THE NUMBER LINE: A "WESTERN" TEACHING AID

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Abstract

This paper aims at discussing a very popular teaching aid, the so-called *number line,* where whole numbers are introduced as labels on unit marks by means of a measuring process and where additions and subtractions can be realized, as operators, with jumps forwards and backwards. Traces of this early approach can be found in the teaching practices of most Western countries, but, surprisingly, not in the most popular Chinese textbooks. A question arises: where does the difference come from? In the following, I review some Western literature to sketch out the analysis of the number line as a teaching aid, from the historic-epistemological, cognitive and didactical perspectives. Later some paradigmatic practices from different countries are presented.

Key words: measuring, mental number line, number line, addition, subtraction.

Introduction

The intention of writing this paper arose after a dialogue with Sun Xuhua about the most popular teaching approaches to whole numbers. When discussing uses of the number line in the tow cultures (Italian and Chinese) I realized that it is not used in China as much as in the West. This brought me to reflect on how some Western popular approaches are not mandatory, but dependent on culture. As has happened for me in the past (Bartolini Bussi and Martignone, 2013; Bartolini Bussi, Ramploud and Anna Baccaglini-Frank, 2014), noticing this difference was a stimulus towards the reconstruction of the roots of our tradition (from the historic-epistemological, cognitive and didactical perspectives), or, following Jullien (2008), towards the "discovery of our own unthought".

Historic-epistemological perspectives.

Epistemological perspective

Hans Freudenthal, a past ICMI President (1967-1970), has devoted several volumes to the epistemological foundation of mathematics education. In the *Didactical Phenomenology of Mathematical Structures* (1999), Freudenthal introduces magnitudes, criticizing traditional teaching where measuring is delayed until children are ready to learn common and decimal fractions.

"The first step in analysing a magnitude, where measuring the magnitude is articulated by the natural multiples of a unit, is possible and desirable at an early age; counting can and must immediately be transferred from discrete quantities, represented by sets, to magnitudes. Modern textbooks start measuring much earlier than tradition allows, but unfortunately this kind of measuring is not yet sufficiently integrated with the operations on natural numbers. The device beyond praise that visualises magnitudes and at the same time the natural numbers articulating them is the number line, where initially only the natural numbers are individualised and named. In the didactics of secondary instruction the number line has been accepted, though it is often still imperfectly and inexpertly exploited; in primary education it makes progress little by little. […]. It seems to me a disadvantage of the number line that it is so easily drawn and that it cannot be sold together with the textbook as teaching material. […] The number line eclipses the Cuisinaire rods in many respects: The virtual infinity is better expressed by the number line. The number line knows no compulsory scale; number lines on different scales – on the blackboard and on paper – are immediately identified, notwithstanding their incongruency" (Freudenthal, 1999, p. 101, my emphasis).

Historical perspective

The concept of magnitude goes back to Euclid (Euclid, Book 5, in Heath vol. 2 p. 113 ff.). Although the idea of magnitude is more general (and also includes areas, volumes and similar), magnitudes are represented by straight lines in all the Euclid's editions after Heiberg. Interestingly, in Book 7 about the study of whole numbers, numbers themselves are represented as straight lines with no relation with the number size (Euclid, Book 7, in Heath vol. 2 p. 277 ff.). It may seem strange to represent whole numbers as straight lines, in contrast with the Greek tradition of using configurations of pebbles (as in the case of square numbers). Netz (1999) gives an interesting interpretation about the meaning of diagrams and links it to the issue of generality:

"Often the proof is about "any integer", a quantity floating freely through the entire space of integers, where it has no foothold, no barriers. […] A dot representation implies a specific number, and therefore immediately gives rise to the problem of the generalisation from that particular to a general conclusion, from the finite to the infinite. Greek mathematicians need, therefore, a representation of a number which would come close to the modern variable. This variable […] is the line itself. The line functions as a variable because nothing is known about the real size of the number it represents" (Netz,1999, p. 268).

The first number line with unit marks dates back to John Wallis (1685).

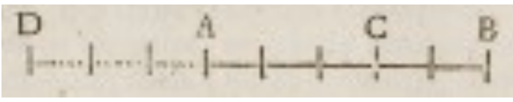

Fig. 1: Wallis's drawing

"Supposing a man to have advanced or moved forward (from A to B) 5 Yards, and then to retreat (from B to C) 2 Yards: if it be asked how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when at A? I find (by Subducting 2 from 5) that he is Advanced 3 Yards. (Because +5 – 2 = + 3). But if, having Advanced 5 Yards to B, he thence Retreat 8 Yards to D; and it be then asked, How much he is Advanced when at D, or how much Forwarder than when he was at A: I say -3 Yards. (because +5 – 8 = -3). That is to say, he is advanced 3 Yards less than nothing. Which in property of Speech, cannot be, (since there cannot be less than nothing). And therefore as to the Line AB Forward, the case is Impossible. But if (contrary to the Supposition) the Line from A, be continued *Backward,* we shall find D, 3 Yards *Behind* A. (which was presumed to be *Before* it). And thus to say, he is *Advanced* -3 Yards; is but what we should say (in ordinary form of Speech) he is *Retreated* 3 Yards; or he wants 3 Yards of being so Forward as he was at A" (ibid., p. 265).

Wallis' number line does not show numerals, but reference to units (Yards). The marks are drawn at regular intervals. The starting point A is the origin of the mark ("zero") whilst the other points are reached with a positive (forward) or negative (backwards) number of steps. Actually "zero" as a label on a number line does not entail the same difficulty and incomprehension raised by zero as either a number or a digit in place value (see below); also the understanding of negative numbers, very difficult until the end of the 18th century, is facilitated.

(Neuro)cognitive perspectives.

The mental number line

(Neuro)cognitive scientists study the representation of numbers in a spatial format along the so-called "mental number line," whereby smaller numbers occupy relatively leftward locations (in the case of horizontal representations) or lower locations (in the case of vertical representations) compared with larger numbers. This idea dates back to Galton (1880) and is now focused in many experimental studies (for a short review, Butterworth, 1999). There is evidence in recent studies that blindness alters the direction of mental number line (Pasqualotto, Shichiro, Proulx, 2014), suggesting that the left-to-right organization can be affected by the enviromental factors and visual perception. Actually a few studies show some correlation between the direction of writing and the direction of the number line (Bender and Beller, 2011).

The number line as a conceptual metaphor

There is a function of the number line in cognitive linguistics. In their program to understand where mathematics comes from, i. e. what is the cognitive structure of sophisticated mathematical ideas, Lakoff & Nunez (2000) have taken an embodied approach, assuming the motion along a path as one of the grounding metaphor for arithmetic. They argue that abstract mathematical notions have their origins in our specific embodiment and could not have been construed differently. Conceptual metaphors work as projections from a domain (in this case the spatial experience of motion along a path) to another domain (in this case the arithmetic of whole numbers). The idea is not new (see Wallis, above) and was already used by Herbst (1997) who referred to Black's theory of conceptual metaphor (1962), that inspired the modern development of cognitive linguistics, in order to analyze the way of introducing and using the number line in a set of textbooks from Argentina.

A short interlude: towards classroom practices.

It is worthwhile to highlight the potential of the number line as a teaching aid. The number line hints at the relationship between whole numbers and magnitudes, initiated in the classical age, and fosters the extension to fractions and rational numbers, by means of measuring. Either real or evoked motion on the line hints at the generation of infinitely many whole numbers, by iterating the action of one step forward. In a similar way, it eases the extension to negative numbers (by iterating the action of one step backward). The so called mental number line is widely accepted by (neuro)cognitive scientists as the mental representation of whole numbers in a spatial format.

Addition and subtraction on the number line are easily interpreted as the opposite of each other (forwards and backwards). This approach to addition and subtraction is consistent with the "counting on" and "counting back" strategies and quite different from approaches involving cardinal numbers (e.g. putting together and taking away). In particular, on the number line, the two addenda of an addition are distinguished (status – operator). Activity on the number line is suitable for both high and low achievers. For the former the challenge is to discover the commutative or associative properties of addition (that seems trivial in the cardinal approach); for the latter (see below) the usefulness is to introduce and reinforce automatic procedures for easy additions and subtractions.

Didactical issues.

Reviewing the literature on the didactical use of number line is far too ambitious for this short contribution. Only some approaches are mentioned below in order to represent the variety and the richness of the use of the number line in the mathematics classroom in many countries.

Davydov's curriculum

The Russian elementary program of V. V. Davydov is the result of the application of Vygotskian theory to school mathematics in order to solve the dichotomies between empirical and theoretical thinking, between arithmetic and algebra, that were (and still are in most countries) typical of primary vs secondary school. A paradigmatic example in this curriculum is given by the number line.

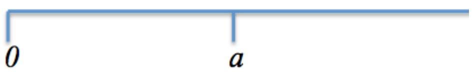

Fig. 2: Davydov's problem

"The number line arises in Davydov's curriculum from a consideration of simple medicinal dosage calculation using a graduated cylinder which is then tipped ninety degrees to create a horizontal gradient. The elements that are essential to the creation of a number line are also explored thoroughly, including direction, starting point and choice of a unit. If any one of these is unspecified, it is impossible to determine where specific numbers will be located even if the other two elements are provided. When the number line is presented as a ready made representation with these elements in place *a priori*, as is typically the case in the U.S., the arbitrary nature of these determinations remains undetected, since they are usually never explored. When children are presented with a number line such as in Figure 2, and asked to mark "a + 1" on the number line, they must notice that no unit has been provided and therefore it is impossible to complete this task." (Schmittau, 2010, p. 273)

Actually the understanding of unit is problematized in the whole curriculum, that aims at introducing numbers by means of measuring rather than counting. Davydov's curriculum was very well known and influential in the West, giving the impression that it was *the* Russian curriculum. Actually it was only *a* curriculum in Russia, and not even the most popular at that time (Ivashova, 2011).

Realistic Mathematics Education: the empty number line

Klein, Beishuizen and Treffers (1998) reported about a new format for the old number line: the empty number line up to 100. They observed that the introduction from the very beginning of a structured number line with ready-made marks for every number fostered only counting and passive reading of the answer on the number line. They therefore opted for an empty number line on which the pupils can draw marks by themselves. They claimed that marking the steps on the number line for the realization of an addition or a subtraction functions as a kind of scaffolding, as it shows which part of the operation is carried out and what remains to be done. The empty format stimulates a mental representation of numbers and number operations (addition and subtraction). Students using the empty number line are cognitively involved in their actions. These were the hopes, but a further analysis (van den Heuvel-Panhuizen, 2008) carried out ten years later by consulting children about the effectiveness of the empty number line, showed that a rigid application and an improper implementation of this teaching aid worked against the hopes of the designers and suggested the need to further research on this issue.

Other approaches

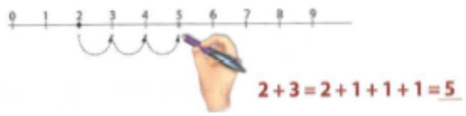

Fig. 3: Addition on a graphic number line

In most cases the number line has not been used to introduce the measuring aspects involved in whole numbers but rather the sequence of whole numbers and additions and subtractions by means of steps forwards or backwards. The relationships between numbers as labels and their distance from the origin have been seldom addressed. In the paper and pencil context, additions and subtractions are usually represented by small curved arrows, whilst in the context of large number lines traced on the floor, students can be asked to move or even jump from one number to another. Today, there are many software where the same operations are carried out on the computer screen (Ginzburg, Jamalian and Creighan, 2013).

Examples from Italy

A few decades of curriculum development: the number line

Arzarello & Bartolini Bussi (1998) have described the development of research in mathematics education in Italy in the last decades. One of the trends was represented by the strong cooperation between researchers and school teachers for the innovation in the mathematics classroom. An important step in this process was represented by the RICME Project (developed in Rome, under the direction of Michele Pellerey, with a group of teachers including also Emma Castelnuovo). This project supported multiple approaches to numbers, including a measuring approach, and the reference to out-of-school experiences, like traditional games. The *goose game* is a board game with uncertain origins, very popular in Southern Europe. The board has a track with (usually 63) consecutively numbered spaces for the pawns to stand in. Each pawn is moved as many steps as one (or two) dice gives. Each pawn is initially placed in the "starting space" (the "zero" space). The attention to a multiple approach to numbers, to the number line and to traditional games is mirrored in the Italian Standards (1985) and, later, in the New Standards (2012), a much shorter document, where some traces of the number line still exist. The number line is used as a teaching aid in most Italian schools. Two examples are reported, both framed by the semiotic mediation approach (Bartolini Bussi and Mariotti, 2008)

The BAMBINI CHE CONTANO Project: the time tube

Fig. 4: The time pipe

The time tube is a vertical empty "number line" designed within the pre-school project "Bambini che contano" (Bartolini Bussi, 2013), to complement the standard counting approach with some meaningful experiences of estimation and measuring. The activity was carried out with 4 and 5 years old children in more than twenty schools. The children are familiar with the number line in the form of a monthly calendar. A day-by-day tear-off calendar is also introduced. A cylinder tube of plexiglass with no graduation is gradually filled with small balls made by crumpling tightly each day-sheet: every day, a child tears off the old sheet (yesterday), cramples it very carefully and throws it into the time tube. The *past* goes into the tube, the *present* is visible on the front of the pad and the *future* is still hidden (in the calendar on the wall). If the teacher suggests to mark the level after one month, it may define the unit in approximate way. Guessing games can be played, like "What will the level be on Christmas day?" and the conjectures can be checked some weeks later.

The PerContare Project

PerContare (http://percontare.asphi.it/) is an innovative Italian project, built upon a collaboration between cognitive psychologists and mathematics educators, that aims at elaborating teaching strategies for preventing and addressing early learning difficulties in arithmetic. The Italian school system is totally inclusive, hence one of the aims of the project was to design teaching activities that could have a positive effect on *all* students, including low achievers and dyscalculics (if any). In this project some artefacts have been chosen to introduce whole numbers through multiple approaches. One of the artefacts is the ruler, used from the very beginning of primary school to evoke the number line. In this short report, I am describing the way of fostering the construction of automatic processes in dyscalculic children, by means of the number line. The following is the prototype of a dialogue (one-to-one interaction) between a low achiever (a child during the process of rehabilitation from dyscalculia). The child can read numbers but cannot retrieve from the memory simple arithmetic facts. There is a number line drawn (in analogy with the goose game) as a linear sequence of square spaces numbered from 0 to 10. The pawn to be moved is Tweety. The task is to calculate 4 + 3.

Adult: “Put Tweety on the 4” ;(done); Adult: “Keep Tweety steady and count on 3 with your finger”; (done); Adult: “Read the number”; Child : “Seven”. Adult: “Good job! 4+3=7”.

The activity is very guided, hence it might be criticized as the passive use of the standard number line (see above). The aim is, however, not to foster the creative discovery of the potential of the number line but to construct a very simple procedure to be used by the low achiever first in a guided way and then independently, to acquire autonomy in the construction of simple number facts.

Discussion and conclusion

In this contribution some practices have been reported from different Western countries. The historic-epistemological and (neuro)cognitive foundations have been sketched out in a very short way. I do not claim that the number line is or must be a universal teaching aid. The (neuro)cognitive assumption of the mental number line as a spontaneous and natural model has been criticized by Núñez (2011) and related to cultural aspects. The same author however claims that “the number-to-line mapping, although ubiquitous in the modern world, is not universally spontaneous but rather seems to be learned through – and continually reinforced by – specific cultural practices (p. 611)”. For teachers, who may (must) exploit this ubiquity, this seems enough.

But now my question is: *Why not in China*?

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