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BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL INCLUSIONS IN FRÉCHET SPACES WITH MULTIPLE SOLUTIONS OF THE HOMOGENEOUS PROBLEM

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Abstract. The paper deals with the multivalued boundary value problem $x' \in A(t,x)x + F(t,x)$ for a.a. $t \in [a,b]$, $Mx(a) + Nx(b) = 0$, in a separable, reflexive Banach space E. The nonlinearity F is weakly upper semicontinuous in x. We prove the existence of global solutions in the Sobolev space $W^{1,p}([a,b],E)$ with $1 < p < \infty$ endowed with the weak topology. We consider the case of multiple solutions of the associated homogeneous linearized problem. An example completes the discussion.

Keywords: multivalued boundary value problem, differential inclusion in Banach space, compact operator, fixed point theorem

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1. Introduction

The paper deals with the two-point boundary value problem (b.v.p.) associated with semilinear multivalued evolution equations. More precisely, assuming $(E, \|\cdot\|)$ is a reflexive, separable Banach space, we consider the problem

$$\begin{cases} x' \in A(t,x)x + F(t,x), & \text{for a.a. } t \in [a,b], \\ Mx(a) + Nx(b) = 0, \end{cases} \tag{1.1}$$

where $M, N \in \mathcal{L}(E)$, the space of linear bounded operators from E into itself, and $M, N \neq 0$.

We denote by E' the dual of E, by $\langle \cdot \rangle$ the duality between E and E', and by E_σ the topological vector space obtained when the topology acting on E is induced from the weak topology. We study (1.1) under the following assumptions:

$A\colon [a,b] \times E \to \mathcal{L}(E)$ and

$$\text{(1.2)} \qquad \begin{aligned} A(\cdot,x)\colon [a,b] \to \mathcal{L}(E) \text{ is measurable for a.a. } x \in E;\\ A(t,\cdot)\colon E_\sigma \to \mathcal{L}(E) \text{ is continuous for all } t \in [a,b]; \end{aligned}$$

$F\colon [a,b] \times E_\sigma \multimap E_\sigma$ is globally measurable with nonempty, bounded, closed, convex values and

$$\text{(1.3)} \qquad F(t,\cdot) \text{ is (weakly) upper semicontinuous (u.s.c.) for a.a. } t \in [a,b],$$

there exists $\alpha \in L^p([a,b],\mathbb{R})$ with $1 < p < \infty$, such that

$$\text{(1.4)} \qquad \|A(t,x)\| + \|F(t,x)\| \leqslant \alpha(t)$$

for a.a. $t \in [a,b]$ and $x \in E$.

We are interested in the existence of classical solutions of (1.1), i.e. absolutely continuous functions $x\colon [a,b] \to E$ satisfying the required condition.

We refer to [1], [2], [8], and [9] and the references therein for the investigation of boundary conditions similar to those in (1.1). In particular, in [8, Corollary 6.1.2] and in [2] the existence of mild solutions was proved; the solutions should satisfy a periodic condition with period T in the first paper and a nonlinear multivalued boundary condition in the second. In both works the authors consider problems with a constant linear part A, the generator of a C_0-semigroup e^{At}. They also assume the χ-regularity of F, where χ represents the Hausdorff measure of non-compactness, which, in particular, implies that F is compact valued. Moreover, in [8] the condition (1.4) with $\alpha \in L^1([a,b],\mathbb{R})$ is required. In [1, Theorem 5.2], the existence of classical solutions was proved when A is independent of x and M and N are invertible, assuming the same compactness condition as in [8] and the sublinearity of F. Finally, in [9] a nonlinear boundary condition is considered. In this setting the author proves the existence of classical solutions under a quite strong compactness assumption on the solution set of the associated linearized problem.

We investigate (1.1) in the Sobolev space $W^{1,p} = W^{1,p}([a,b],E)$ and we always assume $1 < p < \infty$. In this case, E being reflexive and separable, it is known that $W^{1,p}$ is a reflexive, separable Banach space. So, when endowed with its weak topology, it has good compactness properties. Then we assume the regularities on A and F with respect to the appropriate weak topology. To solve this problem we introduce a multivalued solution operator, defined in Section 3, whose fixed points are solutions of problem (1.1). As far as we know, the way we use weak topologies for studying this problem is original; we introduced and discussed it in detail in

[3] for the investigation of the Cauchy problem associated with the same dynamics.

The main aim of this work is to prove an existence result for problem (1.1) (see Theorem 3.2) allowing the possibility to have multiple solutions of the associated homogeneous linearized problem

$$\begin{cases} x' = A(t, q(t))x & \text{for } t \in [a, b], \\ Mx(a) + Nx(b) = 0. \end{cases} \tag{1.5}$$

Given $q \in W^{1,p}$, we denote by U_q the evolution system generated by the family of linear operators $\{A(t, q(t))\}_{t\in[a,b]}$; $U_q\colon \Delta \to \mathcal{L}(E)$ with $\Delta := \{(t, s) \in [a, b] \times [a, b]\colon a \leqslant s \leqslant t \leqslant b\}$. It is known that, in an arbitrary Banach space, the injectivity of $M + NU_q(b, a)$ is equivalent to the assumption that the associated homogeneous linearized problem (1.5) has only the trivial solution (see e.g. Lemma 3.1). Notice that when E is a euclidean space, the last property is indeed equivalent to the invertibility of $M + NU_q(b, a)$. When this operator is invertible for all $q \in W^{1,p}$ it is possible to explicitly write the integral solution operator. Therefore, it is quite usual in literature to assume the invertibility of the operator mentioned above. For instance in [8] the strong contractivity of e^{AT} is assumed, which implies the invertibility of the operator $I - \mathrm{e}^{AT}$. The case of a not necessarily invertible operator $M + NU_q(b, a)$ has been also studied in literature. However, to obtain existence results, quite strong assumptions are usually introduced on the growth of the nonlinear part F, or on the compactness of the solution set of the associated linearized problem, see, e.g., [2, Theorem 2.6] and [9, Theorem 4.1].

With our approach we can require only the surjectivity of the operator $M + NU_q(b, a)$ for all $q \in W^{1,p}$ and not its injectivity, which leads to having multiple solutions of the linear problem (1.5). We are able to completely avoid any assumption of compactness and the use of any measure of non-compactness, moreover, we can treat also the dependence on x of the linear part A. When assuming, in addition, the invertibility of the operator $M + NU_q(b, a)$ our approach can be applied to obtain existence results for solutions of (1.1) under more general growth conditions both on the linear part A and on the nonlinear part F. We refer to the paper [4] for this discussion.

We denote by $\|\cdot\|_1$ the norm in $L^1([a, b], \mathbb{R})$.

2. Preliminary results

Assume $1 < p < \infty$ and consider the Sobolev space

$$W^{1,p} = \left\{ \begin{array}{l} u \in L^p([a,b],E)\colon\ \exists\, g \in L^p([a,b],E) \text{ such that} \\ \displaystyle\int_a^b \varphi'(t)u(t)\,\mathrm{d}t = -\int_a^b \varphi(t)g(t)\,\mathrm{d}t \\ \text{for every } \varphi \in C^1([a,b],\mathbb{R}) \text{ with } \varphi(a) = \varphi(b) = 0 \end{array} \right\}.$$

It is well known that each $u \in W^{1,p}$ admits an absolutely continuous representative and in the sequel u stands for this continuous representative. Moreover, given $H \subset E$ closed and convex and $\beta \in L^p([a,b],\mathbb{R})$, the set

$$Q = \left\{ q \in W^{1,p}\colon \begin{array}{l} q(t) \in H \text{ for all } t \in [a,b] \\ \|q'(t)\| \leqslant \beta(t) \text{ for a.a. } t \in [a,b] \end{array} \right\} \tag{2.1}$$

is closed and convex. If, in addition, H is bounded, Q is also weakly compact.

Given $q \in W^{1,p}$, the measurability of F implies that the set S_q of measurable selections of the multimap $t \longmapsto F(t,q(t))$ with $t \in [a,b]$ is nonempty (see e.g. [5, Theorem III.6]). Moreover, according to (1.2), (1.4) and the continuity of q, the map $t \longmapsto A(t,q(t))$ is measurable and Bochner integrable. Given $q \in W^{1,p}$, let U_q be the evolution system generated by the family of linear operators $\{A(t,q(t))\}_{t\in[a,b]}$. It is well known that (see e.g. [6]), according to (1.4), for each $q \in W^{1,p}$ we have

$$\|U_q(t,s)\| \leqslant \mathrm{e}^{\|\alpha\|_1}, \quad \text{for every } (t,s) \in \Delta. \tag{2.2}$$

We give now a convergence result involving these evolution systems.

Lemma 2.1. *If $q_n \rightharpoonup q$ in $W^{1,p}$ and $\{q_n(a)\}_n$ is bounded, then there exists a subsequence $\{q_{n_k}\}$ such that $U_{q_{n_k}}(t,s) \to U_q(t,s)$ in $\mathcal{L}(E)$ as $k \to \infty$ uniformly for $(t,s) \in \Delta$.*

Proof. Having $q_n \rightharpoonup q$ in $W^{1,p}$, we obtain $q_n \rightharpoonup q$ and $q_n' \rightharpoonup q'$ in $L^p([a,b],E)$. Since $\{q_n(a)\}_n$ is bounded, there exists a subsequence $q_{n_k}(a) \rightharpoonup \bar{q}$ in E. Hence $q_{n_k}(t) = q_{n_k}(a) + \int_a^t q_{n_k}'(s)\,\mathrm{d}s \rightharpoonup \bar{q} + \int_a^t q'(s)\,\mathrm{d}s = \tilde{q}(t)$ in E for all $t \in [a,b]$. Take $\varphi \in (L^p([a,b],E))' = L^{p'}([a,b],E')$ (see e.g. [7]). For all $t \in [a,b]$ it follows that $\langle \varphi(t), q_{n_k}(t)\rangle \to \langle \varphi(t), \tilde{q}(t)\rangle$. Moreover, since $\{q_n(a)\}_n$ is bounded and $\{q_n'\}_n$ is weakly convergent in $L^p([a,b],E)$, there exists a positive constant R such that $|\langle \varphi(t), q_{n_k}(t)\rangle| \leqslant \|\varphi(t)\|[\|q_{n_k}(a)\| + \int_a^b \|q_{n_k}'(s)\|\,\mathrm{d}s] \leqslant R\|\varphi(t)\|$. The Lebesgue dominated convergence theorem then implies that $\int_a^b \langle \varphi(s), q_{n_k}(s)\rangle\,\mathrm{d}s \to$

$\int_a^b \langle \varphi(s), \tilde{q}(s)\rangle \,\mathrm{d}s$, i.e. that $\langle \varphi, q_{n_k}\rangle \to \langle \varphi, \tilde{q}\rangle$ for all $\varphi \in (L^p([a,b],E))'$. We then get that $q_{n_k} \rightharpoonup \tilde{q}$ in $L^p([a,b],E)$ and the uniqueness of the weak limit implies that $q(t) = \tilde{q}(t) = \bar{q} + \int_a^t q'(s)\,\mathrm{d}s$ for a.a. t. It follows that $q_{n_k}(t) \rightharpoonup q(t)$ in E for all t. Now, reasoning for example as in [3, Lemma 3.2], it is possible to prove that there exists a positive constant D, independent of (t,s), such that

$$\|U_{q_{n_k}}(t,s) - U_q(t,s)\| \leqslant D \int_a^b \|A(\tau, q_{n_k}(\tau)) - A(\tau, q(\tau))\|\,\mathrm{d}\tau \to 0 \quad \text{as } k \to \infty.$$

□

3. Main result

In order to solve (1.1) we consider a linearized problem associated with it. That is, given $q \in W^{1,p}$ and $f \in \mathcal{S}_q$, we consider the linear two-point b.v.p.

$$\begin{cases} x' = A(t, q(t))x(t) + f(t) & \text{for a.a. } t \in [a,b], \\ Mx(a) + Nx(b) = 0. \end{cases} \tag{3.1}$$

Now, introducing the solution multioperator

$$T\colon\ Q \multimap W^{1,p}$$

defined by

$$T(q) = \{x_f \text{ is a solution of (3.1) for some } f \in \mathcal{S}_q\},$$

it is clear that every fixed point of T is a solution of (1.1).

The following lemma gives sufficient and necessary conditions for the existence and uniqueness of a solution of (3.1), which yield the well-posedness of the solution operator.

Lemma 3.1. *Given $A\colon [a,b] \to \mathcal{L}(E)$ Bochner integrable, denote by $U(t,s)$ the evolution operator associated with it. Then the two-point b.v.p.*

$$\begin{cases} x' = A(t)x + f(t) & \text{for a.a. } t \in [a,b], \\ Mx(a) + Nx(b) = 0 \end{cases} \tag{3.2}$$

is solvable for all $f \in L^1([a,b],E)$ if and only if $\operatorname{Im} N \subset \operatorname{Im}(M + NU(b,a))$. Moreover, denoting $\Omega = \{x \in E\colon (M + NU(b,a))x \in \operatorname{Im} N\}$, (3.2) is uniquely solvable for all $f \in L^1([a,b],E)$ if and only if $M + NU(b,a)\lfloor_\Omega$ is injective.

P r o o f. Since A is Bochner integrable, it is well known that for each $f \in L^1([a,b],E)$, the linear equation $x' = A(t)x + f(t)$ has a one parameter family of solutions given by $x_c(t) = U(t,a)c + \int_a^t U(t,s)f(s)\,\mathrm{d}s$ with $c = x(a)$ varying in E. It is easy to see that (3.2) is solvable, for every $f \in L^1([a,b],E)$, if and only if there is at least one $c \in E$ such that $Mx_c(a) + Nx_c(b) = 0$, i.e. satisfying $(M + NU(b,a))c + N\int_a^b U(b,s)f(s)\,\mathrm{d}s = 0$. This is equivalent to requiring that, for each $f \in L^1([a,b],E)$, the equation $\bigl(M + NU(b,a)\bigr)c = -N\int_a^b U(b,s)f(s)\,\mathrm{d}s$ is solvable. Since the operator $f \longmapsto \int_a^b U(b,s)f(s)\,\mathrm{d}s$ is surjective in E, this is the same as to ask that $\operatorname{Im} N \subset \operatorname{Im}(M + NU(b,a))$. Finally, when the restriction to Ω of $M + NU(b,a)$ is injective, clearly (3.2) has a unique solution. □

In the following, we assume that

$$M + NU_q(b,a) \text{ is surjective for all } q \in W^{1,p}. \tag{3.3}$$

Since $A(\cdot, q(\cdot))$ is Bochner integrable it follows from Lemma 3.1 that (3.1) has at least a solution belonging to $W^{1,p}$ for each $f \in \mathcal{S}_q$. Moreover, the solution multioperator T has the explicit formulation

$$T(q) = \left\{ \begin{array}{l} U_q(t,a)c + \displaystyle\int_a^t U_q(t,s)f(s)\,\mathrm{d}s \\ f \in \mathcal{S}_q,\ (M + NU_q(b,a))c = -N\displaystyle\int_a^b U_q(b,s)f(s)\,\mathrm{d}s \end{array} \right\}. \tag{3.4}$$

The existence of fixed points of T will be investigated by means of the well known Ky Fan fixed point theorem.

Theorem 3.1 (Ky Fan)**.** *Let X be a Hausdorff locally convex topological vector space, V a closed convex subset of X and $G\colon V \multimap V$ a compact u.s.c. multimap with convex, compact values. Then G has a fixed point.*

To apply our technique we need to estimate $\|x_f(a)\|$ for all q in $W^{1,p}$ and $f \in S_q$.

First consider the case of the linear part A independent of x, i.e. the problem

$$\begin{cases} x' \in A(t)x + F(t,x) & \text{for a.a. } t \in [a,b], \\ Mx(a) + Nx(b) = 0. \end{cases} \tag{3.5}$$

Since $M+NU(b,a)$ is surjective, the open mapping theorem implies the existence of a positive constant C such that for all $e \in E$ there exists $x \in E$ with $(M+NU(b,a))x = e$ and $\|x\| \leqslant C\|e\|$. Therefore, according to (1.4), (2.2), and (3.4), it follows that for all $q \in W^{1,p}$ and $f \in \mathcal{S}_q$, there exists a solution x_f of (3.1) satisfying

$$\|x_f(a)\| \leqslant C\|N\|\mathrm{e}^{\|\alpha\|_1}\|\alpha\|_1. \tag{3.6}$$

When A depends also on x, the open mapping theorem does not allow in general to obtain an estimate of $\|x_f(a)\|$ independent of the choice of q in $W^{1,p}$. However, if we suppose the existence of $C > 0$ such that, for all $e \in E$ and $q \in W^{1,p}$, there exists $x \in E$ satisfying

$$[M + NU_q(b,a)]x = e \quad \text{and} \quad \|x\| \leqslant C\|e\|, \tag{3.7}$$

then again for all $q \in W^{1,p}$, $f \in \mathcal{S}_q$ there exists a solution x_f of (3.1) satisfying (3.6).

Example 3.1. Let N be surjective and $M = cN$, with $c > 0$. An easy example of a linear part that leads to condition (3.3) is the following

$$A(t,x) = \gamma(t,x)I$$

where $\gamma\colon\ [a,b]\times E \to \mathbb{R}$ is a function such that $\gamma(\cdot,x)$ is measurable and $\gamma(t,\cdot)\colon\ E_\sigma \to \mathbb{R}$ is continuous. For instance $\gamma(t,x) = \alpha(t)h(x)$, with $\alpha \in L^1([a,b],\mathbb{R})$ and $h \in E'$. In fact, given $q \in W^{1,p}$, we have that

$$M + NU_q(b,a) = (c + \mathrm{e}^{\int_a^b \gamma(t,q(t))\,\mathrm{d}t})N$$

is a surjective linear operator for all $q \in W^{1,p}$. Moreover, denoted by C_N the constant given by the open mapping theorem applied to N, it is easy to prove that also condition (3.7) is satisfied with $C = C_N/c$.

Now we can state the main result of this paper.

Theorem 3.2. *Problem* (1.1) *under conditions* (1.2), (1.3), (1.4), (3.3) *and* (3.7) *is solvable.*

Proof. According to (1.4), (2.2) and (3.6), there exist two positive constants B, D such that for all $q \in W^{1,p}$ we have $\|U_q(t,s)\| \leqslant D$ for all $(t,s) \in \Delta$ and for all $f \in \mathcal{S}_q$ there exists x_f solution of (3.1) satisfying $\|x_f(a)\| \leqslant B$. Define the closed, convex and weakly compact set

$$Q = \left\{ q \in W^{1,p}\colon \begin{array}{l} \|q(t)\| \leqslant D(B + \|\alpha\|_1) \text{ for all } t \in [a,b] \text{ and} \\ \|q'(t)\| \leqslant \alpha(t)\,[D(B + \|\alpha\|_1) + 1] \ \text{ for a.a. } t \in [a,b] \end{array} \right\}$$

and put

$$\hat{T}(q) = \{x_f \in T(q)\colon\ \|x_f(a)\| \leqslant B\}.$$

According to (1.4) and (3.4), it follows that all $x_f \in \hat{T}(q)$ satisfy

$$\|x_f(t)\| \leqslant D\biggl(\|x_f(a)\| + \int_a^t \alpha(s)\,\mathrm{d}s\biggr) \leqslant D\,(B + \|\alpha\|_1)$$

for all $q \in Q$ and $f \in S_q$, and that $\|x'_f(t)\| \leqslant \alpha(t)(\|x_f(t)\| + 1)$. Hence $\hat{T}(Q_\sigma) \subset Q_\sigma$ and $\hat{T}$ is weakly compact. Since F is convex valued, it is easy to prove that the same property holds also for $\hat{T}$. If we prove that $\hat{T}$ is also closed, then $\hat{T}$ is compact valued and it is u.s.c. (see e.g. [8, Theorem 1.1.5]); hence according to the Ky Fan fixed point theorem (see Theorem 3.1), $\hat{T}$ has a fixed point which is a solution of (1.1). So it remains to prove that $\hat{T}$ is closed. Since Q and $\hat{T}(Q)$ are bounded, hence metrizable when endowed with their weak topologies, it is enough to prove the sequential closedness of $\hat{T}$. Let $q_j, q \in Q$ and $x_j \in \hat{T}(q_j)$ for each $j \in \mathbb{N}$ such that $q_j \rightharpoonup q$ and $x_j \rightharpoonup x$ in $W^{1,p}$. For each $j \in \mathbb{N}$, let $f_j(t) \in \mathcal{S}_{q_j}$ satisfy $x'_j(t) = A(t, q_j(t))x_j(t) + f_j(t)$, for a.a. $t \in [a, b]$. For the sake of simplicity define $U_{q_j} = U_j$. We have $\|U_j(t, s)\|, \|U_q(t, s)\| \leqslant D$ for all j and $(t, s) \in \Delta$ and from Lemma 2.1 we obtain the uniform convergence of a subsequence $U_{j_k}(t, s)$ to $U_q(t, s)$ in $\mathcal{L}(E)$ as $k \to \infty$ for all $(t, s) \in \Delta$. Since $\{f_j\}_j$ is bounded in $L^p([a, b], E)$, we can extract a subsequence, still denoted as the sequence, such that $f_j \rightharpoonup f$ in $L^p([a, b], E)$. Consequently, it is then easy to see that

$$\int_a^t U_{j_k}(t, s) f_{j_k}(s)\,\mathrm{d}s \rightharpoonup \int_a^t U_q(t, s) f(s)\,\mathrm{d}s \quad \text{for all } t \in [a, b].$$

This implies, in particular,

$$[M + NU_{j_k}(b, a)]x_{j_k}(a) = -N \int_a^b U_{j_k}(t, s) f_{j_k}(s)\,\mathrm{d}s \rightharpoonup -N \int_a^b U_q(t, s) f(s)\,\mathrm{d}s.$$

Moreover, for all k we have $\|x_{j_k}(a)\| \leqslant B$. Then there exists $\bar{x} \in E$ and a subsequence, still denoted as the sequence, such that $x_{j_k}(a) \rightharpoonup \bar{x}$ when $k \to \infty$. Therefore

$$[M + NU_{j_k}(b, a)]x_{j_k}(a) \rightharpoonup [M + NU_q(b, a)]\bar{x}.$$

Since $x_j \rightharpoonup x$ in $W^{1,p}$, reasoning e.g. as in Lemma 2.1 it is easy to see that $x_j(t) \rightharpoonup x(t)$ in E for all $t \in [a, b]$; the uniqueness of the weak limit then yields

$$\begin{aligned} x_{j_k}(t) = U_{j_k}(t, a)x_{j_k}(a) + \int_a^t U_{j_k}(t, s) f_{j_k}(s)\,\mathrm{d}s \\ \rightharpoonup U_q(t, a)\bar{x} + \int_a^t U_q(t, s) f(s)\,\mathrm{d}s = x(t) \end{aligned}$$

for all $t \in [a, b]$, with $[M + NU_q(b, a)]\bar{x} = -N \int_a^b U_q(b, s) f(s)\,\mathrm{d}s$ and $\|\bar{x}\| \leqslant B$. According to Mazur's convexity lemma, it is finally possible to show (see e.g. [3, Lemma 3.4]) that $f(t) \in F(t, q(t))$ for a.a. $t \in [a, b]$; hence $x \in \hat{T}(q)$ and the proof is complete. $\square$

Corollary 3.1. *Assuming that A is independent of x, problem* (1.1) *is solvable under conditions* (1.2), (1.3), (1.4), *and* (3.3).

E x a m p l e 3.2. By this example we show that we are able to treat a more general class of boundary value problems than those considered in [1] and [8]. In fact, let $P \in \mathcal{L}(E)$ be surjective but not invertible. Given $\gamma, \delta \in L^p([a,b], \mathbb{R})$, consider the b.v.p.

$$\begin{cases} x' \in \gamma(t)x + \delta(t)h(x)B & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a), \end{cases} \tag{3.8}$$

where $h\colon \mathbb{E} \to \mathbb{R}$ is defined as $h(x) = \min\{1, 1/\|x\|\}$ when $x \neq 0$, $h(0) = 1$, and $M = P + \mathrm{e}^{\int_a^b \gamma(s)\,\mathrm{d}s} I$. Since the nonlinear part $F(t,x) = \delta(t)h(x)B$ is not compact valued and $M - U(b,a) = M - \mathrm{e}^{\int_a^b \gamma(s)\,\mathrm{d}s} I = P$ is not invertible, we cannot apply the results contained in [1] and in [8] in order to solve problem (3.8). On the other hand, since h is u.s.c., we can apply Theorem 3.2 to solve it.

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