On sharply vertex transitive 2-factorizations of the complete graph

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Abstract

We introduce the concept of a 2-starter in a group G of odd order. We prove that any 2-factorization of the complete graph admitting G as a sharply vertex transitive automorphism group is equivalent to a suitable 2-starter in G. Some classes of 2-starters are studied, with special attention given to those leading to solutions of some Oberwolfach or Hamilton-Waterloo problems.

Keywords: (Regular) cycle decomposition; complete graph; 2-factorization; Hamilton-Waterloo Problem; Oberwolfach Problem.

1 Introduction

Throughout the paper K_v will denote the complete graph on v vertices. By $V(K_v)$ and $E(K_v)$ we will respectively denote the vertex-set and the edge-set of K_v. Also, speaking of a cycle or, more generally, of a *closed trail* $A = (a_0, a_1, \ldots, a_{k-1})$, we mean the graph whose edges are $[a_i, a_{i+1}]$, $i = 0, 1, \ldots, k-1$, where the subscripts are defined (mod k).

A *cycle decomposition* $\mathcal{D}$ of K_v is a set of cycles whose edges partition $E(K_v)$ and it is obvious that its existence necessarily implies v to be odd.

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A 2-*factor* F of K_v is a set of cycles whose vertices partition $V(K_v)$. A 2-*factorization* of K_v is a set $\mathcal{F}$ of 2-factors such that any edge of K_v appears in exactly one member of $\mathcal{F}$. Hence the cycles appearing in some factor of $\mathcal{F}$ form, altogether, a cycle decomposition of K_v that we will call the *underlying* cycle decomposition of $\mathcal{F}$.

Observe that a 2-regular subgraph of K_v is a collection of disjoint cycles and it is a 2-factor if and only if its vertex-set coincides with $V(K_v)$.

A 2-factorization of K_v whose cycles all have the same length k is also called a *resolvable k-cycle decomposition of K_v*. In the case of $k = 3$ one also speaks of a *Kirkman triple system of order v* (KTS(v) for short).

Let G be an additive group of odd order v, denote by K_G the complete graph with vertex-set $V(K_G) = G$ and consider the regular action of G on $V(K_G)$ defined by $a \to a + g$, for any $(a, g) \in V(K_G) \times G$. A cycle decomposition $\mathcal{D}$ of K_G is *regular under the action of* G if we have $C + g \in \mathcal{D}$ for any $C \in \mathcal{D}$ and for any $g \in G$.

Again, if $\mathcal{F}$ is a 2-factorization of K_G, we say that $\mathcal{F}$ is *regular under the action of* G, or simply that it is *regular*, or G-*regular*, if we have $F + g \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any $g \in G$.

In some recent papers (see [6], [3], [13], [4]) regular 1-factorizations of K_G were studied for several groups G of even order, despite the fact that the existence is not guaranteed for an arbitrary group G. A regular 1-factorization of K_G has been proved to be equivalent to the concept of a *starter* in a group of even order which was introduced in [6].

In this paper we present a similar method to construct regular 2-factorizations of a complete graph. More precisely, we will introduce the definition of a 2-*starter* in a group G of odd order and we will prove that to give a G-regular 2-factorization of K_G is equivalent to give a suitable 2-starter in G.

We will also observe that a G-regular 2-factorization of K_G exists for any group G of odd order and we will lay emphasis on particular 2-factorizations, which we will call *elementary 2-factorizations*, proving some existence results.

We analyze when our constructions provide solutions to the *Oberwolfach* and to the *Hamilton-Waterloo problem*, or HW-problem for short. These problems relate to seating arrangements at a conference. The first one, [12], asks whether it is possible to seat v people (v odd) on $(v-1)/2$ days at s round tables at which there are $c_1, \ldots, c_s$ seats (with $c_1 + \ldots + c_s = v$, $c_i \geq 3$, $1 \leq i \leq s$) in such a way that each person sits next to every other person exactly once. The HW-problem asks a similar question in case the conference is held for r days at Hamilton and s days at Waterloo, with $r+s=(v-1)/2$, and where the round tables seat $a_1,\ldots,a_t$ people at Hamilton and $b_1,\ldots,b_u$ people at Waterloo (so $a_1+\ldots+a_t=b_1+\ldots+b_u=v$).

In terms of factorizations, the Oberwolfach problem asks for a 2-factorization of K_v in which each factor consists of cycles of length c_1, $\ldots$, c_s. If $c_1=\ldots=c_s=c$ and r is the number of cycles in each factor, the Oberwolfach problem is also denoted by $OP(c;r)$. It is known that $OP(c;r)$ has a solution for all $r\geq 1$ and $c\geq 3$, [2]. The HW-problem asks for a 2-factorization of K_v in which r factors consist of cycles of length $a_1,\ldots,a_t$ and s factors consist of cycles of length $b_1,\ldots,b_u$. If $a_1=\ldots=a_t=m$ and $b_1=\ldots=b_u=n$, for some integers m and n, we will denote the HW-problem by $HWP(v;m,n;r,s)$. See also [1] for a similar notation and for solutions to $HWP(v;m,n;r,s)$ for particular values of m and n.

Our Theorem 5.1 provides a cyclic solution to $HWP(18n+3;3,6n+1;3n,6n+1)$ for each positive integer n.

Some solutions are also given as a consequence of Theorem 2.6.

2 Regular 2-factorizations and 2-starters in groups of odd order

In the rest of the paper when speaking of a group G we will always understand it is additive and of odd order. Also, if H is a subgroup of G, then a system of distinct representatives for the left (respectively right) cosets of H in G will be called a *left transversal* (respectively a *right transversal*) for H in G.

Given a k-cycle $A=(a_0,a_1,...,a_{k-1})$ with vertices in G, the *stabilizer* of A under the action of G is the subgroup G_A of G defined by $G_A=\{g\in G \mid A+g=A\}$. Generalizing what was observed in some previous papers (see, e.g., section 2 of [7]) in the case of G cyclic, we have the following proposition.

Proposition 2.1 *Let $A=(a_0,a_1,...,a_{st-1})$ be a st-cycle with vertices in G and let t be the order of G_A. Then, there is an element $g\in G$ of order t such that the following condition holds:*

$$a_{i+s}-a_i=g \quad \forall\, i \tag{1}$$

or, more explicitly:

$$A = (a_0, a_1, ..., a_{s-1}, a_0 + g, a_1 + g, ..., a_{s-1} + g, ..., \\ a_0 + (t-1)g, a_1 + (t-1)g, ..., a_{s-1} + (t-1)g).$$

Conversely, if g is an element of G of order t, a sequence $A = (a_0, a_1, ..., a_{st-1})$ of st vertices of G satisfying (1) *is a st-cycle with $|G_A| = t$, if the following extra conditions are satisfied:*

- *s is the least divisor of st such that $a_{i+s} - a_i$ does not depend on i;*
- *$a_0, a_1, ..., a_{s-1}$ lie in pairwise distinct left cosets of $< g >$ in G.*

Let A be a cycle as in Proposition 2.1. We define the *list of partial differences* of A to be the multiset

$$\partial A = \pm\{a_{i+1} - a_i \,|\, 0 \leq i < s\}$$

and we set

$$\phi(A) = \{a_0, a_1, ..., a_{s-1}\}.$$

If the stabilizer of A is trivial, then ∂A coincides with ΔA, the list of differences of A in the usual sense. In this case $\phi(A) = V(A)$, the set of vertices of A. More generally, if $\mathcal{A} = \{A_1, \ldots, A_n\}$ is a collection of cycles (in particular, a 2-regular graph) with vertices in G, then we set $\partial\mathcal{A} = \partial A_1 \cup \ldots \cup \partial A_n$ and $\phi(\mathcal{A}) = \phi(A_1) \cup \ldots \cup \phi(A_n)$ (where in the union the elements have to be counted with their multiplicity).

The G-*orbit* of a cycle A is the set $Orb_G(A)$ of all *distinct* cycles in the collection $\{A + g \mid g \in G\}$. Its size (or *length*) is $|G : G_A|$, the index of the stabilizer of A under G and $Orb_G(A) = \{A + t \mid t \in T\}$ where T is a right transversal for G_A in G.

Proposition 2.2 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of cycles with vertices in G. Then $\mathcal{D} = \bigcup_{i=1}^{n} Orb_G(A_i)$ is a cycle decomposition of K_G if and only if $\partial\mathcal{A} = G - \{0\}$.*

Proof. For $i = 1, \ldots, n$, let l_i be the length of A_i and let d_i be the order of the G-stabilizer of A_i. Assume $\mathcal{D}$ is a cycle decomposition of K_G. The size of $Orb_G(A_i)$ is v/d_i so that the number $|E(K_G)| = v(v-1)/2$ of edges covered by $\mathcal{D}$ may be also expressed as $v\sum_{i=1}^{n}\frac{l_i}{d_i}$. It follows that $2\sum_{i=1}^{n}\frac{l_i}{d_i} = v-1$. Now note that the two sides of the last equality are the sizes of $\partial\mathcal{A}$ and $G-\{0\}$, respectively. So, it is enough to show that any $x \in G-\{0\}$ appears at least once in $\partial\mathcal{A}$. Given any non-zero element $x \in G$, we may claim by assumption that $[0,x]$ is an edge of $A_i + t$ for a suitable pair $(i,t) \in \{1,\ldots,n\} \times G$. It follows that $[0,x] = [a+t, b+t]$ where $A_i = (a,b,\ldots)$. This implies that $t = -a$ and hence $x = b+t = b-a \in \partial A_i \subset \partial\mathcal{A}$.

Vice versa, assume that $\partial\mathcal{A} = G-\{0\}$. So we have: $|\partial\mathcal{A}| = 2\sum_{i=1}^{n}\frac{l_i}{d_i} = v-1$, and hence: $v\sum_{i=1}^{n}\frac{l_i}{d_i} = \frac{v(v-1)}{2} = |E(K_G)|$. The left hand side of this equality gives the number of edges covered by the cycles of $\mathcal{D}$. So, to prove that each edge of K_G is covered by the cycles of $\mathcal{D}$ *exactly once*, it is sufficient to prove that this happens *at least once*. Let $[x,y]$ be an edge of K_G. By assumption, there is a suitable i such that $A_i = (a,b,\ldots)$ with $a-b = x-y$. Then we have $[x,y] = [a,b] + (-b+y)$ and we may claim that $[x,y]$ is an edge of $A_i + (-b+y) \in Orb_G(A_i) \subset \mathcal{D}$. □

In what follows we introduce a concept which makes it possible to describe algebraically any G-regular 2-factorization of K_G.

Definition 2.3 *A 2-starter in G is a collection $\Sigma = \{S_1,...,S_n\}$ of 2-regular graphs with vertices in G satisfying the following conditions:*

- $\partial S_1 \cup ... \cup \partial S_n = G-\{0\}$;
- *$\phi(S_i)$ is a left transversal for some subgroup H_i of G containing the stabilizers of all cycles of S_i, $i = 1,...,n$.*

Theorem 2.4 *The existence of a G-regular 2-factorization of K_G is equivalent to the existence of a 2-starter in G.*

Proof. Suppose $\Sigma = \{S_1,...,S_n\}$ is a 2-starter in G. By definition, for $i = 1,...,n$, there is a suitable subgroup H_i of G such that $\phi(S_i)$ is a left transversal for H_i in G. Set $F_i = \bigcup_{A\in S_i} Orb_{H_i}(A)$.

Given a cycle C, let $\ell(C)$ be its length. If $A \in S_i$, in $Orb_{H_i}(A)$ there are exactly $|H_i|/|G_A|$ cycles and each of them has length $\ell(A)$. So we have:

$$\sum_{C\in F_i} \ell(C) = \sum_{A\in S_i} \ell(A)|H_i|/|G_A|. \tag{2}$$

On the other hand, since by assumption $\phi(S_i)$ is a left transversal for H_i in G, we have

$$|\phi(S_i)| = \sum_{A\in S_i} \ell(A)/|G_A| = |G|/|H_i|. \tag{3}$$

¿From (2) and (3) we get

$$\sum_{C\in F_i} \ell(C) = |G|. \tag{4}$$

Now, observe that each $g \in G$ is vertex of at least once cycle of F_i. In fact, since $\phi(S_i)$ is a left transversal for H_i in G, we have $g = x + h$ for a suitable pair $(x, h) \in \phi(S_i) \times H_i$. So, If A is the cycle of S_i such that $x \in \phi(A)$, it is obvious that $A+h$ is a cycle of F_i and that g is a vertex of it. This, together with (4), ensures that each $g \in G$ is vertex of exactly one cycle of F_i, i.e., F_i is a 2-factor of K_G.

Consider the set of 2-factors $\mathcal{F} = Orb_G(F_1) \cup \ldots \cup Orb_G(F_n)$. We prove that $\mathcal{F}$ is a 2-factorization of K_G by proving that $\overline{\mathcal{F}}$, the underlying set of cycles of $\mathcal{F}$, is a cycle decomposition of K_G. Let $\mathcal{A}$ be the collection of cycles of Σ and observe that $Orb_G(\mathcal{A}) = \overline{\mathcal{F}}$. In fact it is obvious that $\overline{\mathcal{F}} \subset Orb_G(\mathcal{A})$ and, vice versa, if $B \in Orb_G(\mathcal{A})$ we have $B = A + g$ for some cycle $A \in \mathcal{A}$ and for some $g \in G$. Therefore $A \in S_i$ for some $S_i \in \Sigma$, and B is a cycle of $Orb_G(F_i)$. By assumption, the set Σ is a 2-starter in G and then by Proposition 2.2, $\overline{\mathcal{F}}$ is a cycle decomposition of K_G. Obviously $\mathcal{F}$ admits G as a sharply vertex transitive automorphism group.

Suppose now $\mathcal{F}$ to be a G-regular 2-factorization of K_G. Let $\{F_1, \ldots, F_n\}$ be a complete system of representatives for the G-factor-orbits of $\mathcal{F}$. For each i, denote by H_i the stabilizer in G of F_i and let S_i be a complete system of representatives for the H_i-cycle-orbits that are contained in F_i. Obviously, if A is a cycle of S_i, then H_i contains G_A and the hence stabilizer of A in H_i coincides with G_A. We prove that $\Sigma := \{S_1, \ldots, S_n\}$ is a 2-starter in G. First of all observe that $F_i = \bigcup_{A\in S_i} Orb_{H_i}(A)$ and $\mathcal{F} = Orb_G(F_1) \cup \ldots \cup Orb_G(F_n)$. Therefore, if $\mathcal{A}$ is the collection of cycles of Σ, then $Orb_G(\mathcal{A})$ is the underlying cycle decomposition $\overline{\mathcal{F}}$ of $\mathcal{F}$. By assumption $\overline{\mathcal{F}}$ is a cycle decomposition of

K_G and by Proposition 2.2 we obtain $\partial\mathcal{A} = \partial S_1 \cup \ldots \cup \partial S_n = G - \{0\}$. It remains to show that for each i, $\phi(S_i)$ is a left transversal for H_i in G. First of all, $\phi(S_i)$ has the right size since we have:

$$|G| = \sum_{A \in F_i} \ell(A) = \sum_{A \in S_i} \ell(A)|Orb_{H_i}(A)| = \sum_{A \in S_i} \ell(A)|H_i|/|G_A| \Longrightarrow$$

$$\Longrightarrow |\phi(S_i)| = \sum_{A \in S_i} \ell(A)/|G_A| = |G|/|H_i|.$$

Hence, it suffices to see that any $g \in G$ may be expressed in the form $g = x+h$ for some $(x, h) \in \phi(S_i) \times H_i$. Since F_i is a 2-factor of K_G, each element $g \in G$ is vertex of a cycle of F_i. This implies the existence of a pair $(A, h) \in S_i \times H_i$ such that g is vertex of the cycle $A + h$, say $g = a + h$, $a \in A$. On the other hand we also have $a = x + h'$ with $x \in \phi(A)$ and $h' \in G_A$ (see (1) in Proposition 2.1). Therefore, recalling that H_i contains G_A, we have $g = x+h''$ with $x \in \phi(S_i)$ and $h'' = h' + h \in H_i$. The assertion follows. $\square$

Example 2.5 *Consider the following three cycles of respective lengths* 14, 7, 7, *and with vertices in* $G = Z_{21}$*:*

$$A = (0, 7, 3, 10, 6, 13, 9, 16, 12, 19, 15, 1, 18, 4);$$

$$B = (2, 5, 8, 11, 14, 17, 20);$$

$$C = (0, 1, 3, 11, 16, 6, 12).$$

The stabilizer of A *and* B *is* $< 3 >$*, i.e., the subgroup of* G *of order* 7. *Instead,* C *has trivial stabilizer. Thus we have:*

$$\partial A = \{\pm 7, \pm 4\}; \quad \partial B = \{\pm 3\}; \quad \partial C = \{\pm 1, \pm 2, \pm 8, \pm 5, \pm 10, \pm 6, \pm 9\}.$$

$$\phi(A) = \{0, 7\}; \quad \phi(B) = \{2\}; \quad \phi(C) = \{0, 1, 3, 11, 16, 6, 12\}.$$

Now note that $\phi(A) \cup \phi(B) \equiv Z_3$ *(mod* 3*) and that* $\phi(C) \equiv Z_7$ *(mod* 7*). So, setting* $S = \{A, B\}$ *and* $T = \{C\}$*, we see that* $\Sigma = \{S, T\}$ *is a* 2*-starter in* G*.*

The base factors of the factorization $\mathcal{F}$ *generated by* Σ *are* $F_1 = \{A, B\}$ *and* $F_2 = \{C, C + 7, C + 14\}$*, and* $\mathcal{F} = Orb_G(F_1) \cup Orb_G(F_2)$*.*

The existence question for cyclic resolvable k-cycle decompositions of K_v has been solved in the case where $v = km$ with k an odd prime and all prime factors of m congruent to 1 (modulo $2k$) (see [10] and [9]). In view of our Theorem 2.4 these factorizations can be described in terms of 2-starters. For instance, the cyclic KTS($3p$) (p a prime congruent to 1 (mod 6)) constructed by Genma, Mishima and Jimbo [10] may be equivalently described as follows:

Let $G = Z_3 \oplus Z_p$ where $p = 6n + 1$ is a prime. Let ρ be a primitive root (modulo p) and for $0 \leq i \leq n - 1$ consider the 3-cycles

$$A_i = ((0, \rho^i), (0, \rho^{2n+i}), (0, \rho^{4n+i})), \quad B_i = ((0, \rho^{3n+i}), (1, \rho^{5n+i}), (2, \rho^{n+i})),$$

$$C_i = ((0, \rho^i), (1, \rho^{2n+i}), (2, \rho^{4n+i})).$$

Let S be the 2-regular graph whose cycles are $((0, 0), (1, 0), (2, 0))$, the A_i's and the B_i's. Then $\Sigma = \{S, C_0, C_1, ..., C_{n-1}\}$ is a 2-starter in G giving rise to a cyclic KTS($3p$).

The question: *For which groups G does a G-regular 2-factorization of K_G exist?* naturally arises. Despite the fact that the analogous question for groups of even order and regular 1-factorizations does not seem easy to solve, [11], [3], [6], [15], [13], the answer to our question is quite simple if no additional restriction is made. In particular a G-regular 2-factorization in which each factor is fixed by G exists as shown below.

Theorem 2.6 *For any group G of odd order, a G-regular 2-factorization of K_G exists.*

Proof. Let $G - \{0\} = X \cup -X$. For any $x \in X$ denote by t_x the order of x in G and by S_x the cycle $(0, x, \ldots, (t_x - 1)x)$. Observe that $\partial S_x = \{\pm x\}$ and that $\phi(S_x) = \{0\}$ is a left transversal for G in G. Therefore the set $\Sigma = \{S_x \mid x \in X\}$ is a 2-starter in G. □

A G-regular 2-factorization of K_G in which each factor is fixed by G, is necessarily obtained in this manner as stated in the Proposition below. We call this factorization the *natural 2-factorization* of K_G and we denote it by $\mathcal{N}(G)$.

Proposition 2.7 *Let $\mathcal{F}$ be a G-regular 2-factorization of K_G such that each factor $F \in \mathcal{F}$ is fixed by G. Then $\mathcal{F}$ is isomorphic to $\mathcal{N}(G)$.*

Proof. Let $F_1, \ldots, F_t$ be the 2-factors of $\mathcal{F}$ and let Σ be the 2-starter in G obtained by $\mathcal{F}$ as in proof of Theorem 2.4. As G fixes each factor F_i, we have $|\Sigma| = |\mathcal{F}|$. Set $\Sigma = \{S_1, \ldots, S_t\}$. As G is transitive on $V(K_G)$, each S_i is a single cycle and we have $F_i = Orb_G(S_i)$. Without loss of generality, suppose that the cycle S_i contains the edge $[0, x]$, $x \in G - \{0\}$, therefore $S_i + x \in F_i$ and $S_i + x = S_i$ as it contains the vertex x. We conclude that $S_i = (0, x, \ldots, (t_x - 1)x)$. □

Observe that the factors of $\mathcal{N}(G)$ are all possible 2-regular *Cayley graphs* of G.

As an easy consequence of the previous proposition, we have:

Proposition 2.8 *If p is an odd prime, then, up to isomorphisms, the only regular 2-factorization of K_p is $\mathcal{N}(Z_p)$.*

Proof. Let $\mathcal{F}$ be a regular 2-factorization of K_{Z_p}. The length of each factor-orbit under the action of Z_p divides p so that each factor of $\mathcal{F}$ is fixed by Z_p. The assertion follows. □

Note that $\mathcal{N}(Z_p)$ (p odd prime) is *Hamiltonian*, i.e., all its factors consist of a single p-cycle. It has been proved [8] that a cyclic Hamiltonian factorization of K_v exists for all odd values of $v \geq 3$ with the only definite exceptions of $v = 15$ and $v = p^\alpha$ with p a prime and $\alpha > 1$.

We finally point out that it has been recently proved [5] that the 2-factorizations of the complete graph admitting a 2-transitive automorphism group are, up to isomorphisms, the natural 2-factorizations associated with an elementary abelian group of odd order, i.e., those of the form $\mathcal{N}(Z_p^n)$ with p an odd prime and n a positive integer. It should be possible to derive the same result from the more general classification of doubly transitive colorings of complete graphs which was recently obtained by T.H. Sibley in [16].

3 Natural 2-factorizations with at most two non-isomorphic factors

Each factor of a natural 2-factorization is uniform, i.e., with all its cycles of the same length. It is also obvious that the number of non-isomorphic 2-factors is equal to the number of distinct orders of the non-zero elements of G.

So the natural 2-factorization of K_G provides a solution to an Oberwolfach problem exactly when G is a group of order p^n (p an odd prime) in which all the non-zero elements have order p. Apart from the elementary abelian p-group Z_p^n, non abelian groups with this property can be found for any $n \geq 3$. As an example with $n = 3$ take the group P having the following defining relations:

$$P =< a, b, c \mid a^p = b^p = c^p = 1, c^{-1}bc = ba, c^{-1}ac = a, b^{-1}ab = a > .$$

For $n > 3$, it suffices to consider the direct sum of P and Z_p^{n-3}.

If just two possible orders m, n are admissible for the non-zero elements of a finite group G, then the natural 2-factorization of K_G gives solutions to $HWP(|G|; m, n; r, s)$ for suitable integers r and s. For the reader's convenience, we prove the following proposition.

Proposition 3.1 *Let G be a finite group in which the non-zero elements have either order m or n, $m > n$. There are two possibilities:*
(i) $|G| = p^r$, *p prime; $m = p^2$, $n = p$.*
(ii) $|G| = pq^j$*; p and q primes with $q \equiv 1$ (mod p); $m = q$, $n = p$.*

Proof. Obviously it is either $m = p^2$ and $n = p$ with p a prime, or $m = q$ and $n = p$ with p and q distinct primes. If the first case occurs, then G is a p-group and (i) follows. Suppose that the second possibility holds. Then the group G has order $p^i q^j$ for suitable integers i and j. By induction on $i + j$, we prove that $i = 1$ and $q \equiv 1 \pmod p$.

If $i + j = 2$, then $|G| = pq$ and G is the Frobenius group; this necessarily implies $q \equiv 1 \pmod p$.

Let $i + j > 2$. The group G is is solvable (by the theorem of Burnside) and it is non-abelian (otherwise it should contain elements of order pq) so that G' (the derived subgroup of G) is not trivial. The group G/G' is abelian and its elements have either order p or q; this implies either $|G'| = p^i q^s$ with $s < j$, or $|G'| = p^r q^j$ with $r < i$.

Suppose $|G'| = p^i q^s$ with $s < j$. By induction, we have $q \equiv 1 \pmod p$ and $i = 1$, that is $|G| = pq^j$.

On the contrary, suppose $|G'| = p^r q^j$ with $r < i$. Then, by induction, we have $q \equiv 1 \pmod p$ and $|G'| = pq^j$. Let M be a Sylow q-subgroup of G'. By Sylow's theorem, the number of Sylow q-subgroups in G' is a divisor of p

congruent to 1 (mod q). So, as $p \not\equiv 1 \pmod{q}$, we necessarily have $M \lhd G'$. Furthermore, it is also $M \lhd G$ since each conjugate of M is still in G'. For each $x \in M$, let y be a non-zero element of $C_G(x)$, the *centralizer of x in G*. The order of y is the same as x, otherwise xy should have order pq. This implies $y \in M$ and then G is a Frobenius group with kernel M (see chapter 12 in [14]). So, if H is a complement of G, then its order is p^i. On the other hand, by [14, 12.6.15, p.356] H must be cyclic so that $i = 1$. Therefore $|G| = pq^j = |G'|$ and this possibility doesn't occur. □

Indeed a group of order pq^j, with p, q odd primes and $q \equiv 1 \pmod{p}$ exists for any j: the semidirect product $S = Z_p \cdot Z_q^j$ defined by

$$(x, y) \cdot (x', y') = (x + x', \epsilon^{x'} y + y')$$

where ϵ is a fixed primitive p-th root of unity in the field of order q^j. Observe that the natural 2-factorization of K_S is a solution to $HWP(pq^j; p, q; q^j(q - 1)/2, (q^j - 1)/2)$.

For each integer e, $1 \le e \le (r - 1)/2$, the group obtained as the direct sum of e copies of the cyclic group Z_{p^2} together with $r - 2e$ copies of the cyclic group Z_p provides a solution to $HWP(p^r; p, p^2; (p^{r-e} - 1)/2, (p^r - p^{r-e})/2)$.

As an example, the natural factorization of Z_9 gives rise to the following solution of $HWP(9; 3, 9; 1, 3)$:

$$(0, 3, 6) \quad (1, 4, 7) \quad (2, 5, 8)$$

$$(0, 1, 2, 3, 4, 5, 6, 7, 8)$$

$$(0, 2, 4, 6, 8, 1, 3, 5, 7)$$

$$(0, 4, 8, 3, 7, 2, 6, 1, 5).$$

4 Elementary 2-starters

Apart from the natural 2-factorization of K_G, many other regular 2-factorizations may be found especially when the lattice of subgroups of G is quite rich. Here we want to fix our attention on regular 2-factorizations such that all cycles of the underlying cycle decomposition have trivial stabilizer under the action of G. There are particular 2-starters, that we call *elementary*, giving rise to factorizations with this property.

We say that a 2-starter $\Sigma = \{S_1, ..., S_t\}$ in G is *elementary* if each S_i is a single cycle with trivial stabilizer. In this case we say that Σ is of type $\{d_1, ..., d_t\}$ if the length of the cycle in S_i is d_i, $i = 1, \dots, t$.

In view of Theorem 5.1, we have the following proposition.

Proposition 4.1 *Let $S_1, \dots, S_t$ be cycles with vertices in Z_v and respective lenghts d_1, ..., d_t. Then $\Sigma = \{S_1, ..., S_t\}$ is an elementary 2-starter of type $\{d_1, ..., d_t\}$ in Z_v if and only if the following conditions are satisfied:*

(a) d_h is a divisor of v for $1 \leq h \leq t$;

(b) $\displaystyle\sum_{h=1}^{t} d_h = (v-1)/2$;

(c) $\displaystyle\bigcup_{h=1}^{t} \Delta S_h = Z_v - \{0\}$;

(d) $V(S_h) = Z_{d_h} \pmod{d_h}$ for $1 \leq h \leq t$;

(e) $lcm(d_1, ..., d_t) = v$.

Proof. If Σ is an elementary starter each cycle S_i has trivial stabilizer. Hence the above relations easily follow observing that $|\phi(S_i)|$ is the length of S_i and $\partial S_i = \Delta S_i$. Condition (e) is a consequence of (c) and (d). Assume that $lcm(d_1, ..., d_t) = m < v$. In this case, by (d), the vertices of each S_h are pairwise distinct (mod m) so that m cannot appear in the list of differences $\displaystyle\bigcup_{h=1}^{t} \Delta S_h$ in contradiction with (c).

The converse is obvious by (c) and (d). □

We give an example of elementary 2-starter.

Example 4.2 *It is straightforward to check that the two cycles* $(0, 4, 14)$ *and* $(0, 1, 9, 4, 6, 12, 3)$ *form an elementary 2-starter of type* $\{3, 7\}$ *in* Z_{21}. *Here is, explicitly, the 2-factorization generated by it:*

$(0, 4, 14)$	$(3, 7, 17)$	$(6, 10, 20)$	$(9, 13, 2)$	$(12, 16, 5)$	$(15, 19, 8)$	$(18, 1, 11)$
$(1, 5, 15)$	$(4, 8, 18)$	$(7, 11, 0)$	$(10, 14, 3)$	$(13, 17, 6)$	$(16, 20, 9)$,	$(19, 2, 12)$
$(2, 6, 16)$,	$(5, 9, 19)$,	$(8, 12, 1)$,	$(11, 15, 4)$	$(14, 18, 7)$	$(17, 0, 10)$	$(20, 3, 13)$

$(0, 1, 9, 4, 6, 12, 3)$	$(7, 8, 16, 11, 13, 19, 10)$	$(14, 15, 2, 18, 20, 5, 17)$
$(1, 2, 10, 5, 7, 13, 4)$	$(8, 9, 17, 12, 14, 20, 11)$	$(15, 16, 3, 19, 0, 6, 18)$
$(2, 3, 11, 6, 8, 14, 5)$	$(9, 10, 18, 13, 15, 0, 12)$	$(16, 17, 4, 20, 1, 7, 19)$
$(3, 4, 12, 7, 9, 15, 6)$	$(10, 11, 19, 14, 16, 1, 13)$	$(17, 18, 5, 0, 2, 8, 20)$
$(4, 5, 13, 8, 10, 6, 7)$	$(11, 12, 20, 15, 17, 2, 14)$	$(18, 19, 6, 1, 3, 9, 0)$
$(5, 6, 14, 9, 11, 17, 8)$	$(12, 13, 0, 16, 18, 3, 15)$	$(19, 20, 7, 2, 4, 10, 1)$
$(6, 7, 15, 10, 12, 8, 9)$	$(13, 14, 1, 17, 19, 4, 16)$	$(20, 0, 8, 3, 5, 11, 2)$

We conjecture that (a), (b) and (e) are sufficient conditions for the existence of an elementary 2-starter of type $\{d_1, ..., d_t\}$ in Z_v.

Recall that a *partition* of an integer m is a multiset P of positive integers whose sum is m.

Conjecture. For any partition $P = \{d_1, ..., d_t\}$ of $(v-1)/2$ into proper divisors of v with $lcm(d_1, ..., d_t) = v$, there exists an elementary 2-starter of type P in Z_v.

In the following section, we give a construction for an elementary 2-starter in Z_v for each $v = 18n + 3$ and we prove the above conjecture to be true when $v = 3p$ with p an odd prime.

5 Construction for an elementary 2-starter of type $\{\underbrace{3, 3, ..., 3}_{n}, 6n+1\}$ in Z_{18n+3}

Theorem 5.1 *Let n be a positive integer and let $v = 18n+3$. An elementary 2-starter of type $\{\underbrace{3, 3, ..., 3}_{n}, 6n+1\}$ exists in Z_v.*

Proof. For the case $n = 1$ see Example 4.2. The assertion is also true for $n = 2, 3$. In fact an elementary 2-starter in Z_{39} is

$$\{(0, 13, 14), (0, 5, 28), (0, 2, 37, 29, 9, 38, 17, 10, 27, 18, 21, 6, 33)\}$$

and an elementary 2-starter in Z_{57} is

$$\{(0, 19, 20), (0, 11, 40), (0, 22, 26), (0, 2, 18, 41, 55, 42, 35, 43, 53, 1, 33, 6, 51, 45, 12, 27, 30, 9, 48)\}.$$

From now on we assume $n \geq 4$ and we identify Z_v with the direct sum $G = Z_3 \oplus Z_m$, $m = 6n + 1$. By Proposition 4.1 we have to find a set $\{T_0, T_1, ..., T_{n-1}, C\}$ of $n+1$ cycles with vertices in G satisfying the following conditions:

(i) For $i = 0, 1, ..., n-1$, the projection of $V(T_i)$ on Z_3 is bijective;

(ii) the projection of $V(C)$ on Z_m is bijective;

(iii) $\Delta T_0 \cup ... \cup \Delta T_{n-1} \cup \Delta C = G - \{0\}$.

Let $(z_1, ..., z_{n-1})$ be a *Skolem sequence* or a *hooked Skolem sequence* of order $n - 1$ (see, e.g., [15]). So we have

$$\bigcup_{i=1}^{n-1} \{z_i, z_i + i\} = \{1, 2, ..., 2n - 1\} - \{\alpha\} \tag{5}$$

with $\alpha = 2n - 2$ or $2n - 1$ according to whether we have $n \equiv 1, 2$ or $n \equiv 0, 3 \pmod 4$, respectively.

Set $T_0 = ((0, 0), (1, 0), (2, n + 2))$ and $T_i = ((0, 0), (1, i), (2, -z_i - n - 2))$ for $i = 1, ..., n - 1$. We have $\Delta T_0 = \pm(\{1\} \times \{0, n + 2, -(n + 2)\})$ and $\Delta T_i = \pm(\{1\} \times \{i, z_i + n + 2, -(z_i + i + n + 2)\})$. By (5), the $n - 1$ pairs $\{z_i + n + 2, z_i + i + n + 2\}$ cover the set $\{n + 3, n + 4, ..., 3n - 1, 3n\}$ or the set $\{n + 3, n + 4, ..., 3n - 1, 3n + 1\}$, according to whether we have $n \equiv 1, 2$ or $n \equiv 0, 3 \pmod 4$, respectively.

Then, observing that $3n + 1 = -3n \pmod m$, we may say that for each $i \in \{0, 1, ..., 3n\} - \{n, n + 1, n + 2\}$ exactly one of the two pairs $(1, i)$ and $(-1, i)$ appears in the list $\Delta \mathcal{T} = \bigcup_{i=0}^{n-1} \Delta T_i$.

Note that we may write $Z_m - \{0\} = \{\pm hn \,|\, 1 \leq h \leq 3n\}$ since we obviously have $\gcd(m, n) = 1$. Also note that $\pm 5n \equiv \mp(n + 1) \pmod m$ and that $\pm 11n \equiv \mp(n + 2) \pmod m$. In view of this, the above paragraph may be reformulated by saying that for any $h \in \{1, 2, ..., 3n\} - \{1, 5, 11\}$ exactly one of the two pairs $(1, hn)$ and $(-1, hn)$ appears in $\Delta \mathcal{T}$. So, we may define a map $f : \{hn \mid 1 \leq h \leq 3n, h \neq 1, 5, 11\} \longrightarrow \{1, -1\}$ in such a way that $f(hn)$ is the unique element of $\{1, -1\}$ such that $(f(hn), hn) \in \Delta \mathcal{T}$.

Let $(y_0, y_1, ..., y_{m-1})$ be the permutation on Z_m defined by

$$(y_0, y_1, ..., y_{10}) = (0, n, -4n, -n, 4n, 2n, -2n, 5n, -5n, 3n, -3n);$$

$$y_i = \begin{cases} n(-1)^{i+1}\lceil \frac{i}{2} \rceil & \text{for } 11 \le i < 3n+\beta \\ n(-1)^{i}\lceil \frac{i}{2} \rceil & \text{for } 3n+\beta \le i \le 6n \end{cases}$$

where $\beta = 0$ or 1 according to whether n is odd or even, respectively.

Set $\theta = -f(3n) - \sum_{i=5}^{3n} f(y_i - y_{i-1})$ and fix $\psi \in Z_3 - \{0, \theta\}$.

Consider the m-cycle $C = (c_0, c_1, ..., c_{m-1})$ with vertices in G and with $c_i = (x_i, y_i)$, the x_i's being defined by the following rules:

$$x_0 = 0; \quad x_1 = \psi; \quad x_2 = \theta; \quad x_3 = \theta - f(3n); \quad x_4 = -\psi - \theta - f(3n);$$

$$x_i = x_{i-1} - f(y_i - y_{i-1}) \quad \text{for } 5 \le i \le 3n;$$

$$x_i = 0 \quad \text{for} \quad 3n+1 \le i \le 6n.$$

We are going to show that any element of $G - \{0\}$ appears in $\Delta\mathcal{T} \cup \Delta C$. First of all, observe that $\{\pm(y_{i+1} - y_i) \mid i = 3n+1, \dots, 6n\} = \{\pm hn \mid 1 \le h \le 3n\}$. Then, considering that the x_i's are all equal to 0 for $i \ge 3n+1$, we have:

$$\{\pm(0, hn) \mid 1 \le h \le 3n\} = \{\pm(c_{i+1} - c_i) \mid i = 3n+1, \dots, 6n\}$$

Given $h \in \{1, 2, ..., 3n\} - \{1, 5, 11\}$, we have $(f(hn), hn) \in \Delta\mathcal{T}$ and

$$(-f(hn), hn) = \begin{cases} c_4 - c_5 & \text{if } h = 2; \\ c_3 - c_2 & \text{if } h = 3; \\ c_5 - c_6 & \text{if } h = 4; \\ c_9 - c_{10} & \text{if } h = 6; \\ c_7 - c_6 & \text{if } h = 7; \\ c_9 - c_8 & \text{if } h = 8; \\ c_{11} - c_{10} & \text{if } h = 9; \\ c_7 - c_8 & \text{if } h = 10; \\ (-1)^h(c_{h-1} - c_h) & \text{if } 12 \le h \le 3n \end{cases}.$$

Now observe that $c_1 - c_0 = (\psi, n)$ and $c_{3n} - c_{3n+1} = (x_{3n} - x_{3n+1}, y_{3n} - y_{3n+1}) = (x_{3n}, y_{3n} - y_{3n+1})$. With the use of the iterating formula giving x_i

for $5 \leq i \leq 3n$, we get $x_{3n} = x_4 - \sum_{i=5}^{3n} f(y_i - y_{i-1}) = x_4 + \theta + f(3n) = -\psi$. Also, checking that $y_{3n} - y_{3n+1} = n$, one obtains:

$$\{(1, n), (-1, n)\} = \{c_1 - c_0, c_{3n} - c_{3n+1}\}.$$

Moreover, since $\psi \neq \theta$, we obtain:

$$\{(1, 5n), (-1, 5n)\} = \{c_1 - c_2, c_4 - c_3\};$$

Finally:

$$(1, 11n), (-1, 11n) \in \Delta T_0.$$

Now note that $\Delta\mathcal{T} \cup \Delta C$ has size $6n + 2m = |G - \{0\}|$ and hence, by the pigeon-hole principle, we may claim that $\Delta\mathcal{T} \cup \Delta C$ covers exactly once $G - \{0\}$. The assertion follows. □

Remark. The above Theorem 5.1 proves our conjecture to be true for $v = 3p$, p an odd prime. In fact, it is immediate to check that in this case P is a partition of $(v-1)/2$ into proper divisors of v if and only if $p = 6n + 1$ and $P = \{\underbrace{3, ..., 3}_{n}, p\}$. The theorem also provides a solution to $HWP(18n + 3; 3, 6n + 1; 3n, 6n + 1)$.

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