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Spinning particles and higher spin fields on (A)dS backgrounds

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Abstract: Spinning particle models can be used to describe higher spin fields in first quantization. In this paper we discuss how spinning particles with gauged O(N) supersymmetries on the worldline can be consistently coupled to conformally flat spacetimes, both at the classical and at the quantum level. In particular, we consider canonical quantization on flat and on (A)dS backgrounds, and discuss in detail how the constraints due to the worldline gauge symmetries produce geometrical equations for higher spin fields, i.e. equations written in terms of generalized curvatures. On flat space the algebra of constraints is linear, and one can integrate part of the constraints by introducing gauge potentials. This way the equivalence of the geometrical formulation with the standard formulation in terms of gauge potentials is made manifest. On (A)dS backgrounds the algebra of constraints becomes quadratic, nevertheless one can use it to extend much of the previous analysis to this case. In particular, we derive general formulas for expressing the curvatures in terms of gauge potentials and discuss explicitly the cases of spin 2, 3 and 4.

Keywords: Gauge Symmetry, Supergravity Models, Sigma Models.

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Contents

1. Introduction

In a previous paper [1] we have discussed the worldline quantization of massless higher spin fields, considering in particular those fields that are described by spinning particle models with gauged O(N) supersymmetries on the worldline [2−4] (which include all $D = 4$ higher spin fields). We calculated the one-loop effective action in flat space, that contains the information on the number of physical degrees of freedom propagating in the loop. This result was achieved by computing the path integral of the O(N) spinning particle on the circle.

To obtain more information on the quantum theory of higher spin fields in a first quantized approach, it is desirable to couple the spinning particles to more general backgrounds other than flat spacetime or, equivalently, to introduce suitable vertex operators to describe couplings to external particles. However, this program has to face with the notorious difficulty of introducing interactions for higher spin fields.[1] This difficulty is evident also from the sigma model point of view. In fact, it was shown in [3] that for $N > 2$ standard supersymmetry transformation rules leave the spinning particle action invariant only if the target spacetime is flat. The standard supersymmetry transformation rules for the worldline supergravity multiplet used in [3] were purely geometrical, and assumed that they would not involve the particle coordinates and corresponding fermionic partners. The situation was improved in [8], where it was realized how to couple the spinning particle to maximally symmetric spaces, namely (A)dS spaces. The construction presented in [8] made use of the conformal invariance of the spinning particle which was discovered by Siegel, who embedded the model in a flat target space with two extra dimensions to keep conformal invariance manifest [9] (this embedding had already been used by Marnelius for the case of $N = 0, 1$ [10]). This construction implied that the susy transformation rule of the supergravity multiplet had to be more general than the one used in [3], and could include the particle coordinates and the corresponding fermionic partners.

In this paper we perform a canonical analysis to study the couplings to curved spaces, and we are able to extend the known results to include couplings to arbitrary conformally flat spaces. This finding can be understood in a simple way: noticing that the spinning particle action is invariant under a Weyl rescaling of the background target space metric is sufficient to guarantee consistent propagation on conformally flat manifolds. The couplings to this class of curved spaces, even if mild, is presumably not negligible, as one may expect some kind of conformal anomaly to give rise to a nontrivial one loop effective action (more general than the one computed in [1]). With this future application in mind, we proceed to study the canonical quantization of the model. A canonical analysis is needed also to provide sufficient data for fixing the counterterms that may arise when computing the corresponding path integral in curved spaces [11, 12], see in particular [13 − 15] for the $N = 0, 1, 2$ spinning particle cases, respectively.

Canonical quantization allows to identify the correct field equations one is describing in first quantization. In the present case it allows to make contact with the classical description of higher spin fields in the so-called geometrical formulation, dynamical equations originally proposed in [16, 17] which make use of the higher spin curvatures constructed in [18, 19] (see [5] for reviews). This relation is seen by recalling that gauge symmetries give rise to first class constraints that select physical states from the Hilbert space. In flat space the constraints of the O(N) spinning particle produce equations of motion written in terms of tensors that are interpreted as generalized curvatures describing higher spin fields. Gauge potentials can be introduced by integrating a subset of these equations (those corresponding to the Bianchi identities). This way one sees how the worldline approach reproduces and unifies various constructions that have appeared in the recent literature on higher spin fields, like the use of compensators to relax trace constraints [17, 20] or the use of generalized Poincaré lemmas to integrate the Bianchi identities [21 − 24] and prove the equivalence with

[1]See for example [5] for a general introduction to the classical theory of higher spin fields, and [6] which reviews and studies the problem of coupling spin 2 to higher spin particles in four dimensions (see also [7] for a recent analysis).

the standard formulation of Fronsdal and Labastida [25, 26] (see [5] for a list of references and discussions of related works). We present the analysis in arbitrary dimensions D, but only for the case of even N, i.e. for particles with integer spin $s = \frac{N}{2}$. Extension to the odd N case should proceed in a similar fashion.

Then we analyze the constraint equations in the case of (A)dS spaces. The algebra of constraints is again first class, but the algebra closes only quadratically. It is interesting to note that this algebra coincides with the zero mode sector of the Bershadsky-Knizhnik SO(N)-extended superconformal algebra in two dimensions [27, 28]. The constraints produce again geometrical equations of motion for the higher spin curvatures on (A)dS spaces. Quadratic closure complicates the algebraic structure, which nevertheless remains of valuable help. In fact, we use it to express the curvatures in terms of higher spin gauge potentials. Then, we consider in detail the cases of spin $s = 2, 3, 4$, with the $s = 2$ case corresponding to the familiar case of the graviton if $D = 4$. Quadratic algebras have appeared before in the description of higher spin fields, see for example [29, 20].

Though not discussed in this paper, one may find in the literature other particle models related to higher spin fields, like the twistor-like particle of refs. [30, 31, 24] or particles that could be constructed using the OSp quantum mechanics of ref. [32]. The same BRST approach of refs. [33] used to describe higher spin field equations can perhaps be related to a particle model. In the following we shall structure our paper as indicated in the table of content.

2. The O(N) spinning particle

In this section we first review the classical formulation of the spinning particle propagating in Minkowski space. Then, we proceed to describe the coupling to conformally flat spaces.

2.1 Minkowski space

It will be useful to present the O(N) spinning particle action directly in phase space. The dynamical variables are given by: the cartesian coordinates x^μ of the particle moving in a D dimensional Minkowski space, their conjugate momenta p_μ, and N real Grassmann variables with spacetime vector indices ψ_i^μ ($i = 1, \ldots, N$). The Minkowski metric $\eta_{\mu\nu} \sim (-, +, \ldots, +)$ is used to raise and lower spacetime indices. In addition, there is an O(N)-extended supergravity on the worldline, whose gauge fields are given by the einbein e, the gravitinos χ_i, and the SO(N) gauge field a_{ij}. The action which defines the model is given by

$$S = \int dt \Big[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{i\mu}\dot{\psi}_i^\mu - e \underbrace{\left(\frac{1}{2}p_\mu p^\mu\right)}_{H} - i\chi_i \underbrace{\left(p_\mu \psi_i^\mu\right)}_{Q_i} - \frac{1}{2}a_{ij} \underbrace{\left(i\psi_i^\mu \psi_{j\mu}\right)}_{J_{ij}} \Big] \tag{2.1}$$

where H, Q_i, J_{ij} denote the first class constraints gauged by the fields e, χ_i, a_{ij}. The kinetic term defines the phase space symplectic form and fixes the graded Poisson brackets: $\{x^\mu, p_\nu\}_{\text{PB}} = \delta_\nu^\mu$ and $\{\psi_i^\mu, \psi_j^\nu\}_{\text{PB}} = -i\eta^{\mu\nu}\delta_{ij}$. With these brackets one can easily compute the constraint algebra at the classical level

$$\{Q_i, Q_j\}_{\text{PB}} = -2i\delta_{ij}H$$

$$
\begin{aligned}
\{J_{ij}, Q_k\}_{\text{PB}} &= \delta_{jk} Q_i - \delta_{ik} Q_j \\
\{J_{ij}, J_{kl}\}_{\text{PB}} &= \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk}
\end{aligned}
\tag{2.2}
$$

which is first class and thus gauged consistently by the fields e, χ_i, a_{ij}. This algebra is known as the O(N)-*extended susy algebra*: it has N susy charges Q_i which close on the Hamiltonian H and which transform in the vector representation of SO(N), whose Lie algebra is described by the last line. We now discuss the various symmetries of the model.

The gauge symmetries are those of the O(N)-extended supergravity on the worldline, whose infinitesimal gauge transformations with parameters $\xi, \epsilon_i, \alpha_{ij}$ are given by

$$
\begin{aligned}
\delta x^\mu &= \{x^\mu, G\}_{\text{PB}} = \xi p^\mu + i\epsilon_i \psi_i^\mu \\
\delta p_\mu &= \{p_\mu, G\}_{\text{PB}} = 0 \\
\delta \psi_i^\mu &= \{\psi_i^\mu, G\}_{\text{PB}} = -\epsilon_i p^\mu + \alpha_{ij} \psi_j^\mu \\
\delta e &= \dot{\xi} + 2i\chi_i \epsilon_i \\
\delta \chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j \\
\delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im} a_{mj} + \alpha_{jm} a_{im}
\end{aligned}
\tag{2.3}
$$

where $G \equiv \xi H + i\epsilon_i Q_i + \frac{1}{2}\alpha_{ij} J_{ij}$ denotes the generator of gauge transformations. One could add trivial symmetries proportional to the equations of motion to present the worldline diffeomorphisms in the standard geometrical form, but this is not so natural in the hamiltonian formalism.

The rigid symmetries include transformations under the Poincaré group of target space, which guarantees the relativistic invariance of model. They are given by

$$
\delta x^\mu = \omega^\mu{}_\nu x^\nu + a^\mu\,, \quad \delta p_\mu = \omega_\mu{}^\nu p_\nu\,, \quad \delta \psi_i^\mu = \omega^\mu{}_\nu \psi_i^\nu \tag{2.4}
$$

where $\omega^\mu{}_\nu$ and a^μ specify infinitesimal Lorentz rotations and spacetime translations, respectively. The worldline gauge fields are left invariant by these symmetries.

In addition, the model is conformal invariant. To prove this we first show that the model has background symmetries[2] corresponding to: (*i*) diffeomorphisms, (*ii*) local Lorentz transformations, (*iii*) Weyl rescalings of the flat target space metric. Then, conformal Killing vectors, which by definition leave invariant the background metric, identify rigid symmetries of the model. They generate the conformal group SO(D, 2).

To discuss these background symmetries we find it convenient to rewrite the action (2.1) using arbitrary coordinates, denoted again by x^μ. We also denote the Minkowski metric in arbitrary coordinates by $g_{\mu\nu}$. Then we introduce an orthonormal tangent frame specified by the vielbein $e_\mu{}^a$ and use $\psi_i^a \equiv \psi_i^\mu e_\mu{}^a(x)$ as independent variables. Given the vielbein one may construct the unique spin connection $\omega_{\mu ab}$, which enters the definition of the covariant momenta

$$
\pi_\mu = p_\mu - \frac{i}{2}\omega_{\mu ab}\psi_i^a \psi_i^b\,. \tag{2.5}
$$

[2]These are symmetries in which also the background fields, like the spacetime metric, transform.

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The coefficient in front of the spin connection is easily fixed by requiring the covariance condition

$$\{\pi_\mu, \pi_\nu\}_{\rm PB} = \frac{i}{2} R_{\mu\nu ab} \psi_i^a \psi_i^b \tag{2.6}$$

so that in flat space the covariant momenta commute. With these tools at hand the action (2.1) can be rewritten in the form

$$S = \int dt \Big[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{ia}\dot{\psi}_i^a - e \underbrace{\left(\frac{1}{2} g^{\mu\nu}\pi_\mu \pi_\nu\right)}_{H} - i\chi_i \underbrace{\left(\psi_i^a e_a{}^\mu \pi_\mu\right)}_{Q_i} - \frac{1}{2} a_{ij} \underbrace{\left(i\psi_i^a \psi_{ja}\right)}_{J_{ij}} \Big] . \tag{2.7}$$

We are now ready to discuss its background symmetries:

(i) Diffeomorphisms of target space are identified quite easily. The coordinates transform as usual, $x^\mu \to x^{\mu\prime}(x)$, the momenta as a 1-form, $p_\mu \to p_\mu{}' = p_\nu \frac{\partial x^\nu}{\partial x^{\mu\prime}}$, and the background fields $g_{\mu\nu}, e_\mu{}^a, \omega_{\mu ab}$ as tensors as indicated by their coordinate indices. The fermions ψ_i^a are left invariant, just like the supergravity gauge fields e, χ_i, a_{ij}. These transformations are easily seen to be an invariance of the action.

(ii) Proving local Lorentz invariance is slightly more difficult. An infinitesimal local Lorentz transformation is specified by the parameters $\lambda^{ab}(x) = -\lambda^{ba}(x)$. It leaves the coordinates x^μ invariant and transforms the worldline fermions as vectors

$$\delta\psi_i^a = \lambda^a{}_b(x)\psi_i^b \ . \tag{2.8}$$

The symplectic term of the action is left invariant if one assigns to the momenta the transformation rule

$$\delta p_\mu = -\frac{i}{2}\partial_\mu \lambda_{ab}(x)\psi_i^a \psi_i^b \ . \tag{2.9}$$

The background fields $g_{\mu\nu}, e_\mu{}^a, \omega_{\mu ab}$ transform as usual under local Lorentz transformations, and in particular the spin connection transforms as the local Lorentz gauge field

$$\delta\omega_\mu{}^{ab} = -\partial_\mu \lambda^{ab} + \lambda^a{}_c\,\omega_\mu{}^{cb} + \lambda^b{}_c\,\omega_\mu{}^{ac} \ . \tag{2.10}$$

As a consequence the covariant momentum π_μ is left invariant. Therefore the full action is invariant.

(iii) Finally, let us prove invariance under Weyl rescalings of the target space metric. Under an infinitesimal Weyl rescaling specified by the local parameter $\phi(x)$, which is a function of target space, the background fields transform as

$$\delta g_{\mu\nu} = 2\phi\, g_{\mu\nu} \ , \quad \delta e_\mu{}^a = \phi\, e_\mu{}^a \ , \quad \delta\omega_\mu{}^{ab} = (e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a)\nabla^\nu\phi \ . \tag{2.11}$$

As a consequence the covariant momentum transforms as

$$\delta\pi_\mu = -i\psi_{\mu i}\psi_i^\nu \partial_\nu \phi \tag{2.12}$$

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and the constraints as

$$\begin{aligned} \delta Q_i &= -\phi\, Q_i - J_{ij}\psi_j^\mu \partial_\mu \phi \\ \delta H &= -2\phi\, H + i\psi_i^\mu \partial_\mu \phi\, Q_i \; . \end{aligned} \tag{2.13}$$

These transformations can be compensated by suitable transformations on the world-line gauge fields

$$\begin{aligned} \delta e &= 2\phi\, e \\ \delta \chi_i &= -e\psi_i^\mu \partial_\mu \phi + \chi_i \phi \\ \delta a_{ij} &= i(\chi_i \psi_j^\mu - \chi_j \psi_i^\mu)\partial_\mu \phi \end{aligned} \tag{2.14}$$

while the variables x^μ, p_μ, ψ_i^a are taken to be invariant. This proves Weyl invariance.

Because of these background symmetries, conformal Killing vectors necessarily produce global symmetries. In fact, the conformal Killing vectors are precisely those vector fields ξ^μ that generate infinitesimal diffeomorphisms whose effect on the metric and on the vielbein can be compensated by suitable Weyl and local Lorentz transformations,

$$\begin{aligned} \delta g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu} + 2\phi g_{\mu\nu} = 0 \\ \delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a + \phi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b = 0 \end{aligned} \tag{2.15}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative acting along the vector field ξ^μ. As the background fields are left untransformed, the conformal Killing vectors induce rigid symmetries of the action (2.7). They generate the conformal group $\mathrm{SO}(D,2)$, which extend the Poincaré group to include scale transformations and conformal boosts.

An additional bonus of the background Weyl symmetry is that it guarantees that the $\mathrm{O}(N)$ spinning particle propagates consistently on arbitrary conformally flat manifolds. These spaces include the class of maximally symmetric spaces, i.e. the (A)dS spaces, which were shown to be consistent backgrounds for the spinning particle in [8], but are more general.

Before closing, let us report the finite Weyl transformations leaving the action invariant. They are given by

$$g'_{\mu\nu} = \mathrm{e}^{2\phi} g_{\mu\nu}\,, \quad e_\mu{}^{a\prime} = \mathrm{e}^{\phi} e_\mu{}^a\,, \quad \omega_\mu{}^{ab\prime} = \omega_\mu{}^{ab} + (e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a)\nabla^\nu \phi\,, \tag{2.16}$$

implying

$$\begin{aligned} Q_i{}' &= \mathrm{e}^{-\phi}\big(Q_i - J_{ij}\psi_j^\mu \partial_\mu \phi\big) \\ H' &= \mathrm{e}^{-2\phi}\bigg(H - iQ_i \psi_i^\mu \partial_\mu \phi - \frac{i}{2} J_{ij}\psi_i^\mu \partial_\mu \phi \psi_j^\nu \partial_\nu \phi\bigg)\,, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} e' &= \mathrm{e}^{2\phi} e \\ \chi_i{}' &= \mathrm{e}^{\phi}\big(\chi_i - e\psi_i^\mu \partial_\mu \phi\big) \\ a_{ij}{}' &= a_{ij} + i(\chi_i \psi_j^\mu - \chi_j \psi_i^\mu)\partial_\mu \phi - ie\psi_i^\mu \partial_\mu \phi \psi_j^\nu \partial_\nu \phi \; . \end{aligned} \tag{2.18}$$

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2.2 Conformally flat spaces

As just discussed, the background Weyl symmetry implies that the spinning particle is consistent on any conformally flat spacetime. In this section we verify this claim by direct canonical analysis.

The form of the action is the same as the one reported in eq. (2.7)

$$S = \int dt \Big[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{ia}\dot{\psi}_i^a - e \underbrace{\left(\frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu\right)}_{H_0} - i\chi_i \underbrace{\left(\psi_i^a e_a{}^\mu \pi_\mu\right)}_{Q_i} - \frac{1}{2}a_{ij} \underbrace{\left(i\psi_i^a\psi_{ja}\right)}_{J_{ij}} \Big] \qquad (2.19)$$

but we have renamed the hamiltonian as H_0 in view of convenient redefinitions to be introduced later. We will start assuming an arbitrary metric $g_{\mu\nu}$, and verify that the constraints H_0, Q_i, J_{ij} continue to form a first class algebra on spaces that are conformally flat, so that by assigning suitable transformation rules to the gauge fields e, χ_i, a_{ij} the action keeps on being gauge invariant.

As anticipated, it is instructive to begin by considering generic curved spaces. Apart from the SO(N) subalgebra generated by the J_{ij}, which remains unmodified, one obtains the following algebra

$$\begin{aligned} \{Q_i, Q_j\}_{\rm PB} &= -2i\delta_{ij}H_0 + \frac{i}{2}R_{abcd}\psi_i^a\psi_j^b\psi^c\cdot\psi^d \\ \{Q_i, H_0\}_{\rm PB} &= -\frac{i}{2}\pi^a R_{abcd}\psi_i^b\psi^c\cdot\psi^d \end{aligned} \qquad (2.20)$$

which generically fails to be first class. Of course, one could try to add new constraints to force the algebra to close, but this may overconstrain the system.

An option, that in the light of the previous analysis is guaranteed to work, is to restrict attention to conformally flat spaces. These spaces have a vanishing Weyl tensor, which allows to solve the Riemann tensor in terms of the Ricci tensor and curvature scalar

$$\begin{aligned} R_{abcd} &= \frac{1}{(D-2)}\big(\eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac}\big) \\ &\quad - \frac{R}{(D-2)(D-1)}\big(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\big)\ . \end{aligned} \qquad (2.21)$$

Substituting this relation into (2.20) produces

$$\begin{aligned} \{Q_i, Q_j\}_{\rm PB} &= -2i\delta_{ij}H_0 - \frac{iR}{(D-2)(D-1)}J_{ik}J_{jk} - \frac{R_{ab}}{(D-2)}\Big(\psi_i^a\psi_k^b J_{jk} + (i\leftrightarrow j)\Big) \\ \{Q_i, H_0\}_{\rm PB} &= \frac{R}{(D-2)(D-1)}Q_k J_{ki} + \frac{R_{ab}}{(D-2)}\Big(\pi^a\psi_k^b J_{ik} + i\psi_i^a\psi_k^b Q_k\Big) \end{aligned} \qquad (2.22)$$

which becomes first class, though with structure functions rather than structure constants. This is enough to guarantee consistency of the gauge system at the classical level, see for example [34].

It may be convenient, especially when considering maximally symmetric spaces, to redefine the hamiltonian as

$$H = H_0 + \Delta H = \frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu - \frac{1}{8}R_{abcd}\psi^a\cdot\psi^b\psi^c\cdot\psi^d \qquad (2.23)$$

so that on general curved spaces the algebra (2.20) takes the form

$$\begin{aligned}
\{Q_i, Q_j\}_{\text{PB}} &= -2i\delta_{ij} H + \frac{i}{2} R_{abcd} \Big(\psi_i^a \psi_j^b - \frac{1}{2}\delta_{ij} \psi^a \cdot \psi^b \Big) \psi^c \cdot \psi^d \\
\{Q_i, H\}_{\text{PB}} &= \frac{1}{8} \psi_i^e \nabla_e R_{abcd} \psi^a \cdot \psi^b \psi^c \cdot \psi^d \ .
\end{aligned} \tag{2.24}$$

Written in this way one sees that the second Poisson bracket vanishes on locally symmetric spaces, but the first one remains second class. Thus, the model is inconsistent on generic curved spaces for $N > 2$ (while for $N \leq 2$ one can show that the offending terms vanish). On conformally flat spaces these relations simplify to

$$\begin{aligned}
\{Q_i, Q_j\}_{\text{PB}} &= -2i\delta_{ij} H + \frac{iR}{(D-2)(D-1)} \Big(\frac{1}{2}\delta_{ij} J_{kl} J_{kl} - J_{ik} J_{jk} \Big) \qquad (2.25) \\
&\quad - \frac{R_{ab}}{(D-2)} \big(\psi_i^a \psi_k^b J_{jk} + \psi_j^a \psi_k^b J_{ik} - \delta_{ij} \psi_k^a \psi_l^b J_{kl} \big) \\
\{Q_i, H\}_{\text{PB}} &= -\frac{1}{4(D-2)(D-1)} \psi_i^c \nabla_c R J_{kl} J_{kl} + \frac{i}{2(D-2)} \psi_i^c \nabla_c R_{ab} \psi_k^a \psi_l^b J_{kl}
\end{aligned}$$

with

$$H = H_0 + \frac{R}{4(D-2)(D-1)} J_{ij} J_{ij} - \frac{iR_{ab}}{2(D-2)} \psi_i^a \psi_j^b J_{ij} \ . \tag{2.26}$$

The corresponding action on conformally flat spaces

$$S = \int dt \Big[p_\mu \dot{x}^\mu + \frac{i}{2} \psi_{ia} \dot{\psi}_i^a - eH - i\chi_i Q_i - \frac{1}{2} a_{ij} J_{ij} \Big] \tag{2.27}$$

is then gauge invariant under suitable transformation rules generated by the constraints and their structure functions. We refrain from presenting them here. Of course this form of the action is equivalent to the one given in (2.19), as can be seen by redefining the gauge field $a_{ij} \to a_{ij} - eF_{ij}(x, \psi)$ which is needed to pass from the form with H to the one with H_0 (the explicit expression of $F_{ij}(x, \psi)$ is easily obtained by using eqs. (2.23) and (2.21)).

All these expressions simplify further on maximally symmetric spaces, the (A)dS spaces, which are a subset of conformally flat spaces. As we are going to treat the canonical quantization of these cases in some detail, it may be useful to report the corresponding classical formulas. The Riemann tensor for maximally symmetric spaces is of the form

$$R_{abcd} = b(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \tag{2.28}$$

where the constant b is related to the curvature scalar by $b = \frac{R}{D(D-1)}$. The improved hamiltonian now reads as

$$H = H_0 + \Delta H = \frac{1}{2} \pi^a \pi_a - \frac{b}{4} J_{ij} J_{ij} \tag{2.29}$$

and the complete gauge algebra, including the J_{ij} charges, has the following nonvanishing Poisson brackets

$$\{Q_i, Q_j\}_{\text{PB}} = -2i\delta_{ij} H + ib \Big(J_{ik} J_{jk} - \frac{1}{2} \delta_{ij} J_{kl} J_{kl} \Big)$$

$$\begin{aligned}
\{J_{ij}, Q_k\}_{\rm PB} &= \delta_{jk}Q_i - \delta_{ik}Q_j \\
\{J_{ij}, J_{kl}\}_{\rm PB} &= \delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk} \ .
\end{aligned} \tag{2.30}$$

It is a quadratic deformation of the linear algebra in (2.2), with b playing the role of deforming parameter. It is interesting to note that this algebra reproduces the (classical version) of the zero mode sector of certain two-dimensional nonlinear superconformal algebras introduced some time ago by Bershadsky and Knizhnik [27, 28]. The corresponding action (2.27) is invariant under transformation rules that can be easily derived using the constraints and their structure functions. We list them here, as they might be useful in discussing gauge fixing issues

$$\begin{aligned}
\delta x^\mu &= \{x^\mu, G\}_{\rm PB} = \xi\pi^\mu + i\epsilon_i\psi_i^\mu \\
\delta p_\mu &= \{p_\mu, G\}_{\rm PB} = (\xi\pi^a + i\epsilon_k\psi_k^a)\Big(\frac{i}{2}\partial_\mu\omega_{abc}\psi_i^b\psi_i^c - p_\nu\partial_\mu e_a{}^\nu\Big) \\
\delta\psi_i^a &= \{\psi_i^a, G\}_{\rm PB} = -(\xi\pi^b + i\epsilon_k\psi_k^b)\omega_{bac}\psi_i^c - \epsilon_i\pi^a + (\alpha_{ij} - \xi b J_{ij})\psi_j^a \\
\delta e &= \dot\xi + 2i\chi_i\epsilon_i \\
\delta\chi_i &= \dot\epsilon_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j \\
\delta a_{ij} &= \dot\alpha_{ij} + \alpha_{im}a_{mj} + \alpha_{jm}a_{im} + ib\big(\chi_k\epsilon_k J_{ij} + \sigma(\epsilon_i\chi_k J_{kj} - \epsilon_j\chi_k J_{ki}) \\
&\qquad + (1-\sigma)(\epsilon_k J_{kj}\chi_i - \epsilon_k J_{ki}\chi_j)\big)
\end{aligned} \tag{2.31}$$

where the free parameter $\sigma \in [0,1]$ labels different choices of splitting the algebra in structure functions and generators.

This hamiltonian formulation of the spinning particle on (A)dS spaces is equivalent to the lagrangian formulation discussed by Kuzenko and Yarevskaya in [8].

3. Canonical quantization

In this section we study canonical quantization of the spinning particle on the class of spaces just discussed. Phase space variables become operators and the problem is to find the correct ordering that preserves the first class property of the constraints. As we shall discuss, this requirement introduces quantum corrections to the classical hamiltonian as well. The quantum constraint equations are then used to select the physical sector of the Hilbert space, and are interpreted as field equations for higher spin fields.

3.1 Minkowski space

Let us briefly review canonical quantization for the O(N) spinning particle in flat space, which is best carried out using cartesian coordinates. The fundamental (anti) commutation relations are obtained from the corresponding classical Poisson brackets and read (from now on all variables are operators)

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \,, \qquad \{\psi_i^\mu, \psi_j^\nu\} = \eta^{\mu\nu}\delta_{ij} \ . \tag{3.1}$$

This operator algebra is realized irreducibly on a Hilbert space which contains also unphysical states. The physical states are obtained à la Dirac-Gupta-Bleuler by requiring the constraints to annihilate them. Of course, the quantum constraints are constructed from the classical ones by specifying a suitable ordering plus possible quantum corrections. In the case of flat spacetime, one only needs to specify the correct ordering in the definition of the SO(N) generators, as there are no other ordering ambiguities. Taking that into account, the quantum constraint are given by

$$H = \frac{1}{2} p_\mu p^\mu \,, \qquad Q_i = p_\mu \psi_i^\mu \,, \qquad J_{ij} = \frac{i}{2}[\psi_i^\mu, \psi_{j\mu}] \tag{3.2}$$

and satisfy the quantum algebra

$$\{Q_i, Q_j\} = 2\delta_{ij} H \tag{3.3}$$

$$[J_{ij}, Q_k] = i\delta_{jk} Q_i - i\delta_{ik} Q_j \tag{3.4}$$

$$[J_{ij}, J_{kl}] = i\delta_{jk} J_{il} - i\delta_{ik} J_{jl} - i\delta_{jl} J_{ik} + i\delta_{il} J_{jk} \tag{3.5}$$

which is first class. The corresponding constraints give rise to higher spin field equations [2 – 4], in the form originally developed by Bargmann and Wigner. These equations are described by a multispinor $\Psi_{\alpha_1,\ldots,\alpha_N}$ that satisfies a Dirac equation in each index and, in addition, suitable algebraic constraints which project onto the irreducible spin $\frac{N}{2}$ components [35]. We shall discuss these equations in a different basis for the case of even N (integer spin) in section 4. The alternative BRST quantization for this model is described in refs. [36] and [37]. In particular in [37] one finds its use to construct second quantized actions for any spin in flat spaces of arbitrary dimensions.

3.2 Conformally flat spaces

The classical structure presented in section 2.2 carries over to the quantum theory after specifying the correct orderings that preserve the symmetries of the model. It is again useful to discuss first the case of generic curved spaces, and then restrict to conformally flat spaces which will be shown to admit a first class constraint algebra.

The quantum algebra of the fundamental operators now reads as

$$[x^\mu, p_\nu] = i\delta_\nu^\mu \,, \qquad \{\psi_i^a, \psi_j^b\} = \eta^{ab}\delta_{ij} \tag{3.6}$$

since worldline fermions with flat indices are taken as fundamental variables. The correct ordering of the SO(N) currents is again immediate

$$J_{ij} = \frac{i}{2}[\psi_i^a, \psi_{ja}] \,. \tag{3.7}$$

The susy charges are also ordered uniquely as follows[3]

$$Q_i = \psi_i^a e_a{}^\mu \left(p_\mu - \frac{i}{2} \omega_{\mu bc} \psi_j^b \psi_j^c \right) \,. \tag{3.8}$$

[3]For notational simplicity we use nonhermitian operators Q_i. Hermiticity is obtained by a similarity transformation $A \to g^{\frac{1}{4}} A g^{-\frac{1}{4}}$ on the quantum variables, so that hermitian operators Q_i (as well as H) are obtained by substituting $p_\mu \to g^{\frac{1}{4}} p_\mu g^{-\frac{1}{4}}$, see for example [11].

To understand why this covariantization is unique, one may recall that it corresponds to the unique covariant derivative acting on a multispinorial wave function.

Before proceeding, it may be useful to introduce the hermitian Lorentz generators

$$M^{ab} = \frac{i}{2}[\psi_j^a, \psi_j^b] \tag{3.9}$$

which satisfy the Lorentz algebra and commute with the SO(N) generators

$$\begin{aligned}
[M^{ab}, M^{cd}] &= i\eta^{bc}M^{ad} - i\eta^{bd}M^{ac} - i\eta^{ac}M^{bd} + i\eta^{ad}M^{bc} \\
[M^{ab}, J_{ij}] &= 0 \ .
\end{aligned} \tag{3.10}$$

Then one can write the covariant momentum in the form $\pi_\mu = p_\mu - \frac{1}{2}\omega_{\mu ab}M^{ab}$ and the susy charges as $Q_i = \psi_i^a e_a{}^\mu \pi_\mu = \psi_i^a \pi_a$.

At this point one may start checking the algebra on generic curved spaces and identify a suitable hamiltonian operator. Equations (3.4) and (3.5) are left unmodified, but the other (anti)commutators produce

$$\{Q_i, Q_j\} = 2\delta_{ij}H_0 + \frac{i}{2}\psi_i^a\psi_j^b R_{abcd}M^{cd} \tag{3.11}$$

$$[Q_i, H_0] = \frac{1}{2}R_{ab}\psi_i^a\pi^b + \frac{i}{2}R_{abcd}\psi_i^a M^{cd}\pi^b - \frac{1}{2}\nabla_a R_{bc}\psi_i^c M^{ab} \tag{3.12}$$

where

$$H_0 = \frac{1}{2}\big(\pi^a\pi_a - i\omega^a{}_{ab}\pi^b\big) \tag{3.13}$$

corresponds to the minimal quantum covariantization of the classical operator appearing in (2.19): in particular, the second term in H_0 is a quantum correction which guarantees covariance. As in the classical case, also in the quantum case the algebra fails to be first class, implying a generic inconsistency on arbitrary spaces.

Thus, we restrict to conformally flat spaces. Using the relation (2.21) for the Riemann tensor on conformally flat spaces, we obtain the quantum version of (2.22) which takes the form

$$\begin{aligned}
\{Q_i, Q_j\} &= 2\delta_{ij}H - \frac{i}{(D-2)}R_{ab}\big(\psi_i^a\psi_k^b J_{jk} + \psi_j^a\psi_k^b J_{ik} - \delta_{ij}\psi_k^a\psi_l^b J_{kl}\big) \\
&\quad + \frac{1}{2(D-1)(D-2)}R\big(J_{ik}J_{jk} + J_{jk}J_{ik} - \delta_{ij}J_{kl}J_{kl}\big) \\
[Q_i, H] &= \frac{1}{4(D-1)}\nabla_a R\,\psi_k^a J_{ik} - \frac{i}{4(D-1)(D-2)}\nabla_a R\,\psi_i^a J_{jk}J_{jk} \\
&\quad - \frac{1}{2(D-2)}\nabla_a R_{bc}\,\psi_i^a\psi_j^b\psi_k^c\ J_{jk}
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
H &= H_0 + \frac{1}{8}R_{abcd}M^{ab}M^{cd} - \frac{(N-2)(D+N-2)}{16(D-1)}\,R \\
&= H_0 + \frac{1}{4(D-1)(D-2)}\,RJ_{jk}J_{jk} - \frac{i}{2(D-2)}R_{ab}\,\psi_j^a\psi_k^b\,J_{jk} + \frac{(D+N-2)}{8(D-1)}\,R
\end{aligned} \tag{3.15}$$

with H_0 as in (3.13). The result is that, with a suitable quantum redefinition of the hamiltonian H, the algebra closes and becomes first class. The last term in both expressions of H, proportional to the scalar curvature, is a quantum effect that did not appear in the corresponding classical expressions (2.23) and (2.26). This final result proves the quantum consistency of the model on conformally flat spaces.

3.3 (A)dS spaces

The subset of maximally symmetric spaces, characterized by a Riemann tensor of the form $R_{abcd} = b(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$, is much simpler. In fact, the above algebra simplifies further and we summarize here the set of quantum constraints appropriate for (A)dS spaces

$$\begin{aligned} J_{ij} &= \frac{i}{2}[\psi_i^a, \psi_{ja}] \\ Q_i &= \psi_i^a e_a{}^\mu \left(p_\mu - \frac{1}{2}\omega_{\mu bc} M^{bc}\right) \\ H &= \frac{1}{2}\big(\pi^a \pi_a - i\omega^a{}_{ab}\pi^b\big) - \frac{b}{4} J_{ij} J_{ij} - bA(D) \end{aligned} \tag{3.16}$$

where $A(D) \equiv (2-N)\frac{D}{8} - \frac{D^2}{8}$, and the corresponding quantum algebra

$$\begin{aligned} [J_{ij}, J_{kl}] &= i\delta_{jk} J_{il} - i\delta_{ik} J_{jl} - i\delta_{jl} J_{ik} + i\delta_{il} J_{jk} \\ [J_{ij}, Q_k] &= i\delta_{jk} Q_i - i\delta_{ik} Q_j \\ \{Q_i, Q_j\} &= 2\delta_{ij} H - \frac{b}{2}(J_{ik}J_{jk} + J_{jk}J_{ik} - \delta_{ij} J_{kl} J_{kl}) \ . \end{aligned} \tag{3.17}$$

Note, in particular, that $[Q_i, H]$ vanishes. This is not a Lie algebra, but rather a quadratically deformed Lie algebra with b playing the role of deforming parameter. Of course, as b is proportional to the (A)dS scalar curvature, in the limit $b \to 0$ one reobtains the flat space constraint algebra. One may check that this quadratic algebra coincides with the zero mode algebra in the Ramond sector of the nonlinear SO(N)-extended superconformal algebras discovered by Bershadsky and Knizhnik in two dimensions [27, 28]. The above construction gives the quantization of the model obtained at the classical level by Kuzenko and Yarevskaya in [8].

4. Geometrical equations for higher spin fields

We now study the quantum constraints that define the quantization of the O(N) spinning particle and use them to derive equations of motion for higher spin fields. The case in flat space is well-known, as the constraints generate the equations of motion of Bargmann and Wigner. We review this in section 4.1, though in different language and notations, to show how the spinning particle reproduces many of the results in higher spin theory, derived previously from field theory. More importantly, it indicates how to extend those results to (A)dS and conformally flat spaces. We discuss the extension to (A)dS spaces in section 4.2. For the sake of concreteness, we consider only the case of even $N = 2s$, i.e. massless particles of integer spin s.

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4.1 Minkowski space

In flat space the equations that select the physical states from the Hilbert space are given by $T_A|R\rangle = 0$, where $T_A = (H, Q_i, J_{ij})$ are the constraints in (3.2) and $|R\rangle$ is a physical state. We consider even $N = 2s$, so that the constraints can be analyzed by taking complex combinations (in a Lorentz invariant way) of the operators ψ_i^μ, and representing half of them as (Grassmann) coordinates and the other half as momenta. Then, one can represent the wave function $|R\rangle$ in a coordinate basis and expand it in terms of tensors of flat space. The only tensor surviving the constraints lives in even dimensions $D = 2d$, has "s" blocks of "d" indices

$$R_{\mu_1^1\ldots\mu_d^1,\ldots,\mu_1^s\ldots\mu_d^s} \tag{4.1}$$

and satisfies the following three sets of properties:

(i) it is symmetric under exchanges of the s blocks, antisymmetric in the d indices of each block, traceless, and satisfies the algebraic Bianchi identities (J constraints); this part is summarized by saying that the tensor R is an irreducible representation of the Lorentz group specified by the Young tableau with d rows and s columns

$$R_{\mu_1^1..\mu_d^1,\ldots,\mu_1^s..\mu_d^s} \sim d\left\{\underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}}_{s}\right. \quad \text{of} \quad SO(D-1,1) \tag{4.2}$$

(ii) it satisfies "differential Bianchi identities" (from half of the Q constraints)

$$\partial_{[\mu} R_{\mu_1^1\ldots\mu_d^1],\ldots,\mu_1^s\ldots\mu_d^s} = 0\,, \tag{4.3}$$

(iii) it satisfies "Maxwell equations" (from the other half of the Q constraints)

$$\partial^{\mu_1^1} R_{\mu_1^1\ldots\mu_d^1,\ldots,\mu_1^s\ldots\mu_d^s} = 0\ . \tag{4.4}$$

The H constraint is automatically satisfied. These are geometrical equations for conformal free fields of integer spin s, and are equivalent to the Bargmann-Wigner equations when $D = 4$ [35]. Up to an overall power of the D'Alembertian operator they coincide with the geometrical equations introduced in [16], that can also be recovered from the compensator extension of Fronsdal's equations of [17].

To derive these equations in more detail, we take complex combinations of the $\mathrm{SO}(N) = \mathrm{SO}(2s)$ indices and define (for $I, i = 1, \ldots, s$)

$$\psi_I = \frac{1}{\sqrt{2}}(\psi_i + i\psi_{i+s}) \tag{4.5}$$

$$\bar\psi_{\bar I} = \frac{1}{\sqrt{2}}(\psi_i - i\psi_{i+s}) \equiv \bar\psi^I \tag{4.6}$$

so that

$$\{\psi_I^\mu, \bar\psi^{J\nu}\} = \eta^{\mu\nu}\delta_I^J\ . \tag{4.7}$$

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In the "coordinate" representation one can realize ψ_I^μ as multiplication by Grassmann variables and $\bar{\psi}_\mu^I = \frac{\partial}{\partial \psi_I^\mu}$ (we use left derivatives). This realization keeps manifest only the U(s) $\subset$ SO($2s$) subgroup of the internal symmetry group, but will be quite useful in classifying the constraints and their solutions.

The susy charges in the U(s) basis take the form $Q_I = \psi_I^\mu p_\mu$ and $\bar{Q}^I = \bar{\psi}^{I\mu} p_\mu$, and the susy algebra (3.3) breaks up into

$$\{Q_I, \bar{Q}^J\} = 2\delta_I^J H \,, \qquad \{Q_I, Q_J\} = \{\bar{Q}^I, \bar{Q}^J\} = 0 \,. \tag{4.8}$$

Similarly, the SO(N) generators split as $J_{ij} \sim (J_{I\bar{J}}, J_{IJ}, \bar{J}_{\bar{I}\bar{J}}) \sim (J_I{}^J, K_{IJ}, \bar{K}^{IJ})$, which we normalize as

$$J_I{}^J = \psi_I \cdot \bar{\psi}^J - d\,\delta_I^J \,, \quad K_{IJ} = \psi_I \cdot \psi_J \,, \quad \bar{K}^{IJ} = \bar{\psi}^I \cdot \bar{\psi}^J \,, \tag{4.9}$$

so that $J_I{}^J$ for $I = J$ is a hermitian operator with real eigenvalues. The SO(N) algebra (3.5) breaks up into

$$\begin{aligned}
[J_I{}^J, J_K{}^L] &= \delta_K^J J_I{}^L - \delta_I^L J_K{}^J \\
[J_I{}^J, K_{KL}] &= \delta_K^J K_{IL} + \delta_L^J K_{KI} \\
[J_I{}^J, \bar{K}^{KL}] &= -\delta_I^K \bar{K}^{JL} - \delta_I^L \bar{K}^{KJ} \\
[K_{IJ}, \bar{K}^{KL}] &= \delta_J^K J_I{}^L - \delta_J^L J_I{}^K - \delta_I^K J_J{}^L + \delta_I^L J_J{}^K
\end{aligned} \tag{4.10}$$

where the first line identifies the U(s) subalgebra. Finally, it is useful to list in the same basis the remaining part of the constraint algebra corresponding to eq. (3.4)

$$\begin{aligned}
[J_I{}^J, Q_K] &= \delta_K^J Q_I \\
[J_I{}^J, \bar{Q}^K] &= -\delta_I^K \bar{Q}^J \\
[\bar{K}^{IJ}, Q_K] &= \delta_K^J \bar{Q}^I - \delta_K^I \bar{Q}^J \\
[K_{IJ}, \bar{Q}^K] &= \delta_J^K Q_I - \delta_I^K Q_J \,.
\end{aligned} \tag{4.11}$$

Let us now analyze the constraint equations, and derive the geometrical equations for fields of integer spin s, briefly summarized above. A general wave function is a function of the coordinates (x^μ, ψ_I^μ) with a finite Taylor expansion in the Grassmann variables ψ_I^μ (with a slight abuse of notation we indicate with ψ_I^μ both the operator and its eigenvalues, but it will be clear from the context which is which)

$$|R\rangle \sim \sum_{A_i=0}^{D} R_{\mu_1 \ldots \mu_{A_1}, \ldots, \nu_1 \ldots \nu_{A_s}}(x)\, \psi_1^{\mu_1} \ldots \psi_1^{\mu_{A_1}} \ldots \psi_s^{\nu_1} \ldots \psi_s^{\nu_{A_s}} \,. \tag{4.12}$$

We start by analyzing the consequences of the constraints $J_{ij} \sim (J_I{}^J, K_{IJ}, \bar{K}^{IJ})$. In the coordinate representation these operators take the form

$$J_I{}^J = \psi_I \cdot \frac{\partial}{\partial \psi_J} - d\,\delta_I^J \,, \quad K_{IJ} = \psi_I \cdot \psi_J \,, \quad \bar{K}^{IJ} = \frac{\partial}{\partial \psi_I} \cdot \frac{\partial}{\partial \psi_J} \tag{4.13}$$

and we find

$$
\begin{aligned}
&{J_I}^I|R\rangle = 0 \quad (I \text{ fixed}) && \Rightarrow \quad |R\rangle \sim R_{\mu_1\ldots\mu_d,\ldots,\nu_1\ldots\nu_d}(x)\,\psi_1^{\mu_1}\ldots\psi_1^{\mu_d}\ldots\psi_s^{\nu_1}\ldots\psi_s^{\nu_d} && (4.14)\\
&{J_I}^J|R\rangle = 0 \quad (I \neq J) && \Rightarrow \quad R \text{ satisfies algebraic Bianchi identities} && (4.15)\\
&\bar{K}^{IJ}|R\rangle = 0 && \Rightarrow \quad R \text{ traceless} && (4.16)\\
&K_{IJ}|R\rangle = 0 && \Rightarrow \quad R \text{ traceless (in dual basis)}\ . && (4.17)
\end{aligned}
$$

Similarly, the constraints $Q_i = (Q_I, \bar{Q}^I)$ produce

$$
\begin{aligned}
&Q_I|R\rangle = 0 \quad \Rightarrow \quad R \text{ closed (Bianchi identities)} && (4.18)\\
&\bar{Q}^I|R\rangle = 0 \quad \Rightarrow \quad R \text{ co}-\text{closed (Maxwell equations)}\ . && (4.19)
\end{aligned}
$$

The constraint H is automatically satisfied as a consequence of $\{Q_I, \bar{Q}^J\} = 2\delta_I^J H$.

Let us comment in more depth some of these equations. The constraints (4.14) and (4.15) correspond to the generators of the subgroup $\mathrm{U}(s) \subset \mathrm{SO}(2s)$, which is manifestly realized in the complex basis. The curvature R that solves these constraints has "s" symmetric blocks of "d" antisymmetric indices each, and satisfies the algebraic Bianchi identities

$$R_{[\mu_1\ldots\mu_d,\nu_1]\ldots\nu_d,\ldots} = 0 \qquad (4.20)$$

where $[\ldots]$ indicates antisymmetrization. Antisymmetry in each block is manifest. Symmetry between blocks can be proved by using finite $\mathrm{SO}(s) \subset \mathrm{U}(s)$ rotations. For example, consider the rotation that exchanges $\psi_I \to \psi_J$ and $\psi_J \to -\psi_I$ for fixed I and J. This proves symmetry under exchange of the block relative to the fermions ψ_I with the block relative to the fermion ψ_J. As these transformations are connected to the identity, they are obtained by exponentiating the infinitesimal generators used in (4.15), so that this symmetry must be a consequence of (4.15), i.e. of the algebraic Bianchi identities. As an aside, we note that the fermionic Fock vacuum $|\Omega\rangle \sim \Omega(x)$ is not invariant under the subgroup $[\mathrm{U}(1)]^s \subset \mathrm{U}(s)$, as the generator ${J_I}^I$ at fixed I transforms it by an infinitesimal phase (${J_I}^I|\Omega\rangle = d|\Omega\rangle$). It is the vector $|R\rangle$ of eq. (4.14) that is left invariant. Thus, the constraint ${J_I}^J$ selects an irreducible representation of the general linear group $\mathrm{GL}(D)$ depicted by a Young tableau with d rows and s columns. Note that traces are not removed at this stage.

The constraint $\bar{K}^{IJ}$ removes all possible traces from this tensor, and thus reduces it to an irreducible representation of the Lorentz group $\mathrm{SO}(D-1,1)$. One may notice that (4.17) (which removes the traces in the dual tensor) is not independent from (4.16). This does not seem to be a consequence of the algebra, but it can be viewed as a consequence of a duality symmetry enjoyed by the spinning particle. One can realize the Hodge operator $\star_I$ which takes the dual in the I-th block of indices by the operation

$$\star_I : \psi_I \leftrightarrow \bar{\psi}^I\,, \qquad (\star_I)^2 = 1\ . \qquad (4.21)$$

This operation can be obtained by a discrete $\mathrm{O}(N)$ symmetry transformation (a reflection on one real ψ_i coordinate). Denote by $\star_{IJ} = \star_I\star_J$ (this combined transformation can be done within $\mathrm{SO}(N)$). Then

$$K_{IJ}|R\rangle = 0 \quad \Rightarrow \quad (\star_{IJ}\,K_{IJ}\,\star_{IJ})\,(\star_{IJ}\,|R\rangle) = \bar{K}^{IJ}|R^{(\star_{IJ})}\rangle = 0\,, \qquad (4.22)$$

which implies that $R^{(\star_{IJ})}$ is traceless when contracting an index of the block I with an index of the block J. Of course, by $R^{(\star_{IJ})}$ we indicate the tensor dual to R both in the set of indices of the block I and of the block J. Then, using $\epsilon\epsilon \sim \delta \dots \delta$ implies tracelessness of R as well. More generally, invariance under duality implies selfduality, which is an expected characterization of conformal field equations in higher dimensions, that are precisely those produced by the $\mathrm{O}(N)$ spinning particle. Finally, note that (4.19) is a consequence of (4.18) and (4.16) (since $[\bar{K}^{IJ}, Q_K] = \delta^J_K \bar{Q}^I - \delta^I_K \bar{Q}^J$).

4.1.1 Gauge potentials

The previous equations can be partially solved and cast in terms of gauge potentials for higher spin fields. An independent set of constraints that describe the geometrical equations is given by (4.18), (4.14)–(4.15), and (4.16), corresponding to the constraints $Q_I, J_I{}^J, \bar{K}^{IJ}$, respectively, and we can try to solve them precisely in that order.

Before starting, it is useful to define the operator

$$q = Q_1 Q_2 \dots Q_s \tag{4.23}$$

that satisfies $Q_I q = q\, Q_I = 0$ for any I. In fact, powers of the Q_I's may be nonvanishing up to the s-th power, since an additional application of any of the Q_I's makes it vanish as a consequence of the algebra (4.8).

Constraint (4.18) (i.e. $Q_I|R\rangle = 0$) can be solved by setting

$$|R\rangle = q|\phi\rangle\ . \tag{4.24}$$

Constraints (4.14)–(4.15) (i.e. $J_I{}^J|R\rangle = 0$) are solved by selecting a tensor $R_{\mu^1_1 \dots \mu^1_d, \dots, \mu^s_1 \dots \mu^s_d}$ with the symmetries described previously, but not traceless. It corresponds to a tensor of $\mathrm{GL}(D)$ with a Young tableau of the form

$$R \sim d \left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \end{array}}_{s} \right. \tag{4.25}$$

To keep (4.14)–(4.15) satisfied by (4.24), one imposes the vanishing of

$$J_I{}^J q|\phi\rangle = ([J_I{}^J, q] + qJ_I{}^J)|\phi\rangle = q(\delta_I{}^J + J_I{}^J)|\phi\rangle = 0 \tag{4.26}$$

that is implemented by setting

$$J_I{}^J|\phi\rangle = -\delta_I{}^J|\phi\rangle \tag{4.27}$$

which says that $|\phi\rangle$ must have the form

$$|\phi\rangle \sim \phi_{\mu_1 \dots \mu_{d-1}, \dots, \nu_1 \dots \nu_{d-1}}(x)\, \psi_1^{\mu_1} \dots \psi_1^{\mu_{d-1}} \dots \psi_s^{\nu_1} \dots \psi_s^{\nu_{d-1}} \tag{4.28}$$

and must satisfy corresponding algebraic Bianchi identities. In particular, the tensor ϕ is symmetric under block exchanges. In short, it corresponds to a Young tableau of $\mathrm{GL}(D)$ of the form

$$\phi \sim d-1 \left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \end{array}}_{s} \right. \tag{4.29}$$

It remains to implement (4.16) (i.e. $\bar{K}^{IJ}|R\rangle = 0$). To do this, let us consider

$$
\begin{aligned}
\bar{K}^{12}\, q|\phi\rangle \;&=\; \bar{K}^{12}\, Q_1 Q_2 Q_3 \ldots Q_s|\phi\rangle = \underbrace{Q_3 \ldots Q_s}_{q^{12}} \bar{K}^{12}\, Q_1 Q_2|\phi\rangle \\
&= q^{12}\Big[[\bar{K}^{12}, Q_1]Q_2 + Q_1[\bar{K}^{12}, Q_2] + Q_1 Q_2 \bar{K}^{12}\Big]|\phi\rangle \\
&= q^{12}\Big[-\bar{Q}^2 Q_2 + Q_1\bar{Q}^1 + Q_1 Q_2 \bar{K}^{12}\Big]|\phi\rangle \\
&= q^{12}\Big[-2H + Q_2\bar{Q}^2 + Q_1\bar{Q}^1 + Q_1 Q_2 \bar{K}^{12}\Big]|\phi\rangle \\
&= q^{12}\Big[-2H + Q_I\bar{Q}^I + \frac{1}{2}Q_I Q_J \bar{K}^{IJ}\Big]|\phi\rangle \\
&= q^{12} G|\phi\rangle
\end{aligned}
\tag{4.30}
$$

where we have defined the Fronsdal-Labastida operator[4]

$$
G = -2H + Q_I\bar{Q}^I + \frac{1}{2}Q_I Q_J \bar{K}^{IJ} \tag{4.31}
$$

which is manifestly U(s) invariant (one may check that $[J_I{}^J, G] = 0$). A similar expression holds for $\bar{K}^{12} \to \bar{K}^{IJ}$, so that imposing (4.16) produces (in an obvious notation)

$$
q^{IJ}\, G|\phi\rangle = 0\ . \tag{4.32}
$$

It is convenient to eliminate the operator q^{IJ} form this equation. Recalling that the product of $s+1$ Q_I's must vanish, one finds the following general solution

$$
G|\phi\rangle = Q_I Q_J Q_K \bar{W}^K \bar{W}^J \bar{W}^I|\rho\rangle \tag{4.33}
$$

which depends on an arbitrary vector field contained in $\bar{W}^I \equiv W^\mu \bar{\psi}^I_\mu$, and on $|\rho\rangle$ that satisfies $J_I{}^J|\rho\rangle = -\delta^J_I|\rho\rangle$ (so that it belongs to the same space of $|\phi\rangle$ and $|\xi\rangle$, i.e. it has the same Young tableau appearing in eq. (4.29)). Eq. (4.33) gives the equations of motion for higher spin fields, written in the form that makes use of the compensator fields described by $|\rho^{IJK}\rangle \equiv \bar{W}^K \bar{W}^J \bar{W}^I|\rho\rangle$, see [17, 20, 22, 24].

To familiarize with the meaning of the present notation, note that the effect of $\bar{W}^I$ acting on $|\rho\rangle$ is to saturate one index belonging to the block I of the tensor sitting in $|\rho\rangle$ with the vector field W^μ, so that $|\rho^{IJK}\rangle$ contains a tensor with $s-3$ blocks with $d-1$ indices, and the remaining 3 blocks (block I, block J, block K) with $d-2$ indices, so that it corresponds to a Young tableau of GL(D) of the form

$$
\rho^{IJK} \sim d-1\left\{ \underbrace{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}}_{s} \right. \tag{4.34}
$$

Let us now discuss gauge symmetries in this language. Using an arbitrary vector field $V^\mu(x)$ we define

$$
\bar{V}^I \equiv V^\mu \bar{\psi}^I_\mu \tag{4.35}
$$

[4]It corresponds to the Fronsdal kinetic operator for higher spin fields in $D = 4$ [25], extended to higher dimensions for generic tensors of mixed symmetry by Labastida [26].

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and use it to define the gauge transformation

$$\delta|\phi\rangle = Q_K \bar{V}^K |\xi\rangle \ . \tag{4.36}$$

It is a gauge symmetry of $|R\rangle = q|\phi\rangle$, the solution of the Bianchi identities that expresses the curvature in terms of the gauge potentials. Since $[J_I{}^J, Q_K \bar{V}^K] = 0$, one requires that the gauge parameters satisfy $J_I{}^J|\xi\rangle = -\delta_I^J|\xi\rangle$ to guarantee that $|\phi\rangle$ and $\delta|\phi\rangle$ are tensors with the same Young tableau.

To study how the gauge symmetries act on equation (4.33), one may compute the gauge variation of $G|\phi\rangle$ using (4.36)

$$G\delta|\phi\rangle = -\frac{1}{2} Q_I Q_J Q_K \bar{V}^K \bar{K}^{JI} |\xi\rangle \ . \tag{4.37}$$

Thus, defining the gauge transformation on the compensators as follows

$$\delta(\bar{W}^K \bar{W}^J \bar{W}^I |\rho\rangle) = -\frac{1}{2} \bar{V}^{[K} \bar{K}^{JI]} |\xi\rangle \tag{4.38}$$

guarantees gauge invariance of eq. (4.33).

One can use part of the gauge symmetry to set to zero the compensator fields described by $\bar{W}^K \bar{W}^J \bar{W}^I |\rho\rangle$, and obtain the equation of motion in the Fronsdal-Labastida form

$$G|\phi\rangle = 0 \ . \tag{4.39}$$

Inspection of eq. (4.33) indicates that the gauge symmetries surviving this partial gauge fixing are those with traceless gauge parameters $|\xi\rangle$, i.e. $\bar{K}^{IJ}|\xi\rangle = 0$, as $\bar{K}^{IJ}$ in the operator that computes the trace. For consistency, the gauge potential $|\phi\rangle$ must be double traceless. This can be seen by applying the operator $\bar{Q}^I - \frac{1}{2} Q_J \bar{K}^{JI}$ on eq. (4.39)

$$\left(\bar{Q}^I - \frac{1}{2} Q_J \bar{K}^{JI}\right) G|\phi\rangle = -\frac{1}{4} Q_J Q_M Q_N \bar{K}^{IJ} \bar{K}^{MN} |\phi\rangle = 0 \tag{4.40}$$

which is consistent only if $\bar{K}^{IJ} \bar{K}^{MN} |\phi\rangle = 0$, i.e. if $|\phi\rangle$ is double traceless.

In appendix A one finds a dictionary for translating our present notation to the standard tensorial notation. In particular, one may verify that in $D = 4$ the gauge potential $|\phi\rangle$ corresponds to a symmetric tensor $\phi_{\mu_1 \ldots \mu_s}$, the Fronsdal equation $G|\phi\rangle \equiv (-2H + Q_I \bar{Q}^I + \frac{1}{2} Q_I Q_J \bar{K}^{IJ})|\phi\rangle = 0$ translates to

$$\partial_\alpha \partial^\alpha \phi_{\mu_1 \ldots \mu_s} - (\partial_{\mu_1} \partial^\alpha \phi_{\alpha \mu_2 \ldots \mu_s} + \cdots) + (\partial_{\mu_1} \partial_{\mu_2} \phi^\alpha{}_{\alpha \mu_3 \ldots \mu_s} + \cdots) = 0 \tag{4.41}$$

where the brackets contain s and $\frac{1}{2}s(s-1)$ terms, respectively, needed for symmetrizing the μ_i indices, and the condition $\bar{K}^{IJ} \bar{K}^{MN} |\phi\rangle = 0$ corresponds to the double traceless condition $\phi^\alpha{}_\alpha{}^\beta{}_{\beta \mu_5 \ldots \mu_s} = 0$.

The analysis presented here makes use of the natural quantum mechanical operators of the spinning particle and corresponds to a translation in the present notations of the analyses performed in [22 − 24]. Geometrical equations for conformal field theories and their link with spinning particles were also discussed in [38], though from a different perspective which emphasized manifest conformal invariance.

4.2 (A)dS spaces

The solutions to the geometrical equations described in the previous section for Minkowski backgrounds can be deformed to other maximally symmetric spaces with non-vanishing cosmological constant, thus producing conformal invariant field equations (see [39] for an analysis of conformal representations on AdS). In fact the corresponding constraint algebra, given in eqs. (3.16) and (3.17), defines a quadratic deformation of the linear algebra which describes the propagation on flat space, and is used to produce the geometrical equations for higher spin fields on (A)dS spaces. These equations can be worked out, and correspond to the simple covariantization of the flat space ones, eqs. (4.1), (4.3), (4.4). They read

$$
\begin{aligned}
R_{\mu_1^1\dots\mu_d^1,\dots,\mu_1^s\dots\mu_d^s} &\sim d\left\{\underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}}_{s}\right. \qquad \text{of } \mathrm{SO}(D-1,1) \\
\nabla_{[\mu} R_{\mu_1^1\dots\mu_d^1],\dots,\mu_1^s\dots\mu_d^s} &= 0 \\
\nabla^{\mu_1^1} R_{\mu_1^1\dots\mu_d^1,\dots,\mu_1^s\dots\mu_d^s} &= 0
\end{aligned}
\tag{4.42}
$$

where ∇_μ is the covariant derivative on (A)dS spaces. To analyze them it is again useful to employ a U(s) notation. The deformed susy algebra reads

$$
\{Q_I, Q_J\} = b\Big(K_{IL} J_J{}^L + K_{JL} J_I{}^L\Big) \tag{4.43}
$$

$$
\{\bar{Q}_I, \bar{Q}_J\} = -b\Big(\bar{K}_{IL} J_J{}^L + \bar{K}_{JL} J_I{}^L\Big) \tag{4.44}
$$

$$
\begin{aligned}
\{Q_I, \bar{Q}^J\} = {}& 2\delta_I^J\Big(H_0 - bA_s(D)\Big) \\
& -\frac{b}{2}\Big(J_I{}^K J_K{}^J + J_K{}^J J_I{}^K - K_{IK}\bar{K}^{JK} - \bar{K}^{JK} K_{IK}\Big)
\end{aligned}
\tag{4.45}
$$

with $A_s(D) = (1-s)\frac{D}{4} - \frac{D^2}{8}$ being the ordering constant given in (3.16) for the case $N = 2s$, while all other algebraic relations remain unchanged. Note that in (3.16) we preferred to use H as hamiltonian to make contact with the zero mode sector of the Bershadsky-Knizhnik superconformal algebra, but now we find it more convenient to use H_0, which is allowed since the difference is proportional to the J_{ij} constraints and the algebra remains first class. An independent set of constraint is again given by the set $Q_I, J_I{}^J, \bar{K}^{IJ}$. We shall discuss in full generality the first two constraints, Q_I and $J_I{}^J$, which can be solved by the introduction of higher spin gauge potentials. The main difference with respect to the flat space case is that the Q_I operators are no longer anticommuting with one another, so that $Q_1 Q_2 \cdots Q_s|\phi\rangle$ does not solve the "Bianchi identity" constraint anymore (the Q_I constraint).

Since $Q_1 Q_2 \cdots Q_s|\phi\rangle$ does solve the Bianchi identity in the flat space limit $b \to 0$, we use it as a starting point to integrate the higher spin curvature. We find it convenient to use an explicitly U(s) covariant formulation (actually SU(s) invariant) and rewrite the above leading order (in powers of b) state as

$$
|R_0\rangle = q_0|\phi\rangle\,, \quad \text{with} \quad q_0 \equiv \frac{1}{s!}\epsilon^{I_1\cdots I_s} Q_{I_1}\cdots Q_{I_s} \tag{4.46}
$$

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with the gauge potential $|\phi\rangle$ still satisfying eq. (4.27) to solve the $J_I{}^J$ constraint. Hence, by acting on the previous state with Q_I and by making repeated use of the anticommutator (4.43), produces on the right hand side only higher order terms, in powers of b. In particular, it is not difficult to convince oneself that only operators of the form $Q_I\,\epsilon^{I_1\cdots I_s}K_{I_1I_2}\cdots K_{I_{2n-1}I_{2n}}Q_{I_{2n+1}}\cdots Q_{I_s}$ are involved. Therefore the higher spin curvature is solved by the expression

$$|R\rangle = \sum_{n=0}^{[s/2]}(-b)^n r_n(s) q_n(s)\ |\phi\rangle \tag{4.47}$$

where the operators $q_n(s)$ are given by

$$q_n(s) \equiv \frac{1}{s!}\epsilon^{I_1I_2\cdots I_s}K_{I_1I_2}\cdots K_{I_{2n-1}I_{2n}}Q_{I_{2n+1}}\cdots Q_{I_s} \tag{4.48}$$

and the coefficients $r_n(s)$ are uniquely fixed by imposing the Bianchi identity (we give a more detailed description of our derivation in the appendix) and can be written recursively in terms of the Pochhammer function $P(s,k) \equiv s(s-1)(s-2)\cdots(s-k)$ as follows

$$r_n(s) = \frac{1}{2n}\sum_{k=1}^{n} r_{n-k}(s)\, a_{2k}(s-2(n-k)+1)\,, \qquad r_0(s) \equiv 1 \tag{4.49}$$

where

$$a_{2k}(s) = f(k)P(s,2k) = f(k)\prod_{l=0}^{2k}(s-l) \tag{4.50}$$

and the s-independent function $f(k)$ is defined by the recursive formula

$$f(k) = (-)^k\left[\frac{1}{(2k+1)!} - \sum_{l=0}^{k-1}\frac{(-)^l}{(2(k-l))!}f(l)\right], \qquad f(0) = 1\ . \tag{4.51}$$

We have checked numerically that these coefficients are generated by the Taylor expansion of the tangent function, $\tan(z) = \sum_{k=0}^{\infty} f(k) z^{2k+1}$. This solves the problem of expressing the higher spin curvature in terms of gauge potentials on (A)dS spaces.

Note that, alternatively, one may find it more convenient to express the coefficients (4.49) in a way that a common Pochhammer function gets factored out, namely

$$r_n(s) = \rho_n(s)\ P(s+1,2n) \tag{4.52}$$

with the prefactor $\rho_n(s)$ given by

$$\begin{aligned}\rho_n(s)\ =&\ \frac{f(n)}{2n} + \sum_{k_1=1}^{n-1}\frac{f(k_1)f(n-k_1)}{2^2 n(n-k_1)}(s-2n+2k_1+1)\\ &+\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-1-k_1}\frac{f(k_1)f(k_2)f(n-k_1-k_2)}{2^3 n(n-k_1)(n-k_1-k_2)}(s-2n+2k_1+1)(s-2n+2k_1+2k_2+1)\end{aligned}$$

$$+\cdots+\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-1-k_1}\cdots\sum_{k_{n-1}=1}^{n-1-k_1\cdots-k_{n-2}}\frac{f(k_1)f(k_2)\cdots f(n-k_1\cdots-k_{n-1})}{2^n n(n-k_1)\cdots(n-k_1\cdots-k_{n-1})}$$
$$\times(s-2n+2k_1+1)\cdots(s-2n+2k_1\cdots+2k_{n-1}+1)\ . \tag{4.53}$$

It remains to study the $\bar{K}^{IJ}$ constraint, which however seems rather involved algebraically and we have not attempted to find a general formula for it. Nevertheless in the next section we shall treat explicitly the first few cases, i.e. for spin $s \leq 4$. Analyses of the geometrical equations for higher spin fields on (A)dS have been presented also in [40, 41], though in the case of totally symmetric potentials that coincide with our conformal models only in $D = 4$.

Let us conclude this section reporting the explicit expressions for the higher spin curvatures for the cases $s \leq 4$. We have

$$\begin{aligned}
r_0(s) &= 1 \\
r_1(s) &= \frac{1}{2}a_2(s+1) = \frac{1}{6}(s+1)s(s-1) \\
r_2(s) &= \frac{1}{4}\left(a_4(s+1)+\frac{1}{2}a_2(s+1)a_2(s-1)\right) \\
&= \frac{5s+7}{360}(s+1)s(s-1)(s-2)(s-3)
\end{aligned}$$

which provide the following expressions for $s = 2, 3, 4$

$$|R\rangle = \frac{1}{2!}\epsilon^{I_1I_2}\Big[Q_{I_1}Q_{I_2} - b\,K_{I_1I_2}\Big]|\phi\rangle\,, \tag{4.54}$$
$$|R\rangle = \frac{1}{3!}\epsilon^{I_1I_2I_3}\Big[Q_{I_1}Q_{I_2}Q_{I_3} - 4b\,K_{I_1I_2}Q_{I_3}\Big]|\phi\rangle\,, \tag{4.55}$$
$$|R\rangle = \frac{1}{4!}\epsilon^{I_1I_2I_3I_4}\Big[Q_{I_1}Q_{I_2}Q_{I_3}Q_{I_4} - 10b\,K_{I_1I_2}Q_{I_3}Q_{I_4} + 9b^2K_{I_1I_2}K_{I_3I_4}\Big]|\phi\rangle\ . \tag{4.56}$$

5. Explicit examples on (A)dS

In this section we prove explicitly the gauge invariance on (A)dS backgrounds of the higher spin curvatures, expressed in terms of gauge potentials, for the special cases of spin 2, 3, 4, and impose the remaining constraints (due to $\bar{K}^{IJ}$) that lead to higher derivative equations of motion for the potentials. Then we make contact with the standard (quadratic in derivatives) formulation by introducing compensator fields to maintain the gauge invariance of the equations of motion. Finally we obtain the Fronsdal-Labastida equation for the double-traceless potentials by gauging to zero the compensators.

5.1 Spin 2

The starting point is the SU(2) invariant expression

$$|R\rangle = \frac{1}{2!}\epsilon^{I_1I_2}\Big[Q_{I_1}Q_{I_2} - bK_{I_1I_2}\Big]|\phi\rangle \tag{5.1}$$

for the spin 2 curvature.

Gauge invariance. Let us consider the transformation

$$\delta|\phi\rangle = Q_K \bar{V}^K |\xi\rangle \tag{5.2}$$

where $\bar{V}^K = V^a \bar{\psi}^K_a$ and $|\xi\rangle$ is the gauge parameter. Both $|\phi\rangle$ and $|\xi\rangle$ are described by a rectangular Young tableau of GL(D) of the type

$$\frac{D}{2}-1\left\{\underbrace{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \end{array}}_{2}\right. \tag{5.3}$$

Now one can easily compute

$$\delta\Big(Q_1 Q_2|\phi\rangle\Big) = b\ K_{12}\ Q_K \bar{V}^K|\phi\rangle \qquad \Longrightarrow \qquad \delta|R\rangle = 0\ .$$

This proves that the spin 2 curvature is invariant with respect to the gauge transformation (5.2).

Equations of motion. The gauge-invariant curvature $|R\rangle$ given above is expressed in terms of the gauge potential $|\phi\rangle$. Imposing the left over trace constraint $\bar{K}^{IJ}|R\rangle = 0$ produces the equations of motion for the potential. We find that

$$\bar{K}^{12}|R\rangle = G_2^{(A)dS}|\phi\rangle = 0 \tag{5.4}$$

where we recognize the spin 2 Fronsdal-Labastida kinetic operator on (A)dS

$$G_2^{(A)dS} = \underbrace{-2H_0 + Q_I\bar{Q}^I + \frac{1}{2}Q_I Q_J \bar{K}^{IJ}}_{G} - bK_{IJ}\bar{K}^{IJ} + b\alpha_2(D) \tag{5.5}$$

and

$$\alpha_2(D) = 4 - \frac{D}{2}\left(\frac{D}{2}+1\right)\ . \tag{5.6}$$

The operator G looks formally as the one in flat space, but of course it is the minimally covariantized version of it. By expressing the equation of motion (5.4) in components it is easy to see that, for $D = 4$, it reduces to the linearized Einstein equation on (A)dS, $R^{(1)}_{\mu\nu}(g+\phi) = 3b\,\phi_{\mu\nu}$, i.e.

$$\nabla^2\phi_{\mu\nu} - \nabla_\mu\nabla^\rho\phi_{\rho\nu} - \nabla_\nu\nabla^\rho\phi_{\rho\mu} + \nabla_\mu\nabla_\nu\phi^\rho{}_\rho + 2b(g_{\mu\nu}\phi^\rho{}_\rho - \phi_{\mu\nu}) = 0\ . \tag{5.7}$$

In even dimension $D = 2d > 4$ it corresponds to

$$\begin{aligned}
&\nabla^2\phi_{\mu_1\ldots\mu_{d-1},\nu_1\ldots\nu_{d-1}} - (d-1)\Big(\nabla_{\mu_1}\nabla^\rho\phi_{\rho\mu_2\ldots\mu_{d-1},\nu_1\ldots\nu_{d-1}} + \nabla_{\nu_1}\nabla^\rho\phi_{\mu_1\mu_2\ldots\mu_{d-1},\rho\nu_2\ldots\nu_{d-1}}\Big)\\
&+(d-1)^2\nabla_{(\mu_1}\nabla_{\nu_1)}\phi^\rho{}_{\mu_2\ldots\mu_{d-1},\rho\nu_2\ldots\nu_{d-1}} + 2b(d-1)^2 g_{\mu_1\nu_1}\phi^\rho{}_{\mu_2\ldots\mu_{d-1},\rho\nu_2\ldots\nu_{d-1}}\\
&+b\Big(4-d(d+1)\Big)\phi_{\mu_1\ldots\mu_{d-1},\nu_1\ldots\nu_{d-1}} = 0
\end{aligned} \tag{5.8}$$

where a weighted antisymmetrization in the μ and ν groups of indices is implied and with the round bracket around indices denoting a weighted symmetrization.

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5.2 Spin 3

We start from the SU(3) invariant expression

$$|R\rangle = \frac{1}{3!}\epsilon^{I_1 I_2 I_3}\Big[Q_{I_1}Q_{I_2}Q_{I_3} - 4bK_{I_1 I_2}Q_{I_3}\Big]|\phi\rangle \tag{5.9}$$

for the spin 3 curvature.

Equations of motion. Similarly to the spin 2 case we obtain the equation for the spin 3 potential by imposing tracelessness of its curvature, $\bar{K}^{IJ}|R\rangle = 0$. Using the quadratic algebra described in the previous section, we obtain an elegant U(3) covariant result

$$0 = \epsilon_{IKL}\bar{K}^{KL}|R\rangle = Q_I G_3^{(A)dS}|\phi\rangle \tag{5.10}$$

where

$$G_3^{(A)dS} = \underbrace{-2H_0 + Q_I\bar{Q}^I + \frac{1}{2}Q_IQ_J\bar{K}^{IJ}}_{G} - bK_{IJ}\bar{K}^{IJ} + b\alpha_3(D) \tag{5.11}$$

is the spin 3 Fronsdal-Labastida kinetic operator on (A)dS and

$$\alpha_3(D) = 9 - \frac{D}{2}\left(\frac{D}{2}+2\right) . \tag{5.12}$$

Note that the equations of motion (5.10) for the spin 3 potential are higher derivative ones. This is well-known to be correct for geometrical equations satisfied by curvatures for spin $s > 2$.

Gauge invariance and Fronsdal-Labastida equation. Using the experience inherited from the flat case, we now study the gauge invariance and describe the appearance of the compensator field $\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle$. First of all, eq. (5.10) shows that $G_3^{(A)dS}|\phi\rangle$ is closed with respect the operator Q_I; hence, in analogy with the spin 3 Damour-Deser identity [42], one can integrate the Q_I by using the compensator to parametrize an element of the kernel of Q_I and obtain the searched for second order differential equation

$$G_3^{(A)dS}|\phi\rangle = \big(Q_IQ_JQ_K - 4b\,K_{IJ}Q_K\big)\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle . \tag{5.13}$$

The gauge transformation

$$\delta|\phi\rangle = Q_K\bar{V}^K|\xi\rangle \tag{5.14}$$

is a symmetry of the generalized curvature (5.9), whereas the left hand side of (5.13) transforms as

$$G_3^{(A)dS}\delta|\phi\rangle = -\frac{1}{2}(Q_IQ_JQ_K - 4b\,K_{IJ}Q_K)\bar{V}^{[K}\bar{K}^{JI]}|\xi\rangle . \tag{5.15}$$

Hence, the differential equation with compensator is fully gauge-invariant provided the compensator transforms as

$$\delta(\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle) = -\frac{1}{2}\bar{V}^{[K}\bar{K}^{JI]}|\xi\rangle . \tag{5.16}$$

The latter can be used at once to gauge fix the compensator to zero yielding

$$G_3^{(A)dS}|\phi\rangle = 0 \tag{5.17}$$

that is the second order spin 3 Fronsdal-Labastida equation on (A)dS. The left over gauge symmetry must keep the left hand side of (5.16) equal to zero, $\bar{V}^{[K}\bar{K}^{JI]}|\xi\rangle = 0$. Hence, the gauge parameter must be traceless.

5.3 Spin 4

We start from the manifestly SU(4) invariant expression

$$|R\rangle = \frac{1}{4!}\epsilon^{I_1I_2I_3I_4}\Big[Q_{I_1}Q_{I_2}Q_{I_3}Q_{I_4} - 10b\ K_{I_1I_2}Q_{I_3}Q_{I_4} + 9b^2\ K_{I_1I_2}K_{I_3I_4}\Big]|\phi\rangle \tag{5.18}$$

for the spin 4 curvature.

Equations of motion. The traceless condition (in the form $\epsilon_{IJKL}\bar{K}^{KL}|R\rangle = 0$) produces the higher order equations of motion

$$\big(Q_{[I}Q_{J]} - bK_{IJ}\big)G_4^{(A)dS}|\phi\rangle = 0 \tag{5.19}$$

where

$$G_4^{(A)dS} = \underbrace{-2H_0 + Q_I\bar{Q}^I + \frac{1}{2}Q_IQ_J\bar{K}^{IJ}}_{G} - bK_{IJ}\bar{K}^{IJ} + b\alpha_4(D) \tag{5.20}$$

is the second order Fronsdal-Labastida differential operator on (A)dS and

$$\alpha_4(D) = 16 - \frac{D}{2}\left(\frac{D}{2} + 3\right)\ . \tag{5.21}$$

Gauge invariance and Fronsdal-Labastida equation. Once again the higher order equations of motion (5.19) are fully gauge invariant under $\delta|\phi\rangle = Q_K\bar{V}^K|\xi\rangle$. On the other hand it is straightforward to check that, identically to the spin 3 case, one gets

$$G_4^{(A)dS}\delta|\phi\rangle = -\frac{1}{2}(Q_IQ_JQ_K - 4b\,K_{IJ}Q_K)\bar{V}^{[K}\bar{K}^{JI]}|\xi\rangle \tag{5.22}$$

so that the "compensated" second order equation

$$G_4^{(A)dS}|\phi\rangle = \big(Q_IQ_JQ_K - 4b\,K_{IJ}Q_K\big)\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle \tag{5.23}$$

is invariant, provided the compensator transforms as in (5.16). The Fronsdal-Labastida equation

$$G_4^{(A)dS}|\phi\rangle = 0 \tag{5.24}$$

is again obtained by gauge fixing the compensator to zero. It is invariant under gauge transformations parametrized by a traceless parameter and requires a gauge potential with vanishing double trace.

5.4 Spin $s > 4$

The results obtained above suggest us that, for every integer spin s in arbitrary (even) dimensions D, the Fronsdal-Labastida kinetic operator on (A)dS becomes

$$G_s^{(A)dS} = \left[-2H_0 + Q_I \bar{Q}^I + \frac{1}{2} Q_I Q_J \bar{K}^{IJ} - b K_{IJ} \bar{K}^{IJ} + b\alpha_s(D) \right] \tag{5.25}$$

where

$$\alpha_s(D) = s^2 - \frac{D}{2}\left(\frac{D}{2} + s - 1\right) = s^2 + 2A_s(D) \; . \tag{5.26}$$

One can check that the gauge transformation of $G_s^{(A)dS}|\phi\rangle$ is identical to the ones obtained above in (5.15) and (5.22) for spin 3 and spin 4, respectively, and it is gauge invariant provided the gauge parameter is traceless. Moreover, in $D = 4$ this operator reproduces the extension of the Fronsdal operator to (A)dS spaces.

6. Conclusions

We have discussed classical and quantum properties of the O(N) spinning particles and studied their relation to the equations of motion for fields of spin $s = \frac{N}{2}$. After a review of the model, we have shown how these spinning particles can be coupled to conformally flat spaces, both classically and quantum mechanically, thus extending the result of [8], where the coupling to (A)dS spaces was obtained at the classical level. One of our results, worth mentioning, is that on (A)dS the algebra of quantum constraints closes quadratically and reproduces the zero mode sector of the 2D Bershadsky-Knizhnik SO(N)-extended nonlinear superconformal algebra [27, 28].

Furthermore, we have analyzed the constraint equations that select the physical states from the particle Hilbert space. We have shown that in flat space these equations reproduce the so-called geometrical equations for higher spin curvatures. Using the quantum mechanical operators we have described how to integrate the "Bianchi identities" to express curvatures in term of gauge potentials, and obtained various well-known forms of the equations of motion for higher spin fields [5, 17, 20, 22 – 26, 43, 44].

Then we have studied the spinning particles on (A)dS spaces and obtained corresponding geometrical equations. To our knowledge generalized Poincaré lemmas are not known for this case, but using the constraint algebra we have shown how to integrate the "Bianchi identities" in terms of gauge potentials. Finally, we have analyzed in detail the equations of motion and the gauge invariances for the cases of spin $s \leq 4$.

Having established the precise connection between the quantum theory of the O(N) spinning particles and the conformal higher spin field equations on (A)dS, one can now use the equivalent path integral quantization to obtain further results on the quantum theory of higher spin fields.

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A. Dictionary

For the reader's convenience, we present a dictionary between our compact notation and the more conventional tensorial notation. Building blocks are the superalgebra constraints that lead to the geometrical equations

$$Q_I = -i\psi_I^a e_a^\mu \left(\partial_\mu + \omega_{\mu bc}\psi_J^b \frac{\partial}{\partial\psi_{Jc}}\right), \qquad \bar{Q}^I = -i\frac{\partial}{\partial\psi_{Ia}} e_a^\mu \left(\partial_\mu + \omega_{\mu bc}\psi_J^b \frac{\partial}{\partial\psi_{Jc}}\right)$$
$$J_I{}^J = \psi_I^a \frac{\partial}{\partial\psi_J^a} - d\delta_I^J\,, \quad K_{IJ} = \psi_I^a\psi_{Ja}\,, \qquad \bar{K}^{IJ} = \frac{\partial}{\partial\psi_I^a}\frac{\partial}{\partial\psi_{Ja}}\,.$$

As an example, let us consider a state corresponding to a rectangular tensor

$$|X\rangle = X_{a_1\ldots a_n, b_1\ldots b_n, \ldots, c_1\ldots c_n}\psi_1^{a_1}\ldots\psi_1^{a_n}\ \psi_2^{b_1}\ldots\psi_2^{b_n}\ \ldots\ \psi_s^{c_1}\ldots\psi_s^{c_n} \sim n\left\{\underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline\end{array}}_{s}\right.$$

with n arbitrary (and similar expansions for more general tensors). A set of correspondences that allows to obtain Fronsdal-Labastida equations in components is given by table 1 and table 2, where a weighted antisymmetrization in each of the s groups of indices $a_i, b_i, \ldots, c_i$ is implied. In the last two expressions the dots in parenthesis indicate a sum over all pairs of indices corresponding to $I < J$ and the round brackets around indices denote a weighted symmetrization.

B. Solution to the "Bianchi identities" on (A)dS

We give here a detailed derivation of the solution to the "Bianchi identities" equations for the higher spin curvatures on (A)dS. In the spinning particle language such equations read

$$J_I{}^J|R\rangle = 0 \tag{B.1}$$
$$Q_I|R\rangle = 0\,, \quad I, J = 1, \ldots, s\,. \tag{B.2}$$

As explained in the main text the first relation selects an irreducible GL(D) tensor represented by a rectangular Young tableau with s rows and $D/2$ columns. The "differential Bianchi identity" is instead encoded in the second relation, and can be solved by expressing the curvature $|R\rangle$ in terms of a potential $|\phi\rangle$

$$|R\rangle = q|\phi\rangle \tag{B.3}$$

where the operator q must reduce in the flat space limit to

$$q \quad \xrightarrow{\text{flat space}} \quad Q_1Q_2\cdots Q_s = \frac{1}{s!}\epsilon^{I_1\cdots I_s}Q_{I_1}\cdots Q_{I_s} \equiv q_0 \tag{B.4}$$

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Compact notation	Tensorial notation
$\|X\rangle$	$X_{a_1\ldots a_n,b_1\ldots b_n,\ldots,c_1\ldots c_n}$
$Q_1\|X\rangle$	$-i\nabla_{a_1}X_{a_2\ldots a_{n+1},b_1\ldots b_n,\ldots,c_1\ldots c_n}$
$\bar{Q}^1\|X\rangle$	$-in\nabla^l X_{la_2\ldots a_n,b_1\ldots b_n,\ldots,c_1\ldots c_n}$
$\bar{K}^{12}\|X\rangle$	$(-)^n n^2\, X^l{}_{a_2\ldots a_n,lb_2\ldots b_n,\ldots,c_1\ldots c_n}$
$K_{12}\|X\rangle$	$(-)^n\eta_{a_1b_1}X_{a_2\ldots a_{n+1},b_2\ldots b_{n+1},\ldots,c_1\ldots c_n}$
$J_1{}^1\|X\rangle$	$(n-d)X_{a_1\ldots a_n,b_1\ldots b_n,\ldots,c_1\ldots c_n}$

Table 1: Dictionary for elementary constraints.

and, since $[J_I{}^J, Q_K] = \delta_K^J\, Q_I$, the potential must satisfy

$$J_I{}^J|\phi\rangle = -\delta_I^J|\phi\rangle \tag{B.5}$$

so that it is represented by a Young tableau with s columns and $D/2-1$ rows. Above and in what follows we express the differential operator q in an explicitly SU(s) invariant form. We construct q by imposing the conditions

$$Q_I|R\rangle = 0 \tag{B.6}$$

and use its flat space limit q_0 as our starting point. In particular, thanks to the SU(s)-invariance it will suffice to require $Q_1|R\rangle = 0$. In order to achieve such a task we shall need a few recursive relations that we derive using the commutation relations

$$\{Q_I, Q_J\} = b\left(K_{IL}J_J{}^L + K_{JL}J_I{}^L\right) \tag{B.7}$$

$$[J_I{}^J, Q_K] = \delta_K^J\, Q_I \tag{B.8}$$

$$[K_{IJ}, K_{KL}] = [K_{IJ}, Q_K] = 0 \tag{B.9}$$

and the condition (B.5). We find it convenient to split the s indices into a “time-like” index 1 and $s-1$ “space-like” indices i

$$I = (1, i)\,, \quad i = 2, \ldots, s\ . \tag{B.10}$$

Let us define a shortcut notation that will prove to be extremely useful

$$\epsilon^{i_1\cdots i_{s-1}}\ Q_{i_1}\cdots Q_{i_n}Q_1Q_{i_{n+1}}\cdots Q_{i_{s-1}} \longrightarrow Q_{[n]}Q_1Q_{[s-1-n]}$$
$$\epsilon^{i_1\cdots i_{s-1}}\ K_{1i_1}\ Q_{i_3}\cdots Q_{i_n}Q_1Q_{i_{n+1}}\cdots Q_{i_{s-1}} \longrightarrow K_{1i_1}\ Q_{[n-2]}Q_1Q_{[s-1-n]}$$

and whenever we encounter a K_{ab} tensor we use the commutation rules above and the antisymmetry provided by the ϵ tensor to bring it in front of everything and give it the first indices of the set $i_1, i_2, \ldots$. It is thus not difficult to prove the relation

$$\begin{aligned}(-)^n Q_{[n]}Q_1Q_{[s-1-n]}|\phi\rangle &= Q_1Q_{[s-1]}|\phi\rangle + b\left(n(s-2) - \frac{n(n-1)}{2}\right)K_{1i_1}Q_{[s-2]}|\phi\rangle \\ &\quad -bK_{i_1i_2}\sum_{m=1}^{n}\sum_{k=m-1}^{s-3}(-)^k Q_{[k]}Q_1Q_{[s-k-3]}|\phi\rangle\end{aligned} \tag{B.11}$$

that can be iterated by noting that the last term is just equal to the left hand side provided one performs the substitution $s \to s-2$. The iteration process thus yields

$$
\begin{aligned}
\sum_{n=0}^{s-1}(-)^n Q_{[n]}Q_1Q_{[s-1-n]}|\phi\rangle = \; & sQ_1Q_{[s-1]}|\phi\rangle \\
& -(-b)\,a_2(s)\Big(K_{1i_1}Q_{[s-2]} - K_{i_1i_2}Q_1Q_{[s-3]}\Big)|\phi\rangle \\
& -(-b)^2a_4(s)\Big(K_{1i_1}K_{i_2i_3}Q_{[s-4]} - K_{i_1i_2}K_{i_3i_4}Q_1Q_{[s-5]}\Big)|\phi\rangle \\
& . \\
& . \\
& -(-b)^p a_{2p}(s)K_{1i_1}K_{i_2i_3}\cdots K_{i_{2(p-1)}i_{2p-1}}Q_{[s-2p]}|\phi\rangle \\
& +(-b)^p\sum_{k_0=1}^{s-1}\sum_{m_1=1}^{k_0}\sum_{k_1=m_1-1}^{s-3}\cdots\sum_{m_p=1}^{k_{p-1}}\sum_{k_p=m_p-1}^{s-2p-1}(-)^{k_p} \\
& \qquad K_{1i_1}K_{i_2i_3}\cdots K_{i_{2(p-1)}i_{2p-1}}Q_{[k_p]}Q_1Q_{[s-2p-1-k_p]}|\phi\rangle
\end{aligned} \tag{B.12}
$$

with

$$
a_{2n}(s) \equiv \sum_{k_0=1}^{s-1}\sum_{m_1=1}^{k_0}\sum_{k_1=m_1-1}^{s-3}\cdots\sum_{m_n=1}^{k_{n-1}}\sum_{k_n=m_n-1}^{s-2n-1} 1 = f(n)\,P(s,2n) \tag{B.13}
$$

where $P(s,2n) = s(s-1)\cdots(s-2n)$ is the Pochhammer function and the s-independent function $f(n)$ is given be the recursive formula (equivalent to (4.51))

$$
\sum_{k=0}^{n}\frac{(-)^k}{(2k)!}f(n-k) = \frac{(-)^n}{(2n+1)!}\ . \tag{B.14}
$$

Note that the iterative expression (B.12) stops at the last-but-one entry if $s=2p$, whereas it stops at the last entry if $s=2p+1$. Another helpful relation that can be obtained with the help of (B.12) and with implied antisymmetrization of the indices "i", reads

$$
\begin{aligned}
Q_1^2Q_{[s-1]}|\phi\rangle &= bK_{1i_1}\sum_{n=0}^{s-2}(-)^nQ_{[n]}Q_1Q_{[s-2-n]}|\phi\rangle \\
&= bK_{1i_1}\Big(a_0(s-1)\,Q_1Q_{[s-2]} - b\,a_2(s-1)\,K_{i_2i_3}Q_1Q_{[s-4]} \\
&\qquad + \cdots + (-b)^{p-1}\,a_{2(p-1)}(s-1)\,K_{i_2i_3}\cdots K_{i_{2(p-1)}i_{2p-1}}Q_1\Big)|\phi\rangle\ .
\end{aligned} \tag{B.15}
$$

It is easy now to convince oneself that the zero-th order operator $q_0(s)$ can be written as

$$
s\ q_0(s) = \sum_{n=0}^{s-1}(-)^nQ_{[n]}Q_1Q_{[s-1-n]} \tag{B.16}
$$

Compact notation	Tensorial notation
$H_0\|X\rangle$	$-\frac{1}{2}\nabla^2 X_{a_1\ldots a_n,b_1\ldots b_n,\ldots,c_1\ldots c_n}$
$Q_I\bar{Q}^I\|X\rangle$	$-n\Big(\nabla_{a_1}\nabla^l X_{la_2\ldots a_n,b_1\ldots b_n,\ldots,c_1\ldots c_n}$ $+\nabla_{b_1}\nabla^l X_{a_1\ldots a_n,lb_2\ldots b_n,\ldots,c_1\ldots c_n}$ $+\cdots+\nabla_{c_1}\nabla^l X_{a_1\ldots a_n,b_1\ldots b_n,\ldots,lc_2\ldots c_n}\Big)$
$Q_IQ_J\bar{K}^{IJ}\|X\rangle$	$2n^2\Big(\nabla_{(a_1}\nabla_{b_1)}X^l{}_{a_2\ldots a_n,lb_2\ldots b_n,\ldots,c_1\ldots c_n}$ $+\cdots+\nabla_{(a_1}\nabla_{c_1)}X^l{}_{a_2\ldots a_n,b_1\ldots b_n,\ldots,lc_2\ldots c_n}$ $+\cdots+\nabla_{(b_1}\nabla_{c_1)}X_{a_1\ldots a_n,}{}^l{}_{b_2\ldots b_n,\ldots,lc_2\ldots c_n}+\cdots\Big)$
$K_{IJ}\bar{K}^{IJ}\|X\rangle$	$-2n^2\Big(\eta_{a_1b_1}X^l{}_{a_2\ldots a_n,lb_2\ldots b_n,\ldots,c_1\ldots c_n}$ $+\cdots+\eta_{a_1c_1}X^l{}_{a_2\ldots a_n,b_1\ldots b_n,\ldots,lc_2\ldots c_n}$ $+\cdots+\eta_{b_1c_1}X_{a_1\ldots a_n,}{}^l{}_{b_2\ldots b_n,\ldots,lc_2\ldots c_n}+\cdots\Big)$

Table 2: Dictionary for some composite operators.

so that making use of (B.12), (B.15), and assuming for definiteness that $s = 2p$, one gets

$$s!\; Q_1 q_0(s)|\phi\rangle \;=\; -\sum_{n=1}^{s/2}(-b)^n a_{2n}(s+1)Q_1 I_n(s)|\phi\rangle \tag{B.17}$$

where

$$I_n(s) \equiv K_{1i_1}K_{i_2i_3}\cdots K_{i_{2(n-1)}i_{2n-1}}Q_{[s-2n]} \tag{B.18}$$

and we have used the identity

$$\sum_{k=0}^{n} a_{2k}(s)a_{2(n-1-k)}(s-1-2k) = a_{2n}(s+1)\,, \qquad a_{-2}(s)\equiv 1 \tag{B.19}$$

that can be proved by induction. This completes the first step. The next step is to rewrite expression (B.17) in terms of U(s)-covariant tensors. The covariantization of the tensors $I_n(s)$ is again an iterative process. Note in fact that one can write

$$I_n(s)|\phi\rangle = \frac{s!}{2n}q_n(s)|\phi\rangle + \frac{1}{2n}\sum_{m=n+1}^{s/2}(-b)^{m-n}\; a_{2(m-n)}(s-2n+1)\; I_m(s)|\phi\rangle$$

that finally yields

$$Q_1 \sum_{n=0}^{[s/2]}(-b)^n r_n(s)q_n(s)|\phi\rangle = 0 \tag{B.20}$$

with

$$r_n(s) = \frac{1}{2n}\sum_{k=1}^{n} r_{n-k}(s)\; a_{2k}(s-2(n-k)+1)\;. \tag{B.21}$$

Finally, note that in (B.20) we have replaced $s/2$ with its integer part: it is in fact not difficult to check that the latter holds for odd s as well, with that precise replacement.

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