal{C}_{ik} \mathcal{C}_{kj} \mathcal{P}_{ij} - i \frac{b}{4} \left( \mathcal{C}_i \mathcal{C}_i J_{kl} \mathcal{P}_{kl} - 2 \mathcal{C}_i \mathcal{C}_j J_{ik} \mathcal{P}_{jk} \right) \end{aligned} \quad (2.14)$$

and

$$\overset{(3)}{\Omega} = \frac{b^2}{24} \left( \mathcal{C}_i \mathcal{C}_j \mathcal{C}_k \mathcal{C}_l \mathcal{P}_{ij} \mathcal{P}_{km} \mathcal{P}_{lm} - 3 \mathcal{C}_m \mathcal{C}_m \mathcal{C}_i \mathcal{C}_j \mathcal{P}_{ik} \mathcal{P}_{jl} \mathcal{P}_{kl} + \mathcal{C}_k \mathcal{C}_k \mathcal{C}_i \mathcal{C}_l \text{Tr}(\mathcal{P}_{ij}^3) \right) \quad (2.15)$$

$$\overset{(2)}{\Omega} = \overset{(p)}{\Omega} = 0, \quad p > 3. \quad (2.16)$$

One can thus write the quantum gauge-fixed hamiltonian operator as

$$\mathbb{H}_{qu} = H_{BRST} - i\{K, \Omega\}$$

where the first term is a BRST-invariant hamiltonian and  $K$  a gauge fixing fermion: the latter is BRST-invariant for any choice of  $K$  thanks to the nilpotency of  $\Omega$ . In the present case since  $H$  itself enters as a constraint we can set  $H_{BRST} = 0$  and thus have

$$\mathbb{H}_{qu} = -i\{K, \Omega\}. \quad (2.17)$$

Let us now use the gauge-fixing fermion

$$K = -\hat{E}^A \mathcal{P}_A, \quad \hat{E}^A = \left( \beta, 0, \frac{\theta_{ij}}{2} \right) \quad (2.18)$$

with  $\theta_{ij}$  a  $N \times N$  skew diagonal matrix, dependent on  $[S] = [N/2] := n$  angular variables  $\theta_k$ , with  $k = 1, \dots, n$ . Here  $S = N/2$  is the ‘‘spin’’ of the particle. With this choice one obtains the hamiltonian operator

$$\mathbb{H}_{qu} = \beta H + \frac{1}{2} \theta_{ij} J_{ij} - \theta_{ij} \mathcal{C}_i \mathcal{P}_j - 2 \theta_{ij} \mathcal{C}_{im} \mathcal{P}_{jm} \quad (2.19)$$

and consequently the gauge-fixed path integral can be written as

$$\Gamma[g] = K_N \int_0^\infty \frac{d\beta}{\beta} \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} \int_{S^1} Dz DC D\mathcal{P} e^{iS_{qu}[z, \mathcal{C}, \mathcal{P}, \hat{E}; g]} \quad (2.20)$$

with phase space action

$$S_{qu}[z, \mathcal{C}, \mathcal{P}, \hat{E}; g] = \int_0^1 dt \left[ p_\mu \dot{x}^\mu + \frac{i}{2} \psi_i^a \dot{\psi}_{ia} + \dot{\mathcal{C}}^A \mathcal{P}_A - H_{qu} \right] \quad (2.21)$$

$$H_{qu} = \beta \left( \frac{1}{2} g_{\mu\nu}(x) \pi^\mu \pi^\nu - \frac{b}{4} J_{ij} J_{ij} - bA(D) \right) + \frac{1}{2} \theta_{ij} J_{ij} - \theta_{ij} \mathcal{C}_i \mathcal{P}_j - 2\theta_{ij} \mathcal{C}_{im} \mathcal{P}_{jm} \quad (2.22)$$

and  $\pi^\mu = p^\mu - \frac{i}{2} \omega^\mu{}_{ab} \psi_i^a \psi_i^b$ . Above  $K_N$  is a normalization factor that implements the reduction to a fundamental region of moduli space

$$K_N = \begin{cases} \frac{2}{2^n n!}, & N = 2n \\ \frac{1}{2^n n!}, & N = 2n + 1 \end{cases} \quad (2.23)$$

as discussed in [1]. Integrating out particle momenta leads to a configuration space path integral that involves the action

$$S_{qu}[y, \mathcal{C}, \mathcal{P}, \hat{E}; g] = \int_0^1 dt \left[ \frac{1}{2\beta} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} \psi_{ai} D_t \psi_i^a + \beta \left( \frac{b}{4} J_{ij} J_{ij} + bA(D) \right) - \frac{1}{2} \theta_{ij} J_{ij} - \mathcal{P} \dot{\mathcal{C}} + \mathcal{P}_i (\delta_{ij} \partial_t - \theta_{ij}) \mathcal{C}_j - \mathcal{P}_{ij} (\delta_{im} \delta_{jp} \partial_t - \theta_{im} \delta_{jp} + \theta_{jm} \delta_{ip}) \mathcal{C}_{mp} \right] \quad (2.24)$$

where  $D_t \psi_i^a = \dot{\psi}_i^a + \dot{x}^\mu \omega_\mu{}^a{}_b \psi_i^b$ . A Wick rotation to euclidean time yields

$$\Gamma[g] = K_N \int_0^\infty \frac{d\beta}{\beta} \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} \int_{S^1} Dy DC D\mathcal{P} e^{-S_{qu}[y, \mathcal{C}, \mathcal{P}, \hat{E}; g]} \quad (2.25)$$

with the euclidean version of the action given by

$$S_{qu}[y, \mathcal{C}, \mathcal{P}, \hat{E}; g] = \frac{1}{\beta} \int_0^1 d\tau \left[ \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi_{ai} (\delta_{ij} D_\tau - \theta_{ij}) \psi_j^a - \frac{b}{4} J_{ij} J_{ij} - \beta^2 bA(D) - \mathcal{P} \dot{\mathcal{C}} + \mathcal{P}_i (\delta_{ij} \partial_t - \theta_{ij}) \mathcal{C}_j + \mathcal{P}_{ij} (\delta_{im} \delta_{jp} \partial_t - \theta_{im} \delta_{jp} + \theta_{jm} \delta_{ip}) \mathcal{C}_{mp} \right]. \quad (2.26)$$

where we have Wick-rotated the  $O(N)$  fields  $\theta_{ij} \rightarrow i\theta_{ij}$  and the ghost momenta  $\mathcal{P}_A \rightarrow i\mathcal{P}_A$ . Here  $D_\tau$  is represented by the same covariant derivative as given above, with “dot” now representing derivative with respect to  $\tau$ . Fermions and ghosts have been suitably rescaled in order to have a common  $\frac{1}{\beta}$  in front of the action. In the following we perturbatively compute the above path integral. Although the latter is defined on (A)dS spaces, for convenience we keep the geometry arbitrary and only at the end do we fix it to (A)dS. In essence, we replace  $\frac{b}{4} J_{ij} J_{ij} + \beta^2 bA(D)$  by  $\frac{1}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d + \beta^2 \frac{(N-2)(D+N-2)}{16(D-1)} R$  in the



above action. Integrating over the ghost fields yields

$$\Gamma[g] = K_N \int_0^\infty \frac{d\beta}{\beta} \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} \left( \text{Det}(\partial_\tau - \theta_{\text{vec}})_{ABC} \right)^{-1} \text{Det}'(\partial_\tau - \theta_{\text{adj}})_{PBC} \int_{S^1} \mathcal{D}x D\psi \exp \left( -\frac{1}{\beta} \int_0^1 d\tau \left[ \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi_{ai} (\delta_{ij} D_\tau - \theta_{ij}) \psi_j^a - \frac{1}{8} R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d - \beta^2 \frac{(N-2)(D+N-2)}{16(D-1)} R \right] \right) \quad (2.27)$$

where  $\theta_{\text{vec}}$  and  $\theta_{\text{adj}}$  denote the gauge-fixed  $O(N)$  potentials in the vector and adjoint representation, respectively.  $\text{Det}'$  indicates a determinant without its zero modes, and  $\mathcal{D}x$  is the reparametrization invariant measure. Below we consider a short-time perturbative approach to the above nonlinear sigma model path integral.

## 2.2 Regularization of supersymmetric nonlinear sigma models

For a particle in curved space, the passage between the operatorial representation of the transition amplitude and its path integral counterpart is in general not straightforward, as the latter involves a nonlinear sigma model that perturbatively gives rise to superficial divergences. These divergences are compensated by vertices arising from the nontrivial path integral measure, but finite ambiguities remain that need to be dealt with by specifying a regularization scheme. This is well studied for models with global (super)symmetries (see [60] for a review). However it is clear that gauging does not introduce further divergences. Indeed upon gauge fixing, the gauged model reduces essentially to the ungauged one. Moreover the ghosts do not couple to the target space geometry and just produce the correct measure for integration over the moduli space.

In [61] we considered the regularization of the spinning particle model with hamiltonian

$$H = H_0 + \alpha R_{abcd} \psi_i^a \psi_i^b \psi_j^c \psi_j^d + V \quad (2.28)$$

with  $H_0$  given by (2.5). The corresponding euclidean classical action in configuration space is given by

$$S = \frac{1}{\beta} \int_0^1 d\tau \left[ \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi_{ai} D_\tau \psi_i^a + \alpha R_{abcd} \psi_i^a \psi_i^b \psi_j^c \psi_j^d + \beta^2 V \right] \quad (2.29)$$

and, for  $\alpha = -\frac{1}{8}$ , is nothing but the ungauged version of the nonlinear sigma model of the previous section. We found that such path integral reproduces the transition amplitudes that satisfies the Schrödinger equation with hamiltonian (2.28) provided we add the counterterm

$$V_{CT} = \begin{cases} -\left(\frac{1}{8} + \frac{\alpha N}{2}\right) R + \frac{1}{8} g^{\mu\nu} \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\rho}^\lambda + \frac{N}{16} \omega_{\mu ab} \omega^{\mu ab}, & TS \\ -\left(\frac{1}{8} + \frac{\alpha N}{2}\right) R - \frac{1}{24} (\Gamma_{\mu\lambda}^\rho)^2 + \frac{N}{24} \omega_{\mu ab} \omega^{\mu ab}, & MR \\ -\left(\frac{1}{8} + \frac{\alpha N}{2}\right) R, & DR \end{cases} \quad (2.30)$$

Since the process of gauging does not introduce further ambiguities than those already taken into account in [61], we conclude that the regularization there discussed is suitable for the model of the previous section, provided one sets  $\alpha = -\frac{1}{8}$  and

$$V = V_{CT} - \frac{(N-2)(D+N-2)}{16(D-1)}R. \quad (2.31)$$

Above *TS* refers to Time Slicing regularization [50, 51], *MR* refers to Mode Regularization [52–56] and *DR* refers to Dimensional Regularization [10, 12, 57–59], that are the three regularization schemes developed in the past to treat one-dimensional nonlinear sigma models (particles in curved space). In the present work we adopt *DR* to compute the short time perturbative expansion of (2.27). We parametrize the coordinates of the circle as  $x^\mu(\tau) = x^\mu + q^\mu(\tau)$ , where  $x^\mu$  is the initial/final point of the circle and  $q^\mu(\tau)$  are quantum fluctuations with Dirichlet boundary conditions  $q^\mu(0) = q^\mu(1) = 0$ . Fermions have antiperiodic boundary conditions on the circle and have no zero modes. We then expand the metric and the spin connection about the point  $x^\mu$  using Riemann normal coordinates, and get

$$g_{\mu\nu}(x(\tau)) = g_{\mu\nu} + \frac{1}{3}R_{\alpha\mu\nu\beta}q^\alpha q^\beta + \frac{1}{6}\nabla_\gamma R_{\alpha\mu\nu\beta}q^\alpha q^\beta q^\gamma + R_{\alpha\beta\mu\nu\gamma\delta}q^\alpha q^\beta q^\gamma q^\delta + O(q^5) \quad (2.32)$$

$$\omega_{\mu ab}(x(\tau)) = \frac{1}{2}R_{\alpha\mu ab}q^\alpha + \frac{1}{3}\nabla_\alpha R_{\beta\mu ab}q^\alpha q^\beta + \frac{1}{8}\nabla_\alpha \nabla_\beta R_{\gamma\mu ab}q^\alpha q^\beta q^\gamma + \frac{1}{30}\nabla_\alpha \nabla_\beta \nabla_\gamma R_{\delta\mu ab}q^\alpha q^\beta q^\gamma q^\delta + O(q^5) \quad (2.33)$$

where  $R_{\alpha\beta\mu\nu\gamma\delta} = \frac{1}{20}\nabla_\delta \nabla_\gamma R_{\alpha\mu\nu\beta} + \frac{2}{45}R_{\alpha\mu}{}^\sigma{}_\beta R_{\gamma\sigma\nu\delta}$ . All the tensors are here evaluated at the initial point  $x^\mu$ . Above we only give the terms that are needed to obtain a perturbative expansion to order  $\beta^2$ . We thus get

$$\begin{aligned} \Gamma[g] &= K_N \int d^D x \int_0^\infty \frac{d\beta}{\beta} \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} \left( \text{Det}(\partial_\tau - \theta_{\text{vec}})_{ABC} \right)^{-1} \text{Det}'(\partial_\tau - \theta_{\text{adj}})_{PBC} \\ &\times \int_{DBC} Dq Da Db Dc D\bar{\psi} D\psi D\eta e^{-\frac{1}{\beta} \int_0^1 d\tau \left( \frac{1}{2}g_{\mu\nu}(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \sum_k \bar{\psi}_{\alpha k}(\partial_\tau + i\theta_k)\psi_k^\alpha + \frac{1}{2}\eta_\alpha \partial_\tau \eta^\alpha \right)} \\ &\times e^{-S_{\text{int}}} \end{aligned} \quad (2.34)$$

where we have exponentiated the reparametrization invariant measure by means of measure ghosts  $a, b, c$  [52, 53] and have complexified the  $2n$  fermions  $\psi$ ; the leftover uncomplexified Majorana fermion  $\eta$  is only present when the number of supersymmetries  $N$  is odd — i.e. for half-integer spin. From the quadratic part of the action one gets the path integral normalization and the propagators for all fields, that are reported in appendix B, whereas

higher order terms form the interacting action

$$\begin{aligned}
S_{\text{int}} = & \frac{1}{\beta} \int_0^1 d\tau \\
& \left[ \left( \frac{1}{6} R_{\alpha\mu\nu\beta} q^\alpha q^\beta + \frac{1}{12} \nabla_\gamma R_{\alpha\mu\nu\beta} q^\alpha q^\beta q^\gamma + \frac{1}{2} R_{\alpha\beta\mu\nu\gamma\delta} q^\alpha q^\beta q^\gamma q^\delta \right) (\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu) \right. \\
& + \left( \frac{1}{2} R_{\alpha\mu ab} q^\alpha + \frac{1}{3} \nabla_\alpha R_{\beta\mu ab} q^\alpha q^\beta + \frac{1}{8} \nabla_\alpha \nabla_\beta R_{\gamma\mu ab} q^\alpha q^\beta q^\gamma \right. \\
& \quad \left. + \frac{1}{30} \nabla_\alpha \nabla_\beta \nabla_\gamma R_{\delta\mu ab} q^\alpha q^\beta q^\gamma q^\delta \right) \dot{q}^\mu \left( \sum_{k=1}^n \bar{\psi}_k^a \psi_k^b + \frac{1}{2} \eta^a \eta^b \right) \\
& + \alpha \left( R_{abcd} + q^\alpha \nabla_\alpha R_{abcd} + \frac{1}{2} q^\alpha q^\beta \nabla_\alpha \nabla_\beta R_{abcd} \right) \psi^a \cdot \bar{\psi}^b \left( \psi^c \cdot \bar{\psi}^d + \eta^c \eta^d \right) \\
& \left. + \beta^2 \left( V + q^\alpha \nabla_\alpha V + \frac{1}{2} q^\alpha q^\beta \nabla_\alpha \nabla_\beta V \right) \right] \tag{2.35}
\end{aligned}$$

whose path integral average is computed using the Wick theorem. We thus get

$$\Gamma[g] = \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x \sqrt{|g|}}{(2\pi\beta)^{D/2}} \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} d(\theta; D, S) \langle e^{-S_{\text{int}}} \rangle \tag{2.36}$$

with  $\frac{\sqrt{|g|}}{(2\pi\beta)^{D/2}}$  being the normalization of the bosonic path integral in  $D$  dimensions with Dirichlet boundary conditions, whereas the fermionic normalization contributes to the moduli integrand

$$\begin{aligned}
d(\theta; D, N) = & K_N \left( \text{Det}(\partial_\tau - \theta_{\text{vec}})_{ABC} \right)^{\frac{D}{2}-1} \text{Det}'(\partial_\tau - \theta_{\text{adj}})_{PBC} \tag{2.37} \\
= & \begin{cases} \frac{2}{2^n n!} \prod_{k=1}^n \left( 2 \cos \frac{\theta_k}{2} \right)^{D-2} \prod_{k<l} \left[ \left( 2 \cos \frac{\theta_l}{2} \right)^2 - \left( 2 \cos \frac{\theta_k}{2} \right)^2 \right]^2, & N = 2n \\ \frac{2^{\frac{D}{2}-1}}{2^n n!} \prod_{k=1}^n \left( 2 \cos \frac{\theta_k}{2} \right)^{D-2} \left( 2 \sin \frac{\theta_k}{2} \right)^2 \prod_{k<l} \left[ \left( 2 \cos \frac{\theta_l}{2} \right)^2 - \left( 2 \cos \frac{\theta_k}{2} \right)^2 \right]^2, & N = 2n + 1 \end{cases}
\end{aligned}$$

that integrated gives

$$\text{Dof}(D, N) = \prod_{k=1}^n \int_0^{2\pi} \frac{d\theta_k}{2\pi} d(\theta; D, N) := a_0, \tag{2.38}$$

the number of degrees of freedom for the higher spin field described by the locally supersymmetric spinning particle model with  $N$  supersymmetries [1], i.e. the physical polarizations of a particle of spin  $S = N/2$ . By factoring out the number of degrees of freedom, we can finally write the above effective action in a compact way as

$$\Gamma[g] = a_0 \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x \sqrt{|g|}}{(2\pi\beta)^{D/2}} \langle\langle e^{-S_{\text{int}}} \rangle\rangle = \int_0^\infty \frac{d\beta}{\beta} Z(\beta) \tag{2.39}$$

with  $\langle\langle \dots \rangle\rangle$  representing the average over the path integral and over the moduli space. Hence, for the effective action density in proper time we get

$$Z(\beta) = a_0 \int \frac{d^D x \sqrt{|g|}}{(2\pi\beta)^{D/2}} \langle\langle e^{-S_{\text{int}}} \rangle\rangle = \int \frac{d^D x \sqrt{|g|}}{(2\pi\beta)^{D/2}} (a_0 + a_1\beta + a_2\beta^2 + O(\beta^3)) \quad (2.40)$$

and we parametrize the Seley-DeWitt coefficients  $a_i$  as follows

$$a_0 \left( 1 + v_2 R \beta + (v_3 R_{abcd}^2 + v_4 R_{ab}^2 + v_5 R^2 + v_6 \nabla^2 R) \beta^2 + O(\beta^3) \right). \quad (2.41)$$

Next we compute the numerical coefficients  $v_i$ .

### 3 Heat kernel expansion for higher spin fields in (A)dS

Equipped with the results of the previous sections we can now compute the heat kernel in a perturbative expansion for higher spin fields on (A)dS spaces, using the  $O(N)$  spinning particle representation discussed above. Although in the previous sections we gauge fixed the locally supersymmetric action for maximally symmetric spaces only, here we compute the expansion keeping an unspecified metric in the sigma model and only at the end of the section will we specialize to (A)dS spaces. This we do mostly for future convenience, as intermediate results might be useful when considering more general spacetimes, such as the conformally flat spaces. Since in the following we adopt dimensional regularization, the total potential acquires the form:

$$V = w R, \quad \text{with } w(D, N, \alpha) := w_{CT}(N, \alpha) + w_{(A)dS}(D, N)$$

where

$$w_{CT}(N, \alpha) = - \left( \frac{1}{8} + \frac{\alpha N}{2} \right), \quad w_{(A)dS}(D, N) = - \frac{(N-2)(D+N-2)}{16(D-1)}, \quad (3.1)$$

as follows from (2.30), (2.31).

#### 3.1 Integer spins

For this case we set  $N = 2n$ . One can complexify fermions and, with the help of propagators given in appendix B, one gets for the perturbative average

$$\begin{aligned} \langle e^{-S_{\text{int}}} \rangle = \exp \left\{ -\beta \left[ \frac{1}{24} + \alpha \left( n - \sum_k \cos^{-2} \frac{\theta_k}{2} \right) + w \right] R \right. \\ + \beta^2 \left[ -\frac{1}{720} R_{\alpha\beta}^2 + \left( \frac{1}{720} - \frac{1}{192} \sum_k \cos^{-2} \frac{\theta_k}{2} \right) R_{\alpha\mu\nu\beta}^2 - \left( \frac{1}{480} + \frac{w}{12} \right) \nabla^2 R \right] \\ - \frac{\alpha\beta^2}{12} \left( n - \sum_k \cos^{-2} \frac{\theta_k}{2} \right) \nabla^2 R \\ + (\alpha\beta)^2 \left[ \left( \left( \sum_k \cos^{-2} \frac{\theta_k}{2} \right)^2 - \frac{1}{2} \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{\alpha\mu\nu\beta}^2 \right. \\ \left. + 2 \left( \sum_k \cos^{-2} \frac{\theta_k}{2} - \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{\alpha\beta}^2 \right] + O(\beta^3) \left. \right\}, \quad (3.2) \end{aligned}$$

that, for  $\alpha = -1/8$  reduces to

$$\begin{aligned}
 \langle e^{-S_{\text{int}}} \rangle &= 1 - \beta \left( \frac{1-3n}{24} + \frac{1}{8} \sum_k \cos^{-2} \frac{\theta_k}{2} + w \right) R \\
 &+ \beta^2 \left\{ \frac{1}{2} \left( \frac{1-3n}{24} + \frac{1}{8} \sum_k \cos^{-2} \frac{\theta_k}{2} + w \right)^2 R^2 \right. \\
 &+ \left( -\frac{1}{720} - \frac{1}{32} \sum_k \cos^{-4} \frac{\theta_k}{2} + \frac{1}{32} \sum_k \cos^{-2} \frac{\theta_k}{2} \right) R_{ab}^2 \\
 &+ \left( \frac{1}{720} - \frac{1}{192} \sum_k \cos^{-2} \frac{\theta_k}{2} + \frac{1}{64} \left( \sum_k \cos^{-2} \frac{\theta_k}{2} \right)^2 \right. \\
 &\quad \left. - \frac{1}{128} \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{abcd}^2 - \left( \frac{1-5n}{480} + \frac{1}{96} \sum_k \cos^{-2} \frac{\theta_k}{2} + \frac{w}{12} \right) \nabla^2 R \left. \right\} \\
 &+ O(\beta^3), \tag{3.3}
 \end{aligned}$$

with

$$w = w(D = 2d, N = 2n, \alpha = -\frac{1}{8}) = -\frac{(N-2)(N-1)}{16(D-1)} = -\frac{(2n-1)(n-1)}{8(2d-1)}.$$

We are ready now to extract the Seeley-DeWitt coefficients for arbitrary integer spin  $S = n$  in arbitrary even dimension  $D = 2d$ ; to this aim we integrate (3.3) against the modular measure given in (2.36), (2.37) and get:

$$a_0 = \begin{cases} 1, & n = 0 \\ 2^{n-1} \frac{(2d-2)!}{[(d-1)!]^2} \prod_{k=1}^{n-1} \frac{k(2k-1)!(2k+2d-3)!}{(2k+d-2)!(2k+d-1)!}, & n > 0 \end{cases} \tag{3.4}$$

and

$$\begin{aligned}
 v_2 &= \frac{3n-1}{24} - \frac{1}{8} \mathbf{I}_1 - w \\
 v_3 &= \frac{1}{720} - \frac{n(n+1)}{256} + \frac{3n+1}{384} \mathbf{I}_1 + \frac{3}{256} \mathbf{I}_2 + \frac{1}{256} \mathbf{I}_3 \\
 v_4 &= -\frac{1}{720} + \frac{n(n+1)}{64} - \frac{n}{32} \mathbf{I}_1 + \frac{1}{64} \mathbf{I}_2 - \frac{1}{64} \mathbf{I}_3 \\
 v_5 &= \frac{1}{2} \left( \frac{9n^2 - 21n + 2}{1152} - \frac{w(3n-1)}{12} + w^2 \right) + \frac{1}{2} \left( \frac{5-3n}{192} + \frac{w}{4} \right) \mathbf{I}_1 + \frac{1}{256} (\mathbf{I}_2 + \mathbf{I}_3) \\
 v_6 &= \frac{5n-1}{480} - \frac{w}{12} - \frac{1}{96} \mathbf{I}_1 \tag{3.5}
 \end{aligned}$$

with

$$\begin{aligned}
 \mathbf{I}_1 &= \frac{2n(n+d-2)}{2d-3} \\
 \mathbf{I}_2 &= \frac{4n(n-1)(n+d-1)(n+d-2)}{(2d-3)(2d-1)} \\
 \mathbf{I}_3 &= \frac{n(n+1)(4n^2-1)}{(2d-3)(2d-5)} \tag{3.6}
 \end{aligned}$$

Detailed computation of modular integrals is given in appendix C. Let us now briefly comment on the results described above in (3.4), (3.5):

- For  $n = 0$ , the formalism describes a conformally coupled scalar field and the expected results are easily obtained.
- For  $n = 1$ , (3.4), (3.5) reproduce the well known Seeley-DeWitt coefficients for a degree  $(d - 1)$  differential form (vector field in  $D = 4$ ) [12] on a general background.
- For  $n \geq 2$ , the spinning particle consistently propagates on conformally flat manifolds. However, for this case, in the previous sections we limited the computation of the BRST charge to (A)dS spaces. Hence the structure of the Seeley-DeWitt coefficients reduces to

$$a_0 \left( 1 + v_2 R \beta + v R^2 \beta^2 \right) \quad \text{with} \quad v = \frac{1}{d(2d-1)} v_3 + \frac{1}{2d} v_4 + v_5.$$

**Example:  $D = 4$ , spin  $n$**

In 4-dimensional space-time the model describes completely symmetric tensors of spin  $n$ , and the Seeley-DeWitt coefficients are given by:

$$a_0 = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}, \quad v_2 = -\frac{n^2}{6}, \quad v_3 = \frac{1}{720} - \frac{n^2}{96} \tag{3.7}$$

$$v_4 = -\frac{1}{720} - \frac{n^2}{48} + \frac{n^4}{12}, \quad v_5 = \frac{1}{96} n^2 - \frac{1}{36} n^4, \quad v_6 = \frac{1}{720} - \frac{1}{72} n^2,$$

When  $n \geq 2$  the restriction to (A)dS yields:

$$v = \frac{1}{6} v_3 + \frac{1}{4} v_4 + v_5 = -\frac{1}{8640} + \frac{1}{288} n^2 - \frac{1}{144} n^4 \tag{3.8}$$

We again recognize for  $n = 0, 1$  the known coefficients for a conformally improved scalar and an ordinary spin one vector field. For  $n > 0$  the first coefficient  $a_0$  represents the two polarizations of massless particles of spin  $n$ .

The case of  $n = 2$  corresponds to a linearized graviton on a fixed background, but this is true only in  $D = 4$ . In other dimensions one has a different field content compatible with conformal invariance.

**3.2 Half-integer spins**

In such a case one can only complexify  $2n$  fermions. The left-over one has no  $\theta$ , and one thus gets

$$\begin{aligned} \langle e^{-S_{\text{int}}} \rangle = & \exp \left\{ -\beta \left[ \frac{1}{24} + w + \alpha \left( n - \sum_k \cos^{-2} \frac{\theta_k}{2} \right) \right] R \right. \\ & + \beta^2 \left[ -\frac{1}{720} R_{\alpha\beta}^2 - \left( \frac{7}{5760} + \frac{1}{192} \sum_k \cos^{-2} \frac{\theta_k}{2} \right) R_{\alpha\mu\nu\beta}^2 - \left( \frac{1}{480} + \frac{w}{12} \right) \nabla^2 R \right] \\ & \left. - \frac{\alpha\beta^2}{12} \left( n - \sum_k \cos^{-2} \frac{\theta_k}{2} \right) \nabla^2 R \right\} \end{aligned}$$

$$\begin{aligned}
& + (\alpha\beta)^2 \left[ \left( \left( \sum_k \cos^{-2} \frac{\theta_k}{2} \right)^2 + \sum_k \cos^{-2} \frac{\theta_k}{2} - \frac{1}{2} \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{\alpha\mu\nu\beta}^2 \right. \\
& \quad \left. + 2 \left( \sum_k \cos^{-2} \frac{\theta_k}{2} - \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{\alpha\beta}^2 \right] + O(\beta^3) \Big\}, \tag{3.9}
\end{aligned}$$

that, for  $\alpha = -1/8$ , reduces to

$$\begin{aligned}
\langle e^{-S_{\text{int}}} \rangle &= 1 - \beta \left( \frac{1-3n}{24} + \frac{1}{8} \sum_k \cos^{-2} \frac{\theta_k}{2} + w \right) R \\
&+ \beta^2 \left\{ \frac{1}{2} \left( \frac{1-3n}{24} + \frac{1}{8} \sum_k \cos^{-2} \frac{\theta_k}{2} + w \right)^2 R^2 \right. \\
&\quad + \left( -\frac{1}{720} - \frac{1}{32} \sum_k \cos^{-4} \frac{\theta_k}{2} + \frac{1}{32} \sum_k \cos^{-2} \frac{\theta_k}{2} \right) R_{ab}^2 \\
&\quad + \left( -\frac{7}{5760} + \frac{1}{96} \sum_k \cos^{-2} \frac{\theta_k}{2} + \frac{1}{64} \left( \sum_k \cos^{-2} \frac{\theta_k}{2} \right)^2 \right. \\
&\quad \quad \left. - \frac{1}{128} \sum_k \cos^{-4} \frac{\theta_k}{2} \right) R_{abcd}^2 - \left( \frac{1-5n}{480} + \frac{w}{12} + \frac{1}{96} \sum_k \cos^{-2} \frac{\theta_k}{2} \right) \nabla^2 R \Big\} \\
&+ O(\beta^3) \tag{3.10}
\end{aligned}$$

where now we use

$$w = w(D = 2d, N = 2n + 1, \alpha = -\frac{1}{8}) = -\frac{(N-2)(N-1)}{16(D-1)} = -\frac{n(2n-1)}{8(2d-1)}.$$

We compute, in analogy with the previous section, the Seeley-DeWitt coefficients for arbitrary half-integer spin  $S = n + \frac{1}{2}$  in arbitrary even dimension  $2d$ , represented by spin-tensors corresponding to potentials with rectangular Young tableaux of  $n$  columns and  $d-1$  rows; we get:

$$a_0 = \frac{2^{d-2+n}}{d} \frac{(2d-2)!}{[(d-1)!]^2} \prod_{k=1}^{n-1} \frac{(k+d-1)(2k+1)!(2k+2d-3)!}{(2k+d-1)!(2k+d)!} \tag{3.11}$$

and

$$\begin{aligned}
v_2 &= \frac{3n-1}{24} - \frac{1}{8} \tilde{\mathbf{I}}_1 - w \\
v_3 &= -\frac{7}{5760} - \frac{n(n+1)}{256} + \frac{3n+7}{384} \tilde{\mathbf{I}}_1 + \frac{3}{256} \tilde{\mathbf{I}}_2 + \frac{1}{256} \tilde{\mathbf{I}}_3 \\
v_4 &= -\frac{1}{720} + \frac{n(n+1)}{64} - \frac{n}{32} \tilde{\mathbf{I}}_1 + \frac{1}{64} \tilde{\mathbf{I}}_2 - \frac{1}{64} \tilde{\mathbf{I}}_3 \\
v_5 &= \frac{1}{2} \left( \frac{9n^2 - 21n + 2}{1152} - \frac{w(3n-1)}{12} + w^2 \right) + \frac{1}{2} \left( \frac{5-3n}{192} + \frac{w}{4} \right) \tilde{\mathbf{I}}_1 + \frac{1}{256} (\tilde{\mathbf{I}}_2 + \tilde{\mathbf{I}}_3) \\
v_6 &= \frac{5n-1}{480} - \frac{w}{12} - \frac{1}{96} \tilde{\mathbf{I}}_1 \tag{3.12}
\end{aligned}$$

with

$$\begin{aligned}
 \tilde{\mathbf{I}}_1 &= \frac{2n(n+d-1)}{2d-3} \\
 \tilde{\mathbf{I}}_2 &= \frac{4n(n-1)(n+d-1)(n+d)}{(2d-3)(2d-1)} \\
 \tilde{\mathbf{I}}_3 &= \frac{n(n+1)(2n+1)(2n+3)}{(2d-3)(2d-5)}.
 \end{aligned}
 \tag{3.13}$$

The modular integrals are again computed in details in appendix C.

In the half-integer spin case the spinning particle model we start with is consistent on any background only if  $n = 0$  (i.e. spin  $\frac{1}{2}$ ). When  $n \geq 1$  we restrict our analysis to (A)dS spaces and at order  $\beta^2$  in the expansion of the effective action the only term that survives is  $a_0 v R^2$  where, again,  $v = \frac{1}{d(2d-1)}v_3 + \frac{1}{2d}v_4 + v_5$ .

**Example:  $D = 4$ , spin  $n + \frac{1}{2}$**

In 4-dimensional space-time we describe spinor-tensors with  $n$  completely symmetric vector indices and one spinor index (i.e. spin  $n + \frac{1}{2}$ ). The Seeley-DeWitt coefficients we find are:

$$\begin{aligned}
 a_0 &= 2, & v_2 &= -\frac{(2n+1)^2}{24}, & v_3 &= -\frac{7}{5760} - \frac{n}{96} - \frac{n^2}{96} \\
 v_4 &= -\frac{1}{720} + \frac{n}{48} + \frac{5n^2}{48} + \frac{n^3}{6} + \frac{n^4}{12}, & v_5 &= \frac{1}{1152} + \frac{1}{144}n - \frac{5}{96}n^2 - \frac{19}{288}n^3 - \frac{1}{144}n^4, \\
 v_6 &= -\frac{1}{480} - \frac{1}{72}n - \frac{1}{72}n^2.
 \end{aligned}
 \tag{3.14}$$

When  $n = 0$  the previous formulas reproduce the well know Seeley-DeWitt coefficients for a spinor field [10], while for  $n \geq 1$  in (A)dS we get:

$$v = \frac{11}{34560} + \frac{n}{96} - \frac{n^2}{36} - \frac{7n^3}{288} + \frac{n^4}{72}.$$

Let us stress again that in 4 dimension we recognize in  $a_0 = 2$  the two polarizations of a massless half-integer spin field.

**Acknowledgments**

The work of FB was supported in part by the MIUR-PRIN contract 2009-KHZKRX. The work of OC was partly funded by SEP-PROMEP/103.5/11/6653. EL acknowledges partial support of SNF Grant No. 200020-131813/1. OC and EL are grateful to the Dipartimento di Fisica and INFN Bologna for hospitality and support while parts of this work were completed.

**A Hamiltonian BRST quantization**

The hamiltonian BRST formalism is a construction that allows to convert the local (gauge) symmetry of the unfixed action (in hamiltonian form) to a global symmetry of the gauge-fixed action. It makes use of the double aspect that first-class generators have, as restrictions on the phase-space and generators of gauge transformations (see for examples [62]).



One defines a differential  $\delta$  (the Koszul-Tate differential) that acts as a derivative in the directions orthogonal to the constrained phase-space manifold and is nilpotent,  $\delta^2 = 0$ . Hence the definition

$$\delta z^\alpha = 0, \quad z^\alpha = (p_\mu, x^\mu, \psi_i^a). \quad (\text{A.1})$$

Moreover, one extends the phase space defining ghosts  $\mathcal{C}^A$  and ghost momenta  $\mathcal{P}_A$ , such that  $\{\mathcal{P}_A, \mathcal{C}^B\} = -\delta_A^B$  and

$$\delta \mathcal{C}^A = 0, \quad \delta \mathcal{P}_A = -G_A \quad (\text{A.2})$$

with  $G_A$  first class constraints. The operator  $\delta$  thus defines a natural grading, characterized by the antighost number

$$\overline{\text{gh}}(\delta) = -1, \quad \overline{\text{gh}}(z) = 0 = \overline{\text{gh}}(\mathcal{C}), \quad \overline{\text{gh}}(\mathcal{P}) = 1. \quad (\text{A.3})$$

Note that the bracket itself in the ghost sector has antighost number  $-1$ . Another grading is the Grassmann parity

$$\varepsilon_A := \varepsilon(G_A) \quad (\text{A.4})$$

so that, since  $\varepsilon(\delta) = 1$ , we have

$$\varepsilon(\mathcal{C}^A) = \varepsilon(\mathcal{P}_A) = \varepsilon_A + 1, \quad \text{mod } 2. \quad (\text{A.5})$$

One also introduces another derivative  $d$  that acts parallel to the gauge orbits. It is defined on functions of the original phase space,  $\phi(z)$ , as

$$d\phi = \{\phi, \mathcal{C}^A G_A\} = \{\phi, G_A\} \mathcal{C}^A, \quad \overline{\text{gh}}(d) = 0, \quad \varepsilon(d) = 1. \quad (\text{A.6})$$

Finally one seeks a differential  $s$  that is a graded sum of  $\delta$ ,  $d$  and higher order (in antighost number) derivatives, such that it results nilpotent on the extended phase space involving ghosts

$$s = \delta + d + \text{“higher order terms“}, \quad s^2 = 0. \quad (\text{A.7})$$

Thanks to antighost grading, nilpotency of  $s$  implies

$$\delta^2 = 0 \quad (\text{A.8})$$

$$d\delta + \delta d = 0 \quad (\text{A.9})$$

$$d^2 = -\{\delta, \Delta\} \quad (\text{A.10})$$

...

Equations (A.9), (A.10) mean that  $d$  is a “differential modulo  $\delta$ ”. The first one is satisfied, along with the grading properties, if one defines the following rules for the action of  $d$  on the extended phase space

$$d\mathcal{P}_A = (-)^{\varepsilon_A} \mathcal{C}^C F_{CA}^B \mathcal{P}_B, \quad d\mathcal{C}^A = 0 \quad (\text{A.11})$$

where  $F$ 's are structure functions and only depend upon the original phase space variables

$$\{G_A, G_B\} = F_{AB}^C G_C, \quad F_{AB}^C = F_{AB}^C(z). \quad (\text{A.12})$$

One then seeks a BRST operator  $\Omega$

$$\Omega = \sum_{p \geq 0} \binom{p}{\Omega}, \quad \overline{\text{gh}}\left(\binom{p}{\Omega}\right) = p \quad (\text{A.13})$$

that implements the action of the differential  $s$  as

$$s\Phi = \{\Phi, \Omega\} \quad (\text{A.14})$$

with  $\Phi(z, \mathcal{C}, \mathcal{P})$  a function of the extended phase space variables, where

$$\binom{(0)}{\Omega} = \mathcal{C}^A G_A \quad (\text{A.15})$$

so that

$$\delta\Phi = \left\{ \Phi, \binom{(0)}{\Omega} \right\}_{\mathcal{C}\mathcal{P}} = \{\Phi, \mathcal{C}^A\} G_A \quad (\text{A.16})$$

with the lowerscript  $\mathcal{C}\mathcal{P}$  meaning that the bracket is only taken in the ghost sector. It is trivial to check that (A.16) works correctly on the extended phase space variables. For a function of the original phase space we obviously have  $d\phi(z) = \left\{ \phi, \binom{(0)}{\Omega} \right\}_{\text{orig}} = \{\phi, G_A\} \mathcal{C}^A$ . Finally, thanks to the Jacobi identity, the nilpotency conditions turns into

$$\{\Omega, \Omega\} = 0. \quad (\text{A.17})$$

Higher order operators have the form

$$\binom{(p)}{\Omega} = \mathcal{C}^{B_1} \dots \mathcal{C}^{B_{p+1}} U_{B_1 \dots B_{p+1}}^{A_1 \dots A_p} \mathcal{P}_{A_1} \dots \mathcal{P}_{A_p}, \quad U = U(z) \quad (\text{A.18})$$

so that the nilpotency equation (A.17), with the help of (A.16), allows to write

$$\delta \binom{(0)}{\Omega} = 0 \quad (\text{A.19})$$

$$\delta \binom{(p+1)}{\Omega} + \frac{1}{2} \left( \sum_{k=0}^p \left\{ \binom{(p-k)}{\Omega}, \binom{(k)}{\Omega} \right\}_{\text{orig}} + \sum_{k=0}^{p-1} \left\{ \binom{(p-k)}{\Omega}, \binom{(k+1)}{\Omega} \right\}_{\mathcal{C}\mathcal{P}} \right) = 0, \quad p \geq 0. \quad (\text{A.20})$$

For example, it is easy to find the next-to-leading operator  $\binom{(1)}{\Omega}$  as

$$\delta \binom{(1)}{\Omega} = -\frac{1}{2} \left\{ \binom{(0)}{\Omega}, \binom{(0)}{\Omega} \right\}_{\text{orig}} = \frac{1}{2} (-)^{\varepsilon_A} \mathcal{C}^A \mathcal{C}^B F_{BA}^C G_C = \delta \left( -\frac{1}{2} (-)^{\varepsilon_A} \mathcal{C}^A \mathcal{C}^B F_{BA}^C \mathcal{P}_C \right) \quad (\text{A.21})$$

so that, modulo a  $\delta$ -exact term

$$\binom{(1)}{\Omega} = -\frac{1}{2} (-)^{\varepsilon_A} \mathcal{C}^A \mathcal{C}^B F_{BA}^C \mathcal{P}_C. \quad (\text{A.22})$$

One thus recursively fixes all other terms in the graded expansion. If the constraint algebra is linear (i.e. it is a Lie algebra) the expansion stops at  $p = 1$ .

The gauge fixed action in hamiltonian form reads

$$S_{gf} = \int_0^1 dt \left[ \dot{z}^\alpha a_\alpha + \dot{C}^A \mathcal{P}_A - H_{BRST} - \{K, Q\} \right] \quad (\text{A.23})$$

where  $a_\alpha$  are momenta conjugated to the phase space variables  $z$  (indices are contracted with the symplectic identity matrix),  $H_{BRST}$  is the BRST-invariant extension of the extended hamiltonian and  $K$  an arbitrary gauge-fixing fermion. This action is BRST invariant for any  $K$ . An important example concerns algebraic gauges for which the gauge fields are fixed to  $\hat{E}^A$ : in such a special case

$$K = -\hat{E}^A \mathcal{P}_A \quad (\text{A.24})$$

for which

$$\{K, Q\} = \hat{E}^A G_A - (-)^{\varepsilon_A} \hat{E}^A \mathcal{C}^B F_{BA}^C \mathcal{P}_C + \dots \quad (\text{A.25})$$

The above technique to construct the BRST charge  $\Omega$  is known as Koszul-Tate resolution.

Below we use the Koszul-Tate resolution to study an interesting class of non lie rank 3 superalgebra, and to construct the gauge fixed action for  $O(N)$  spinning particles propagating on (A)dS target spaces. We do it directly at the quantum level where Poisson brackets are replaced by (anti-)commutators, such as  $[\mathcal{P}_A, \mathcal{C}^B] = -i\delta_A^B$  and  $\mathcal{P}_A$  are taken to be (anti-)hermitian when (anti-)commuting whereas  $\mathcal{C}^A$  are always hermitians. The master formula (A.17) is now a nilpotency condition on the BRST charge,  $\Omega^2 = 0$ . We thus have

$$\overset{(0)}{\Omega} = \mathcal{C}^A G_A, \quad \overset{(1)}{\Omega} = \frac{i}{2} (-)^{\varepsilon_A} \mathcal{C}^A \mathcal{C}^B F_{BA}^C \mathcal{P}_C, \dots \quad (\text{A.26})$$

The hamiltonian operator is given by  $\mathbb{H}_{qu} = H_{BRST} - i\{K, \Omega\}$ , with  $H_{BRST}$  a BRST-invariant hamiltonian and  $K$  a gauge-fixing fermion.

## B Propagators

Propagators are obtained by inverting the differential operators appearing in the quadratic action  $\frac{1}{\beta} \int_0^1 d\tau \left( \frac{1}{2} g_{\mu\nu} (\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \sum_k \bar{\psi}_{ak} (\partial_\tau + i\theta_k) \psi_k^a + \frac{1}{2} \eta_a \partial_\tau \eta^a \right)$

$$\langle q^\mu(\tau) q^\sigma(\sigma) \rangle = -\beta g^{\mu\nu} \Delta(\tau, \sigma) \quad (\text{B.1})$$

$$\langle a^\mu(\tau) a^\sigma(\sigma) \rangle = \beta g^{\mu\nu} \Delta_{gh}(\tau, \sigma) \quad (\text{B.2})$$

$$\langle b^\mu(\tau) c^\sigma(\sigma) \rangle = -2\beta g^{\mu\nu} \Delta_{gh}(\tau, \sigma) \quad (\text{B.3})$$

$$\langle \psi_k^a(\tau) \bar{\psi}_{k'}^b(\sigma) \rangle = \beta \delta_{kk'} \delta^{ab} \Delta_{AF}(\tau - \sigma, \theta_k) \quad (\text{B.4})$$

$$\langle \eta^a(\tau) \eta^b(\sigma) \rangle = \beta \delta^{ab} \Delta_{AF}(\tau - \sigma, 0) \quad (\text{B.5})$$

with

$$\Delta(\tau, \sigma) = (\tau - 1)\sigma\theta(\tau - \sigma) + (\sigma - 1)\tau\theta(\sigma - \tau) \quad (\text{B.6})$$

$$\Delta_{gh}(\tau, \sigma) = \bullet\bullet\Delta(\tau, \sigma) = \delta(\tau, \sigma) \quad (\text{B.7})$$

and

$$\left(\partial_\tau + i\theta_k\right)\Delta_{AF}(\tau - \sigma, \theta_k) = \delta_A(\tau - \sigma) \tag{B.8}$$

that yields

$$\Delta_{AF}(\tau - \sigma, \theta_k) = \frac{e^{-i\theta_k(\tau - \sigma)}}{2 \cos \frac{\theta_k}{2}} \left( e^{i\theta_k/2}\theta(\tau - \sigma) - e^{-i\theta_k/2}\theta(\sigma - \tau) \right). \tag{B.9}$$

Hence

$$\Delta_{AF}(0, \theta_k) = \frac{i}{2} \tan \frac{\theta_k}{2} \tag{B.10}$$

$$\Delta_{AF}(\tau - \sigma, 0) = \frac{1}{2}\epsilon(\tau - \sigma). \tag{B.11}$$

## C Modular integrals

In this appendix we are going to show the detailed calculation of the modular integrals required to find the Seelley-DeWitt (SDW) coefficients presented in section 3. We will always consider even dimensional spacetime with  $D = 2d$ , and we shall distinguish the two cases of even and odd  $N$ , although the techniques will be the same.

### C.1 Even $N$

We compute the modular integrals for the even  $N = 2n$  case. First of all, we define the modular average of an arbitrary function  $f(\theta_j)$  of the moduli  $\theta_j$ ; by using the measure given in (2.36) and (2.37), and taking into account that modular integrals are even under  $\theta_i \rightarrow 2\pi - \theta_i$ , we have:

$$\langle\langle f(\theta_j) \rangle\rangle_E := \frac{1}{a_0} \frac{2}{n!} \prod_{i=1}^n \int_0^\pi \frac{d\theta_i}{2\pi} \left(2 \cos \frac{\theta_i}{2}\right)^{D-2} \prod_{k<l} \left[ \left(2 \cos \frac{\theta_k}{2}\right)^2 - \left(2 \cos \frac{\theta_l}{2}\right)^2 \right]^2 f(\theta_j) \tag{C.1}$$

where  $a_0$  is the normalization factor giving the degrees of freedom, that ensures  $\langle\langle 1 \rangle\rangle_E = 1$ , and reads

$$a_0 := \frac{2}{n!} \prod_{i=1}^n \int_0^\pi \frac{d\theta_i}{2\pi} \left(2 \cos \frac{\theta_i}{2}\right)^{D-2} \prod_{k<l} \left[ \left(2 \cos \frac{\theta_k}{2}\right)^2 - \left(2 \cos \frac{\theta_l}{2}\right)^2 \right]^2. \tag{C.2}$$

The result for (C.2) is already known from [1], but will be rederived here. Since all the integrals we need will be expressed as generalizations of the Selberg's integral, it is convenient to change variables as  $x_i = \sin^2 \frac{\theta_i}{2}$ , ranging from 0 to 1. The average of a function  $f(x_j) := f(\theta(x_j))$  becomes

$$\langle\langle f(x_j) \rangle\rangle_E := \frac{\mathcal{N}}{a_0} \prod_{i=1}^n \int_0^1 dx_i x_i^{-1/2} (1-x_i)^{d-3/2} \prod_{k<l} (x_k - x_l)^2 f(x_j), \tag{C.3}$$

where

$$\mathcal{N} = \frac{2^{2(d-1)n+(n-1)(2n-1)}}{\pi^n n!}. \tag{C.4}$$

The averages we need to compute can be read down from (3.3), and are

$$\begin{aligned}
 \mathbf{I}_1 &:= \left\langle \left\langle \sum_{i=1}^n \cos^{-2} \frac{\theta_i}{2} \right\rangle \right\rangle_E = \left\langle \left\langle \sum_{i=1}^n \frac{1}{1-x_i} \right\rangle \right\rangle_E, \\
 \mathbf{J} &:= \left\langle \left\langle \sum_{i,j=1}^n \cos^{-2} \frac{\theta_i}{2} \cos^{-2} \frac{\theta_j}{2} \right\rangle \right\rangle_E = \left\langle \left\langle \sum_{i,j=1}^n \frac{1}{(1-x_i)(1-x_j)} \right\rangle \right\rangle_E, \\
 \mathbf{K} &:= \left\langle \left\langle \sum_{i=1}^n \cos^{-4} \frac{\theta_i}{2} \right\rangle \right\rangle_E = \left\langle \left\langle \sum_{i=1}^n \frac{1}{(1-x_i)^2} \right\rangle \right\rangle_E.
 \end{aligned} \tag{C.5}$$

For notational convenience we gave the names  $\mathbf{J}$  and  $\mathbf{K}$  to the corresponding averages, since they will be found as linear combinations of other quantities named  $\mathbf{I}_2$  and  $\mathbf{I}_3$ , in terms of which the SDW coefficients are presented in the paper.

Let us focus now on the factor  $a_0$ , that gives the degrees of freedom of the model. In the  $x_i$  variables it is given by

$$a_0 = \mathcal{N} \prod_{i=1}^n \int_0^1 dx_i x_i^{-1/2} (1-x_i)^{d-3/2} \prod_{k<l} (x_k - x_l)^2. \tag{C.6}$$

There is a well known result by Selberg [63, 64] for such kind of integrals, that gives:

$$S_n(\alpha, \beta) := \prod_{i=1}^n \int_0^1 dx_i x_i^\alpha (1-x_i)^\beta \prod_{k<l} (x_k - x_l)^2 = \prod_{k=1}^n \frac{k! \Gamma(k+\alpha) \Gamma(k+\beta)}{\Gamma(k+n+\alpha+\beta)}, \tag{C.7}$$

from which we obtain, after inserting the factor (C.4) and rearranging the product in (C.7):

$$a_0 = \mathcal{N} S_n\left(-\frac{1}{2}, d - \frac{3}{2}\right) = 2^{n-1} \frac{(2d-2)!}{[(d-1)!]^2} \prod_{k=1}^{n-1} \frac{k(2k-1)! (2k+2d-3)!}{(2k+d-2)! (2k+d-1)!}, \tag{C.8}$$

that indeed coincides with the result found in [1].

To proceed further, let us consider the following generalization of Selberg's integral by Aomoto [63, 65]:

$$\begin{aligned}
 S_{n,1}(\alpha, \beta; t) &:= \prod_{i=1}^n \int_0^1 dx_i x_i^\alpha (1-x_i)^\beta (x_i - t) \prod_{k<l} (x_k - x_l)^2 \\
 &= S_n(\alpha, \beta) \frac{n!}{\prod_k (k+n+\alpha+\beta)} P_n^{(\alpha, \beta)}(1-2t),
 \end{aligned} \tag{C.9}$$

where  $P_n^{(\alpha, \beta)}(1-2t)$  is the Jacobi polynomial of degree  $n$ . By taking a derivative of (C.9) with respect to  $t$ , and evaluating it at  $t=1$  we get very close to the definition of  $\mathbf{I}_1$ , and precisely we have

$$\mathbf{I}_1 = \frac{\mathcal{N}}{a_0} (-)^n \partial_t S_{n,1}\left(-\frac{1}{2}, d - \frac{5}{2}; t\right)|_{t=1} = (-)^n \frac{\partial_t S_{n,1}\left(-\frac{1}{2}, d - \frac{5}{2}; t\right)|_{t=1}}{S_n\left(-\frac{1}{2}, d - \frac{3}{2}\right)}. \tag{C.10}$$

The basic properties of Jacobi polynomials that we need for such calculation are:

$$\begin{aligned} \frac{d^k}{dz^k} P_n^{(\alpha,\beta)}(z) &= \frac{\Gamma(\alpha + \beta + n + 1 + k)}{2^k \Gamma(\alpha + \beta + n + 1)} P_{n-k}^{(\alpha+k,\beta+k)}(z), \\ P_n^{(\alpha,\beta)}(-1) &= (-1)^n \binom{n + \beta}{n}. \end{aligned} \tag{C.11}$$

We can now compute  $\mathbf{I}_1$  by inserting (C.9) into (C.10), and using the relations (C.11) and the result (C.7) we find a quite compact result:

$$\begin{aligned} \mathbf{I}_1 &= (-1)^{n-1} \frac{S_n(-\frac{1}{2}, d - \frac{5}{2})}{S_n(-\frac{1}{2}, d - \frac{3}{2})} \frac{n!(n + d - 2)}{\prod_k (k + n + d - 3)} P_{n-1}^{(1/2, d-3/2)}(1 - 2t)|_{t=1} \\ &= \frac{2n(n + d - 2)}{2d - 3}. \end{aligned} \tag{C.12}$$

We now turn to compute the average  $\mathbf{I}_2$ , defined as

$$\mathbf{I}_2 := \left\langle \left\langle \sum_{i \neq j} \frac{1}{(1 - x_i)(1 - x_j)} \right\rangle \right\rangle_E = \mathbf{J} - \mathbf{K}. \tag{C.13}$$

From the definition of  $S_{n,1}(\alpha, \beta; t)$  in (C.9), it is easy to see that  $\mathbf{I}_2$  is related to its second  $t$  derivative as

$$\mathbf{I}_2 = \frac{\mathcal{N}}{a_0} (-1)^n \partial_t^2 S_{n,1}(-\frac{1}{2}, d - \frac{5}{2}; t)|_{t=1} = (-1)^n \frac{\partial_t^2 S_{n,1}(-\frac{1}{2}, d - \frac{5}{2}; t)}{S_n(-\frac{1}{2}, d - \frac{3}{2})}, \tag{C.14}$$

and in the same way we computed  $\mathbf{I}_1$  we find for  $\mathbf{I}_2$

$$\begin{aligned} \mathbf{I}_2 &= \frac{S_n(-\frac{1}{2}, d - \frac{5}{2})}{S_n(-\frac{1}{2}, d - \frac{3}{2})} \frac{n!(n + d - 2)(n + d - 1)}{\prod_k (k + n + d - 3)} P_{n-2}^{(3/2, d-1/2)}(1 - 2t)|_{t=1} \\ &= 4n(n - 1) \frac{(n + d - 1)(n + d - 2)}{(2d - 1)(2d - 3)}. \end{aligned} \tag{C.15}$$

We need at this point to introduce one further generalization of Selberg's integral, provided by Kaneko [63]:

$$\begin{aligned} K_n(\alpha, \beta; t) &:= \prod_{i=1}^n \int_0^1 dx_i x_i^\alpha (1 - x_i)^\beta (1 - tx_i)^{-1} \prod_{k < l} (x_k - x_l)^2 \\ &= S_n(\alpha, \beta) {}_2F_1(n, n + \alpha; 2n + \alpha + \beta; t), \end{aligned} \tag{C.16}$$

where  ${}_2F_1(a, b; c; t)$  is the Gauss hypergeometric function. By taking two derivatives with respect to  $t$  in (C.16) and evaluating at  $t = 1$  one finds an average that is related to  $\mathbf{K}$  by linear combinations of  $\mathbf{I}_1$  and  $\mathbf{I}_2$ . We shall then define  $\mathbf{I}_3$  as

$$\mathbf{I}_3 := \frac{\mathcal{N}}{a_0} \partial_t^2 K_n(-\frac{1}{2}, d - \frac{1}{2}; t)|_{t=1} = \frac{S_n(-\frac{1}{2}, d - \frac{1}{2})}{S_n(-\frac{1}{2}, d - \frac{3}{2})} \partial_t^2 {}_2F_1(n, n - \frac{1}{2}; 2n + d - 1; t)|_{t=1}. \tag{C.17}$$

In order to perform the computation we need the following properties of the hypergeometric function:

$$\begin{aligned} \frac{d^k}{dz^k} {}_2F_1(a, b; c; z) &= \frac{(a)_k (b)_k}{(c)_k} {}_2F_1(a+k, b+k; c+k; z), \quad (a_k) := \frac{\Gamma(a+k)}{\Gamma(a)}, \\ {}_2F_1(a, b; c; 1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \end{aligned} \quad (\text{C.18})$$

Using now (C.18) in (C.17), we can compute  $\mathbf{I}_3$  that results

$$\begin{aligned} \mathbf{I}_3 &= \frac{S_n(-\frac{1}{2}, d-\frac{1}{2})}{S_n(-\frac{1}{2}, d-\frac{3}{2})} \frac{(n)_2 (n-\frac{1}{2})_2}{(2n+d-1)_2} {}_2F_1(n+2, n+\frac{3}{2}; 2n+d+1; t)|_{t=1} \\ &= \frac{n(n+1)(4n^2-1)}{(2d-3)(2d-5)}. \end{aligned} \quad (\text{C.19})$$

By using the definition (C.16) and taking the double derivative with respect to  $t$  in  $t=1$  one finds that  $\mathbf{K}$  is given as the following linear combination:

$$\mathbf{K} = \frac{1}{2} \mathbf{I}_3 - \frac{1}{2} \mathbf{I}_2 + (n+1) \mathbf{I}_1 - \frac{n(n+1)}{2}, \quad (\text{C.20})$$

whereas, by means of  $\mathbf{I}_2 = \mathbf{J} - \mathbf{K}$ , one has

$$\mathbf{J} = \frac{1}{2} \mathbf{I}_3 + \frac{1}{2} \mathbf{I}_2 + (n+1) \mathbf{I}_1 - \frac{n(n+1)}{2}. \quad (\text{C.21})$$

This concludes our computations of the modular integrals for even  $N = 2n$ . Although the SDW coefficients can be read off straightforwardly from  $\mathbf{I}_1$ ,  $\mathbf{J}$  and  $\mathbf{K}$ , we choose to present them in the paper in terms of  $\mathbf{I}_1$ ,  $\mathbf{I}_2$  and  $\mathbf{I}_3$ , since they have much more compact expressions.

## C.2 Odd $N$

We turn now to compute the modular integrals required for odd  $N = 2n+1$ . The averages needed will have exactly the same structure as the even  $N$  case, the only difference being the form of the modular measure. In particular, the only changes needed will be in the prefactor  $\mathcal{N}$  and in all the generalized Selberg's formulas, where the parameter  $\alpha$  will switch everywhere from  $-\frac{1}{2}$  to  $+\frac{1}{2}$ . The averages in the odd case are explicitly given by

$$\langle\langle f(x_j) \rangle\rangle_{\mathcal{O}} := \frac{\mathcal{N}}{a_0} \prod_{i=1}^n \int_0^1 dx_i x_i^{1/2} (1-x_i)^{d-3/2} \prod_{k<l} (x_k - x_l)^2 f(x_j), \quad (\text{C.22})$$

where we see that the only difference between (C.22) and (C.3) is the power  $1/2$  instead of  $-1/2$ , that is the  $\alpha$  parameter we used in all the previous computations. In addition, the prefactor now reads

$$\mathcal{N} = \frac{2^{2(d-1)+n(2n+2d-3)}}{\pi^n n!}. \quad (\text{C.23})$$

The averages have the same definition as before, being

$$\begin{aligned}
 \tilde{\mathbf{I}}_1 &:= \left\langle \left\langle \sum_{i=1}^n \cos^{-2} \frac{\theta_i}{2} \right\rangle \right\rangle_O = \left\langle \left\langle \sum_{i=1}^n \frac{1}{1-x_i} \right\rangle \right\rangle_O, \\
 \tilde{\mathbf{J}} &:= \left\langle \left\langle \sum_{i,j=1}^n \cos^{-2} \frac{\theta_i}{2} \cos^{-2} \frac{\theta_j}{2} \right\rangle \right\rangle_O = \left\langle \left\langle \sum_{i,j=1}^n \frac{1}{(1-x_i)(1-x_j)} \right\rangle \right\rangle_O, \\
 \tilde{\mathbf{K}} &:= \left\langle \left\langle \sum_{i=1}^n \cos^{-4} \frac{\theta_i}{2} \right\rangle \right\rangle_O = \left\langle \left\langle \sum_{i=1}^n \frac{1}{(1-x_i)^2} \right\rangle \right\rangle_O.
 \end{aligned} \tag{C.24}$$

Again, we will compute  $\tilde{\mathbf{I}}_2$  and  $\tilde{\mathbf{I}}_3$  instead of  $\tilde{\mathbf{J}}$  and  $\tilde{\mathbf{K}}$ . The degrees of freedom factor  $a_0$  now reads

$$a_0 = \mathcal{N} \prod_{i=1}^n \int_0^1 dx_i x_i^{1/2} (1-x_i)^{d-3/2} \prod_{k<l} (x_k - x_l)^2. \tag{C.25}$$

Everything goes in the same way as it did with even  $N$ , and we easily obtain:

$$\begin{aligned}
 a_0 &= \mathcal{N} S_n \left( \frac{1}{2}, d - \frac{3}{2} \right) = \frac{2^{d-2+n} (2d-2)!}{d [(d-1)!]^2} \prod_{k=1}^{n-1} \frac{(k+d-1)(2k+1)!(2k+2d-3)!}{(2k+d-1)!(2k+d)!}, \\
 \tilde{\mathbf{I}}_1 &= \frac{\mathcal{N}}{a_0} (-)^n \partial_t S_{n,1} \left( \frac{1}{2}, d - \frac{5}{2}; t \right) |_{t=1} = \frac{2n(n+d-1)}{2d-3}, \\
 \tilde{\mathbf{I}}_2 &= \frac{\mathcal{N}}{a_0} (-)^n \partial_t^2 S_{n,1} \left( \frac{1}{2}, d - \frac{5}{2}; t \right) |_{t=1} = \frac{4n(n-1)(n+d)(n+d-1)}{(2d-1)(2d-3)}, \\
 \tilde{\mathbf{I}}_3 &= \frac{\mathcal{N}}{a_0} \partial_t^2 K_n \left( \frac{1}{2}, d - \frac{1}{2}; t \right) |_{t=1} = \frac{n(n+1)(2n+1)(2n+3)}{(2d-3)(2d-5)}.
 \end{aligned} \tag{C.26}$$

Also the relations that give  $\tilde{\mathbf{J}}$  and  $\tilde{\mathbf{K}}$  remain unchanged and are

$$\begin{aligned}
 \tilde{\mathbf{K}} &= \frac{1}{2} \tilde{\mathbf{I}}_3 - \frac{1}{2} \tilde{\mathbf{I}}_2 + (n+1) \tilde{\mathbf{I}}_1 - \frac{n(n+1)}{2}, \\
 \tilde{\mathbf{J}} &= \frac{1}{2} \tilde{\mathbf{I}}_3 + \frac{1}{2} \tilde{\mathbf{I}}_2 + (n+1) \tilde{\mathbf{I}}_1 - \frac{n(n+1)}{2}.
 \end{aligned} \tag{C.27}$$

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