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Computing Matveev's complexity of non-orientable 3-manifolds via crystallization theory ☆

Maria Rita Casali

Dipartimento di Matematica Pura ed Applicata, Università di Modena e Reggio Emilia, Via Campi 213 B, I-41100 Modena, Italy

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Abstract

The present paper looks at *Matveev's complexity* (introduced in 1990 and based on the existence of a *simple spine* for each compact 3-manifold: see [Acta Appl. Math. 19 (1990) 101]) through another combinatorial theory for representing 3-manifolds, which makes use of particular edge-coloured graphs, called *crystallizations*.

Crystallization catalogue $\tilde{\mathcal{C}}^{26}$ for closed non-orientable 3-manifolds (due to [Acta Appl. Math. 54 (1999) 75]) is proved to yield upper bounds for Matveev's complexity of any involved 3-manifold.

As a consequence, an improvement of Amendola and Martelli classification of closed non-orientable irreducible and $\mathbb{P}^2$-irreducible 3-manifolds up to complexity $c = 6$ is obtained.

MSC: 57N20; 57M15; 57M20; 57M50

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1. Introduction

In 1990, Matveev [15] introduced an interesting notion of *complexity* for 3-manifolds, based on the existence, for each compact 3-manifold M^3, of a *simple spine*, i.e., a subpolyhedron $P \subset \operatorname{Int} M^3$ with the property that the link of each of its points can be

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E-mail address: casali@unimore.it (M.R. Casali).

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embedded in Δ (the 1-skeleton of the 3-simplex) and such that M^3 — or M^3 minus an open 3-ball, in case $\partial M^3 = \emptyset$ — collapses to P.

Definition 1 [15]. For each compact 3-manifold M^3, (*Matveev's*) *complexity* $c(M^3)$ of M^3 is defined as the minimal number of vertices (i.e., points whose link is homeomorphic to Δ) of any simple spine of M^3.

As Matveev himself points out in his foundational paper, complexity *measures how complicated a combinatorial description of the manifold must be*; moreover, additivity property and finiteness property are proved to hold for complexity function, at least within the most interesting classes of 3-manifolds (see, for example, [15] for compact orientable irreducible 3-manifolds and [16] for compact irreducible and $\mathbb{P}^2$-irreducible non-orientable 3-manifolds).

In the last 25 years, many results have been obtained, in order to classify (classes of) 3-manifolds with known complexity. In particular:

- as far as closed irreducible orientable 3-manifolds are concerned, complete classification is obtained up to complexity $c = 6$ in [15] (via computer enumeration of all possible minimal spines), and then up to complexity $c = 9$ in [16] (by means of a suitable decomposition into *bricks*, algorithmically performed with the aid of computer);
- the first attempt to classify non-orientable 3-manifolds by means of complexity is due to [1], and concerns closed irreducible and $\mathbb{P}^2$-irreducible non-orientable manifolds up to complexity $c = 6$ (by means of a purely theoretical application of brick-decomposition).

The present paper looks at Matveev's complexity from a slightly different point of view, i.e., through another combinatorial theory for representing 3-manifolds, which makes use of particular edge-coloured graphs, called *crystallizations* (see [11] or [2] for a survey on this representation theory, for PL-manifolds of arbitrary dimension).

The attention is fixed upon the whole class of closed non-orientable 3-manifolds, for which a classification in terms of crystallizations is performed in [5]:

Proposition 1 [5, Theorem 1]. *Exactly seven closed connected prime non-orientable* 3-*manifolds exist, which admit a crystallization of order* 26 *at most*: *they are the four Euclidean non-orientable* 3-*manifolds* (*denoted by* $\frac{\mathbb{E}^3}{\mathcal{B}_i}$, *for* $i \in \{1,2,3,4\}$), *the nontrivial* $\mathbb{S}^2$-*bundle over* $\mathbb{S}^1$ (*denoted by* $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1$), *the topological product between the real projective plane* $\mathbb{RP}^2$ *and* $\mathbb{S}^1$ (*denoted by* $\mathbb{RP}^2 \times \mathbb{S}^1$), *and the torus bundle*[1] $TB\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

[1] For each matrix $A \in GL(2;\mathbb{Z})$, we denote by $TB(A)$ the torus bundle over $\mathbb{S}^1$ with monodromy induced by A, i.e., the quotient $TB(A) = \frac{T\times[0,1]}{\sim_A}$, where the equivalence relation $\sim_A$ is given by $(x,0) \sim_A (\tilde{\phi}_A(x), 1)$, $\forall x \in T$, $\tilde{\phi}_A$ being the punctured homeomorphism $(T, x_0) \to (T, x_0)$ $(x_0 \in T)$ having A as an associated matrix. Note that two torus bundles $TB(A)$ and $TB(A')$ are equivalent if and only if A' is conjugate to either A or A^{-1} in $GL(2;\mathbb{Z})$. Within crystallization theory a procedure exists, which allows to construct, directly from any matrix $A \in GL(2;\mathbb{Z})$, an edge-coloured graph $\Gamma(A)$ representing the torus bundle $TB(A)$ (see [6]).

Since an algorithmic computation (easily implementable via computer) directly allows to give an estimation of Matveev's complexity $c(M^3)$ from any crystallization representing M^3, the above catalogue obviously yields upper bounds for Matveev's complexity of any involved manifold. The interesting fact is that for $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1$ and for the four Euclidean non-orientable 3-manifolds these upper bounds coincide with the precise value of complexity, as computed in [1], while $TB\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ is proved to have complexity $\leqslant 6$ (despite the statement of [1, Theorem 1.2]: see Section 3).

As a consequence, we can state the following improvement of Amendola and Martelli result:

Proposition 2.

- $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1$ *is the only closed non-orientable prime and* $\mathbb{P}^2$*-irreducible* 3*-manifold with complexity* $c = 0$.
- *No closed non-orientable irreducible and* $\mathbb{P}^2$*-irreducible* 3*-manifold admits complexity* c*, with* $1 \leqslant c \leqslant 5$.
- *The only closed non-orientable irreducible and* $\mathbb{P}^2$*-irreducible* 3*-manifolds with complexity* $c = 6$ *are the four Euclidean non-orientable* 3*-manifolds and the torus bundle* (*with geometry Sol*) $TB\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

In particular, note that $TB\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ has a non-Seifert geometry; this fact throws a new light on the comparison between geometric structures of 3-manifolds with increasing complexity, in the orientable and non-orientable case (see [1, paragraph 1]).

The analysis performed in the present paper may be likewise repeated for other existing catalogues of 3-manifolds represented via crystallizations);[2] results obtained in the non-orientable case naturally suggest the following

Open problem. It would be interesting *to find other classes of* 3*-manifolds for which Matveev's complexity may be directly computed from minimal edge-coloured graphs or,* better*, to give a characterization of the classes of* 3*-manifolds for which this happens.*

Finally, we point out that in [5], where the notion of *gem-complexity* for a closed 3-manifold M^3 was introduced, as a measure of the minimum order of a coloured graph representing M^3, it was suggested as an interesting idea *to analyze the existing relationships between Matveev's complexity and gem-complexity of closed* 3*-manifolds*.

As far as this matter is concerned, we can now make the following

Remark 1. Classification of irreducible and $\mathcal{P}^2$-irreducible non-orientable 3-manifolds up to Matveev's complexity $c = 6$ exactly coincides with classification of the same manifolds up to gem-complexity $k = 12$.

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[2] A catalogue concerning the whole class of orientable 3-manifolds is described in [14], while [2, Proposition 8.5] and [3] concern orientable 3-manifolds of Heegaard genus 2.

2. GM-complexity of (non-orientable) 3-manifolds

As already pointed out, the basic objects of crystallization theory are edge-coloured graphs, which are a representation tool for general piecewise linear (PL) manifolds, without assumptions about dimension, connectedness, orientability or boundary properties. In the present work, however, all manifolds are assumed to be closed and connected, of dimension $n = 3$; thus, we will restrict our attention to basic notions and results of the theory, dealing only with this restricted class of PL-manifolds.

Definition 2. A 4-*coloured graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular multigraph of degree four[3] and $\gamma : E(\Gamma) \to \Delta_4 = \{0, 1, 2, 3\}$ is a proper edge-coloration (i.e., $\gamma(e) \neq \gamma(f)$ for every adjacent edges $e, f \in E(\Gamma)$).

The elements of the set $\Delta_4 = \{0, 1, 2, 3\}$ are said to be *colours* of Γ; thus, for every $i \in \Delta_3$, an *i-coloured edge* is an element $e \in E(\Gamma)$ such that $\gamma(e) = i$. For every $i, j \in \Delta_4$ let $\Gamma_{\hat{i}}$ (respectively $\Gamma_{i,j}$) (respectively $\Gamma_{\hat{i},\hat{j}}$) the subgraph obtained from (Γ, γ) by deleting all edges of colour i (respectively by deleting all edges of colour $c \in \Delta_4 - \{i, j\}$) (respectively by deleting all edges of colour $c \in \{i, j\}$). The connected components of $\Gamma_{i,j}$ are said to be $\{i, j\}$-*coloured* cycles of Γ, and their number is denoted by $g_{i,j}$.

A 4-coloured graph (Γ, γ) is said to *represent* a 3-manifold M^3 if M^3 is PL-homeomorphic to $|K(\Gamma)|$, $K(\Gamma)$ being the 3-dimensional ball-complex[4] associated to (Γ, γ) by the following rules:

- for every vertex $v \in V(\Gamma)$, take a 3-ball $\sigma(v)$ abstractly isomorphic to a 3-simplex, and label injectively its four vertices by the colours of Δ_4;
- for every i-coloured edge between $v, w \in V(\Gamma)$, identify the vertices of $\sigma(v)$ and $\sigma(w)$ which are labelled by the same colour $c \in \Delta_4 - \{i\}$, and the spanned bidimensional faces.

According to [14], a 4-coloured graph (Γ, γ) representing a PL 3-manifold M^3 is also called a *gem* (=graph encoded manifold) of M^3. Moreover, it is easy to check that, in case (Γ, γ) being a gem of M^3, then M^3 results to be orientable (respectively non-orientable) iff Γ is bipartite (respectively non-bipartite).

In particular, a gem (Γ, γ) of M^3 is said to be a *crystallization* of M^3 if, for every $i \in \Delta_4$, the subgraph $\Gamma_{\hat{i}}$ is connected (or equivalently, if $K(\Gamma)$ has exactly four vertices); moreover, a crystallization is said to be *rigid* if every pair of equally coloured edges belong to one common bicoloured cycle at most.

Proposition 3 [5, Proposition 4]. *Every closed connected* 3*-manifold* M^3 *admits a rigid crystallization. Moreover, if* M^3 *is handle-free* (*i.e., it admits no connected sum*

[3] For graph theory, we refer to [18].

[4] Note that, in general, $K(\Gamma)$ fails to be a simplicial complex, since its balls may intersect in more than one face (according to [13], it may be defined to be a *pseudocomplex*); notwithstanding this, we will always call *h-simplices* its h-balls, for every $h \leqslant 3$.

decomposition, where one of the factors is an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$) and (Γ, γ) is any gem of M^3, with $\#V(\Gamma) = 2p$, then a rigid crystallization $(\overline{\Gamma}, \bar{\gamma})$ of M^3 exists, with $\#V(\overline{\Gamma}) \leqslant 2p$; in particular, the equality $\#V(\overline{\Gamma}) = 2p$ holds only if (Γ, γ) is itself a rigid crystallization of M^3.

As a consequence, a complete cataloguing of all prime orientable (respectively non-orientable) 3-manifolds may be performed by means of algorithmic construction of all possible bipartite (respectively non-bipartite) rigid crystallizations, with increasing number of vertices.

Moreover, the efficiency of the previous cataloguing may be improved through the definition of a suitable *code* (whose algorithmic calculation may be easily implemented), which allows to effectively recognize the so-called (*colour-*) *isomorphic graphs*, i.e., coloured graphs coinciding up to permutations of the vertex set and/or of the colour set: see [8] for details.

As far as the non-orientable case is concerned, the catalogue has been effectively produced and analyzed in [5] for up to 26 vertices, to reach the complete identification of all involved 3-manifolds (see [5, Proposition 7]).

As a direct consequence, the classification already stated in Proposition 1 follows.

It is well known (see [11] or [2], together with their references) that, if (Γ, γ) is a bipartite (respectively non-bipartite) crystallization of M^3, for every pair $\alpha, \beta \in \Delta_3$, there exists a regular embedding[5] $i_{\alpha,\beta} : \Gamma \to F_{\alpha,\beta}$, $F_{\alpha,\beta}$ being a closed orientable (respectively non-orientable) surface of genus $g_{\alpha,\beta} - 1$. Moreover, the surface $F_{\alpha,\beta}$, together with the images $\mathbf{x}$ (respectively $\mathbf{y}$) of all $\{\alpha, \beta\}$-coloured (respectively $\{\hat{\alpha}, \hat{\beta}\}$-coloured) cycles of (Γ, γ), but one arbitrarily chosen, yields a Heegaard diagram of M^3.

Now, if $\mathcal{D}$ (respectively $\mathcal{D}'$) is an arbitrarily chosen $\{\alpha, \beta\}$-coloured (respectively $\{\hat{\alpha}, \hat{\beta}\}$-coloured) cycle of (Γ, γ), let us denote by $\mathcal{R}_{\mathcal{D},\mathcal{D}'}$ the set of regions of $F_{\alpha,\beta} - (\mathbf{x} \cup \mathbf{y}) = F_{\alpha,\beta} - i_{\alpha,\beta}((\Gamma_{\alpha,\beta} - \mathcal{D}) \cup (\Gamma_{\hat{\alpha},\hat{\beta}} - \mathcal{D}'))$.

The following definition introduces the (purely combinatorial) notion of *Gem–Matveev complexity*, at first for a crystallization Γ of M^3, and then for any closed 3-manifold M^3. The reason of the terminology will appear clearly from the subsequent result.

Definition 3. Let M^3 be a closed 3-manifold, and let (Γ, γ) be a crystallization of M^3. With the above notations, *Gem–Matveev complexity* of Γ is defined as the non-negative integer

$$c_{GM}(\Gamma) = \min\bigl\{\#V(\Gamma) - \#\bigl(V(\mathcal{D}) \cup V\bigl(\mathcal{D}'\bigr) \cup V(\Xi)\bigr)/\mathcal{D} \in \Gamma_{\alpha,\beta},\\ \mathcal{D}' \in \Gamma_{\hat{\alpha},\hat{\beta}},\ \Xi \in \mathcal{R}_{\mathcal{D},\mathcal{D}'}\bigr\},$$

[5] The embedding of a coloured graph into a surface is said to be *regular* if the connected components split by the image of the graph onto the surface are open balls (called *regions* of the embedding) bounded by the image of bicoloured cycles. Note that this property, which holds in arbitrary dimension, is the starting point for the definition of a combinatorial PL-manifold invariant, called *regular genus*, extending the notions of genus of a surface and of Heegaard genus of a 3-manifold (see [12]). Interesting results about classification of PL-manifolds via regular genus may be found, for example, in [10,7,4,9].

while *Gem–Matveev complexity* of M^3 is defined as the minimum value of *Gem–Matveev complexity* of any minimal [6] crystallization of M^3:

$$c_{GM}\big(M^3\big) = \min\big\{c_{GM}(\Gamma)/\Gamma \text{ minimal}, \big|K(\Gamma)\big| = M^3\big\}.$$

Proposition 4. *For every closed* 3*-manifold* M^3*, Gem–Matveev complexity gives an upper bound for Matveev's complexity of* M^3:

$$c\big(M^3\big) \leqslant c_{GM}\big(M^3\big).$$

Proof. Let (Γ, γ) be a crystallization of M^3. As already stated,

$$(F_{\alpha,\beta}, \mathbf{x}, \mathbf{y}) = \big(F_{\alpha,\beta}, i_{\alpha,\beta}(\Gamma_{\alpha,\beta} - \mathcal{D}), i_{\alpha,\beta}\big(\Gamma_{\hat\alpha,\hat\beta} - \mathcal{D}'\big)\big)$$

is an Heegaard diagram for M^3. According to [15, Proposition 3], an associated simple spine P of M^3 may be obtained from any Heegaard diagram $(F_{\alpha,\beta}, \mathbf{x}, \mathbf{y})$ by considering the simple polyhedron union of $F_{\alpha,\beta}$ and the meridional discs of the two handlebodies, and then by removing the 2-component corresponding to an arbitrary region Ξ of $F_{\alpha,\beta} - (\mathbf{x} \cup \mathbf{y})$. Since the number of vertices of P obviously equals $\#(V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi))$, the existence of a simple spine for M^3 having $c \leqslant c_{GM}(\Gamma)$ vertices directly follows. □

Now, we are able to prove results about Gem–Matveev-complexity arising from catalogue $\widetilde{C}^{(26)}$ (i.e., the complete catalogue of non-orientable 3-manifolds admitting a rigid non-bipartite crystallization of order 26 at most). Since Gem–Matveev-complexity turns out to be additive, within $\widetilde{C}^{(26)}$, with respect to connected sum of 3-manifolds[7] we only fix the attention upon prime 3-manifolds.

Proposition 5.

(a) $c_{GM}(\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1) = 0$;
(b) $c_{GM}(\mathbb{RP}^2 \times \mathbb{S}^1) = 1$;
(c) $c_{GM}(\frac{\mathbb{E}^3}{\mathcal{B}_i}) = 6$, $\forall i \in \{1, 2, 3, 4\}$;
(d) $c_{GM}(TB\big(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\big)) = 6$;

Proof. Since the proof is similar for all involved 3-manifolds, we explicitly give it just for one case, i.e., case (d), concerning $M^3 = TB\big(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\big)$.

[6] Here, the notion of minimality is referred to the order of the edge-coloured graph; hence, by Proposition 3, for any handle-free 3-manifold M^3, $c_{GM}(M^3)$ is realized by a rigid crystallization of M^3.

[7] A direct calculation, possibly performed with the aid of computer, allows to easily check additivity property.

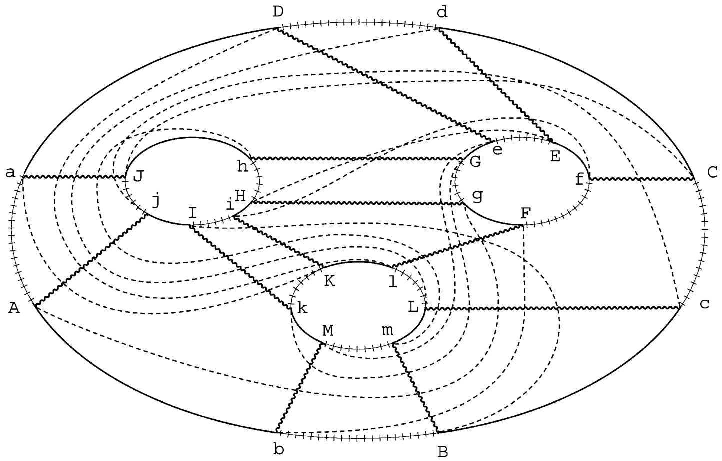

Fig. 1.

According to [5], the minimal rigid crystallization representing $TB\binom{0\ \ 1}{1\ -1}$ is the order twenty-six edge-coloured graph $\Gamma^{(209)}$ depicted in Fig. 1, whose code is

$DABCGEFJHIMKL$
$JMLEDCHGKAIFB$
$KFjLMiAmfcGDh$
$gIJlHbkEBCade.$

A direct check allows us to say that, if $\mathcal{D}$ is the $\{0,1\}$-coloured cycle containing vertices $\{a,A,b,B,c,C,d,D\}$ and $\mathcal{D}'$ is the $\{2,3\}$-coloured cycle containing vertices $\{b,M,e,D,l,F\}$, then by choosing as region Ξ the one bounded by vertices $\{C,f,g,H,i,K,L,c\}\cup\{C,J,j,c\}$, $\#V(\Gamma^{(209)})-\#(V(\mathcal{D})\cup V(\mathcal{D}')\cup V(\Xi))=6$ is obtained.

Moreover, it is easy to prove that, for any $\Gamma\in\widetilde{C}^{(26)}$ representing $TB\binom{0\ \ 1}{1\ -1}$, and for any choice of $\mathcal{D}$, $\mathcal{D}'$ and Ξ, $\#V(\Gamma)-\#(V(\mathcal{D})\cup V(\mathcal{D}')\cup V(\Xi))\geqslant 6$ holds. $\square$

3. Applications to Matveev's complexity of non-orientable 3-manifolds

As already pointed out in the introduction, the only existing result about Matveev's complexity for non-orientable 3-manifolds is due to Amendola and Martelli:

Proposition 6 [1, Theorem 1.2]. *There are no closed non-orientable irreducible and* $\mathbb{P}^2$-*irreducible* 3-*manifolds with complexity* $c\leqslant 5$ *and the only ones with complexity* $c=6$ *are the four euclidean ones.*

The above statement is clearly contradicted—via Proposition 4—by results of the previous section, in particular as far as torus bundle $TB\binom{0\ \ 1}{1\ -1}$ is concerned. Thus, the statement needs to be improved, as it appears in Proposition 2.

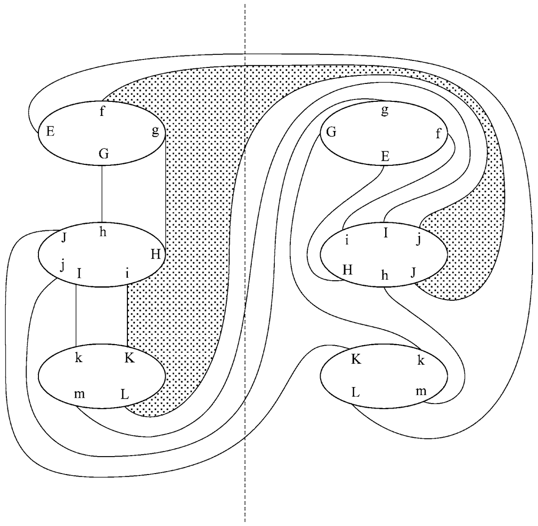

Fig. 2.

Proof of Proposition 2. Note that, as a consequence of Proposition 5(d) and Proposition 4, $c\big(TB\big(\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\big)\big) \leqslant 6$ directly follows. Actually, the original proof of [1, Theorem 1.2] fails exactly in the last line before conclusion: the statement "*... ψ is read as a matrix with trace between -2 and 2. Such a matrix is not hyperbolic, therefore M is flat*" (see [1, p. 169]) is probably based on a similar statement by Scott (see [17, p. 481], part (iii) of the proof of Theorem 5.5: *"If $|a+d|<2$, then ... the eigenvalues are distinct complex numbers and are roots of unity. It follows that A is periodic so that M admits a E^3-structure"*), but it is incorrect, as matrix $\bar{A} = \big(\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\big)$ clearly proves. On the other hand, it is easy to check that any matrix $A \in GL(2,\mathbb{Z})$ with $\det A = -1$ and trace 0 is really periodic, while any matrix $A \in GL(2,\mathbb{Z})$ taking values in the set $\{0, 1, -1\}$ (as it follows from the fact—pointed out by Amendola and Martelli—that $\psi(0), \psi(\infty) \in \{-1, 0, 1, \infty\}$), with $\det A = -1$ and trace -1 (respectively with trace 1) is conjugate to $\bar{A}$ (respectively to $(\bar{A})^{-1} = \big(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\big)$). This proves the third statement, since the associated torus bundle turns out to be either an euclidean non-orientable 3-manifold, or torus bundle $TB\big(\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\big)$. Moreover, according to [17, Theorem 5.3(i)], the fact that $\bar{A}$ is hyperbolic (i.e., neither of its eigenvalues has absolute value 1) directly implies $TB(\bar{A})$ to have geometry Sol.

As far as the first and second statements are concerned, they may be proved by Amendola and Martelli arguments (see [1]). □

Remark 2. As a consequence of our method, minimal spines for each closed non-orientable irreducible and $\mathbb{P}^2$-irreducible 3-manifold with complexity six may be constructively produced. For example, a 6-vertices spine for $TB\big(\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\big)$ is obtained—in virtue of the proof of Proposition 4—from the Heegaard diagram of Fig. 2, by removing the 2-component associated to the selected region.

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