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THE CONCEPT OF THE CROSS SECTION

F. Corni, *Department of Physics, University of Modena (Italy)*
M. Michelini, L. Santi, F. Soramel, A. Stefanel, *Research Unit in Physics Education, University of Udine (Italy)*

When teaching physics in secondary schools, it is possible to use concepts that are related to other fields in order to approach more topics in a comparative or integrated way. Thus, we highlight not only basic concepts, but also their involvement with basic physics, research and applied physics. One such concept is cross section. It is an important concept in classical physics, but it becomes essential in quantum mechanics where it is not possible to define the trajectory of a particle but only the probability of finding it in a certain space. Moreover, the concept of cross section is central to the study of the interactions between elementary particles in nuclear physics at both low and high energies in atomic collisions. Approaching this topic allows us to move from classical physics, to wider and more complex fields using a powerful research tool. These are just some of the reasons for looking for an appropriate pedagogic approach to introduce this concept in secondary school, showing at the same time which associations can be derived from it. To approach the concept of cross section, basic notions on collisions are needed: specifically the importance of the conservation laws in central collisions, a good knowledge of one-dimensional collisions and the basic statements about plane collisions (at least for rigid bodies).

1. The Probabilistic Meaning of Cross Section.

Though a geometrical interpretation of the concept of cross section might seem to be the easiest approach, this is not the case. In fact, this interpretation is consistent only with the classical case of rigid spheres. The concept of cross section is generally a probabilistic one, and this is what needs to be highlighted in our approach. Let us look at some typical cases of collisions, highlighting the general aspects of the phenomena and showing how to relate measurements to interpretations, independently from the traditional force/equation model of motion/trajectory scheme. Let us consider as our first example the elastic collision of two rigid spheres, in relation to the solid center of mass reference system (figure 1). From first principles and an idealization of the system (infinite rigidity, normal collision, identity of the spheres before and after collision), we derive that the rebound of the two spheres is symmetrical with respect to the line through their centers. The scattering angle, θ, is connected to the distance b between the lines on which the centers of the two spheres were moving before the collision (impact parameter), as derived from the following equation:

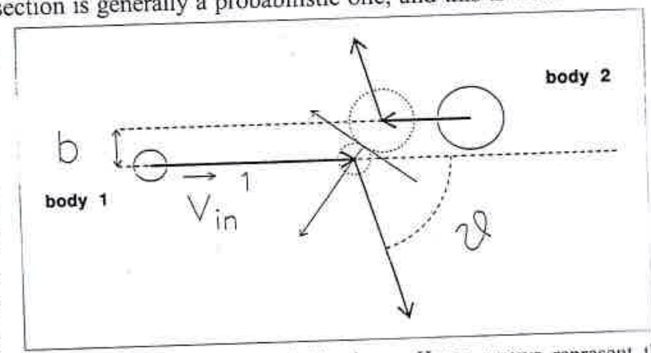

Figure 1. Collision of two rigid spheres. Heavy arrows represent the velocity vectors of the two bodies, before and after collision.

$$b = (R_1+R_2)\cos(\theta/2) \qquad (1)$$

The interaction during the collision is characterized by the impulse vector $\boldsymbol{I}$ acting between the two spheres (e.g. from 2 to 1). Eq.(1) and the comparison between the initial and final states of the system give us all possible information about what happened during the collision. In more complex cases, this straightforward procedure fails. In particular, when we replace sphere 2 with an irregular shaped object, measuring the dependence of the scattering angle θ from the impact parameter b becomes more difficult. Another cause of unpredictability is the finite precision with which the impact parameter is determined. In fact, even a small variation in b can cause variations of the

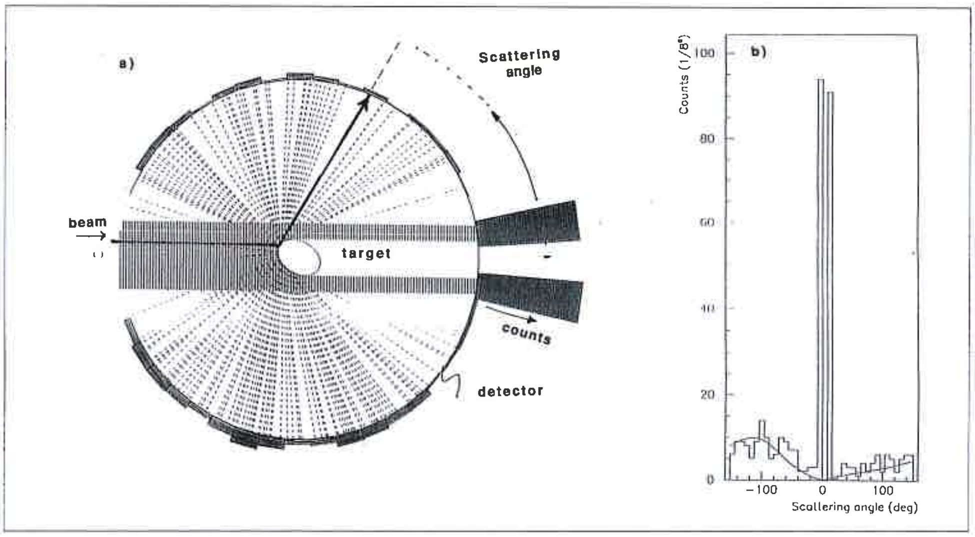

Figure 2. a) Scheme of a scattering 'experiment' of a beam of light particles (from left to right) by an asymmetric rigid body. The trajectories of the scattered particles are represented by dotted lines, up to the detector (the large circle centered on the scattering body): the number of the particles detected in a given angular sector is represented by the height of the strips drawn on the detector. The histogram in b) reports this distribution as a function of the scattering angle (the central peaks correspond to the unperturbed particles).

scattering angle of any order of magnitude. A consequence of this is that by driving the two bodies against each other a number of times with a poor control of initial conditions, the individual results (final states) obtained are significantly different. Does this mean that such an experiment does not give any kind of information about the collision or the geometry of the two bodies? Let us look at figure 2a where the results of a certain number of impacts are depicted. The scattering angles are distributed over the range (-160°,160°), with fluctuations from bin to bin essentially due to limited statistics (400 events). A more accurate examination shows a general asymmetry in the distribution, due to the asymmetrical shape of the scattering body (figure 2b shows the average 'theoretical behavior', superposed on the histogram). This relation between *distribution asymmetry* and *asymmetrical shape* gives a qualitative example of how to extract the basic properties of the collision phenomenon from the scattering angle distribution. In general, a complete characterization of the collision requires the knowledge of the probability P_i for any given measurement outcome S_i; these probabilities can be obtained from the averages $P_i = \langle N_i / N_{tot} \rangle$

where N_i is the number of outcomes S_i for N_{tot} observations carried out. This however has the disadvantage that probability, the element which is more related to the interaction, still depends on the details of the measuring procedure. For example, if we increase the width of the distribution of the impact parameter of body 1, there will be a greater number of cases in which body 1 will not interact at all with body 2, scaling the entire probability distribution. In order to avoid this, it is necessary to explicitly consider the short range character of the interaction (as are the cases so far observed).

Figure 3 shows a uniform beam of (identical) particles colliding with a target of evenly distributed (identical) particles. So an incoming particle, wherever it crosses section (a), finds, on average, the same target distribution. Therefore the total number N_i of scattering events having a certain final state S_i will be proportional to the number of incident particles (there are no border effects due to the 'width' of the beam).

On the other hand, given the limited range of the scattering phenomenon, a certain incident particle will only interact with the particles of the target within the range of action of the force (represented

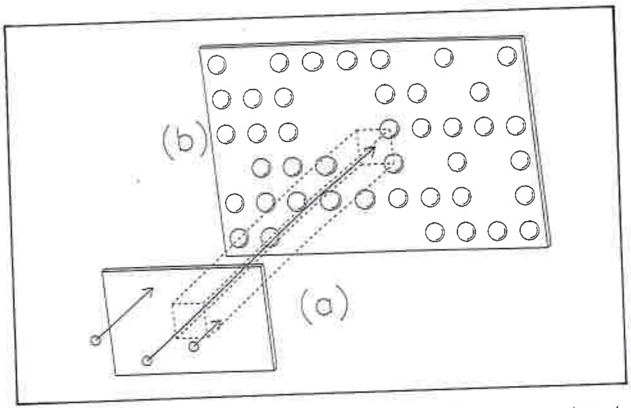

Figure 3. Scheme of a beam of particles colliding on a target, constituted by particles arranged in a bi-dimensional array (section b). The dotted parallelogram represents the active volume of a beam particle (e.g., the region in which it can interact with a target particle).

in the figure by the small square of area A on section (b)).
The probability that any incident particle of the beam interacts with a target particle within area A, originating a certain S_i result, will be:

$$P = P_1 \; P_2 \; P_{int}$$

where P_1 is the probability that the incident particle will cross A, P_2 is the probability that there will be a target particle in that area and P_{int} the probability for that type of interaction. If the two distributions are sufficiently sparse (a very important hypothesis which holds in all the following), P_1 and P_2 are equal to the average number of particles in that area:

$$P_1=\langle N_1\rangle=n_1\,A,\; P_2=\langle N_2\rangle=n_2\,A$$

where n_1 and n_2 represent the surface densities in a projection transversal to the axis of the beam. If we sum up all the small squares in which the section A_{tot} effectively crossed by the beam can be divided, we derive the average number of interactions with outcome S_i:

$$\langle N_i\rangle = N_A\, P_1\, P_2\, P_{int} = (A_{tot}/A)\,(n_1\,A)\,(n_2\,A)\,P_{int} = A_{tot}\, n_1\, n_2\,(A\,P_{int}).$$

We can clearly see that the characteristics more closely related to the observed interaction are shown by the factor $\sigma_i=(A\,P_{int})$.
This quantity is not properly a probability as its dimension is that of an area; for this reason it is better to call it cross section (for the reaction channel considered) and to rewrite the previous equation as follows:

$$\langle N_i\rangle=A_{tot}\, n_1\, n_2\, \sigma_i\, .$$

2. Total and differential Cross Sections

Let us go back to the example in figure 1 and ask ourselves which is the cross section of the process in which the two particles are effectively scattered in the final state. Such a section will be called total as it includes all the possible results in which the final state of the system does not coincide with the initial state. The two spheres will interact with each other only if $b\leq R_1+R_2$. In this case, the two spheres will certainly collide (P_{int}=1). Thus:

$$\sigma_{tot} = A\, P_{int} = \pi\,(R_1+R_2)^2$$

This result (valid for any reference system) can be used for a geometrical interpretation of the concept of cross section. Let us assume that $R_1 \ll R_2$: subsequently $\sigma_{tot} \approx \pi R_2^2$, that is to say it is equal to the area of the projection of body 2 on a plane transversal to the direction of the collision. This result does not depend on the shape of body 2, the latter in fact could well be made of a separated set of single bodies. If the single bodies do not overlap along the direction of the collision, the total cross section is equal to the sum of the areas of each constituting body. What kind of information does the total cross section give? As an integral quantity, it gives information only on the strength of the interaction (or, alternatively, on its range); a deeper insight of the phenomenon can be achieved only by considering explicitly each possible result S_i of the collision. Let us consider the histogram in figure 2, that represents the angular distribution of the results of collision. The i-th bin of the histogram contains the number N_i of the outcomes for which the scattering angle is included between the extremes $(\theta, \theta + \Delta\theta)$ of the bin itself. Thus the cross section for the collisions with a scattering angle at such an interval is:

$$\sigma_i = AP_i \approx A\,N_i/(N_i A) = N_i/n_i$$

This value depends on the bin width $\Delta\theta$. To avoid such dependency, the following quantity is introduced: $\frac{d\sigma}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\sigma}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{<N_i>}{n_i}\frac{1}{\Delta\theta}$

This is called differential cross section. The quantity thus introduced is a continuous function of the variable to which it is referred (in this case, the scattering angle). Practically, though, it is preferable to introduce a differential cross section which is not referred to the plane scattering angle but to the solid angle:

$$\frac{d\sigma}{d\Omega} = \lim_{\Delta\Omega\to 0}\frac{<N_i>}{n_i}\frac{1}{\Delta\Omega} \qquad (3)$$

where $\Delta\Omega = \Delta\cos\theta\Delta\phi$ and ϕ is the azimuthal scattering angle, with respect to the axis of the beam. As a practical example, let us calculate the differential section for the scattering of the material point on a sphere with radius R. As the system has cylindrical symmetry around the axis of the beam, the number dN_I of incident bodies which have a collision parameter included between b and $b+db$ is:

$$dN_I = n_I\,(2\pi b\,db)$$

These bodies are all scattered at the same angle θ which is linked to b by:

$$b = R\cos(\theta/2)$$

Substituting in the equation of the cross section, we derive:

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$

which is independent from θ. The integral on all the solid angle gives the total cross section which obviously coincides with the area of the section of the sphere.

The calculation can be extended, in a more complex way but still using the elements of the secondary school syllabus, even to cases of scattering bodies with a more complex profile, for example that generated by the rotation of an ellipse of x_0 and y_0 semi-axes, around the x axis (oriented as the beam). In this case we derive:

$$\frac{d\sigma}{d\Omega} = \frac{y_0^2}{4}\left(\frac{x_0}{y_0}\sin^2\frac{\theta}{2} + \frac{y_0}{x_0}\cos^2\frac{\theta}{2}\right)^{-2}$$

In figure 4, the differential cross section (in arbitrary units) is shown, calculated for three different cases: a) $x_0 = 0.9\,y_0$, b) $x_0 = y_0$ (sphere), c) $x_0 = y_0 / 0.9$. We can see how the dependence of the differential cross section on the scattering angle gives an information on the collision. In case a), where $d\sigma/d\Omega$ is larger at small angles, the interaction proceeds mainly by peripheral collisions; this implies that the scattering body shows an obtuse end to the beam. The opposite happens in case c), where $d\sigma/d\Omega$ is larger at large scattering angles, the interaction is dominated by the back-scattering of the incident particles; thus, keeping in mind the nature of the force involved in the collision, the scattering body shows the end with the least curving radius to the beam. (These properties which seem to be of a geometric nature, are in fact due to the type of interaction which mediates the collision: the geometrical correlations

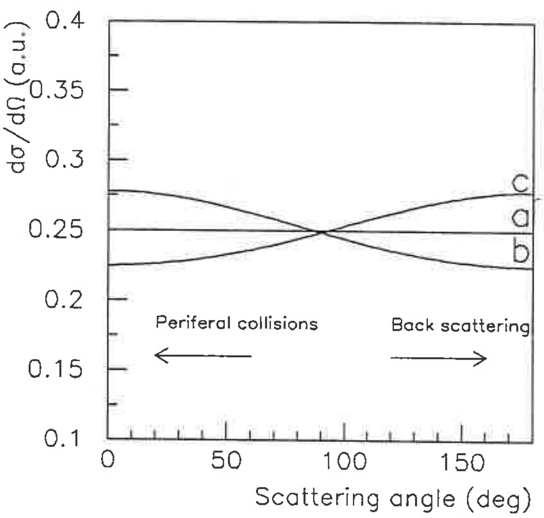

Figure 4 - Differential Cross Section for different shapes of the scattering body. a) $x_0 = 0.9\,y_0$, b) $x_0 = y_0$, c) $x_0 = y_0/0.9$.

derive from the model used to describe the dynamic part of the process).

3. The Coulomb Scattering of pointlike bodies

Let us now examine a case that involves a more complex type of interaction than that taking place between two ideal rigid bodies: the collision of two pointlike bodies with Z, Z' charge, which interactaccording to a repulsive Coulomb potential. Let us also assume that the mass of one of the two bodies (scatterer) is much bigger than the other (incident) so that its position does not vary in the collision process and we can therefore consider it still at rest. The calculation of the differential cross section for this process is done in the same way as for the previous section: calculation of the relationship between the collision parameter and scattering angle and subsequent application of the definition.

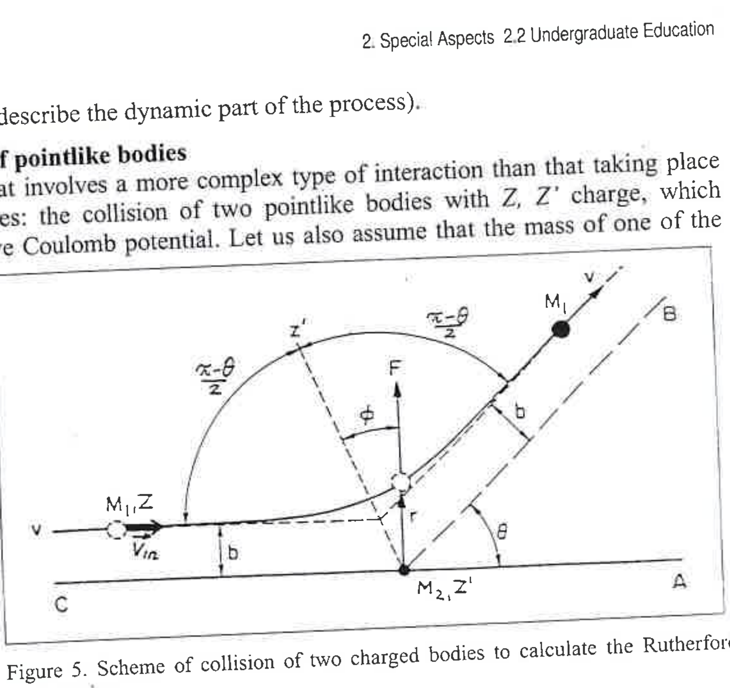

Figure 5. Scheme of collision of two charged bodies to calculate the Rutherford cross section.

The Coulomb force on the incident body is central and conservative, thus the conservation principles of angular momentum and energy apply. We derive that the speed of the incident body is the same before and after the collision, and so the variation of its momentum is equal to:

$$\Delta \boldsymbol{P}_I = 2\, M_I\, v_{in} \sin(\theta/2)\, \boldsymbol{n}$$

where the versor $\boldsymbol{n}$ is directed according to the bisector of the angle between the directions of incidence and scattering (figure 5). Using the theorem of impulse we derive:

$$\Delta P_I = \boldsymbol{n} \bullet \Delta \boldsymbol{P}_I = \int_{-\infty}^{\infty} F_{21} \cos\phi \, dt = \int_{(\theta-\pi)/2}^{(\pi-\theta)/2} \frac{ZZ' e^2}{r^2} \cos\phi \, \frac{dt}{d\phi} d\phi = \int_{(\theta-\pi)/2}^{(\pi-\theta)/2} \frac{ZZ' e^2}{r^2} \cos\phi \, d\phi$$

in which, in order to substitute the factor $dt/d\phi$, we considered the conservation of the angular momentum.

The value of the integral is easy to calculate and we obtain, by comparing the two expressions for ΔP_I,

$$b = \frac{ZZ' e^2}{2E} \mathrm{cotg}\frac{\theta}{2}$$

where E is the initial kinetic energy of the particle. Finally, by applying the definition (3), we obtain:

$$\frac{d\sigma}{d\Omega} = \left(\frac{ZZ' e^2}{4E}\right)^2 \sin^{-4}\left(\frac{\theta}{2}\right) \tag{4}$$

In order to calculate the total cross section, we integrate eq.(4) in the range ($\varepsilon \le \theta \le 180^0$) ($\varepsilon > 0$)

$$\sigma(\varepsilon) = \int_{\varepsilon}^{180} \frac{d\sigma}{d\Omega} 2\pi \, d\cos\theta = 2\pi \left(\frac{ZZ' e^2}{4E}\right)^2 \left(\frac{1}{1-\cos\varepsilon} - \frac{1}{2}\right)$$

Thus, the total cross section diverges as $\varepsilon \to 0$.

This fact reflects the long-range nature of the electromagnetic force; small-angle scattering corresponds to very large impact parameters. However, a cutoff in the smallest possible scattering angle ε is introduced by the experimental conditions: the transverse size of the beam and target, and the screening effect of electrons (in targets constituted by atoms) both limit the largest permissible impact parameter.

4. The 'Rutherford' Experiment

Expression (4) was derived in the historical context of the study of the structure of the atom, carried out by Rutherford and other experimenters at the begining of century. In this period, it was well known that the atom, electrically neutral, contains negatively charged electrons of a mass which is much lower than the mass of the atom itself. The problem therefore was to understand how the greater part of the mass of the atom, which is positively charged, is distributed through the volume of the atom. Beginning with Geiger and Marsden in 1909, in order to choose among different hypotheses, a series of measurements of scattering of α particles (produced by the decay of Po^{210}) on a thin leaf of gold was performed. The experimental results demonstrate the non-applicability of the Thompson model: in particular, the predictions of the Thompson model are in clear contrast with the relatively high fraction of α particles scattered at large angles. Quoting Rutherford: *It seems reasonable to suppose that the deflection through a large angle is due to a single atomic encounter, for the chance of a second encounter of a kind to produce a large deflection must in most cases be exceedingly small. A simple calculation shows that the atom must be a seat of an intense electric field in order to produce such a large deflection.* Passing to a more quantitative level, Rutherford demonstrated that the results obtained were consistent with the predictions derived from the hypothesis that the mass of the atom, positively charged, was concentrated in a small area of the volume of the atom: in practice, eq.(4). How small? The data then available agreed with (4) assuming a nuclear radius less than 4.5 10^{-14}m; subsequent measurements have shown deviations from the $\sin^4(\theta/2)$ law, corresponding to impact parameters less than 10^{-14}m, under which another potential, due to the strong interaction, becomes effective in addition to Coulomb potential: the structure of the nucleus reveals itself.

5. 'Rutherford' Scattering inside the Nucleus

Dealing with reactions where the impact parameter b is close to the nucleus radius R or even smaller, the two nuclei will now be able to 'touch' each other during the collision and therefore the result of the interaction will be governed by nuclear forces rather than Coulomb forces (nuclear forces are stronger than Coulomb forces for $b \le R$). The result of such a collision will depend on how much the two nuclei touch, or, in other words, how big is the overlap between the cross section areas of the two nuclei. If the nuclei come very close but do not touch, they will just be able to exchange some energy and they will be left in an excited state after the interaction; if only the peripheral parts of the nuclei overlap they will exchange some nucleons too (protons and neutrons) and the result will be two new nuclei in an excited state with masses close to the original values; moving to bigger overlapping areas the nuclei could be able to fuse forming final nuclei with masses very close to the sum of the original nuclei, high excitation energy and high angular momentum (of course mass, energy and angular momentum must be conserved during all these processes). As for all processes in physics the three scenarios briefly described (they are not the only possible ones), do not happen in a sharply defined b range but are the dominant processes for specific b ranges. This implies that, to know the total cross section, one has to measure all the processes in a specific b range. The cross section, though remaining an area, will then give the probability for different nuclear processes. In a similar way one can understand that Rutherford scattering will be observed too even if $b \le R$. How do we measure a cross section for a nuclear reaction?

One has to keep in mind a basic thing: due to the high number of nucleons involved in a nuclear reaction (for example for ^{58}Ni + ^{64}Ni there are 122 nucleons involved: 56 protons and 66 neutrons), we are not able to write down a simple expression for the nuclear force and can not derive a simple equation for the cross section like in Rutherford's case. To overcome the problem one remembers that we could have, at the same time, some nuclei interact via the nuclear field and some others via the Coulomb field and then write the following equation :

$$\sigma_{NUC} = A\ \sigma_R\ N_{NU} / N_R$$

with σ_R the Rutherford cross section, N_{NUC} the number of nuclear events, N_R the number of Rutherford events and A a parameter depending on detectors' solid angles and intrinsic efficiencies.

The only thing we do not know yet is how do we measure N_{NUC}. This number depends on the nuclear process we want to observe, so let's think about a nuclear reaction where the two nuclei come close enough to fuse and form final nuclei at high excitation energy and angular momentum. Let's pick one of these nuclei: it has to dissipate the extra energy and angular momentum and will achieve it removing nucleons from high energy levels to lower energy ones and emitting photons with energy equal to the difference of the two levels (these photons are called γ rays).
The deexcitation pattern is then a cascade of γ rays. For each nuclear fusion event we will have a γ ray cascade. The cascades could have different origins (in energy and angular momentum), but they will all end up at the same point (called the ground state level of the nucleus with zero excitation energy and the lowest possible angular momentum) as if they were going down a funnel. At the bottom all γ cascades will follow a common path; this means that every time we measure a γ ray which makes the deexcitation from the first excited state to the ground state (usually denoted as the $2^+ \rightarrow 0^+$ transition), we can say we had a fusion event. If we now denote the total number of $2^+ \rightarrow 0^+$ transitions with N_γ and with σ_{fus} the nuclear fusion cross section we can finally write

$$\sigma_{fus} = A\sigma_R N_\gamma / N_R \, .$$

We can now conclude that measuring both γ rays and Rutherford scattering events allows to deduce the cross section for the nuclear process called fusion. In a similar way it is possible to measure the cross section for other nuclear processes.

6. Closing Remarks

In basic physics courses the concept of the cross section is often introduced as a semi-qualitative definition, mostly on a geometrical basis, as required or after the study of impact in mechanics. It is rarely studied in detail and its significance as a parameter of measurement in many areas of physics (from the field of solids, from the point of view of the analysis techniques, to the nuclear physics at high and low energies). A cultural analysis for teacher training which may produce the necessary knowledge for a pedagogic follow-up suitable to the role that such a parameter plays in all experimental research, gives us some suggestions for a teaching proposal.

Reference

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CREATION AND DETECTION OF "THERMAL" WAVES THROUGH ABSORPTION OF MODULATED LIGHT BY SOLIDS

F.J. Macedo, J.A. Ferreira, *Department of Physics, University of Minho, Braga (Portugal)*

1. Introduction

Place the substance to be experimented within a glass test-tube, connect a rubber tube with the mouth of the test-tube, placing the other end of the pipe to the ear. Then focus the intermittent beam upon the substance in the tube. I have tried a large number of substances in this way with great success, although it is extremely difficult to get a glimpse of the sun here, and when it does shine the intensity of the light is not to be compared with that to be obtained in Washington. I got splendid effects from crystals of bichromate of potash, crystals of sulphate of copper and from tobacco-smoke. A whole cigar placed in the test-tube produced a very loud sound....[1]

The text presented above is the transcription of part of a letter, written by A. G. Bell to an American colleague in 1880. It represents a proof of his accidental discovery of the photoacoustic effect while he was investigating the photophone. Although the discovery was classified as "extremely interesting" by many scientists at that time, it was very difficult for them to quantify any results because they had no acoustic detectors better than their ears. Because of this limitation, there was not much interest in the photoacoustic effect for many years. Some studies took place in the 1940s