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Extended SUSY quantum mechanics: transition amplitudes and path integrals

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Abstract: Quantum mechanical models with extended supersymmetry find interesting applications in worldline approaches to relativistic field theories. In this paper we consider one-dimensional nonlinear sigma models with $\mathrm{O}(N)$ extended supersymmetry on the worldline, which are used in the study of higher spin fields on curved backgrounds. We calculate the transition amplitude for euclidean times (i.e. the heat kernel) in a perturbative expansion, using both canonical methods and path integrals. The latter are constructed using three different regularization schemes, and the corresponding counterterms that ensure scheme independence are explicitly identified.

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Contents

1 Introduction

Quantum mechanical models with extended supersymmetry find useful applications in the worldline description of relativistic field theories. Indeed fields with spin S can be described in four dimensions by quantizing particle actions with $N = 2S$ extended local supersymmetry on the worldline [1–3]. While complete in four dimensions, these models describe only conformal invariant fields in other dimensions [4, 5]. They can be consistently defined not only in Minkowski space, but also in maximally symmetric spacetimes [6] and, more generally, in conformally flat spacetimes [7]. Upon gauge fixing, the worldline actions of the spinning particles moving on such spaces give rise to an interesting class of one-dimensional nonlinear sigma models possessing extended rigid supersymmetries. The goal of this paper is to analyze the corresponding quantum mechanics.

In particular, we are interested in computing the transition amplitude for the O(N) extended supersymmetric nonlinear sigma models in a euclidean short time expansion since, in applications to higher spin field theory, the transition amplitude can be used to study ultraviolet properties of propagators and one-loop effective actions [8]. We will achieve this result using two different methods. The first method employs canonical quantization, and starting from the operatorial definition of the transition amplitude we compute it perturbatively in the euclidean time β by using the commutation properties of the various operators. The final result yields a perturbative solution of the heat equation (the Schrödinger equation with imaginary time) and identifies a benchmark for equivalent calculations in terms of path integrals. This canonical approach has been already employed in [9] for the $N = 0, 1, 2$

supersymmetric quantum mechanics, that can be defined on any curved manifold (see [10] for a review of the method in the bosonic case). We extend that computation to arbitrary N for the $\mathrm{O}(N)$ extended supersymmetric nonlinear sigma models. For $N > 2$ the extended supersymmetry may be broken by a generic curved target space, though it can be maintained on locally symmetric spaces. Nevertheless, we use an arbitrary target space geometry since for the present purposes we do not need to gauge supersymmetry, and can view the latter just as an accidental symmetry emerging on particular backgrounds.

The second method we employ for computing the transition amplitude makes use of path integrals. Our reason for considering this approach is that, in typical first-quantized applications, canonical methods are used first to identify the precise operators of interest and path integrals are used next to perform more extensive calculations and manipulations. A classical example is the calculation of chiral anomalies and the proof of index theorems by worldline methods [11–13]. It is therefore important to be able to reproduce the transition amplitude with path integrals. Path integrals for particles moving on curved spaces can be quite subtle, and their consistent definition needs the specification of a regularization scheme which carry finite counterterms to guarantee scheme independence, see [10] for an extensive treatment and [14] for a short discussion. We will use three different schemes, for completeness and because each one carries its own advantages. The first scheme, time slicing (TS), can be deduced directly by using operatorial methods [15, 16] and can be related to the lattice approximation of the propagation time. A second scheme, mode regularization (MR), is related to a momentum cut-off (or, more properly, energy cut-off in quantum mechanics) and allows the introduction of a regulated functional space to integrate over [17–21]. Finally, dimensional regularization (DR) is defined as a purely perturbative regularization but has the advantage of carrying only covariant counterterms [22–26]. For each of the three regularization schemes we find the corresponding counterterms that ensure scheme independence, making them ready for extending to curved backgrounds the worldline approach to higher spin fields initiated in [27].

Let us now describe the precise form of the (supersymmetric) quantum mechanics we are interested in. We consider a particle moving in a curved space $\mathcal{M}$ of dimensions D and metric $g_{\mu\nu}$. It is described in phase space by bosonic coordinates and momenta (x^μ, p_μ), where $\mu = 1, \dots, D$ is a curved spacetime index, and by Majorana worldline fermions (i.e. real Grassmann variables) ψ_i^a, where $i = 1, \dots, N$ is a flavour index and $a = 1, \dots, D$ is a flat spacetime index related to curved indices by the vielbein e_μ^a, defined by $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ with η_{ab} the flat tangent space metric (which can be taken either minkowskian or euclidean, according to the desired applications). Quantum mechanically the bosonic variables satisfy the usual commutation relations $[x^\mu, p_\nu] = i\delta_\nu^\mu$, and the fermionic ones give rise to a multi Clifford algebra $\{\psi_i^a, \psi_j^b\} = \eta^{ab}\delta_{ij}$. The general hamiltonian operator we wish to consider involves a free covariant kinetic term H_0, a four-Fermi interaction depending on the Riemann tensor and carrying a coupling constant α, and a contribution from an arbitrary scalar potential V which depends only on the spacetime coordinates x^μ (and which in most applications is proportional to the curvature scalar R)

$$H = H_0 + \alpha R_{abcd}\psi_i^a \psi_i^b \psi_j^c \psi_j^d + V \tag{1.1}$$

where

$$H_0 = \frac{1}{2}\Big(\pi^a - i\omega_b{}^{ba}\Big)\pi_a$$
$$\pi_a = e^\mu_a \pi_\mu \ , \quad \pi_\mu = g^{1/4} p_\mu g^{-1/4} - \frac{i}{2}\omega_{\mu ab}\psi_i^a\psi_i^b \tag{1.2}$$

with $\omega_{\mu ab}$ the spin connection and $\omega_{abc} = e^\mu_a \omega_{\mu bc}$. Note the appearance of the spin connection, required by covariance, and the powers of g next to the momentum operator that ensure hermiticity of the hamiltonian. This hamiltonian is general enough to allow future applications to the first-quantized theory of higher spin fields on curved backgrounds.

The corresponding euclidean classical action in configuration space is given by

$$S = \int_0^\beta d\tau \left[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\psi_{ai}D_\tau\psi_i^a + \alpha R_{abcd}\psi_i^a\psi_i^b\psi_j^c\psi_j^d + V\right] \tag{1.3}$$

where $D_\tau\psi_i^a = \partial_\tau\psi_i^a + \dot{x}^\mu\omega_\mu{}^a{}_b\psi_i^b$. It describes the particle propagation for a euclidean time β and will be used in the path integral quantization. For notational simplicity we do not make explicit distinction between quantum operators and classical variables, as it will be clear from the context which one is used.

Having defined the model, we proceed with the rest of the paper and start describing the computation of the transition amplitude. In section 2 we use operatorial methods. In section 3 we consider path integrals in various regularization schemes, namely TS, MR and DR, calculating the corresponding counterterms and finding complete agreement with the expression of the transition amplitude found in section 2. Finally, we present our conclusions and perspectives in section 4.

2 Transition amplitude from operatorial methods

The aim of this section is the explicit computation of the transition amplitude for a euclidean time β determined by the quantum mechanical hamiltonian H given in (1.1). We compute the matrix elements of the evolution operator $e^{-\beta H}$ between position eigenstates for the bosonic variables and suitable coherent states for the fermionic ones. We consider a short time expansion, and using systematically the fundamental (anti)-commutation relations of the basic operators x, p and ψ we obtain the final perturbative answer. The calculation is tailored after similar ones performed in [9] for the $N = 0, 1, 2$ supersymmetric quantum mechanics, explained carefully in [10] for the bosonic case, and in [28] for a class of supersymmetric sigma models on Kähler spaces which arise from the study of certain higher spin equations on complex manifolds [29]. As typical in semiclassical approximations, the result can be cast as the product of three terms: i) the exponential of the classical action evaluated on the (perturbative) classical solution, ii) a standard leading prefactor depending on the propagation time as $\beta^{-\frac{D}{2}}$, usually arranged in the so-called Van Vleck determinant, iii) a power series in positive powers of the propagation time, which identify the heat-kernel coefficients [8]. In our calculation we keep the approximation up to the first non trivial heat kernel coefficient, i.e. up to order β in the power series, keeping in mind that the bosonic displacement $z^\mu \equiv y^\mu - x^\mu$ can be considered of order $\sqrt{\beta}$, as explained later on.

In this section we restrict the calculation to even $N \equiv 2S$, with S an integer, which allows us to introduce complex combinations of the fermionic operators and consider the corresponding coherent states as a (over)-complete basis of the associated Hilbert space. This is appropriate for applications to fields of integer spin. The method can be easily extended to the case of odd N, appropriate for applications to fields with half-integer spin: one way is to introduce another set of auxiliary Majorana fermions ("doubling trick") to be able to consider complex (Dirac) fermions and their coherent states. However, we refrain from doing that at this stage, as we employ it in the path integral approach of the following section.

Thus we consider $N = 2S$, and introduce S Dirac worldline fermions Ψ_I^a out of the $2S$ Majorana fermions ψ_i^a

$$\Psi_I^a = \frac{1}{\sqrt{2}}\left(\psi_I^a + i\psi_{I+S}^a\right), \quad \bar{\Psi}^{aI} = \frac{1}{\sqrt{2}}\left(\psi_I^a - i\psi_{I+S}^a\right), \quad I = 1, \ldots, S \tag{2.1}$$

so that the new variables display the usual creation-annihilation algebra

$$\{\Psi_I^a, \bar{\Psi}^{bJ}\} = \eta^{ab}\delta_I^J \ .$$

With these complex fields at hand we can readily construct coherent states such that $\Psi_I^a|\eta\rangle = \eta_I^a|\eta\rangle$ and $\langle\bar{\lambda}|\bar{\Psi}^{aI} = \langle\bar{\lambda}|\bar{\lambda}^{aI}$, with usual normalization $\langle\bar{\lambda}|\eta\rangle = e^{\bar{\lambda}\cdot\eta}$ (other properties of fermionic coherent states can be found in appendix A). Denoting by $|y\,\eta\rangle \equiv |y\rangle \otimes |\eta\rangle$ where $|y\rangle$ is the usual position eigenstate, the euclidean heat kernel we want to compute is

$$\langle x\,\bar{\lambda}|e^{-\beta H}|y\,\eta\rangle \tag{2.2}$$

where the quantum hamiltonian (1.1) and the covariant momentum, written in terms of the new complex variables, read

$$\begin{aligned} H &= H_0 + 4\alpha R_{abcd}\,\bar{\Psi}^a\cdot\Psi^b\,\bar{\Psi}^c\cdot\Psi^d + V \\ \pi_\mu &= g^{1/4}(p_\mu - i\omega_{\mu ab}\bar{\Psi}^a\cdot\Psi^b)g^{-1/4} \end{aligned} \tag{2.3}$$

with $\bar{\Psi}^a\cdot\Psi^b \equiv \bar{\Psi}^{aI}\Psi_I^b$. In close analogy with the procedure employed in [9] we first focus on the mixed amplitude containing momentum eigenstates on the right hand side

$$\langle x\,\bar{\lambda}|e^{-\beta H}|p\,\eta\rangle = \sum_{k=0}^{\infty}\frac{(-)^k}{k!}\,\beta^k\langle x\,\bar{\lambda}|H^k|p\,\eta\rangle \ . \tag{2.4}$$

From the above expansion in β one could naively think to retain only the linear term, $e^{-\beta H} = 1 - \beta H + \mathcal{O}(\beta^2)$, to get the answer up to the desired order. This is not the case for nonlinear sigma models, as well-known (see [9]). In fact it is necessary to take into account contributions for every order k, the only truncation being the number of (anti)-commutators to keep track of. In order to evaluate (2.4), we push all p's and Ψ's in each factor H^k to the right hand side and all x's and $\bar{\Psi}$'s to the left hand side, taking into account all (anti)-commutators, and substitute with the corresponding eigenvalues. Since the hamiltonian is quadratic in momenta, the matrix element of H^k is a polynomial of degree $2k$ in p, so that in general one finds structures of the form

$$\langle x\,\bar\lambda|H^k|p\,\eta\rangle = \sum_{l=0}^{2k} B^k_l(x,\eta,\bar\lambda)p^l\langle x|p\rangle\langle\bar\lambda|\eta\rangle\ , \tag{2.5}$$

where p^l stands for the part of such polynomial with precisely l-th powers of p eigenvalues, and the coefficients B^k_l are computed in appendix B. For plane waves we use the normalization

$$\langle x|p\rangle = (2\pi)^{-D/2}g^{-1/4}(x)\,e^{ip\cdot x}$$

so, inserting in (2.2) a momentum completeness relation, $\mathbb{1} = \int d^D p\,|p\rangle\langle p|$, and rescaling momenta by $p_\mu = \frac{1}{\sqrt\beta}\,q_\mu$, we obtain for the transition amplitude

$$\begin{aligned}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle = (4\pi^2\beta)^{-D/2}[g(x)g(y)]^{-1/4}\int d^Dq\,e^{\frac{i}{\sqrt\beta}q\cdot(x-y)}e^{\bar\lambda\cdot\eta}\\ \times\sum_{k=0}^{\infty}\frac{(-)^k}{k!}\sum_{l=0}^{2k}\beta^{k-l/2}B^k_l(x,\eta,\bar\lambda)\,q^l\ .\end{aligned} \tag{2.6}$$

Let us look at this formula: it is well-known that the leading free particle contribution to (2.2) contains the exponential $\exp\{-\frac{(x-y)^2}{2\beta}\}$, that gives the effective order of magnitude $(x-y)\sim\sqrt\beta$. Therefore, we see that $q\sim\beta^0$ and it is immediate to realize that, for all k, only the terms B^k_{2k}, B^k_{2k-1} and B^k_{2k-2} will contribute up to order β. Taking for B^k_l the expressions given in appendix B, it is immediate to sum the series in k

$$\begin{aligned}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle &= (4\pi^2\beta)^{-D/2}[g(x)g(y)]^{-1/4}\int d^Dq\,e^{-\frac12 g^{\mu\nu}q_\mu q_\nu+\frac{i}{\sqrt\beta}q\cdot(x-y)}e^{\bar\lambda\cdot\eta}\Big\{1+\sqrt\beta\\ &\times\Big[\frac{i}{2}g^\mu q_\mu-\frac{i}{4}g^{\mu\nu\lambda}\,q_\lambda\,q_\mu\,q_\nu+ig^{\mu\nu}\omega_\mu q_\nu\Big]+\beta\Big[-\frac{1}{32}\partial_\mu\ln g\partial^\mu\ln g-\frac18\partial_\mu\partial^\mu\ln g\\ &-\frac18 g^\mu\partial_\mu\ln g-\Big(\frac14\partial^\mu g^\nu+\frac18 g^\mu g^\nu+\frac18 g^\sigma g^{\mu\nu}_\sigma+\frac18 g^{\mu\nu\sigma}_\sigma\Big)\,q_\mu\,q_\nu+\Big(\frac{1}{12}g^{\mu\nu\sigma\lambda}+\frac18 g^{\sigma\lambda\mu}g^\nu\\ &+\frac{1}{12}g^{\kappa\sigma\lambda}g^{\mu\nu}_\kappa+\frac{1}{24}g^{\sigma\lambda}_\kappa g^{\mu\nu\kappa}\Big)\,q_\sigma\,q_\lambda\,q_\mu\,q_\nu-\Big(\frac{1}{32}g^{\mu\nu\lambda}g^{\sigma\rho\kappa}\Big)\,q_\lambda\,q_\mu\,q_\nu\,q_\kappa\,q_\sigma\,q_\rho\\ &-\frac12\partial^\nu(g^{\lambda\sigma}\omega_\lambda)q_\nu\,q_\sigma-\frac14 g^{\lambda\sigma\mu}\omega_\mu\,q_\lambda q_\sigma-\frac12 g^{\mu\nu}\omega_\mu\,g^\sigma\,q_\nu q_\sigma+\frac14 g^{\mu\nu}\omega_\mu\,g^{\alpha\beta\sigma}\,q_\nu q_\sigma q_\alpha q_\beta\\ &-\frac14 g^{\mu\nu}\omega_\mu\partial_\nu\ln g+\frac{1}{2\sqrt g}\partial_\mu[\sqrt g g^{\mu\nu}\omega_\nu]+\frac12(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd}-8\alpha\,R_{abcd})(\bar\lambda^a\!\cdot\,\eta^d\eta^{bc}\\ &+\bar\lambda^a\!\cdot\,\eta^b\bar\lambda^c\!\cdot\,\eta^d)-\frac12 g^{\mu\nu}\omega_{\mu ab}g^{\lambda\sigma}\omega_{\lambda cd}(\bar\lambda^a\!\cdot\,\eta^d\eta^{bc}+\bar\lambda^a\!\cdot\,\eta^b\bar\lambda^c\!\cdot\,\eta^d)q_\nu q_\sigma-V\Big]\Big\}\ ,\end{aligned} \tag{2.7}$$

where we remind that all the geometric quantities are evaluated at the final point x, unless otherwise specified, and we used the following compact notations

$$\omega_\mu=\omega_{\mu ab}\bar\lambda^a\!\cdot\,\eta^b\ ,\quad g^{\nu\sigma}_{\mu\ldots\lambda}=\partial_\mu\ldots\partial_\lambda g^{\nu\sigma}\ ,\quad g^{\lambda\sigma\mu}=g^{\mu\nu}g^{\lambda\sigma}_\nu\ ,\quad g^\mu=g^{\mu\nu}_\nu\ ,\quad \partial^\mu g^\lambda=g^{\mu\nu}\partial_\nu g^{\lambda\sigma}_\sigma\ .$$

Now we are ready to perform the q integration, that reduces to a bunch of gaussian integrals. Defining the coordinate displacement as $z^\mu\equiv y^\mu-x^\mu$, one obtains the transition amplitude, expanded up to order β, in a form that hides manifest covariance

$$
\begin{aligned}
\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle = (2\pi\beta)^{-D/2}\left[g(x)/g(y)\right]^{1/4} e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu}\, e^{\bar\lambda_a\cdot\eta^a}\Big\{&1+z^\mu\,g^{-1/4}\,\partial_\mu\,g^{1/4}\\
&-\frac{1}{4\beta}\,\partial_\lambda g_{\mu\nu}\,z^\mu z^\nu z^\lambda+\frac{1}{2}\,z^\mu z^\nu\,g^{-1/4}\,\partial_\mu\partial_\nu g^{1/4}-\frac{1}{4\beta}\,z^\mu g^{-1/4}\partial_\mu g^{1/4}\partial_\lambda g_{\sigma\rho}\,z^\lambda z^\sigma z^\rho\\
&+\frac{1}{2}\left[\frac{1}{4\beta}\,\partial_\lambda g_{\mu\nu}\,z^\mu z^\nu z^\lambda\right]^2-\frac{1}{12\beta}\left[\partial_\lambda\partial_\sigma g_{\mu\nu}-\frac{1}{2}g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\right]z^\mu z^\nu z^\lambda z^\sigma\\
&+\frac{1}{12}\,R_{\mu\nu}\,z^\mu z^\nu+z^\mu\omega_\mu+\frac{1}{2}z^\mu z^\nu\partial_\mu\omega_\nu+\frac{1}{4}z^\mu\omega_\mu\,z^\nu g^{\lambda\sigma}\partial_\nu g_{\lambda\sigma}\\
&+\frac{1}{2}z^\mu z^\nu\omega_{\mu ab}\omega_\nu{}^b{}_c\bar\lambda^a\cdot\eta^c+\frac{1}{2}(z^\mu\omega_\mu)^2-z^\mu\omega_\mu\left(\frac{1}{4\beta}z^\nu z^\lambda z^\sigma\partial_\nu g_{\lambda\sigma}\right)\\
&+\beta\Big[-4\alpha\,R_{abcd}\bar\lambda^a\cdot\eta^b\bar\lambda^c\cdot\eta^d+4\alpha\,R_{ab}\bar\lambda^a\cdot\eta^b+\frac{1}{12}R-V\Big]\Big\}\,.
\end{aligned}
\tag{2.8}
$$

This form is quite explicit. However, it can be simplified and cast in alternative and more suggestive forms. Keeping in mind that the exponent of the on-shell action should appear in the final result, one can factorize (2.8) in the following way[1]

$$
\begin{aligned}
&\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle=(2\pi\beta)^{-D/2}\\
&\times\exp\left\{-\frac{1}{\beta}\left[\frac{1}{2}\,g_{\mu\nu}\,z^\mu z^\nu+\frac{1}{4}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda+\frac{1}{12}\left(\partial_\mu\partial_\nu g_{\lambda\sigma}-\frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\right)z^\mu z^\nu z^\lambda z^\sigma\right]\right\}\\
&\times\exp\left\{\bar\lambda^a\cdot\eta_a+z^\mu\omega_{\mu ab}\bar\lambda^a\cdot\eta^b+\frac{1}{2}z^\mu z^\nu\Big(\partial_\mu\omega_{\nu ab}+\omega_{\mu ac}\omega_\nu{}^c{}_b\Big)\bar\lambda^a\cdot\eta^b-4\alpha\beta\,R_{abcd}\bar\lambda^a\cdot\eta^b\bar\lambda^c\cdot\eta^d\right\}\\
&\times\left[1+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(4\alpha R_{ab}\bar\lambda^a\cdot\eta^b+\frac{1}{12}R-V\Big)\right]\,.
\end{aligned}
\tag{2.9}
$$

The first exponential is the expansion up to order β of the on-shell bosonic action evaluated with the boundary conditions $x^\mu(0)=y^\mu$ and $x^\mu(\beta)=x^\mu$

$$
\begin{aligned}
S_x&=\int_0^\beta d\tau\,\left[\frac{1}{2}g_{\mu\nu}\dot x^\mu\dot x^\nu\right]_{\text{on shell}}\\
&=\frac{1}{\beta}\left[\frac{1}{2}\,g_{\mu\nu}\,z^\mu z^\nu+\frac{1}{4}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda+\frac{1}{12}\left(\partial_\mu\partial_\nu g_{\lambda\sigma}-\frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\right)z^\mu z^\nu z^\lambda z^\sigma\right]
\end{aligned}
\tag{2.10}
$$

where all functions in the second line are evaluated at the point x. Similarly, one can see that the second exponential in (2.9) is the expansion, up to order β, of the fermionic action evaluated on-shell, with boundary conditions $\Psi^a_I(0)=\eta^a_I$ and $\bar\Psi^{aI}(\beta)=\bar\lambda^{aI}$, and with the usual boundary term added

$$
\begin{aligned}
S_\Psi&=\left(\int_0^\beta d\tau\,\left[\bar\Psi^I_a D_\tau\Psi^a_I+4\alpha\,R_{abcd}\bar\Psi^a\cdot\Psi^b\bar\Psi^c\cdot\Psi^d\right]-\bar\Psi_a(\beta)\cdot\Psi^a(\beta)\right)\Bigg|_{\text{on shell}}\\
&=-\left\{\bar\lambda^a\cdot\eta_a+z^\mu\omega_{\mu ab}\bar\lambda^a\cdot\eta^b+\frac{1}{2}z^\mu z^\nu\Big(\partial_\mu\omega_{\nu ab}+\omega_{\mu ac}\omega_\nu{}^c{}_b\Big)\bar\lambda^a\cdot\eta^b-4\alpha\beta\,R_{abcd}\bar\lambda^a\cdot\eta^b\bar\lambda^c\cdot\eta^d\right\}
\end{aligned}
\tag{2.11}
$$

where the covariant time derivative reads $D_\tau\Psi^a_I=\dot\Psi^a_I+\dot x^\mu\omega_\mu{}^a{}_b\,\Psi^b_I$. Similar calculations of on-shell actions can be found, for instance, in [9]. Once the expansions of the on-shell

[1] The factor $[g(x)/g(y)]^{1/4}$ cancel against its inverse, whose Taylor expansion around x can be factored out from (2.8).

classical actions have been recognized, the transition amplitude can be cast in the following covariant form

$$\begin{aligned}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle &= \frac{1}{(2\pi\beta)^{D/2}}\ \exp\{-(S_x+S_\Psi)\}\\ &\times\Big[1+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(4\alpha R_{ab}\bar\lambda^{a}\cdot\eta^b+\frac{1}{12}R-V\Big)+\mathcal{O}(\beta^2)\Big]\end{aligned}\tag{2.12}$$

with all functions evaluated at point x. This is our final result for the transition amplitude.

For comparison purposes, it may be useful to present the result after tracing over the fermionic Hilbert space

$$\begin{aligned}\langle x|\mathrm{Tr}_\Psi(e^{-\beta H})|y\rangle &= \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}}\ \exp\{-S_x\}\Big[1-\frac{N}{16}z^\mu z^\nu\omega_{\mu ab}\omega_\nu{}^{ab}\Big]\\ &\times\Big[1+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(\frac{1+6\alpha N}{12}R-V\Big)+\mathcal{O}(\beta^2)\Big]\,.\end{aligned}\tag{2.13}$$

Evaluated at coinciding points ($z^\mu=0$) it reads as

$$\langle x|\mathrm{Tr}_\Psi(e^{-\beta H})|x\rangle = \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}}\Big[1+\beta\Big(\frac{1+6\alpha N}{12}R-V\Big)+\mathcal{O}(\beta^2)\Big]\tag{2.14}$$

which shows the first heat kernel coefficient at coinciding points for our model. As we shall see, the last two formulas remain valid also for odd N.

3 Transition amplitude from path integrals

In the present section we compute the transition amplitude by making use of path integral methods. To define the path integrals we fix a suitable regularization scheme, and compute the transition amplitude. Then, by comparing with the previous section or alternatively by imposing the Schrödinger equation, we identify the corresponding counterterms that ensure scheme independence.

Unlike the previous section here we treat together both the cases with even and odd numbers of Majorana variables. In order to do so, we found it convenient to use the so-called "doubling trick" that consists in doubling the number of fermionic variables by adding "spectator" Majorana fermions $\psi_i'^a$ that satisfy free anticommutation relations. These new fermions are spectators in that they do not enter the interactions. With the help of these new variables one can consider Dirac fermions

$$\Psi_i^a=\frac{1}{\sqrt2}\Big(\psi_i^a+i\psi_i'^a\Big)\,,\quad \bar\Psi_i^a=\frac{1}{\sqrt2}\Big(\psi_i^a-i\psi_i'^a\Big)\tag{3.1}$$

that satisfy

$$\{\Psi_i^a,\bar\Psi_j^b\}=\eta^{ab}\delta_{ij}\tag{3.2}$$

giving rise to a set of fermionic harmonic oscillators, whose Hilbert space can be represented in terms of bra and ket coherent states

$$\Psi_i^a|\eta\rangle=\eta_i^a|\eta\rangle\,,\quad \langle\bar\lambda|\bar\Psi_i^a=\langle\bar\lambda|\bar\lambda_i^a\tag{3.3}$$

and whose properties are briefly recalled in appendix A. The wave function of the system $\Phi(x,\bar\lambda)=\langle x\,\bar\lambda|\Phi\rangle$, with $\langle x\,\bar\lambda|\equiv\langle x|\otimes\langle\bar\lambda|$, evolves in euclidean time as

$$\begin{aligned}\Phi(x,\bar\lambda;\beta) &= \langle x\,\bar\lambda|e^{-\beta H}|\Phi\rangle \\ &= \int d^Dy\sqrt{g(y)}\int d\bar\zeta d\eta\; e^{-\bar\zeta\cdot\eta}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle\Phi(y,\bar\zeta;0) \end{aligned}\tag{3.4}$$

where we have used the spectral decomposition of unity for the bosonic and fermionic sectors

$$\mathbb{1}_b=\int d^Dy\sqrt{g(y)}|y\rangle\langle y|\,,\quad \mathbb{1}_f=\int d\bar\zeta d\eta\; e^{-\bar\zeta\cdot\eta}|\eta\rangle\langle\bar\zeta|\;. \tag{3.5}$$

The evolution is generated by the basic transition amplitude that can be written in terms of a path integral as

$$\begin{aligned}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle &= \langle x\,\bar\lambda;\beta|y\,\eta;0\rangle \\ &= \int_{x(-1)=y}^{x(0)=x} DxDaDbDc\int_{\Psi(-1)=\sqrt{\beta}\eta}^{\bar\Psi(0)=\sqrt{\beta}\bar\lambda} D\bar\Psi D\Psi\; e^{-S[x,a,b,c,\Psi,\bar\Psi]} \end{aligned}\tag{3.6}$$

where

$$\begin{aligned}S[x,a,b,c,\Psi,\bar\Psi] &= \frac{1}{2\beta}\int_{-1}^{0}d\tau\, g_{\mu\nu}(x(\tau))(\dot x^\mu\dot x^\nu+a^\mu a^\nu+b^\mu c^\nu) \\ &+\frac{1}{\beta}\int_{-1}^{0}d\tau\,\bar\Psi_{ai}\dot\Psi^a_i-\frac{1}{\beta}\bar\Psi_{ai}(0)\Psi^a_i(0)+\frac{1}{2\beta}\int_{-1}^{0}d\tau\,\dot x^\mu\omega_{\mu ab}(x(\tau))\psi^a_i\psi^b_i \\ &+\frac{\alpha}{\beta}\int_{-1}^{0}d\tau\,R_{abcd}(x(\tau))\psi^a_i\psi^b_i\psi^c_j\psi^d_j+\beta\int_{-1}^{0}d\tau\,V(x(\tau))\;. \end{aligned}\tag{3.7}$$

Here, and in the following, we use a shifted and rescaled euclidean time $\tau\in[-1,0]$ to make comparison with the literature easier. We use bosonic (a^μ) and fermionic (b^μ, c^μ) ghosts to reproduce the reparametrization invariant measure $\mathcal{D}x=\prod_\tau\sqrt{g(x(\tau))}d^Dx(\tau)$. We also rescaled fermionic "coordinates" and ghosts so that all propagators are of order β, and added a fermionic boundary term to be able to set boundary conditions at initial time for Ψ and at final time for $\bar\Psi$. Finally, let us note that the arbitrary potential V will eventually be modified by the addition of a counterterm V_{CT} related to the regularization chosen. Apart form these modifications, and the inclusion of the spectator fermions, this is the same action given in eq. (1.3).

We are interested in the short-time perturbative expansion of the transition amplitude. Thus we expand all geometric expressions about the final point x^μ, and split the action into a quadratic part plus interactions, $S=S_2+S_{\rm int}$, with

$$S_2=\frac{1}{2\beta}g_{\mu\nu}\int_{-1}^{0}d\tau\,(\dot x^\mu\dot x^\nu+a^\mu a^\nu+b^\mu c^\nu)+\frac{1}{\beta}\int_{-1}^{0}d\tau\,\bar\Psi_{ai}\dot\Psi^a_i-\frac{1}{\beta}\bar\Psi_{ai}(0)\Psi^a_i(0)\;. \tag{3.8}$$

Here and in the following, geometric quantities with no explicit functional dependence are meant to be evaluated at the final point $x(0)=x$, such as $g_{\mu\nu}=g_{\mu\nu}(x)$. We can thus

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split the fields into backgrounds, satisfying the free equations of motion with corresponding boundary conditions, and quantum fluctuations; namely

$$x^\mu(\tau) = \tilde{x}^\mu(\tau) + q^\mu(\tau)\,, \qquad \tilde{x}^\mu(\tau) = x^\mu - z^\mu\tau\,, \qquad q^\mu(0) = q^\mu(-1) = 0 \tag{3.9}$$

$$\Psi_i^a(\tau) = \tilde{\Psi}_i^a(\tau) + Q_i^a(\tau)\,, \qquad \tilde{\Psi}_i^a(\tau) = \sqrt{\beta}\eta_i^a \qquad Q_i^a(-1) = 0 \tag{3.10}$$

$$\bar{\Psi}_i^a(\tau) = \tilde{\bar{\Psi}}_i^a(\tau) + \bar{Q}_i^a(\tau)\,, \qquad \tilde{\bar{\Psi}}_i^a(\tau) = \sqrt{\beta}\bar{\lambda}_i^a\,, \qquad \bar{Q}_i^a(0) = 0 \tag{3.11}$$

where $z^\mu \equiv y^\mu - x^\nu$. The free on-shell classical action reads (henceforth we use a dot to indicate the contraction of whatever type of free flat indices)

$$\tilde{S}_2 = \frac{1}{2\beta} g_{\mu\nu} z^\mu z^\nu - \bar{\lambda}\cdot\eta \tag{3.12}$$

and the free propagators for the quantum fields, together with their Feynman diagrams, are

$$\langle q^\mu(\tau) q^\nu(\sigma)\rangle = -\beta g^{\mu\nu}\Delta(\tau,\sigma) = \tag{3.13}$$

$$\langle a^\mu(\tau) a^\nu(\sigma)\rangle = \beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma) = \tag{3.14}$$

$$\langle b^\mu(\tau) c^\nu(\sigma)\rangle = -2\beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma) = \tag{3.15}$$

$$\langle \bar{Q}_i^a(\tau) Q_j^b(\sigma)\rangle = \beta\eta^{ab}\delta_{ij} G(\tau,\sigma) \tag{3.16}$$

where the right-hand sides are given, at the unregulated level, by the following distributions

$$\begin{aligned} \Delta(\tau,\sigma) &= \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau) \\ \Delta_{gh}(\tau,\sigma) &= \delta(\tau-\sigma) \\ G(\tau,\sigma) &= -\theta(\sigma-\tau) \end{aligned} \tag{3.17}$$

which obey the Green equations ${}^{\bullet\bullet}\Delta(\tau,\sigma) = \Delta_{gh}(\tau,\sigma) = \delta(\tau-\sigma)$ and ${}^\bullet G(\tau,\sigma) = \delta(\tau-\sigma)$, with boundary conditions $\Delta(0,\sigma) = \Delta(\tau,0) = \Delta(-1,\sigma) = \Delta(\tau,-1) = 0$ and $G(0,\sigma) = G(\tau,-1) = 0$. Dots to the left (right) indicate derivatives with respect to the first (second) variable. These propagators have to be regulated in order to deal with divergences and ambiguities present in some diagrams. However, for each regularization scheme, one is able to cast all expressions in a way that can be unambiguously computed by using directly the expressions (3.17).

Since fermions enter the interactions only through the combination $\psi_i^a = \frac{1}{\sqrt{2}}(\Psi_i^a + \bar{\Psi}_i^a)$ it is convenient to write backgrounds and propagators for these fields as well. We split them as $\psi_i^a(\tau) = \tilde{\psi}_i^a + \chi_i^a(\tau)$ with

$$\tilde{\psi}_i^a = \sqrt{\frac{\beta}{2}}(\eta_i^a + \bar{\lambda}_i^a) \tag{3.18}$$

and

$$\langle \chi_i^a(\tau)\chi_j^b(\sigma)\rangle = \beta\eta^{ab}\delta_{ij}\Delta_F(\tau,\sigma) = \tag{3.19}$$

satisfying ${}^\bullet\Delta_F(\tau,\sigma) = -\Delta_F^\bullet(\tau,\sigma) = \delta(\tau-\sigma)$ and given at the unregulated level by

$$\Delta_F(\tau,\sigma) = \frac{1}{2}(\theta(\tau-\sigma) - \theta(\sigma-\tau)) = \frac{1}{2}\epsilon(\tau-\sigma)\,. \tag{3.20}$$

We only wrote propagators for unprimed fermionic fields since only they enter the interactions. The primed fields instead only contribute to the one-loop semiclassical factor that normalizes the path integral and drop immediately out of the computation.

The interaction vertices that enter the perturbative expansion can be obtained by Taylor expanding the action (3.7) about the final point $x(0) = x$ and read $S_{\text{int}} = \sum_{n=3}^{\infty} S_n$, with

$$
\begin{aligned}
S_3 =\; & \frac{1}{2\beta}\omega_{\mu ab}\int_{-1}^{0} d\tau (\dot q^\mu - z^\mu)(\tilde\psi^a + \chi^a)\cdot(\tilde\psi^b + \chi^b) \\
& + \frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\int_{-1}^{0} d\tau (q^\lambda - z^\lambda\tau)\Big((\dot q^\mu - z^\mu)(\dot q^\nu - z^\nu) + a^\mu a^\nu + b^\mu c^\nu\Big)
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
S_4 =\; & \beta V + \frac{1}{2\beta}\partial_\lambda\omega_{\mu ab}\int_{-1}^{0} d\tau (q^\lambda - z^\lambda\tau)(\dot q^\mu - z^\mu)(\tilde\psi^a + \chi^a)\cdot(\tilde\psi^b + \chi^b) \\
& + \frac{\alpha}{\beta}R_{abcd}\int_{-1}^{0} d\tau (\tilde\psi^a + \chi^a)\cdot(\tilde\psi^b + \chi^b)(\tilde\psi^c + \chi^c)\cdot(\tilde\psi^d + \chi^d) \\
& + \frac{1}{4\beta}\partial_\lambda\partial_\sigma g_{\mu\nu}\int_{-1}^{0} d\tau (q^\lambda - z^\lambda\tau)(q^\sigma - z^\sigma\tau)\Big((\dot q^\mu - z^\mu)(\dot q^\nu - z^\nu) + a^\mu a^\nu + b^\mu c^\nu\Big)\,.
\end{aligned}
\tag{3.22}
$$

Higher order terms are not reported because we compute the transition amplitude to order β, for which only the previous two terms are needed; all fields, classical and quantum, count as $\beta^{1/2}$.

The transition amplitude can now be computed perturbatively using Wick-contractions of the quantum fields

$$
\begin{aligned}
\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle &= A\ e^{-\tilde S_2}\left\langle e^{-S_{\text{int}}}\right\rangle \\
&= A\ e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu + \bar\lambda\cdot\eta}\exp\Big[-\langle S_3\rangle - \langle S_4\rangle + \frac{1}{2}\langle S_3^2\rangle_c\Big]
\end{aligned}
\tag{3.23}
$$

with the suffix c indicating connected diagrams only and with the prefactor A soon to be commented upon. The various Wick-contractions produce

$$
\begin{aligned}
\exp\Big[-\langle S_3\rangle - \langle S_4\rangle + \frac{1}{2}\langle S_3^2\rangle_c\Big] = \exp\Bigg[& -\frac{1}{4\beta}\partial_\lambda g_{\mu\nu}z^\lambda z^\mu z^\nu + \frac{1}{2\beta}\omega_{\lambda ab}z^\lambda\tilde\psi^a\cdot\tilde\psi^b \\
& - z^\lambda\left(\frac{1}{2}\partial_\lambda g\,\mathbf{I}_1 + g_\lambda\,\mathbf{I}_2\right) - \frac{1}{12\beta}\partial_\lambda\partial_\sigma g_{\mu\nu}z^\lambda z^\sigma z^\mu z^\nu - \frac{\beta}{4}\partial^2 g\,\mathbf{I}_3 - \frac{\beta}{2}\partial^\lambda\partial^\sigma g_{\lambda\sigma}\,\mathbf{I}_4 \\
& + \frac{1}{4}\partial_\lambda\partial_\sigma g z^\lambda z^\sigma\,\mathbf{I}_5 + \frac{1}{4}\partial^2 g_{\lambda\sigma}z^\lambda z^\sigma\,\mathbf{I}_6 + \partial_\lambda g_\sigma z^\lambda z^\sigma\,\mathbf{I}_7 + \frac{1}{4\beta}\partial_\lambda\omega_{\sigma ab}\tilde\psi^a\cdot\tilde\psi^b z^\lambda z^\sigma \\
& + \frac{1}{2}\partial^\lambda\omega_{\lambda ab}\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_2 - \frac{\alpha}{\beta}R_{abcd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d - \beta V \\
& - \frac{\beta}{4}(\partial_\lambda g_{\mu\nu})^2\,\mathbf{I}_9 - \frac{\beta}{2}\partial_\lambda g_{\mu\nu}\partial_\mu g_{\lambda\nu}\,\mathbf{I}_{10} - \frac{\beta}{8}(\partial_\lambda g)^2\,\mathbf{I}_{11} - \frac{\beta}{2}g_\lambda\partial_\lambda g\,\mathbf{I}_{12} - \frac{\beta}{2}g_\lambda^2\,\mathbf{I}_{13} \\
& + \frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\lambda g_{\mu'\nu}z^\mu z^{\mu'}\,\mathbf{I}_{14} + \frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\nu g_{\mu'\lambda}z^\mu z^{\mu'}\,\mathbf{I}_{15} + \frac{1}{4}\partial_\lambda g_{\mu\nu}\partial_{\lambda'}g_{\mu\nu}z^\lambda z^{\lambda'}\,\mathbf{I}_{16} \\
& + \partial_\lambda g_{\mu\nu}\partial_\mu g_{\mu'\nu}z^\lambda z^{\mu'}\,\mathbf{I}_{17} + \frac{1}{4}\partial_\lambda g_{\mu\nu}\partial_\lambda g z^\mu z^\nu\,\mathbf{I}_{18} + \frac{1}{2}\partial_\lambda g_{\mu\nu}g_\lambda z^\mu z^\nu\,\mathbf{I}_{19} \\
& + \frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\mu g z^\lambda z^\nu\,\mathbf{I}_{20} + \partial_\lambda g_{\mu\nu}g_\mu z^\lambda z^\nu\,\mathbf{I}_{21} - \frac{1}{8\beta}\partial_\lambda g_{\mu\nu}\partial_\lambda g_{\mu'\nu'}z^\mu z^\nu z^{\mu'}z^{\nu'}\,\mathbf{I}_{22}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\partial_{\lambda'} g_{\lambda\nu'} z^\mu z^\nu z^{\lambda'} z^{\nu'}\, \mathbf{I}_{23} - \frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\partial_{\lambda'} g_{\mu\nu'} z^\lambda z^\nu z^{\lambda'} z^{\nu'}\, \mathbf{I}_{24} \\
&+ \frac{\beta N}{4}\omega_{\lambda ab}^2\, \mathbf{I}_{25} - \frac{N}{4}\omega_{\lambda ab}\,\omega_\sigma{}^{ab} z^\lambda z^\sigma\, \mathbf{I}_{26} + \frac{1}{2}\omega_{\lambda ac}\,\omega_{\lambda b}{}^c \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{27} \\
&- \frac{1}{2\beta}\omega_{\lambda ac}\,\omega_{\sigma b}{}^c z^\lambda z^\sigma \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{28} - \frac{1}{8\beta}\omega_{\lambda ab}\,\omega_{\lambda cd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d\, \mathbf{I}_{29} \\
&+ \frac{1}{4}\omega_{\lambda ab}\partial_\lambda g \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{30} + \frac{1}{2}\omega_{\lambda ab} g_\lambda \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{31} - \frac{1}{4\beta}\omega_{\lambda ab}\partial_\lambda g_{\mu\nu} z^\mu z^\nu \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{32} \\
&- \frac{1}{2\beta}\omega_{\mu ab}\partial_\lambda g_{\mu\nu} z^\lambda z^\nu \tilde\psi^a\cdot\tilde\psi^b\, \mathbf{I}_{33}\Bigg]
\end{aligned}
\tag{3.24}
$$

where we made use of several shortcut notations, including $g_\mu = g^{\lambda\nu}\partial_\lambda g_{\mu\nu}$, $\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$, $\partial_\lambda\partial_\sigma g = g^{\mu\nu}\partial_\lambda\partial_\sigma g_{\mu\nu}$ and $\partial^2 = g^{\lambda\sigma}\partial_\lambda\partial_\sigma$. The integrals $\mathbf{I}_k$ are reported in appendix C along with the pictorial representation of the diagrams they belong to (an integral named $\mathbf{I}_8$ is absent, but we kept the same notation as in [10], where such an integral arose from the coupling to external gauge fields, to allow easy comparison). We compute them in the following subsections, using the different regularization schemes discussed in the introduction. The purely bosonic contributions ($k \leq 24$) are well-known from many previous computations, see [10] for example; the remaining ones have been computed, though only for $N \leq 2$, in the time slicing regularization technique [16], in dimensional regularization [25, 26], and in mode regularization [21]. However, in the last two schemes fermions were traced out to obtain directly heat kernel coefficients and trace anomalies. In the present case we are interested in the full transition amplitude and wish to keep N generic, so that we need to reconsider all such integrals with fermionic contributions. Finally, the prefactor A can be fixed by requiring that, in the limit $\beta \to 0$, the transition amplitude reduces to $\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle \to \delta(x-y)\,e^{\bar\lambda\cdot\eta}$, which in MR and DR gives

$$
A = \frac{1}{(2\pi\beta)^{D/2}}\,. \tag{3.25}
$$

In TS such prefactor can be deduced directly starting from operatorial methods, and reads

$$
A = \left[\frac{g(x)}{g(y)}\right]^{1/4} \frac{1}{(2\pi\beta)^{D/2}}\,. \tag{3.26}
$$

We are now ready to consider the various regularization schemes.

3.1 Time slicing regularization

Time sliced path integrals in curved space were extensively discussed in [15, 16]. In essence time slicing regularization consists in studying the discretized version of the path integral, as derived form the operatorial picture by using Weyl ordering and midpoint rule, and recognizing the action with the correct counterterms together with the rules how to compute Feynman graphs that must be used in the continuum limit. These rules state that the Dirac delta functions should always be implemented as if they were Kronecker delta's, using the value $\theta(0) = \frac{1}{2}$ for the discontinuous step function. We do not need to repeat here many computations, namely the bosonic ones, though one can easily compute them using the explicit expressions collected in appendix C, or extract them directly from [10]. It is enough to focus on the graphs containing fermionic lines, i.e. $\mathbf{I}_k$ with $k \geq 25$.

Thus, let us start with $\mathbf{I}_{25}$, that is the only diagram that depends heavily on the regularization chosen. Following the TS prescriptions, we have

$$
\begin{aligned}
\mathbf{I}_{25} &= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\!\Delta^\bullet \; \Delta_F^2 = \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \Big(1-\delta(\tau-\sigma)\Big)\Big(\frac{1}{2}\epsilon(\tau-\sigma)\Big)^2 \\
&= \frac{1}{4}\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \, \epsilon^2(\tau-\sigma) = \frac{1}{4}\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma = \frac{1}{4}
\end{aligned} \tag{3.27}
$$

since ${}^\bullet\!\Delta^\bullet = 1-\delta(\tau-\sigma)$ and $\epsilon(0)=0$, as follows form $\theta(0)=\frac{1}{2}$ and eq. (3.20). The regular $\mathbf{I}_{26}$ does not need any prescription and corresponds to the last line above, giving $\mathbf{I}_{26}=\frac{1}{4}$. Similarly, one finds that the integrals $\mathbf{I}_k$ with $k \geq 27$ yield a vanishing result (those with $k \geq 28$ actually contain only bosonic propagators, but depend on the external fermionic backgrounds).

The transition amplitude then reads

$$
\begin{aligned}
\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle = &\frac{e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu+\bar\lambda\cdot\eta}}{(2\pi\beta)^{D/2}}\exp\Big[\langle\text{bosonic}\rangle_{TS}+\frac{1}{2\beta}\omega_{\lambda ab}z^\lambda\tilde\psi^a\cdot\tilde\psi^b \\
&+\frac{1}{4\beta}\partial_\lambda\omega_{\sigma ab}z^\lambda z^\sigma\tilde\psi^a\cdot\tilde\psi^b-\frac{\alpha}{\beta}R_{abcd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d-\frac{N}{16}\omega_{\lambda ab}\omega_{\sigma ab}z^\lambda z^\sigma \\
&+\frac{\beta N}{16}\omega^2_{\lambda ab}\Big]
\end{aligned} \tag{3.28}
$$

where $\langle\text{bosonic}\rangle_{TS}$ contains the purely bosonic contributions of (3.24), including the metric factors appearing in (3.26). It can be extracted from the literature, or easily recomputed with the set-up described above, and reads

$$
\begin{aligned}
\langle\text{bosonic}\rangle_{TS} = &-\frac{1}{4\beta}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda-\frac{1}{12\beta}\Big(\partial_\mu\partial_\nu g_{\lambda\sigma}-\frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\Big)\,z^\mu z^\nu z^\lambda z^\sigma \\
&+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(\frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}-\frac{1}{24}R-V\Big)\,.
\end{aligned} \tag{3.29}
$$

This transition amplitude is the one computed with a TS regularization of the Feynman diagrams, and in general differs form those computed with other regularizations if no counterterms are introduced. In particular, eq. (3.28) satisfies a Schrödinger equation with a non-covariant hamiltonian that differs from the one given in eq. (1.1). One expects different regularizations to be related by finite local counterterms, so we need to identify the correct counterterm that make sure that we are discussing the same quantum theory as the one associated to the hamiltonian H of eq. (1.1). To achieve this we can either compare with the transition amplitude obtained by operatorial methods, or compute directly the hamiltonian associated with the transition amplitude above. We shall follow both methods as a check of our calculations.

As it stands the transition amplitude calculated above cannot be compared directly with the result from canonical methods in eq. (2.12): the latter is valid for even N and is written in terms of fermionic coherent states that correspond to the physical fermions only, while the present result, valid for any integer N, contains also the Majorana spectator fields introduced in the fermion doubling trick. To overcome these differences, we can take a trace over the fermionic Hilbert space and eliminate the contribution of the decoupled spectator variables by subtracting their degrees of freedom. Thus, let us compute the trace in the fermionic sector of eq. (3.28) by integrating over the Grassmann variables with measure $\int d\eta d\bar{\lambda}\, e^{\bar{\lambda}\cdot\eta}$, see appendix A. The final result can be obtained, for instance, using the standard Wick-contractions associated to the gaussian integral $\int d\eta d\bar{\lambda}\, e^{2\bar{\lambda}\cdot\eta}$ (note the factor 2 in the exponent arising form the trace measure and the leading part of (3.28)), which gives the following propagators

$$\langle \bar{\lambda}_i^a \eta_j^b \rangle = \frac{1}{2}\eta^{ab}\delta_{ij} \qquad \rightarrow \qquad \langle \tilde{\psi}_i^a \tilde{\psi}_j^b \rangle = 0 \tag{3.30}$$

where we used the definition in (3.18). Note that the normalization factor $\int d\eta d\bar{\lambda}\, e^{2\bar{\lambda}\cdot\eta} = 2^{ND}$ has to be renormalized to $2^{\frac{ND}{2}}$ to undo the fermion doubling. Taking all this into account, and setting $z \equiv y - x = 0$ for simplicity, we obtain

$$2^{-\frac{ND}{2}}\int d\eta d\bar{\lambda}\, e^{\bar{\lambda}\cdot\eta}\langle x\, \bar{\lambda}; \beta | x\, \eta; 0\rangle = \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}}\left[1+\beta\Big(\frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \frac{1}{24}R - V\Big) + \frac{\beta N}{16}\omega^2_{\lambda ab} + \dots\right] \tag{3.31}$$

where the first term in round brackets is due to the purely bosonic contributions. Comparing with (2.14) one recognizes the counterterm that needs to be added to the potential V in the path integral action to achieve equality

$$V_{TS}^{(N)} = -\Big(\frac{1}{8} + \frac{\alpha N}{2}\Big)R + \frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \frac{N}{16}\omega_{\mu ab}\omega^{\mu ab}\, . \tag{3.32}$$

Power counting shows that no higher order corrections to the counterterm are to be expected.

Alternatively, one may compute the hamiltonian appearing in the Schrödinger equation satisfied by the amplitude (3.28). To do this we insert the latter into (3.4), Taylor-expand to order β all terms in the right hand side about the final point, and identify the Schrödinger equation associated to it. Comparing with the Schrödinger equation due to the hamiltonian (1.1) one deduces eventual counterterms. Let us describe this computation. We perform the Gaussian integrals over $d^D y = d^D z$ and the integrals over the fermionic coherent states using the properties summarized in appendix A. The purely bosonic contributions of the diagrammatic expansion yield the standard TS result

$$\Phi(x, \bar{\lambda}; \beta) = \Big(1 - \beta\partial_t - \beta H_B + O(\beta^2)\Big)\Phi(x, \bar{\lambda}; \beta) \tag{3.33}$$

with

$$H_B = -\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu + V + \frac{1}{8}\Big(R - g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\Big)\, . \tag{3.34}$$

that by itself requires the addition of the familiar counterterm $V_{TS}^{(0)} = -\frac{1}{8}\Big(R - g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\Big)$ into the path integral, in order to get the covariant $H = -\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu + V$. We need to dress this result with the fermionic contributions. One way to perform the integrals over the fermionic coherent states is to use the formulas involving fermionic delta functions derived at the end of section A. By defining the full hamiltonian H_F as

$$H_F = H_B + \Delta \tag{3.35}$$

and using (3.33), we obtain

$$\begin{aligned}\Delta \,=&\, -\frac{1}{4}\partial^\lambda \omega_{\lambda ab} M^{ab} + \frac{1}{4} g^{\lambda\sigma}\Gamma^\mu_{\lambda\sigma}\omega_{\mu ab}M^{ab} - \frac{1}{2}\omega^\mu{}_{ab}M^{ab}\partial_\mu - \frac{1}{8}\omega^\lambda{}_{ab}\omega_{\lambda cd}M^{ab}M^{cd}\\ &+\alpha R_{abcd}M^{ab}M^{cd} + \frac{\alpha N}{2}R - \frac{N}{16}(\omega_{\lambda ab})^2\end{aligned}\tag{3.36}$$

with $M^{ab} = \frac{1}{2}\Big(\bar\lambda^a\cdot\bar\lambda^b + \bar\lambda^a\cdot\frac{\partial}{\partial\bar\lambda_b} - \bar\lambda^b\cdot\frac{\partial}{\partial\bar\lambda_a} + \frac{\partial}{\partial\bar\lambda_a}\cdot\frac{\partial}{\partial\bar\lambda_b}\Big)$ being Lorentz generators. We observe that the noncovariant terms in the line above are those necessary to render the bosonic hamiltonian H_B local-Lorentz covariant, namely

$$\begin{aligned}H_F \,=&\, H_B + \Delta = -\frac{1}{2}g^{\mu\nu}D_\mu D_\nu + \alpha R_{abcd}M^{ab}M^{cd} + V\\ &+\Big(\frac{1}{8}+\frac{\alpha N}{2}\Big)R - \frac{1}{8}g^{\lambda\sigma}\Gamma^\tau_{\lambda\rho}\Gamma^\rho_{\sigma\tau} - \frac{N}{16}(\omega_{\lambda ab})^2\end{aligned}\tag{3.37}$$

where D_μ is the fully covariant derivative (with both Lorentz and Christoffel connections), so that in order to have[2]

$$H \,=\, -\frac{1}{2}g^{\mu\nu}D_\mu D_\nu + \alpha R_{abcd}\psi^a\cdot\psi^b\psi^c\cdot\psi^d + V \tag{3.38}$$

one needs to add to the path integral the same counterterm found before in eq. (3.32). Thus we found complete agreement.

The above expression for the counterterm $V_{TS}^{(N)}$ matches all the previously known cases [15, 16]: the purely bosonic case ($N=0$) is obviously reproduced. For $N=1$ supersymmetry fixes $\alpha=-1/4$ and $V=0$ so that $V_{TS}^{(1)} = \frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \frac{1}{16}(\omega_{\mu ab})^2$ comes out correctly. Note that in this case the relation $R_{abcd}\psi^a\psi^b\psi^c\psi^d = -\frac{1}{2}R$ allows to use $\alpha=0$ and $V=\frac{1}{8}R$ as well to identify the same supersymmetric hamitonian. For $N=2$, supersymmetry requires $\alpha=-\frac{1}{8}$ and $V=0$, so that again $V_{TS}^{(2)} = \frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \frac{1}{8}(\omega_{\mu ab})^2$ is correctly reproduced.

3.2 Mode regularization

In this section we approach the previous computation using mode regularization (MR), that can be considered as the worldline equivalent of a cut-off regularization in momentum space of standard quantum field theories. Mode regularization starts by expanding all fields in Fourier sums, thus identifying a suitable functional space to integrate over in the path integral. The regularization is achieved by truncating the Fourier sums at a fixed mode M, so that all distributions appearing in Feynman graphs become well-behaved functions.

[2] Here we use $\langle\bar\lambda|\psi^{ai}|\Phi\rangle = \frac{1}{\sqrt2}\left(\frac{\partial}{\partial\bar\lambda^i_a} + \bar\lambda^{ai}\right)\Phi(\bar\lambda)$, so that the Lorentz generators can be written as $M^{ab} = \frac{1}{2}\left(\psi^a\cdot\psi^b - \psi^b\cdot\psi^a\right)$.

Eventually one takes the limit $M \to \infty$ to obtain a unique (finite) result. Often one can proceed faster: performing manipulations that are guaranteed to be valid at the regulated level one may cast the integrands in alternative forms that can be computed directly in the continuum limit, without encountering any ambiguity.

The boundary conditions for the bosonic variables are as in (3.9), so that the bosonic quantum fluctuations, as well as the ghosts, are naturally expanded in a Fourier sine series

$$q^\mu(\tau) = \sum_{m=1}^{M} q^\mu_m \sin(\pi m \tau) \tag{3.39}$$

where the mode M is the regulating cut-off that is eventually sent to infinity, as in [17]. This choice preserves the boundary conditions imposed at initial and final times. On the other hand, fermionic fields satisfy first order differential equation and carry boundary conditions only at initial or final times, but not at both, see eqs. (3.10) and (3.11). Thus we find it useful to expand the fermionic quantum fields in half integers modes as follows

$$\bar{Q}^{ai}(\tau) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \bar{Q}^{ai}_r \sin(\pi r \tau)\ , \qquad Q^{ai}(\tau) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} Q^{ai}_r \cos(\pi r \tau)\ . \tag{3.40}$$

This choice preserves the boundary conditions and provides us with a regulated functional space to integrate over also for the fermions. In addition, it produces a simple regulated kinetic term that is easily inverted to obtain the propagators. Finally, the path integral is defined as a regulated integration over the Fourier coefficients of the various fields.

Perturbatively, the propagators are as in eqs. (3.13)–(3.16) and are regulated as follows

$$\Delta(\tau,\sigma) = -\sum_{m=1}^{M} \frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \tag{3.41}$$

$$\Delta_{gh}(\tau,\sigma) = \sum_{m=1}^{M} 2 \sin(\pi m \tau) \sin(\pi m \sigma) \tag{3.42}$$

$$G(\tau,\sigma) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \frac{2}{\pi r} \sin(\pi r \tau) \cos(\pi r \sigma)\ . \tag{3.43}$$

As a consequence

$$\Delta_F(\tau,\sigma) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \frac{1}{\pi r} \sin(\pi r(\tau-\sigma)) \tag{3.44}$$

for (3.19) that turns out to be translational invariant. Note that in the limit $M \to \infty$ the previous formulas reproduce eqs. (3.17) and (3.20).

We are now ready to compute the Feynman integrals with fermionic contributions in MR (the purely bosonic ones are standard and can be extracted from [10]). Using the regularized expressions one obtains $\mathbf{I}_{27} = \mathbf{I}_{28} = \ldots = \mathbf{I}_{33} = 0$. Also, one finds $\mathbf{I}_{26} = \frac{1}{4}$, as it is unambiguous and gives the same result in all regularization schemes. The only tricky integral is $\mathbf{I}_{25}$. Using the mode regulated propagators we can manipulate it using an integration by parts as follows

$$\begin{aligned}\mathbf{I}_{25} &= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^{\bullet}\Delta^{\bullet}\,\Delta_F^2 = \int_{-1}^{0} d\tau \Big[\Delta^{\bullet}\Delta_F^2\Big]_{\sigma=-1}^{\sigma=0} - \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\,\Delta^{\bullet\bullet}\,\Delta_F^2 = \\ &= \frac{1}{2}\int_{-1}^{0} d\tau\Big[\Delta^{\bullet}\Delta_F^2\Big]_{\sigma=-1}^{\sigma=0} + \frac{1}{2}\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\,({}^{\bullet}\Delta^{\bullet} - \Delta^{\bullet\bullet})\,\Delta_F^2\end{aligned} \tag{3.45}$$

The first term is unambiguous and can be computed directly upon removing the cut-off to give $\frac{1}{8}$. In the second term one can use the following identity valid at the regulated level (with $x = \tau - \sigma$)

$${}^{\bullet}\Delta^{\bullet}(\tau,\sigma) - \Delta^{\bullet\bullet}(\tau,\sigma) = -2\cos\left(\frac{\pi x}{2}\right){}^{\bullet}\Delta_F(x) + 1 - \cos(\pi M x)\,. \tag{3.46}$$

The integral involving the last term ($\cos(\pi M x)$) produces a vanishing result in the limit $M \to \infty$, thanks to the Riemann-Lebesgue lemma, while the rest can be computed using an integration by parts and removing the regularization to yield the final result

$$\mathbf{I}_{25} = \frac{1}{8} + \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \left[\frac{1}{2} - \cos\left(\frac{\pi x}{2}\right){}^{\bullet}\Delta_F(x)\right]\Delta_F^2(x) = \frac{1}{6}\,. \tag{3.47}$$

The transition amplitude computed in mode regularization hence reads

$$\begin{aligned}\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle = \frac{e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu + \bar\lambda\cdot\eta}}{(2\pi\beta)^{D/2}}\exp\Big[&\langle\text{bosonic}\rangle_{MR} + \frac{1}{2\beta}\omega_{\mu ab}\,z^\mu\,\tilde\psi^a\cdot\tilde\psi^b \\ &+\frac{1}{4\beta}\,\partial_\mu\omega_{\nu ab}\,z^\mu z^\nu\,\tilde\psi^a\cdot\tilde\psi^b - \frac{\alpha}{\beta}R_{abcd}\,\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d - \frac{N}{16}\,\omega_{\mu ab}\,\omega_\nu{}^{ab}\,z^\mu z^\nu \\ &+\frac{\beta N}{24}\,\omega_{\mu ab}^2\Big]\,.\end{aligned} \tag{3.48}$$

where $\langle\text{bosonic}\rangle_{MR}$ is the purely bosonic contribution that can be found in [10]. Comparing the above result with the TS one given in eq. (3.28), we notice that the only difference coming from the fermionic sector sits in the coefficients of the last ω^2 term, which is due to $\mathbf{I}_{25}$. As the bosonic part of the MR calculation requires the counterterm

$$V_{MR}^{(0)} = -\frac{1}{8}R - \frac{1}{24}\left(\Gamma^\mu_{\nu\lambda}\right)^2$$

to reproduce the heat kernel for $H = -\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu$, where $\left(\Gamma^\mu_{\nu\lambda}\right)^2 \equiv g^{\nu\nu'}g^{\lambda\lambda'}g_{\mu\mu'}\Gamma^\mu_{\nu\lambda}\Gamma^{\mu'}_{\nu'\lambda'}$, we see that, in order to obtain the correct amplitude for (1.1), we need the counterterm

$$V_{\rm MR}^{(N)} = -\left(\frac{1}{8} + \frac{\alpha N}{2}\right)R - \frac{1}{24}\left(\Gamma^\mu_{\nu\lambda}\right)^2 + \frac{N}{24}\omega_{\mu ab}\omega^{\mu ab} \tag{3.49}$$

that again matches the results known for $N = 0, 1, 2$ [19, 21].

3.3 Dimensional regularization

Finally, we reconsider the previous calculations using dimensional regularization. This is a perturbative regularization that uses an adaptation of standard dimensional regularization to regulate the distributions defined on the one dimensional compact space $I = [0, \beta]$. One adds d extra non-compact dimensions and perform all computations of ambiguous Feynman graphs in $d+1$ dimensions. Extra dimensions act as a regulator when d is extended analytically to the complex plane, as in the usual QFT case. After evaluation of the various integrals one should take the $d \to 0$ limit [24]. This is quite difficult in general, since the compact space I produces sums over discrete momenta, and standard formulas of dimensional regularization do not include that situation. However one may use manipulations valid at the regulated level, like differential equations satisfied by the Green functions and partial integration, to cast the integrands in equivalent forms that can be unambiguously computed in the $d \to 0$ limit. While purely perturbative, this method carries a covariant counterterm that simplify extensive calculations, as the one performed in [30] to obtain trace anomalies for a scalar field in six dimensions using worldline methods.

In DR the computation turns out to be quite simple for most diagrams: $\mathbf{I}_{27} = \mathbf{I}_{28} = 0$ because the integrand is odd, whereas $\mathbf{I}_{29} = \cdots = \mathbf{I}_{33} = 0$ as can be shown by integrating by parts. Also $\mathbf{I}_{26} = \frac{1}{4}$, as it is regular and can be safely evaluated by using the unregulated expression for the propagator. As usual, more care is needed to compute $\mathbf{I}_{25}$ since the integral is ambiguous (products of distributions). By dimensionally extending the cubic vertex

$$\dot{x}^\mu \psi_i^a \psi_i^b \quad \to \quad \partial_A x^\mu \bar{\psi}_i^a \gamma^A \psi_i^b\,, \quad A = 1, \dots, d+1 \tag{3.50}$$

the above integral becomes

$$\mathbf{I}_{25} = \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\Delta^\bullet\ \Delta_F^2 \quad \to \quad \int\int dt dt'\ {}_A\Delta_B\ \mathrm{tr}[\gamma^A \Delta_F \gamma^B \Delta_F] \tag{3.51}$$

where $\Delta(t,t') = -\beta^{-1}\langle q(t) q(t')\rangle$ and $\Delta_F(t,t') = \beta^{-1}\langle \chi(t)\bar{\chi}(t')\rangle$ (the bar indicates Majorana conjugation, see [24]) are the dimensionally regulated propagators where $\partial\!\!\!/ \Delta_F = -\Delta_F \overleftarrow{\partial\!\!\!/'} = \delta^{(d+1)}(t,t')$, and ${}_A\Delta_B(t,t') = \partial_A \partial'_B \Delta(t,t')$. One can partially integrate the above integral without picking boundary terms since in the compact direction bosonic fields vanish at the boundary, and in the extended directions Poincaré invariance allows partial integration as usual. Hence the above expression becomes

$$\begin{aligned}
&-\int\int dt dt'\ \Delta_B\ \partial_A \mathrm{tr}[\gamma^A \Delta_F \gamma^B \Delta_F] \\
&= -\int\int dt dt'\ \Delta_B\ \mathrm{tr}\left[(\partial\!\!\!/ \Delta_F)\gamma^B \Delta_F - \Delta_F \gamma^B (\Delta_F \overleftarrow{\partial\!\!\!/'})\right] \\
&= -2\int dt\ \Delta_B\ \mathrm{tr}[\gamma^B \Delta_F(t,t)] \quad \to \quad \mathbf{I}_{25} = 2\int_{-1}^{0} d\tau\ {}^\bullet\Delta\ \Delta_F|_\tau = 0
\end{aligned} \tag{3.52}$$

where we have used the regulated Green equation, integrated the delta function, and used that Δ_F vanishes for coinciding points.

The transition amplitude in DR thus reads

$$\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle = \frac{e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu+\bar\lambda\cdot\eta}}{(2\pi\beta)^{D/2}}\exp\Big[\langle\text{bosonic}\rangle_{DR}+\frac{1}{2\beta}\omega_{\lambda ab}z^\lambda\tilde\psi^a\cdot\bar{\tilde\psi}^b \\ +\frac{1}{4\beta}\partial_\lambda\omega_{\sigma ab}z^\lambda z^\sigma\tilde\psi^a\cdot\bar{\tilde\psi}^b-\frac{\alpha}{\beta}R_{abcd}\tilde\psi^a\cdot\bar{\tilde\psi}^b\tilde\psi^c\cdot\bar{\tilde\psi}^d-\frac{N}{16}\omega_{\lambda ab}\omega_{\sigma ab}z^\lambda z^\sigma\Big]\,. \tag{3.53}$$

where the bosonic contribution $\langle\text{bosonic}\rangle_{DR}$ can be extracted from [10]. Comparing this result with the ones obtained before in other regularizations we note that, thanks to the vanishing of $\mathbf{I}_{25}$, the diagrammatic expansion does not produce the term proportional to ω^2 which previously had to be canceled by the counterterms. Thus, the standard bosonic counterterm $V_{DR}^{(0)}=-\frac{1}{8}R$ is dressed up to

$$V_{DR}^{(N)}=-\Big(\frac{1}{8}+\frac{\alpha N}{2}\Big)R\,. \tag{3.54}$$

This matches known results for $N=0,1,2$ [24–26].

4 Conclusions

In this paper we have studied the quantum mechanics of one dimensional nonlinear sigma models possessing a O(N) extended supersymmetry on suitable target space backgrounds. Considering an arbitrary background geometry, we have computed the transition amplitude for short propagation times using both canonical and path integrals methods, obtaining in the latter case the correct counterterms associated to various regularization schemes which are needed to define unambiguously the path integrals.

A possible use of our results is in the discussion of higher spin fields in a first quantized picture. Worldline approaches are useful in finding efficient ways of computing amplitudes for relativistic processes both in flat space [31] and curved spaces [32]. The quantum mechanical nonlinear sigma models discussed here arose precisely in an attempt to use worldlines methods to describe one-loop effective action due to higher spin fields in a curved background [7, 27, 33]. In future works we plan indeed to use the path integrals constructed here to study effective actions induced by higher spin fields and compute corresponding heat kernel coefficients. For that purpose one must consider a gauged version of the sigma models considered here, and then take into account the gauge fixing of the extended local supersymmetry. The latter is not related to a standard Lie superalgebra when the target spaces is curved, but contains structure functions, so that enough care must be used to identify the corresponding ghost system and related integration over modular parameters. Finally, one might wish to extend our results to the OSp$(N,2M)$ quantum mechanics of ref. [34], which have also found applications to higher spin theories [35–37].

A Fermionic coherent states

The even-dimensional Clifford algebra

$$\{\psi^M,\psi^N\}=\delta^{MN}\,,\quad M,N=1,\dots,2l \tag{A.1}$$

can be written as a set of l fermionic harmonic oscillators (the index M may collectively denote a set of indices that may involve internal indices as well as a space-time index), by simply taking complex combinations of the previous operators

$$a^m = \frac{1}{\sqrt{2}}\left(\psi^m + i\psi^{m+l}\right) \tag{A.2}$$

$$a^\dagger_m = \frac{1}{\sqrt{2}}\left(\psi^m - i\psi^{m+l}\right) , \qquad m = 1, \dots, l \tag{A.3}$$

$$\{a^m, a^\dagger_n\} = \delta^m_n \tag{A.4}$$

and it can be thus represented in the vector space spanned by the 2^l orthonormal states $|\mathbf{k}\rangle = \prod_m (a^\dagger_m)^{k_m}|0\rangle$ with $a_m|0\rangle = 0$ and the vector $\mathbf{k}$ has elements taking only two possible values, $k_m = 0, 1$. This basis (often called spin-basis) yields a standard representation of the Clifford algebra, i.e. of the Dirac gamma matrices.

An alternative overcomplete basis is given by the coherent states that are eigenstates of creation or annihilation operators

$$|\xi\rangle = e^{a^\dagger_m \xi^m}|0\rangle \quad \to \quad a^m|\xi\rangle = \xi^m|\xi\rangle = |\xi\rangle\xi^m \tag{A.5}$$

$$\langle\bar\eta| = \langle 0|e^{\bar\eta_m a^m} \quad \to \quad \langle\bar\eta|a^\dagger_m = \langle\bar\eta|\bar\eta_m = \bar\eta_m\langle\bar\eta| \ . \tag{A.6}$$

Below we list some of the useful properties satisfied by these states. Using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^X e^{[X,Y]}$, valid if $[X,Y] = c$-number, one finds

$$\langle\bar\eta|\xi\rangle = e^{\bar\eta\cdot\xi} \tag{A.7}$$

that in turn implies

$$\langle\bar\eta|a^m|\xi\rangle = \xi^m\langle\bar\eta|\xi\rangle = \frac{\partial}{\partial\bar\eta_m}\langle\bar\eta|\xi\rangle \tag{A.8}$$

$$\langle\bar\eta|a^\dagger_m|\xi\rangle = \bar\eta_m\langle\bar\eta|\xi\rangle \tag{A.9}$$

so that $\{\frac{\partial}{\partial\bar\eta_m}, \bar\eta_n\} = \delta^m_n$. Defining

$$d\bar\eta = d\bar\eta_l \cdots d\bar\eta_1 \ , \qquad d\xi = d\xi^1 \cdots d\xi^l \tag{A.10}$$

so that $d\bar\eta d\xi = d\bar\eta_1 d\xi^1 d\bar\eta_2 d\xi^2 \cdots d\bar\eta_l d\xi^l$, one finds the following relations

$$\int d\bar\eta d\xi \ e^{-\bar\eta\cdot\xi} = 1 \tag{A.11}$$

$$\int d\bar\eta d\xi \ e^{-\bar\eta\cdot\xi} \ |\xi\rangle\langle\bar\eta| = \mathbb{1} \tag{A.12}$$

where $\mathbb{1}$ is the identity in the Fock space. One can also define a fermionic delta function with respect to the measure (A.10) by

$$\delta(\bar\eta - \bar\lambda) \equiv (\bar\eta^1 - \bar\lambda^1)\cdots(\bar\eta^l - \bar\lambda^l) = \int d\xi \ e^{(\bar\lambda-\bar\eta)\cdot\xi} \ . \tag{A.13}$$

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Finally, the trace of an arbitrary operator can be written as

$$\mathrm{Tr}\, A = \int d\bar\eta d\xi\ e^{-\bar\eta\cdot\xi}\langle -\bar\eta|A|\xi\rangle = \int d\xi d\bar\eta\ e^{\bar\eta\cdot\xi}\langle\bar\eta|A|\xi\rangle\ . \tag{A.14}$$

As a check one may compute the trace of the identity

$$\mathrm{Tr}\,\mathbb{1} = \int d\xi d\bar\eta\ e^{\bar\eta\cdot\xi}\langle\bar\eta|\xi\rangle = \int d\xi d\bar\eta\ e^{2\bar\eta\cdot\xi} = 2^l\ . \tag{A.15}$$

Let us end this section by deriving a few expressions involving fermionic delta functions that are helpful in the computation of section 3.1. Using $\tilde\psi^a_i = \sqrt{\frac{\beta}{2}}(\eta^a_i + \bar\lambda^a_i)$, as defined in (3.18) (here fermions are labelled by two indices, a tangent space index a and an SO(N) R-symmetry index i), we may compute

$$\begin{aligned}\int d\bar\zeta d\eta\ e^{(\bar\lambda-\bar\zeta)\cdot\eta}\tilde\psi^a_i &= \sqrt{\frac{\beta}{2}}\Big(\frac{\partial}{\partial\bar\lambda^i_a} + \bar\lambda^a_i\Big)\int d\bar\zeta d\eta\ e^{(\bar\lambda-\bar\zeta)\cdot\eta}\\ &= \sqrt{\frac{\beta}{2}}\Big(\frac{\partial}{\partial\bar\rho^i_a} + \bar\lambda^a_i\Big)\int d\bar\zeta\ \delta(\bar\zeta-\bar\rho)\Big|_{\rho=\lambda}\end{aligned} \tag{A.16}$$

where the above apparently baroque notation is used because the fermionic derivatives only act upon the delta-function and not on eventual $\bar\lambda$-dependent expression that may appear next to it. We thus have

$$\int d\bar\zeta d\eta\ e^{(\bar\lambda-\bar\zeta)\cdot\eta}\tilde\psi^a\cdot\tilde\psi^b = \beta\tilde M^{ab}\int d\bar\zeta\ \delta(\bar\zeta-\bar\rho)\Big|_{\rho=\lambda} \tag{A.17}$$

with

$$\tilde M^{ab} = \frac{1}{2}\Big(\bar\lambda^a\cdot\bar\lambda^b + \bar\lambda^a\cdot\frac{\partial}{\partial\bar\rho_b} - \bar\lambda^b\cdot\frac{\partial}{\partial\bar\rho_a} + \frac{\partial}{\partial\bar\rho_a}\cdot\frac{\partial}{\partial\bar\rho_b}\Big)\ . \tag{A.18}$$

One can then switch to the Lorentz generators

$$M^{ab} = \frac{1}{2}\Big(\bar\lambda^a\cdot\bar\lambda^b + \bar\lambda^a\cdot\frac{\partial}{\partial\bar\lambda_b} - \bar\lambda^b\cdot\frac{\partial}{\partial\bar\lambda_a} + \frac{\partial}{\partial\bar\lambda_a}\cdot\frac{\partial}{\partial\bar\lambda_b}\Big) \tag{A.19}$$

by suitably subtracting those terms that appear when derivatives on $\bar\lambda$ act on eventual functions of $\bar\lambda$ that may show up before the wave function, such as M^{cd} in (3.36).

B B^k_l coefficients

The coefficients $B^k_l(x,\bar\eta,\xi)$, defined in (2.5), can be computed following the strategy described in detail in [9] for $N = 0, 1, 2$, and in [28] for the complex U$(N|M)$ sigma model. First of all we divide the hamiltonian (2.3) into three pieces contributing at most two, one or no p eigenvalues

$$\begin{aligned} H &= H_B + H_1 + H_2\ , \quad \text{where}\\ H_B &= \frac{1}{2}g^{-1/4}p_\mu g^{1/2}g^{\mu\nu}p_\nu g^{-1/4}\\ H_1 &= -i\, g^{\mu\nu}\omega_{\mu ab}\bar\Psi^a\cdot\Psi^b\left(g^{1/4}p_\nu g^{-1/4}\right)\\ H_2 &= -\frac{1}{2}\, g^{-1/2}\partial_\mu\big(g^{1/2}g^{\mu\nu}\omega_{\nu ab}\big)\bar\Psi^a\cdot\Psi^b\\ &\quad -\frac{1}{2}\left(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd} - 8\alpha\, R_{abcd}\right)\bar\Psi^a\cdot\Psi^b\bar\Psi^c\cdot\Psi^d + V\ . \end{aligned} \tag{B.1}$$

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First of all, notice that H_B is precisely the usual bosonic quantum hamiltonian, carefully studied in the literature [9, 10]. Let us start with B^k_{2k}: the only way to have $2k$ p eigenvalues is from k factors of H_B and no commutators taken into account, giving simply

$$B^k_{2k}\, p^{2k} = \left(\frac{p^2}{2}\right)^k . \tag{B.2}$$

For B^k_{2k-1} we can have two kinds of terms. The first comes from k factors of H_B with one p acting as a derivative; this gives the corresponding B^k_{2k-1} of the purely bosonic model, whose computation is explained in detail in [9, 10]. The other term comes from $k-1$ factors of H_B and one H_1, by substituting all operators with the corresponding eigenvalues. Putting things together we obtain

$$B^k_{2k-1}\, p^{2k-1} = -\frac{ik}{2}\left(\frac{p^2}{2}\right)^{k-1} g^{\mu}\, p_\mu - i\binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\frac{1}{2}g^{\nu\lambda\mu}\, p_\mu\, p_\nu\, p_\lambda - i\,k\left(\frac{p^2}{2}\right)^{k-1} g^{\mu\nu}\omega_\mu\, p_\nu\, . \tag{B.3}$$

For B^k_{2k-2} four types of term contribute: *i)* k factors of H_B, giving the coefficient of the corresponding bosonic model, *ii)* $k-1$ factors of H_B and one H_1, with one p acting as a derivative. This contribution gives four terms: the derivative acting from one H_B to H_1, from H_1 to one H_B, within H_1 or within the $k-1$ H_B's. *iii)* $k-1$ factors of H_B and one H_2, substituting all operators with their eigenvalues, and *iv)* $k-2$ factors of H_B and two H_1, substituting all with eigenvalues. Remember that in *iii)* and *iv)* $\{\Psi, \bar{\Psi}\}$ anticommutators have to be taken into account in order to obtain eigenvalues on the coherent states. Altogether it results in

$$\begin{aligned}
B^k_{2k-2}\, p^{2k-2} = {} & k\left(\frac{p^2}{2}\right)^{k-1}\left[\frac{1}{32}\partial_\mu \ln g\,\partial^\mu \ln g + \frac{1}{8}\partial_\mu\partial^\mu \ln g + \frac{1}{8}g^\mu\partial_\mu \ln g\right] \\
& - \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\left[\frac{1}{2}\partial^\mu g^\nu + \frac{1}{4}g^\mu g^\nu + \frac{1}{4}g^\lambda g^{\mu\nu}_{\lambda} + \frac{1}{4}g^{\mu\nu\lambda}_{\lambda}\right] p_\mu\, p_\nu \\
& - \binom{k}{3}\left(\frac{p^2}{2}\right)^{k-3}\left[\frac{1}{2}g^{\lambda\sigma\mu\nu} + \frac{3}{4}g^{\mu\nu\lambda}g^\sigma + \frac{1}{2}g^{\rho\mu\nu}g^{\lambda\sigma}_{\rho} + \frac{1}{4}g^{\mu\nu}_{\rho}g^{\lambda\sigma\rho}\right] p_\mu\, p_\nu\, p_\lambda\, p_\sigma \\
& - \binom{k}{4}\left(\frac{p^2}{2}\right)^{k-4}\left[\frac{3}{4}g^{\nu\lambda\mu}g^{\rho\tau\sigma}\right] p_\mu\, p_\nu\, p_\lambda\, p_\sigma\, p_\rho\, p_\tau \\
& - \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\left[\partial^\mu\left(g^{\nu\lambda}\omega_\lambda\right) - \frac{1}{2}g^{\mu\nu\lambda}\omega_\lambda\right] p_\mu\, p_\nu - k\left[\frac{1}{2}(k-1)\left(\frac{p^2}{2}\right)^{k-2} g^\mu p_\mu \right. \\
& \left. + \binom{k-1}{2}\left(\frac{p^2}{2}\right)^{k-3}\frac{1}{2}g^{\nu\lambda\mu}\, p_\mu\, p_\nu\, p_\lambda\right] g^{\sigma\rho}\omega_\sigma\, p_\rho + \frac{1}{4}k\left(\frac{p^2}{2}\right)^{k-1} g^{\mu\nu}\omega_\mu\partial_\nu \ln g \\
& - \frac{1}{2}k\left(\frac{p^2}{2}\right)^{k-1}\left\{g^{-1/2}\partial_\mu\left(g^{1/2}g^{\mu\nu}\omega_\nu\right) + \left(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd} - 8\alpha\, R_{abcd}\right)\left[\bar{\eta}^a\!\cdot\xi^d\eta^{bc} \right.\right. \\
& \left.\left. + \bar{\eta}^a\!\cdot\xi^b\bar{\eta}^c\!\cdot\xi^d\right] - 2V\right\} \\
& - \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2} g^{\mu\nu}\omega_{\mu ab}g^{\lambda\sigma}\omega_{\lambda cd}\left[\bar{\eta}^a\!\cdot\xi^d\eta^{bc} + \bar{\eta}^a\!\cdot\xi^b\bar{\eta}^c\!\cdot\xi^d\right] p_\nu\, p_\sigma
\end{aligned} \tag{B.4}$$

where again we use the compact notations for tensors introduced below eq. (2.7).

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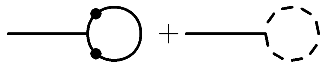
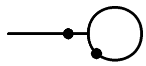
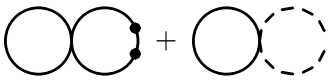
C Feynman diagrams

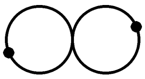
We list the set of Feynman diagrams and the associated worldline integrals that enter the computation of the transition amplitude to order β in section 3. We are not reporting here those diagrams that involve fermionic self-contractions as with the rules of section 3 such self-contrations are trivially vanishing. Hence, the a priori non-trivial diagrams entering the contribution $\langle S_3 \rangle$ are

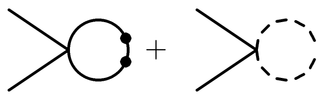
$\mathbf{I}_1 =$  $= \int_{-1}^{0} d\tau \; \tau ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.1)

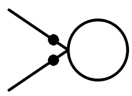
$\mathbf{I}_2 =$  $= \int_{-1}^{0} d\tau \; {}^\bullet\Delta|_\tau$ (C.2)

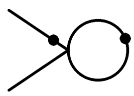
Those contributing to $\langle S_4 \rangle$ are

$\mathbf{I}_3 =$  $= \int_{-1}^{0} d\tau \; \Delta|_\tau ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.3)

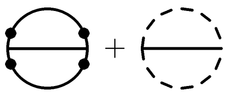
$\mathbf{I}_4 =$  $= \int_{-1}^{0} d\tau \; {}^\bullet\Delta^2|_\tau$ (C.4)

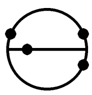
$\mathbf{I}_5 =$  $= \int_{-1}^{0} d\tau \; \tau^2 ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.5)

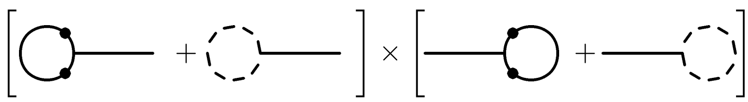
$\mathbf{I}_6 =$  $= \int_{-1}^{0} d\tau \; \Delta|_\tau$ (C.6)

$\mathbf{I}_7 =$  $= \int_{-1}^{0} d\tau \; \tau \Delta^\bullet|_\tau \; .$ (C.7)

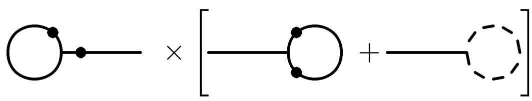
The remaining ones contributing to $\langle S_3^2 \rangle_c$ can be devided into purely bosonic contributions

$\mathbf{I}_9 =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta({}^\bullet\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2)$ (C.8)

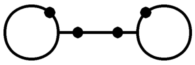
$\mathbf{I}_{10} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet\Delta^\bullet\Delta^\bullet$ (C.9)

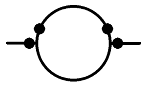
$\mathbf{I}_{11} =$ 

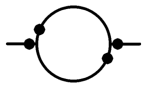
$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau \; \Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma$ (C.10)

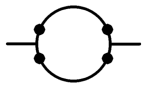
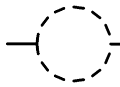
$\mathbf{I}_{12} =$ 

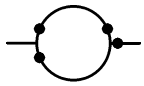
$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta|_\tau \; {}^\bullet\Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma$ (C.11)

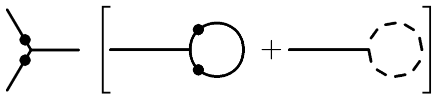
$$\mathbf{I}_{13} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta|_\tau \; {}^\bullet\Delta^\bullet \; \Delta^\bullet|_\sigma \tag{C.12}$$

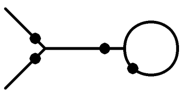
$$\mathbf{I}_{14} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta \tag{C.13}$$

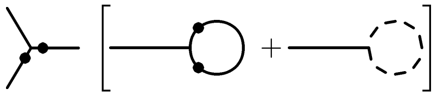
$$\mathbf{I}_{15} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta \; \Delta^\bullet \tag{C.14}$$

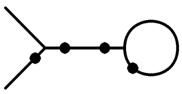
$$\mathbf{I}_{16} =$$  $$+$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \tau\sigma({}^\bullet\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2) \tag{C.15}$$

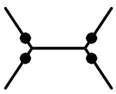
$$\mathbf{I}_{17} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; {}^\bullet\Delta \tag{C.16}$$

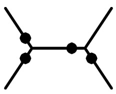
$$\mathbf{I}_{18} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma \tag{C.17}$$

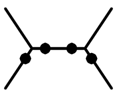
$$\mathbf{I}_{19} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta^\bullet \; \Delta^\bullet|_\sigma \tag{C.18}$$

$$\mathbf{I}_{20} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \tau \; {}^\bullet\Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma \tag{C.19}$$

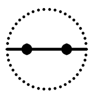
$$\mathbf{I}_{21} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \tau \; {}^\bullet\Delta^\bullet \; \Delta^\bullet|_\sigma \tag{C.20}$$

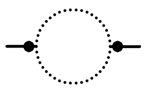
$$\mathbf{I}_{22} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta \tag{C.21}$$

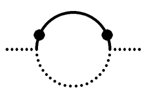
$$\mathbf{I}_{23} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \sigma \; \Delta^\bullet \tag{C.22}$$

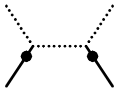
$$\mathbf{I}_{24} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \tau\sigma \; {}^\bullet\Delta^\bullet \tag{C.23}$$

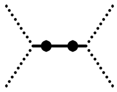
and those with mixed bosonic-fermionic contributions

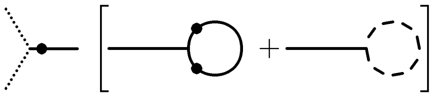
$$\mathbf{I}_{25} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta_F^2 \tag{C.24}$$

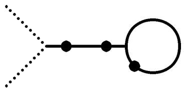
$$\mathbf{I}_{26} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta_F^2 \tag{C.25}$$

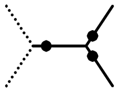
$$\mathbf{I}_{27} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta_F \tag{C.26}$$

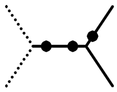
$$\mathbf{I}_{28} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta_F \tag{C.27}$$

$$\mathbf{I}_{29} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\Delta^\bullet \tag{C.28}$$

$$\mathbf{I}_{30} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\Delta\ ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma \tag{C.29}$$

$$\mathbf{I}_{31} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\Delta^\bullet\ \Delta^\bullet|_\sigma \tag{C.30}$$

$$\mathbf{I}_{32} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^\bullet\Delta \tag{C.31}$$

$$\mathbf{I}_{33} = $$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \sigma\ {}^\bullet\Delta^\bullet \tag{C.32}$$

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