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Numerical Methods in Thermal Problems

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SECTION 9

COUPLED CONDUCTION AND CONVECTION

LAMINAR FLOW HEAT TRANSFER WITH AXIAL CONDUCTION IN A CIRCULAR TUBE: A FINITE DIFFERENCE SOLUTION.

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Summary

A finite difference technique is used for the evaluation of the rate of heat transfer in the thermal entrance region of ducts with axial conduction. The velocity profile is fully developed and the pipe is assumed to extend from minus to plus infinite. An uniform heat flux is imposed for $z \geq 0$, while the wall temperature is kept uniform and equal to its $-\infty$ value for $z < 0$. The results, given for Péclet numbers as low as 1., show that axial conduction and heat losses from the unheated section of the pipe markedly affect the temperature profile at the inlet of the heated part of the pipe at low Pe, so that the wall temperatures can result lower than the local mixed mean temperatures immediately downstream the start of heating. A new parameter is then proposed as an alternative to the usual Nusselt number to describe the performance of this kind of heat exchanger.

1 INTRODUCTION

This paper deals with laminar heat transfer in completely developed flow close to start of heating sections of cylindrical pipes and can be added to numerous pubblications extending the classical works of Graetz and Nusselt. It is well known that in most of the engineering applications the heat conducted in flow direction is negligibile as compared to the heat convected axially and conducted radially, but even that it is not always possible to disregard the axial conduction with fluids having a good conductivity.

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The boundary condition here considered is that of uniform heating flux at the wall for $z \geqslant 0$ commonly designed as UHF or Ⓗ condition, while internal heat generation is disregarded and physical properties are held constant. Three fundamental sets of boundary conditions for this thermal entrance problem have been recognized in the well known Shah and London source book |1|. They are specified in Fig. 1.

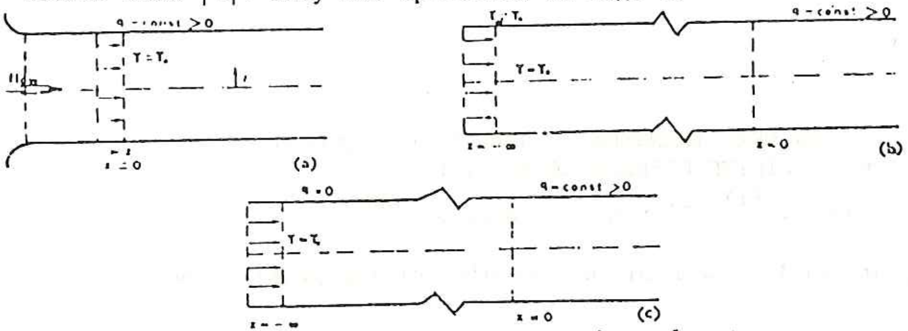

Fig.1 - Boundary conditions for UHF thermal entrance problem with axial heat conduction.

Hennecke |2| has pointed out firstly, condition (a) is not realistic expecially when a fully developed velocity profile is considered. The thermal conduction upstream cannot in any way stopped at $z = 0$ and then the temperature profile at the origin will never be uniform. The conditions (b) and (c) seem more useful for practical evaluation of the influence of the axial conduction along the fluid. The case (c) has been considered in |2,3|, while the case (b) seems to have received no attention up now in spite of the fact that it is at least as realistic as case (c) for real heat exchangers. This problem is tackled here using a finite difference method build up to solve a wider class of axi-symmetrical problems in laminar heat transfer |4|.

2 RELEVANT EQUATIONS

Assuming fully developed flow, costancy of the thermophysical properties and disregarding internal heat generation and viscous dissipation, the steady-state energy equation may be written:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{\rho C_p}{K}\, u \frac{\partial T}{\partial z} \qquad (1)$$

where the Hagen-Poiseuille profile is assumed for u:

$$u = 2\, u_m (1 - (r/r_w)^2) \qquad (2)$$

The following boundary conditions are imposed (Fig. 1-b):

$$z = -\infty \qquad T = T_o \qquad (a)$$

$$z = +\infty \qquad \frac{\partial T}{\partial z} = \frac{2\dot{q}}{\rho C_p u_m r_w} \qquad (b)$$

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \qquad (c)$$

$$r = r_w,\ z < 0 \qquad T = T_o \qquad (d) \qquad (3)$$

$$r = r_w,\ z \geqslant 0 \qquad \frac{\partial T}{\partial r} = \frac{\dot{q}_w}{K} \qquad (e)$$

In dimensionless form the eqs. (1) and (2) become

$$\frac{1}{Pe}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Theta}{\partial R}\right) + \frac{\partial^2 \Theta}{\partial Z^2}\right) = U\frac{\partial \Theta}{\partial Z} \qquad (4)$$

$$U = 2\ (1 - R^2) \qquad (5)$$

and the boundary conditions

$$Z = -\infty \qquad \Theta = 0 \qquad (a)$$

$$Z = +\infty \qquad \frac{\partial \Theta}{\partial Z} = \frac{4}{Pe} \qquad (b)$$

$$R = 0 \qquad \frac{\partial \Theta}{\partial R} = 0 \qquad (c) \qquad (6)$$

$$R = 1,\ Z < 0 \qquad \Theta = 0 \qquad (d)$$

$$R = 1,\ Z \geqslant 0 \qquad \frac{\partial \Theta}{\partial R} = 1 \qquad (e)$$

where

$$\Theta = \frac{T - T_o}{\dot{q}\ r_w/K} \qquad (7)$$

$$Z = z/r_w,\ R = r/r_w \qquad (8)$$

$$U = u/u_m \qquad (9)$$

The temperature solutions of eqs. (4)÷(6) permit the computation of the thermal quantities of practical interest. The bulk fluid temperature is defined as

$$\Theta_b = \int_0^1 RU\Theta dR / \int_0^1 RUdR \qquad (10)$$

and the Nusselt number as

$$Nu = \frac{2}{\Theta_w - \Theta_b} \qquad (11)$$

for $Z \geqslant 0$, and

$$Nu = \frac{2(\partial \Theta/\partial R)_w}{\Theta_w - \Theta_b} \qquad (12)$$

for $Z < 0$

3 NUMERICAL SOLUTION

A finite difference method was chosen to solve the eq.(4). It is described in greater detail in |4| and shall be outlined briefly here. As a first step the domain of integration was bounded using the following variable transformations

$$\eta = -\ \mathrm{tgh}\ k_1 |Z|^{\alpha_1} \quad , \quad Z < 0$$
$$\eta = \mathrm{tgh}\ k_2\ Z^{\alpha_2} \quad , \quad Z > 0 \qquad (13)$$

with $0 < \alpha_1, \alpha_2 \leq 1$ so that $-1 \leq \eta \leq +1$.

$k_1, k_2, \alpha_1, \alpha_2$ are constants whose values can be fixed suitably to fit the physical problem studied. The results given afterwards have been obtained for $\alpha_1 = \alpha_2 = .375$ and $k_1 = k_2 = 0.33 \div 1$. A different variable transformation is used for the radial coordinate, so we have

$$\rho = a\ R^2 + bR \qquad (14)$$

and $0 < \rho < 1$.
The equation (4) was then rewritten as follows

$$L(\Theta) \equiv A \frac{\partial^2 \Theta}{\partial \rho^2} + B \frac{\partial \Theta}{\partial \rho} + C \frac{\partial \Theta}{\partial \eta} + D \frac{\partial^2 \Theta}{\partial \eta^2} - G \frac{\partial \Theta}{\partial \eta} = 0 \qquad (15)$$

where

$$A = \left(\frac{d\rho}{dR}\right)^2 , \quad B = \left(\frac{d^2\rho}{dR^2} + \frac{1}{R}\frac{d\rho}{dR}\right) ,$$
$$C = \left(\frac{d^2\eta}{dZ^2}\right) , \quad D = \left(\frac{d\eta}{dZ}\right)^2 , \quad G = \frac{Pe}{R}\ \frac{d\eta}{dZ}\ U \qquad (16)$$

The operator $L(\Theta)$ is evaluated at the nodal points of the rectangular grid having a constant mesh size of $\Delta\eta$ by $\Delta\rho$. In this way, the (13) allows to get the points of the grid in the original or physical domain are very close near the origin and, departing from it,space out themselves more and more. This characteristic becomes relevant in the case considered here, as the Neumann condition (6b) is evaluated by a finite difference, and must be imposed at a distance from the origin, large but finite and not at $\eta = 1$ where Θ takes infinite values. With the (14) the nodal points are closer near the wall, where Θ is expected to have higher gradients. As a general procedure, the standard five point Laplace operator was used for second order derivatives while first order derivatives were replaced with central differences in the diffusive terms of (15) and with forward or backward differences according to the sign of G in the convective terms, in order to increase the dominance of the elements in the main diagonal of the coefficients matrix. In the case considered here, G is always positive and backward differences are used so that, solving eq. (15) at the nodal point and with reference to Fig. 2, one has:

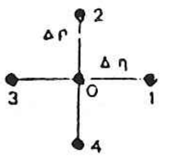

Fig.2 - Molecule of computation.

$$C_o\Theta_o + C_1\Theta_1 + C_2\Theta_2 + C_3\Theta_3 + C_4\Theta_4 = 0 \qquad (17)$$

where

$$\begin{aligned}
C_o &= -2\left(\left(\frac{d\rho}{dR}/\Delta\rho\right)^2 + \left(\frac{d\eta}{dZ}/\Delta\eta\right)^2 + G/\Delta\eta\right) \\
C_1 &= \frac{d^2\eta}{dZ^2}/2\Delta\eta + \left(\frac{d\eta}{dZ}/\Delta\eta\right)^2 \\
C_2 &= \left(\frac{d\rho}{dR}/\Delta\rho\right)^2 + \left(\frac{d^2\rho}{dR^2} + \frac{d\rho}{dR}/R\right)/2\,\Delta\rho \\
C_3 &= \left(\frac{d\eta}{dZ}/\Delta\eta\right)^2 - \frac{d^2\eta}{dZ^2}/2\,\Delta\eta + G/\Delta\eta \\
C_4 &= \left(\frac{d\rho}{dR}/\Delta\rho\right)^2 - \left(\frac{d^2\rho}{dR^2} + \frac{d\rho}{dR}/R\right)/2\,\Delta\rho
\end{aligned} \qquad (18)$$

The boundary conditions (6a,6d) need no particular treatise, the condition 6b has been written in finite different form, while a parabolic interpolation was used to express conditions (6c,6e), preliminary assays had shown that this scheme gives better results than the usual finite difference form of the derivatives at the boundary. The coefficients matrix of the algebraic linear equations obtained by the implicit finite difference method above described is a band-matrix; in this case only the elements of the main diagonal and those of a reduced number of the off-diagonals can be non-zero, so that the Gauss elimination method with partial pivoting can be profitably used. Bulk temperatures and thermal gradients at the wall for $Z < 0$, are obtained by a natural cubic spline. As the(13) has discontinous derivatives at $Z = 0$, a suitable form of eq. (17) in the original axial coordinate Z is employed at that location.

4 PRELIMINARY TESTS

Tests had been performed solving the classical Graetz problem disregarding axial conduction and with the UHF boundary condition. The results of these preliminary assays and the related comments have been reported extensively in |4|.
The values of the constants k and α appearing in eq.(13) may affect the precision of the results expecially in the proximity of the origin. Values of k in the range .25 ÷ 1. and α in the range .375 ÷ .5 have proven to give the better results in terms of the Nusselt number.
Various grid sizes have been tryed and good solutions were

obtained also with a 10 radial and 10 axial steps grid (n > 0)
Errors on the Nusselt number were less then 2.% near to the inlet ($Z^{+} \simeq 2.10^{-5}$) using 20 x 30 and 20 x 40 grids, while the asymptotic value of 4.364 was approached with accuracy better then .4%.
$Z^{+} = Z/2Pe$ is the usual dimensionless axial coordinate for the thermal entrance problems.

5 RESULTS

Final computations have been performed with a 20 x 30 grid both in the positive and the negative part of the domain and for Pe = 1,2,5,10,20,50.
The results are presented graphically in Figs. 3 through 7.
Fig. 3 demonstrates the influence of axial conduction on the temperature profile at Z = 0.
It may be observed that the smaller is Pe the more the tempera ture distribution deviates from the flat profile and the higher becomes the value of the bulk temperature. Only for Pe $\geq$ 20 the inlet temperature may be considered nearly uniform.
Some of these profile show a characteristic slope, Pe = 2 and 5 for instance, with two minima and a maximum. Fig. 4, depicting the temperature profiles at various axial locations for Pe = 1, points out that this behaviour is an effect of the ze ro temperature condition at the wall for Z < 0, wich implies an heat flow from the fluid toward the boundary.
In fact the temperature profile takes on a negative concavity upstream the inlet and, as the fluid reaches the start of hea ting, a dip must be formed near to the wall, owing to the UHF condition there imposed.
It is worthy to observe that with this particular set of boun dary conditions the bulk temperature may exceed the value at the wall at the thermal inlet and just downward it. In fact Θ_b has no upper bound as in the UWT case, even if the tempera ture profiles are similar in trend for $Z^{+} < 0$ (see f.e.|2,5|).
Figs. 5 and 6 depict the bulk temperature upstream and downstream the inlet respectively. They are strikingly affected by axial conduction at least for Pe < 20.
From eq.(12) the values of Nu have been computed for $Z^{+} < 0$, they are indicated on Fig. 5 for the lowest Pe. These curves intersect at about $Z^{+} = 4.4\ 10^{-3}$ while the asymptotic value is about 4.3.
On Fig. 6 the plot of Θ_w is seen to cross that of Θ_b for Pe=1 and 2. Negative values of the Nusselt number should then be found, i.e. Nu, as defined by eq.(11), no longer gives indication of the rate of heat transfer.
The following definition of Nu has been assumed:

$$Nu_w = \frac{\dot{q}_w}{T_w - T_o} = \frac{2}{\Theta_w} \qquad (19)$$

The values of Nu_w have been plotted on Fig. 7.
Also on Fig. 7 is depicted the ratio

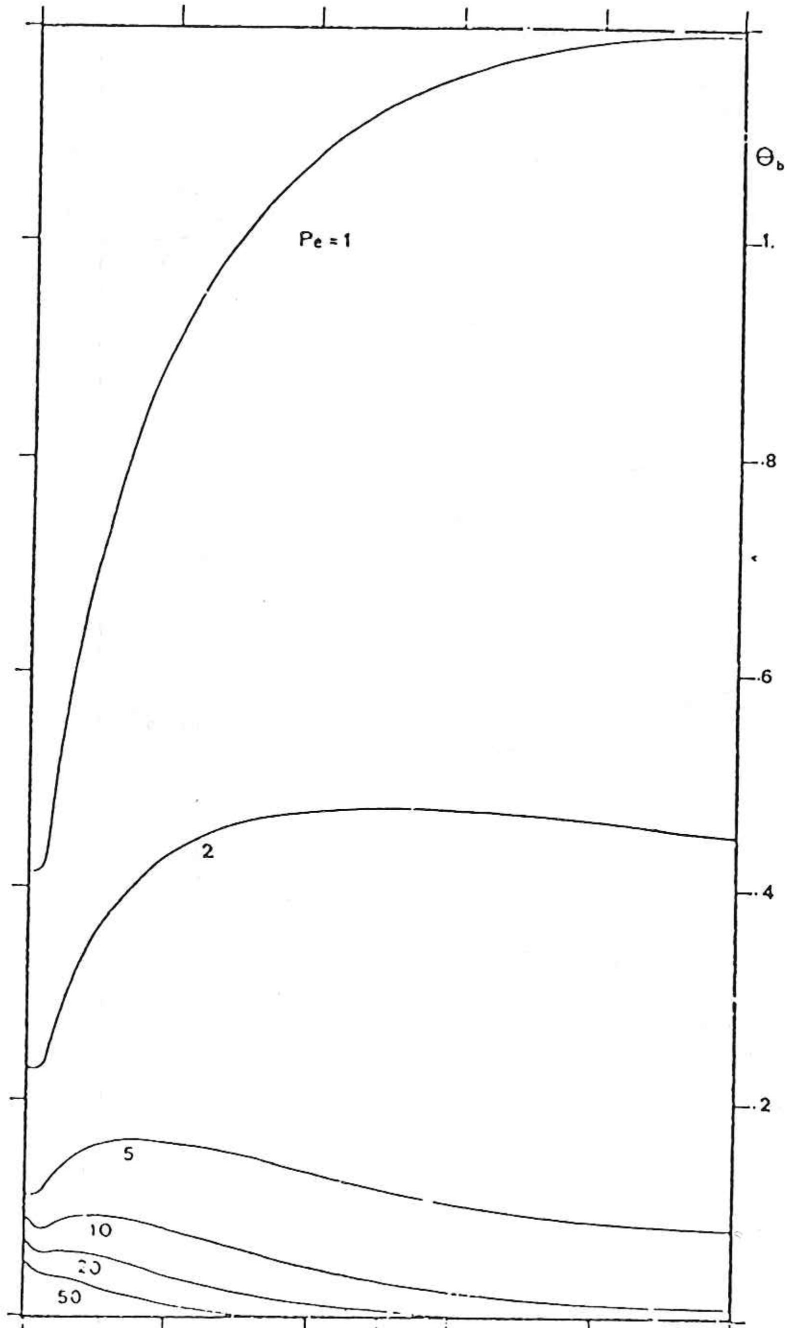

Fig.3 - Temperature profiles at the entrance.

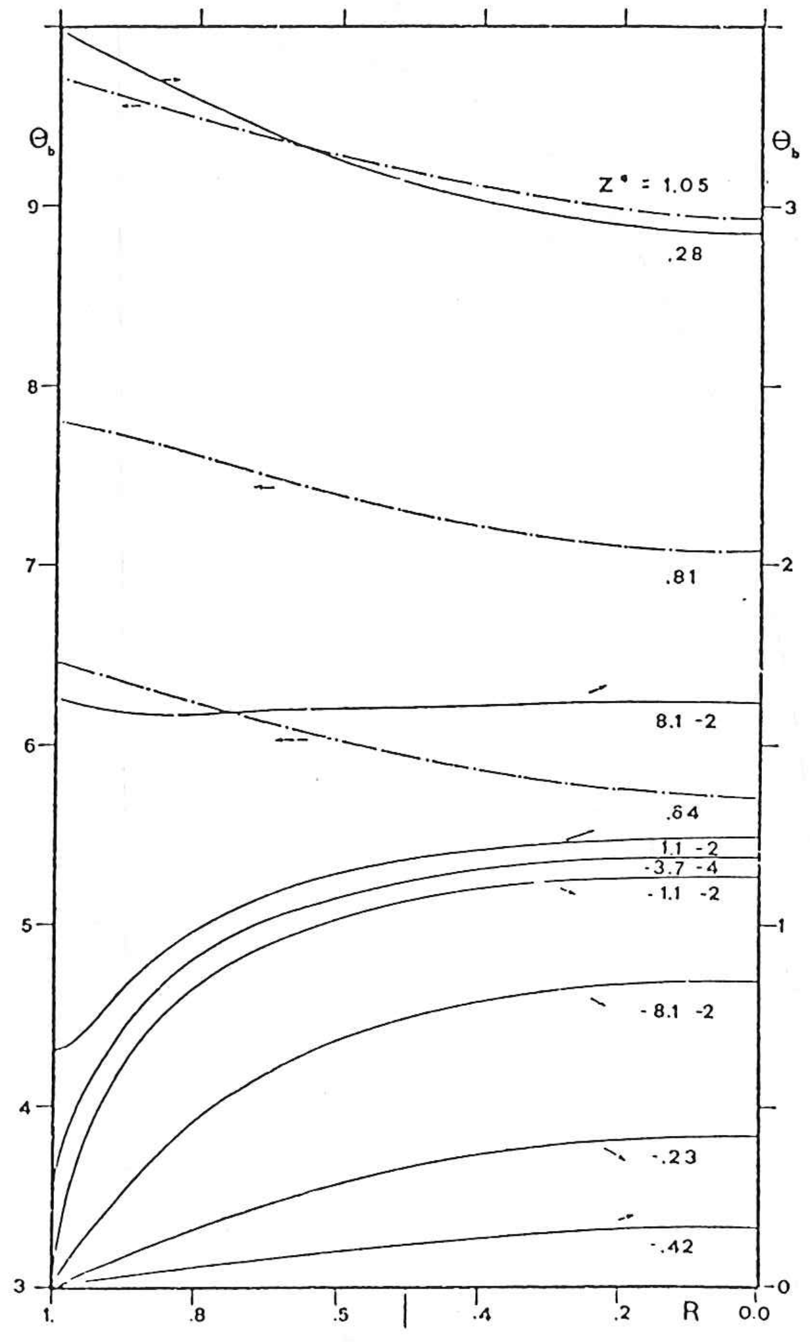

Fig.4 - Temperature distribution for Pe=1.

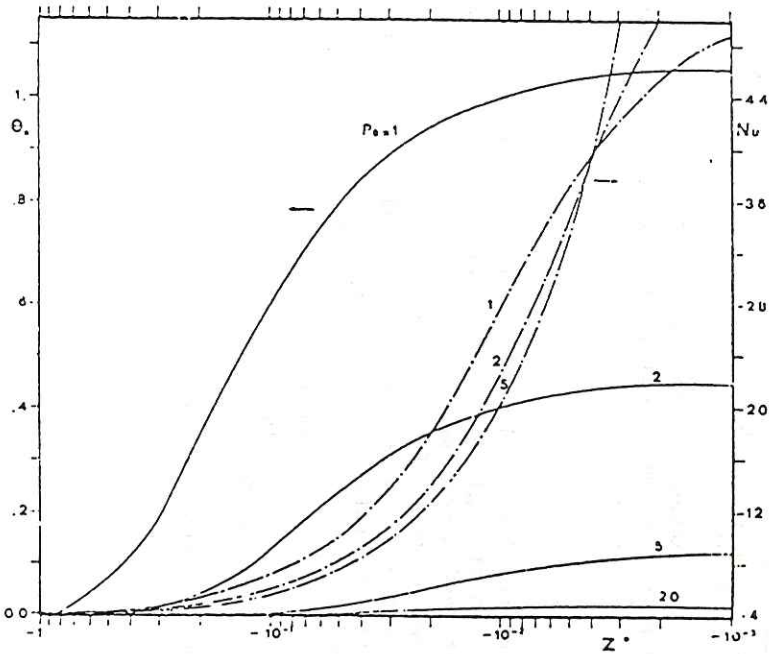

Fig.5 - Bulk and wall temperature vs.Z^+ ($Z^+<0$)

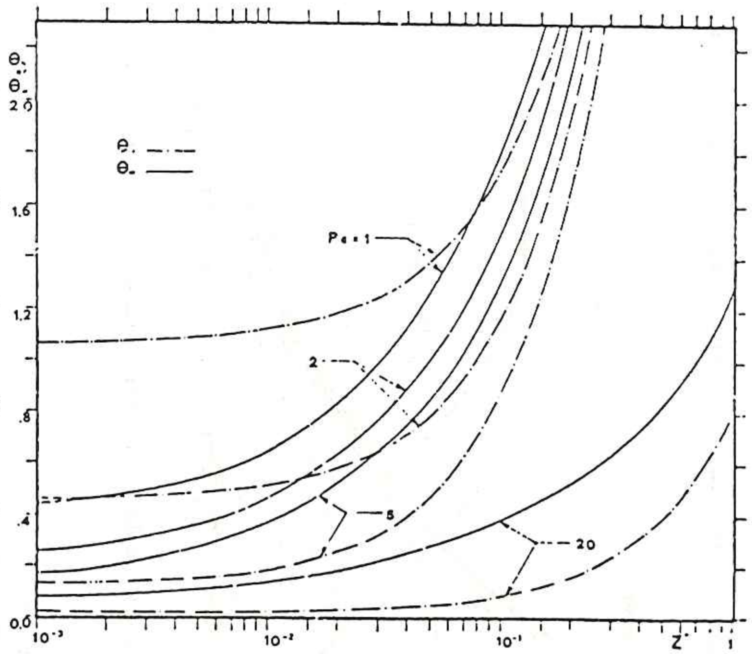

Fig.6 - Bulk temperature and Nusselt number vs.Z^+ ($Z^+>0$)

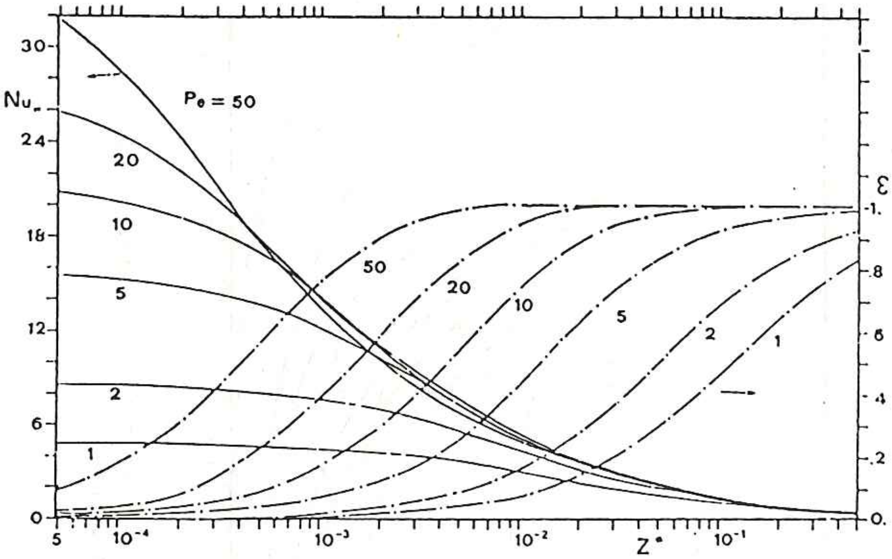

Fig.7 - Nusselt number and performance ratio vs.Z (Z 0)

$$\varepsilon = \frac{2\pi \dot{q}_w r_w z}{u_m \pi r_w^2 \rho C_p (T_b - T_o)} = \frac{8Z^+}{\Theta_b} \qquad (20)$$

In absence of axial conduction one has $\varepsilon = 1$.
At all Pe, ε goes to 1 asymptotically, as expected, and faster and faster as Pe increases.

6 CONCLUSIONS

The present results may be summarized as follows:

- axial conduction strongly affects the heat transfer in the thermal inlet region for small Pe,
- the inlet temperature profile is always not uniform,
- the bulk temperature becomes very large at the inlet and exceeds the wall temperature for Pe < 5,
- the usual definition of Nu seems to fall in defect at least in the neighborwood of the start of heating, owing to the particular trend of the temperature profiles upstream the inlet. Different parameters should then be used for this set of boundary conditions,
- for practical purposes axial conduction effects may be disregarded only for Pe ⩾ 20.

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