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QQ-system and Weyl-type transfer matrices in integrable $\mathrm{SO}(2r)$ spin chains

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Abstract: We propose the full system of Baxter Q-functions (QQ-system) for the integrable spin chains with the symmetry of the D_r Lie algebra. We use this QQ-system to derive new Weyl-type formulas expressing transfer matrices in all symmetric and antisymmetric (fundamental) representations through $r + 1$ basic Q-functions. Our functional relations are consistent with the Q-operators proposed recently by one of the authors and verified explicitly on the level of operators at small finite length.

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1 Introduction

Baxter Q-operators play an important role in the theory of integrable spin chains [1], in 2D integrable quantum field theory and sigma models [2], in integrable examples of higher dimensional CFTs, such as QCD in BFKL limit [3–5], $\mathcal{N} = 4$ super Yang-Mills theory and ABJM theory [6] where Q-functional approach has led to the elegant description of spectrum of the systems in terms of the quantum spectral curve (QSC) [7, 8], the ODE/IM correspondence [9], the fermionic basis [10], stochastic processes [11, 12] and pure mathematics [13]. In particular, they provide a natural formulation for the Bethe ansatz equations (BAE) whose solutions (Bethe roots) yield the spectrum of energy for the Heisenberg-type spin chains and are at the heart of Sklyanin's separation of variables (SoV) construction [14]. They also allow for natural representations of transfer matrices (T-operators), encoding all quantum conserved charges of the system.

All these operators, T and Q, commute due to the underlying integrable structure, so that on a given eigenstate we can operate with their eigenvalues — the functions of a spectral parameter: $T(x)$ and $Q(x)$. For A-type spin chains all these operators can be built within the framework of the quantum inverse scattering method [15] from solutions of the Yang-Baxter equation. The transfer matrices are built from Lax matrices of finite dimension while, as noted in [2, 16, 17], the construction of Q-operators is related to an infinite-dimensional Hilbert space. These methods were further developed in [18–27]. For us the most relevant articles are [28–30] for Q-operators of A-type spin chains and the recent generalisation to some Q-operators of D-type spin chains [31]. An alternative approach, based on the formalism of co-derivatives [32], was proposed in [33] and further developed in [34] in relation to the interplay between quantum and classical integrability of A-type spin chains.

T-functions represent a quantum generalisation of the characters for the symmetry algebra of the spin chain. They depend on the representation f in the auxiliary space and, generically, on the twist τ — a group element introduced into the spin chain in the form of twisted, quasi-periodic boundary conditions or, alternatively, as generalized

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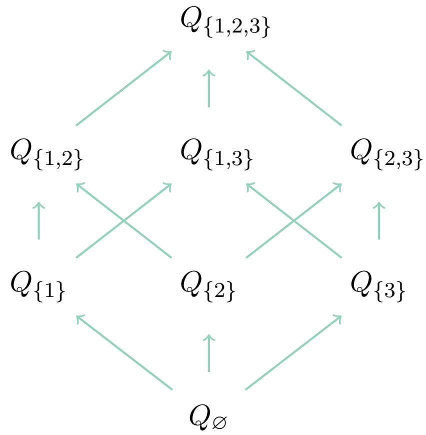

Figure 1. Hasse diagram for A_2.

"magnetic" fields. That is why we will denote the T-functions as $T_f^{(\tau)}$.[1] Generally, one has an infinite number of different T-functions since there exists an infinite number of inequivalent representations. However, most of them are not independent quantities. The most constructive way to see that is to represent T-functions in terms of Baxter Q-functions since the latter always form a finite variety. Say, for A_r algebra, the 2^{r+1} Q-functions are usually labeled by subsets of integers $I \subset \{1, 2, \dots, r, r+1\}$ where r is the rank of the algebra (for instance $I = \{1, 3, 4\} \subset \{1, 2, 3, 4, 5\}$). This QQ-system[2] can be conveniently depicted as a Hasse diagram in the shape of an $r + 1$-dimensional hypercube with the vertices labeling the corresponding Q-functions [35], see figure 1 for the example of A_2.

As we will see, in D_r algebra the labeling is similar but slightly different. Moreover, only $r + 1$ Q-functions are algebraically independent as in the A_r case [19, 36–38], see also the supersymmetric generalisation [8, 35, 39, 40]. The system of all Q-functions, which we will call here QQ-system, is endowed with a Graßmannian structure. The remaining Q-functions can thus be expressed through a chosen basis of $r + 1$ Q's by various Plücker QQ-relations, often in the form of Wronskian determinants (Casoratians).

For the Heisenberg spin chains with A_r symmetry, the most traditional representation of T-functions is given in terms of the basis of Q-functions of the type $Q_{\{1\}}, Q_{\{1,2\}}, Q_{\{1,2,3\}}, \dots, Q_{\{1,2,\dots,r+1\}}$ (or different re-labelings of the same basis), cf. figure 1. The same functions enter into the formulation of the standard nested system of BAE's. The T-functions in this basis are usually represented by the so-called tableaux formulas which are direct generalizations of Schur polynomials for characters [37, 41, 42].

The other well-known, so-called Cherednik-Bazhanov-Reshetikhin (CBR), formulas for T-functions in an arbitrary finite dimensional representation ℓ, are given in terms of determinants of T-functions in the simplest symmetric or antisymmetric representations [43, 44].

[1]Though the explicit τ dependence will often be omitted.

[2]We decided to call it QQ-system, to avoid the confusion with the "Q-system" established in the mathematical literature to denote the quadratic, Hirota-type relations for characters of "rectangular" representations. This hints on Plücker QQ-relations or on "Quantum Q"-relations.

They have been proven in [32], including the supersymmetric $A_{r|s}$ algebra. They represent the quantum generalization of the Jacobi-Trudi formulas for characters. For the D_r algebra the corresponding determinant representations have been found in [45]. For both algebras, the CBR type formulas appear to be solutions [46], with appropriate boundary conditions, of Hirota finite difference equations for T-functions [42, 47, 48] (TT-system).

The most natural representation of T-functions in terms of Q-functions, using the basis of the single-index Q's, $\{Q_{\{1\}}, Q_{\{2\}}, \dots, Q_{\{r+1\}}\}$, was constructed only for the A_r algebra [8, 19, 29, 35, 36, 40, 49–52]. Irreducible representations of A_r are labelled by highest weights $(\lambda_1, \cdots, \lambda_r) \in \mathbb{N}^r$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r$ and, assuming $Q_\emptyset(x) = 1$, the T-functions read

$$T_\lambda^{(\tau)}(x) = Q_{\{1,\dots,r+1\}}(x) \frac{\det\limits_{1\leqslant i,j\leqslant r+1} Q_{\{i\}}(x+\mu_j)}{\det\limits_{1\leqslant i,j\leqslant r+1} Q_{\{i\}}(x+r+1-j)} \equiv Q_{\{1,\dots,r+1\}}(x) \frac{\left|Q_{\{i\}}^{[2\mu_j]}\right|_{r+1}}{\left|Q_{\{i\}}^{[2(r+1-j)]}\right|_{r+1}} . \tag{1.1}$$

Here we introduced the shifted weights $\mu_j = \lambda_j + r - j + 1$ for $j = 1, \dots, r$ and $\mu_{r+1} = 0$ as well as the twist matrix $\mathrm{diag}(\tau_1, \tau_2, \dots, \tau_{r+1})$. We set $\prod_i \tau_i = 1$ to restrict to $\mathrm{SL}(r+1)$. In order to shorten the formulas we shall use the following notations throughout the article: $|M_{i,j}|_p \equiv \det\limits_{1\leqslant i,j\leqslant p} M_{i,j}$ and $M^{[k]} \equiv M(x + \frac{k}{2})$, where x is the spectral parameter. The single-index Q-functions in (1.1) are polynomials up to an exponential prefactor:

$$Q_{\{i\}}(x) = (\tau_i)^x \left(x^{m_i} + C_{i,m_i-1}\, x^{m_i-1} + \dots + C_{i,0}\right) . \tag{1.2}$$

The representation (1.1) is the direct generalization of Weyl's formula for characters:

$$\chi_\lambda^{\mathrm{SL}(r+1)}(\tau) = \frac{|\tau_i^{\mu_j}|_{r+1}}{|\tau_i^{r+1-j}|_{r+1}} . \tag{1.3}$$

It is clear that (1.1) behaves as $Q_{\{1,\dots,r+1\}}(x)\chi_\lambda^{\mathrm{SL}(r+1)}(\tau)$ in the "classical" limit $x \to \infty$.

The goal of this article is to construct a similar QQ-system, together with a similar Weyl-type representation for T-matrices, for the D_r algebra. The standard Weyl formula for D_r characters is

$$\chi_f^{\mathrm{SO}(2r)}(\tau) = \frac{|\tau_i^{\ell_j} + \tau_i^{-\ell_j}|_r + |\tau_i^{\ell_j} - \tau_i^{-\ell_j}|_r}{|\tau_i^{r-j} + \tau_i^{-r+j}|_r} , \tag{1.4}$$

see e.g. [53–56], with $\ell_j = f_j + r - j$ and the highest weights $f_1 \geqslant f_2 \geqslant \dots \geqslant f_{r-1} \geqslant |f_r|$ are all integers or all half-integers (the last one can also be negative).

However, in general, the situation for D_r is more complicated than for A_r algebra. The representations of the Lie algebra do not "quantize" trivially, i.e. cannot be lifted to the Yangian algebra (apart from the symmetric and spinorial representations), see [57] and [58] for an instructive example. Instead, in order to construct the T-functions, one has to introduce the representations acting in the so-called Kirillov-Reshetikhin modules [59]. Such modules are known only for rectangular representations (a, s), see appendix A. These representations have highest weights $f_1 = f_2 = \dots = f_a = s$ and $f_{a+1} = \dots = f_r = 0$

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for $a \leqslant r-2$ and $f_1 = \ldots = f_{r-1} = \pm f_r = s/2$ for $a = \pm$. The Kirillov-Reshetikhin characters are linear combinations of the above mentioned Weyl characters. The symmetric and spinorial characters in Kirillov-Reshetikhin representation are not different from the Weyl characters (1.4) but in other representations they do differ.

The generating function for characters in symmetric representations reads

$$K_s(t, \{\tau_i\}) = \frac{1-t^2}{\prod_{i=1}^{r}\left(1 - t(\tau_i + 1/\tau_i) + t^2\right)} = \sum_{k=0}^{\infty} t^k \chi_k(\tau) \tag{1.5}$$

so that they coincide with standard Weyl characters (1.4). On the contrary, the KR-characters for totally antisymmetric representations already differ from usual D_r Weyl characters (1.4). The generating function of KR characters for these representations reads

$$K_a(t, \{\tau_i\}) = \frac{\prod_{i=1}^{r}\left(1 - t(\tau_i + 1/\tau_i) + t^2\right)}{1-t^2} = \sum_{k=0}^{r-2} t^k \Psi_k(\tau) + \ldots \tag{1.6}$$

where only the coefficients of t^k, up to t^{r-2} term, give the KR type antisymmetric characters $\Psi_{a=k}(\tau)$.

In this work, we propose a QQ-system, appropriate for the D_r algebra, and discuss the corresponding Hasse diagram. We will also introduce new QQ$'$-type conditions. From either the QQ-system or these QQ$'$-type conditions, one can derive new Weyl-type formulas for T-functions in the symmetric and antisymmetric representations of Kirillov-Reshetikhin modules: T-functions are then given in terms of ratios of determinants involving a basic set of $r+1$ Q-functions, generalizing the classical Weyl-type formulas (1.5) and (1.6). In particular, we will show that these Weyl-type formulas are consistent with the tableau sum formulas for T-functions. The QQ-relations and QQ$'$-type conditions were checked using explicit expressions for T and Q-operators found in [31] at small lengths of the spin chain.

Note added. Shortly after this preprint appeared on the arXiv, the preprint [60] with partially overlapping results on QQ-systems for D_r algebra was posted on the arXiv.

2 Lax matrix construction and eigenvalues of T-operators

We start by introducing the fundamental R-matrix of $\mathfrak{so}(2r)$ which was written down in [61]. It is a matrix of size $(2r)^2 \times (2r)^2$ and it reads

$$R(x) = x(x+\kappa)\mathrm{I} + (x+\kappa)\mathrm{P} - x\mathrm{Q}\,. \tag{2.1}$$

Here $\kappa = r-1$, the letter I denotes the identity matrix while the permutation P and the trace operator Q are defined by

$$\mathrm{P} = \sum_{i,j=1}^{2r} E_{ij} \otimes E_{ji}\,, \qquad \mathrm{Q} = \sum_{i,j=1}^{2r} E_{i,j} \otimes E_{i',j'}\,. \tag{2.2}$$

The elementary $2r \times 2r$ matrices E_{ij} obey the standard relations $E_{ij}E_{kl} = \delta_{jk}E_{il}$. We use the notation $i' = 2r - i + 1$. The R-matrix in (2.1) is related by a similarity transformation to the one originally obtained in [61], cf. [31], and generates the extended Yangian $X(\mathfrak{so}(2r))$ [62]. It is invariant under transformations

$$[R(x), B \otimes B] = 0\,, \tag{2.3}$$

if B satisfies the orthogonality condition $BB' = \theta\mathrm{I}$ with $\theta \in \mathbb{C}$ and $B'_{ij} \equiv B_{j'i'}$.

2.1 Transfer matrix construction for first fundamental

In the following we focus on spin chains of length N with the defining representation at each site. The quantum space of the spin chain is

$$V = \mathbb{C}^{2r} \otimes \ldots \otimes \mathbb{C}^{2r}\,. \tag{2.4}$$

The R-matrix (2.1) allows to construct the fundamental transfer matrix $T = T_{1,1}$, i.e. with the defining representation in auxiliary space, which contains the Hamiltonian of the spin chain. It is also convenient to introduce the symmetric generalisations $T_{1,s}$ at this point. The required Lax matrix was given in [63]. It reads

$$\mathcal{L}(x) = x^2\mathrm{I} + x\sum_{i,j=1}^{2r} J_{ij} \otimes E_{ji} + \sum_{i,j=1}^{2r} G_{ij} \otimes E_{ji}\,. \tag{2.5}$$

with

$$G_{ij} = \frac{1}{2}\sum_{k=1}^{2r} J_{kj}J_{ik} + \frac{\kappa}{2}J_{ij} - \frac{1}{4}\left((\kappa-1)^2 + 2\kappa s + s^2\right)\delta_{ij}\,. \tag{2.6}$$

Here we introduce the generators J_{ij} of $\mathfrak{so}(2r)$ obeying the commutation relations

$$[J_{ij}, J_{kl}] = \delta_{jk}J_{il} - \delta_{i'k}J_{j'l} - \delta_{jl'}J_{ik'} + \delta_{il}J_{j'k'}\,, \tag{2.7}$$

with $J_{ij} = -J_{j'i'}$. We stress that the formula for the Lax matrix only holds for symmetric representations with generators acting on the highest weight state $|\mathrm{hws}\rangle$ as follows

$$J_{ij}|\mathrm{hws}\rangle = 0\,, \quad \text{for} \quad i<j\,, \qquad J_{ii}|\mathrm{hws}\rangle = s\delta_{1i}|\mathrm{hws}\rangle \tag{2.8}$$

where $s \in \mathbb{N}$ for finite dimensional representations. The generators in such representation satisfy the characteristic identity

$$\sum_{j,k=1}^{2r} (J_{ij} - \delta_{ij})\,(J_{jk} + s\delta_{jk})(J_{kl} - (s+2\kappa)\delta_{kl}) = 0\,, \tag{2.9}$$

which is needed in order to satisfy the Yang-Baxter equation, see also [64] for a recent discussion of such constraints. A realisation of the generators J_{ij} for general s in terms of oscillators can be found in [31]. The defining representation $s=1$ can be realised via

$$J_{ij} = E_{ij} - E_{j'i'}\,. \tag{2.10}$$

We recover the R-matrix $\mathcal{L}(x) = R(x - \frac{\kappa}{2})$.

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The first space in (2.5) with generators J_{ij} serves as our auxiliary space and the quantum space is built from N copies of the second one with matrix elements E_{ij}. The transfer matrix constructed from this monodromy is defined via

$$T_{1,s}(x) = \operatorname{tr} \mathcal{D}\mathcal{L}_1(x)\mathcal{L}_2(x)\cdots\mathcal{L}_N(x) \tag{2.11}$$

where $\mathcal{L}_i(x)$ denotes the Lax matrix acting non-trivially on the ith spin chain site and the trace is taken over the representation with generators J_{ij}. We further introduced a diagonal twist

$$\mathcal{D} = \prod_{k=1}^{r} \tau_k^{J_{kk}}\,, \tag{2.12}$$

with the parameters $\tau \in \mathbb{C}^r$ that we already encountered in the definition of characters. Some symmetries of the transfer matrix constructed via (2.11) can be found in appendix C.1.

The Hamiltonian of the spin chain is obtained from the fundamental transfer matrix T by taking the logarithmic derivative at the permutation point

$$H = \frac{\partial}{\partial x}\ln T(x)\Big|_{x=\frac{\kappa}{2}} = \sum_{i=1}^{N}\mathcal{H}_{i,i+1}\,. \tag{2.13}$$

The Hamiltonian density is obtained from the logarithmic derivative of the R-matrix at the permutation point and it reads

$$\mathcal{H}_{i,i+1} = \kappa^{-1}\left(\mathrm{I} - \mathrm{Q} + \kappa\mathrm{P}\right)_{i,i+1} \tag{2.14}$$

and $\mathcal{D}_N$ the twist (2.12) at site N enters via $\mathcal{H}_{N,N+1} = \mathcal{D}_N\mathcal{H}_{N,1}\mathcal{D}_N^{-1}$. We also remind the reader that $\kappa = r - 1$.

2.2 Diagonalisation of fundamental transfer matrix

As discussed at the end of the previous section, the fundamental transfer matrix $T = T_{1,1}$ with $s = 1$ contains the nearest-neighbour Hamiltonian and higher local charges. It has been diagonalised in [63, 65] using the algebraic Bethe ansatz, see also [66] for a different nesting procedure and [67] for the trigonometric case. One of the key observations is that the transfer matrix can be written as

$$T(x) = T_+(x) + T_-(x) \tag{2.15}$$

where the two terms are related via

$$T_\pm^t(-x)\big|_{\tau_i\to\tau_i^{-1}} = T_\mp(x)\,. \tag{2.16}$$

We note that the twist only slightly modifies the derivation of the spectrum of the transfer matrix in [63, 65]. Following the same logic as in the references above we find the contributions of $T_\pm$ to the eigenvalues of the transfer matrix

$$T_\pm(x) = q_0^{[1-r]}q_0^{[r-1]}\sum_{k=1}^{r}\tau_k^{\mp 1}\frac{q_{k-1}^{[\pm(k-r+2)]}}{q_{k-1}^{[\pm(k-r)]}}\frac{q_k^{[\pm(k-r-1)]}}{q_k^{[\pm(k-r+1)]}}\,. \tag{2.17}$$

with the notation $q^{[k]} \equiv q(x + \frac{k}{2})$. In (2.17) above we introduced the Q-functions along the tail of the Dynkin diagram, cf. figure 2. This equation is valid on the level of operators. In the diagonal form the Q-functions are written in terms of the Bethe roots $x_i^{(j)}$ at level $j \in \{1, 2, \ldots, r-2, +, -\}$ corresponding to the nodes of the Dynkin diagram as given in figure 2. The index i takes values $i \in \{1, 2, \ldots, m_j\}$. Here m_j denotes the magnon numbers $\vec{m} = (m_1, \ldots, m_{r-2}, m_+, m_-)$. They are determined for a given state labelled by weight vector $\vec{n}$ via

$$\vec{n} = \begin{pmatrix} 2m_1 - m_0 - m_2 \\ \vdots \\ 2m_{r-3} - m_{r-4} - m_{r-2} \\ 2m_{r-2} - m_{r-3} - m_+ - m_- \\ 2m_+ - m_{r-2} \\ 2m_- - m_{r-2} \end{pmatrix} \tag{2.18}$$

where $m_0 = N$ is the length of the spin chain, see [63] and $n_i = f_i - f_{i+1}$ for $1 \leqslant i < r$ and $n_r = f_{r-1} + f_r$. The first Q-functions along the tail of the Dynkin diagram are then given by

$$q_0(x) = x^N \,, \qquad q_i(x) = \prod_{j=1}^{m_i} (x - x_j^{(i)}) \,, \quad 1 \leqslant i \leqslant r-2 \,. \tag{2.19}$$

Here q_0 does not depend on any Bethe roots and plays a role similar to that of the A-type full set Q-function. The last two Q-functions factorise:

$$q_{r-1} = s_+ s_- \,, \qquad q_r = s_+^{[+1]} s_+^{[-1]} \,, \tag{2.20}$$

where $s_\pm$ are the Q-functions that correspond to the spinorial nodes. They are polynomials of degree $m_\pm$ in the spectral parameter

$$s_\pm(x) = \prod_{i=1}^{m_\pm} (x - x_i^{(\pm)}) \,. \tag{2.21}$$

It immediately follows that the last term in (2.17) reduces to the more familiar form

$$\frac{q_{r-1}^{[\pm 2]}}{q_{r-1}^{[0]}} \frac{q_r^{[\mp 1]}}{q_r^{[\pm 1]}} = \frac{s_-^{[\pm 2]} s_+^{[\mp 2]}}{s_- s_+} \,. \tag{2.22}$$

From the definition of the Hamiltonian (2.13) and the eigenvalue equation (2.15) of the transfer matrix we obtain the energy formula. The eigenvalues of the Hamiltonian are parametrised by the Bethe roots and read

$$E = \frac{r}{r-1} N + \frac{q_1'\left(-\frac{1}{2}\right)}{q_1\left(-\frac{1}{2}\right)} - \frac{q_1'\left(\frac{1}{2}\right)}{q_1\left(\frac{1}{2}\right)} = \frac{r}{r-1} N - \sum_{k=1}^{m_1} \left(\frac{1}{x_k^{(1)} + \frac{1}{2}} - \frac{1}{x_k^{(1)} - \frac{1}{2}} \right) \,, \tag{2.23}$$

cf. [63]. As for the first fundamental representation of A-type, the energy eigenvalues only depend on the Bethe roots at the first nesting level.

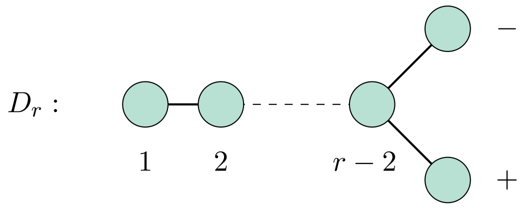

Figure 2. Dynkin diagram for D_r Lie algebra.

3 QQ-relations from Bethe ansatz equations

The Bethe equations can be read off from the eigenvalue equation of the transfer matrix

$$T(x) = q_0^{[1-r]} q_0^{[r-1]} \sum_{k=1}^{r} \left[\tau_k^{-1} \frac{q_{k-1}^{[k-r+2]}}{q_{k-1}^{[k-r]}} \frac{q_k^{[k-r-1]}}{q_k^{[k-r+1]}} + \tau_k \frac{q_{k-1}^{[r-k-2]}}{q_{k-1}^{[r-k]}} \frac{q_k^{[r-k+1]}}{q_k^{[r-k-1]}} \right] , \tag{3.1}$$

which is obtained by combining (2.15) and (2.17). When demanding that the transfer matrix is regular and Bethe roots are distinct the Bethe equations arise as pole cancellation conditions. They are conveniently written in terms of Q-functions as

$$\begin{aligned} -\frac{\tau_{k+1}}{\tau_k} &= \left(\frac{q_{k-1}^{[-1]}}{q_{k-1}^{[+1]}} \frac{q_k^{[+2]}}{q_k^{[-2]}} \frac{q_{k+1}^{[-1]}}{q_{k+1}^{[+1]}} \right)_k , \qquad (k = 1, 2, \ldots, r-3) \\ -\frac{\tau_{r-1}}{\tau_{r-2}} &= \left(\frac{q_{r-3}^{[-1]}}{q_{r-3}^{[+1]}} \frac{q_{r-2}^{[+2]}}{q_{r-2}^{[-2]}} \frac{s_+^{[-1]}}{s_+^{[+1]}} \frac{s_-^{[-1]}}{s_-^{[+1]}} \right)_{r-2} , \\ -\frac{1}{\tau_{r-1}\tau_r} &= \left(\frac{q_{r-2}^{[-1]}}{q_{r-2}^{[+1]}} \frac{s_+^{[+2]}}{s_+^{[-2]}} \right)_+ , \\ -\frac{\tau_r}{\tau_{r-1}} &= \left(\frac{q_{r-2}^{[-1]}}{q_{r-2}^{[+1]}} \frac{s_-^{[+2]}}{s_-^{[-2]}} \right)_- , \end{aligned} \tag{3.2}$$

where $(\ldots)_k$ with $1 \leqslant k \leqslant r-2$ indicates that the expression is taken at a root of q_k and $(\ldots)_\pm$ at a root of $s_\pm$.

Along the tail of the Dynkin diagram, cf. figure 2, we induce the standard A_n type Plücker QQ-relation

$$\frac{\tau_k - \tau_{k+1}}{\sqrt{\tau_k \tau_{k+1}}} q_{k-1} q_{k+1} = \sqrt{\frac{\tau_k}{\tau_{k+1}}} q_k^+ \widetilde{q}_k^- - \sqrt{\frac{\tau_{k+1}}{\tau_k}} q_k^- \widetilde{q}_k^+ \tag{3.3}$$

where q_k and $\widetilde{q}_k$ are two different Q-functions at the same level of the Hasse diagram, see section 5 for that details. The form of the eigenvalue equation (3.1) is unchanged by such transformation. The Bethe ansatz equations can be restored by shifting the argument of the QQ-relation by $\pm 1/2$, evaluating it at a root of q_k and dividing one of the resulting equations by the other. At the fork of the Dynkin diagram, $(r-2)th$ node, the QQ-relation takes the form

$$\frac{\tau_{r-2} - \tau_{r-1}}{\sqrt{\tau_{r-2}\tau_{r-1}}} q_{r-3}\, s_+\, s_- = \sqrt{\frac{\tau_{r-2}}{\tau_{r-1}}} q_{r-2}^+ \widetilde{q}_{r-2}^- - \sqrt{\frac{\tau_{r-1}}{\tau_{r-2}}} q_{r-2}^- \widetilde{q}_{r-2}^+ \,. \tag{3.4}$$

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At the spinorial nodes $\pm$, the QQ-relations are

$$\frac{\tau_{r-1}\tau_r - 1}{\sqrt{\tau_{r-1}\tau_r}} q_{r-2} = \sqrt{\tau_{r-1}\tau_r} s_+^+ \tilde{s}_+^- - \frac{1}{\sqrt{\tau_{r-1}\tau_r}} s_+^- \tilde{s}_+^+ , \tag{3.5}$$

$$\frac{\tau_{r-1} - \tau_r}{\sqrt{\tau_{r-1}\tau_r}} q_{r-2} = \sqrt{\frac{\tau_{r-1}}{\tau_r}} s_-^+ \tilde{s}_-^- - \sqrt{\frac{\tau_r}{\tau_{r-1}}} s_-^- \tilde{s}_-^+ . \tag{3.6}$$

These QQ-relations for spinorial nodes have appeared in [68] in relation to the ODE/IM correspodence [9] and recently in [69]. In section 5 we propose a more general version of the QQ-relations.

4 Basic (extremal) Q-functions

A construction of the Q-operators corresponding to the extremal nodes of the Dynkin diagram, cf. figure 2, was recently proposed in [31]. The latter construction was inspired by the isomorphism $A_3 \simeq D_3$, admits the expected asympotic behavior (2.18) and has been checked by showing some functional relations of $r = 4$ in some examples of finite length. All functional relations in the following sections are consistent with the proposed Q-operators and have been verified explicitly for several examples of finite length.

4.1 Q-operator construction for first fundamental

We construct $2r$ Q-operators Q_i with $1 \leqslant i \leqslant 2r$ corresponding to the first fundamental node. The Lax matrix needed is of the size $2r \times 2r$ with oscillators as entries and its leading order in the spectral parameter is quadratic. It reads

$$L(z) = \left(\begin{array}{c|c|c} z^2 + z(2 - r - \bar{\mathbf{w}}\mathbf{w}) + \frac{1}{4}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J}\mathbf{w} & z\bar{\mathbf{w}} - \frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t \\ \hline -z\mathbf{w} + \frac{1}{2}\mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J}\mathbf{w} & z\mathrm{I} - \mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J} & -\mathrm{J}\bar{\mathbf{w}}^t \\ \hline -\frac{1}{2}\mathbf{w}^t\mathrm{J}\mathbf{w} & \mathbf{w}^t\mathrm{J} & 1 \end{array}\right) . \tag{4.1}$$

The Lax matrix above contains $2(r-1)$ oscillators arranged into the vectors $\bar{\mathbf{w}}$ and $\mathbf{w}$ as follows

$$\bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r'}, \ldots, \bar{\mathbf{a}}_{2'}) , \qquad \mathbf{w} = (\mathbf{a}_2, \ldots, \mathbf{a}_r, \mathbf{a}_{r'}, \ldots, \mathbf{a}_{2'})^t . \tag{4.2}$$

They obey the standard commutation relations

$$[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_{ij} . \tag{4.3}$$

The matrix J is given in (C.2). The Q-operator Q_1 is defined as the regularised trace over the monodromy of the Lax matrices (4.1) which is constructed by taking the N-fold tensor product in the matrix space and multiplying in the auxiliary oscillator space:

$$Q_1(x) = \tau_1^x \,\hat{\mathrm{tr}} \left[D L^{[-1]} \otimes L^{[-1]} \otimes \ldots \otimes L^{[-1]} \right] . \tag{4.4}$$

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The twist matrix D in the auxiliary space depends on the parameters τ_i, cf. (2.12) for the transfer matrix. In the case of the Q-operator Q_1 it reads

$$D = \prod_{i=2}^{r} \left(\tau_i \tau_1^{-1}\right)^{\mathbf{N}_i} \left(\tau_i^{-1} \tau_1^{-1}\right)^{\mathbf{N}_{i'}} , \tag{4.5}$$

with the number operator $\mathbf{N}_i = \bar{\mathbf{a}}_i \mathbf{a}_i$. The trace is defined as

$$\hat{\mathrm{tr}}(DX) = \frac{\mathrm{tr}(DX)}{\mathrm{tr}(D)} . \tag{4.6}$$

By construction of the Q-operators, Q_1 commutes with the transfer matrices defined previously. The Q-operator for the case $N = 1$ is spelled out explicitly in appendix B.

From Q_1 we define the remaining $2r - 1$ Q-operators at the first fundamental node. For that we introduce the transformation

$$\tilde{B}_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{r} (E_{k',k'} + E_{k,k}) + E_{i',j'} + E_{j',i'} + E_{i,j} + E_{j,i} , \tag{4.7}$$

with $1 \leqslant i \neq j \leqslant r$. It belongs to the class of transformations discussed in (2.3) and commutes with the R-matrix. It follows that the Q-operators defined via

$$Q_i(x) = (\tilde{B}_{1,i} \otimes \ldots \otimes \tilde{B}_{1,i})\, Q_1(x) (\tilde{B}_{1,i} \otimes \ldots \otimes \tilde{B}_{1,i})\Big|_{\tau_1 \leftrightarrow \tau_i} , \qquad i = 2, \ldots, r \tag{4.8}$$

and

$$Q_i(x) = (\mathrm{J} \otimes \ldots \otimes \mathrm{J})\, Q_{i'}(x)\, (\mathrm{J} \otimes \ldots \otimes \mathrm{J})\big|_{\tau_i \to \tau_i^{-1}} , \qquad i = r+1, \ldots, 2r \tag{4.9}$$

also belong to the family of commuting operators. This defines $2r$ Q-operators

$$\{Q_1, Q_2, \ldots, Q_{2r}\} . \tag{4.10}$$

Up to the exponential prefactor, we identify the q-function q_1 with the eigenvalues of the Q-operator Q_1. Here we could have chosen any other single-index Q.

4.2 Q-operator construction for spinor representations

Similarly we proceed for the Q-operators corresponding to the spinorial nodes $\pm$ of the Dynkin diagram in figure 2. Here the Lax matrix is a 2×2 block matrix with block size $r \times r$. It reads

$$\check{L}(x) = \left(\begin{array}{c:c} x\,\mathrm{I} + \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hdashline \mathbf{A} & \mathrm{I} \end{array}\right) , \tag{4.11}$$

and contains $\frac{r(r-1)}{2}$ pairs of oscillators $[\mathbf{a}_{i,j}, \bar{\mathbf{a}}_{k,l}] = \delta_{il}\delta_{jk}$. The submatrices $\bar{\mathbf{A}}$ and $\mathbf{A}$ are of the form

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 0 \\ \vdots & \iddots & 0 & -\bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & 0 & \iddots & \vdots \\ 0 & -\bar{\mathbf{a}}_{r-1,r'} & \cdots & -\bar{\mathbf{a}}_{1,r'} \end{pmatrix} , \quad \mathbf{A} = \begin{pmatrix} -\mathbf{a}_{r',1} & \cdots & -\mathbf{a}_{r',r-1} & 0 \\ \vdots & \iddots & 0 & \mathbf{a}_{r',r-1} \\ -\mathbf{a}_{2',1} & 0 & \iddots & \vdots \\ 0 & \mathbf{a}_{2',1} & \cdots & \mathbf{a}_{r',1} \end{pmatrix} . \tag{4.12}$$

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Similar as before we define the Q-operator as the trace of the monodromy built out of the Lax matrix $\check{L}$ above as

$$S(x) = (\tau_1 \cdots \tau_r)^{\frac{x}{2}} \,\hat{\mathrm{tr}} \left[\check{D}\, \check{L}^{[1-r]} \otimes \check{L}^{[1-r]} \otimes \ldots \otimes \check{L}^{[1-r]}\right] . \tag{4.13}$$

Here we introduced the twist in the auxiliary space via

$$\check{D} = \prod_{1\leqslant i<j\leqslant r} (\tau_i\tau_j)^{\bar{\mathbf{a}}_{i,j'}\mathbf{a}_{j',i}} . \tag{4.14}$$

The remaining Q-operators at the spinorial nodes are obtained through the similarity transformation

$$B(\vec{\alpha}) = \frac{1}{2}\sum_{i=1}^{r} \left((1+\alpha_i)(E_{i',i'}+E_{i,i}) + (1-\alpha_i)(E_{i',i}+E_{i,i'})\right) , \tag{4.15}$$

with $\alpha_i = \pm 1$, that commutes with the R-matrix, cf. (2.3), and subsequently inverting the twist parameters. For $\alpha_i = 1$ the matrix $B(\vec{\alpha})$ reduces to the identity. We define

$$S_{\vec{\alpha}}(x) = (B(\vec{\alpha}) \otimes \ldots \otimes B(\vec{\alpha}))S(x)(B(\vec{\alpha}) \otimes \ldots \otimes B(\vec{\alpha}))|_{\tau_i \to \tau_i^{\alpha_i}} , \tag{4.16}$$

labelled by $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_i = \pm 1$. By construction the 2^r operators $S_{\vec{\alpha}}$ commute with one another. We choose to identify $s_\pm$ with $S_{(+1,\ldots,+1,\pm 1)}$ up to the exponential prefactor.

5 The QQ-system for D_r

In this section we introduce the QQ-system. It has been verified at small finite length using the construction [31] that was reviewed in section 4. In total we have $3^r - 2^{r-1}r + 2$ Q-functions, see figure 4 and figure 5 for $r = 3, 4$ examples.

The QQ-relations along the tail of the Dynkin diagram have a structure similar to those for A_r but the labeling of single-index functions is different. We shall say that a subset I of $\{1, \ldots, 2r\}$ is acceptable if for all $1 \leqslant k \leqslant r$, the integers k and $k' = 2r - k + 1$ do not both belong to I. In particular, an acceptable set cannot have more than r elements: $|I| \leqslant r$. A Q-function Q_I is associated to each acceptable I and these functions satisfy the relations

$$Q^{[+1]}_{J\cup\{i\}} Q^{[-1]}_{J\cup\{j\}} - Q^{[-1]}_{J\cup\{i\}} Q^{[+1]}_{J\cup\{j\}} = \frac{\tau_i - \tau_j}{\sqrt{\tau_i\tau_j}} Q_J Q_{J\cup\{i,j\}} \tag{5.1}$$

where $\tau_i = \tau_{i'}^{-1}$ for $i > r$, $\{i, i'\} \cap \{j, j'\} = \emptyset$, J is acceptable of order at most $r-2$ and does not contain i, i', j or j'. We have excluded here the case where k and k' are contained in the same set as the Q-functions defined this way would not have the expected asymptotic behavior. For the D_r spin chains under consideration, the Q-operator of the empty set can be conveniently fixed as

$$Q_\emptyset(x) = x^N , \tag{5.2}$$

though such a choice for a generic D_r QQ-system can be changed by a gauge transformation, see below in this section.

$$\begin{tikzcd}
& Q_{\{i_1,i_2,i_3\}} & \\
Q_{\{i_1,i_2\}} \arrow[ur] & Q_{\{i_1,i_3\}} \arrow[u] & Q_{\{i_2,i_3\}} \arrow[ul] \\
Q_{\{i_1\}} \arrow[u] \arrow[ur] & Q_{\{i_2\}} \arrow[ul] \arrow[ur] & Q_{\{i_3\}} \arrow[ul] \arrow[u] \\
& Q_\varnothing \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Figure 3. Directed Hasse diagram for D_3.

As discussed at the end of the section 2, the Q-operators Q_I with $|I| = r-1$ or $|I| = r$ factorise into spinorial Q-functions. More precisely,

$$Q_{\{i_1,\ldots,i_{r-1}\}} = S_{\{i_1,\ldots,i_{r-1},i_r\}} S_{\{i_1,\ldots,i_{r-1},i'_r\}}\,, \tag{5.3}$$

and

$$Q_{\{i_1,\ldots,i_r\}} = S^{[+1]}_{\{i_1,\ldots,i_r\}} S^{[-1]}_{\{i_1,\ldots,i_r\}}\,. \tag{5.4}$$

The set notation for the Q-operators S_I can be mapped to the notation $S_{\vec{\alpha}}$ in the previous subsection using $\vec{\alpha}$ as follows: to an acceptable set I of order r we associate $\vec{\alpha}$ such that, for $1 \leqslant i \leqslant r$,

$$\alpha_i = \begin{cases} +1 & \text{if} \quad i \in I \\ -1 & \text{if} \quad i' \in I \end{cases}\,. \tag{5.5}$$

We thus obtain a one-to-one correspondence between $S_{\{i_1,\ldots,i_r\}}$ and $S_{\vec{\alpha}}$ as defined in (4.16). We further remark that the polynomial structure of the spinorial Q-functions allows to determine them from the quadratic relations (5.3) and (5.4).

The QQ-relation (5.1) can be summarised in a Hasse diagram which recalls that of the A_r case. The latter is exemplified for D_3 in figure 3. However, the last two levels are nontrivial: the level $|I| = (r-1)$ factorises according to (5.3) and the level $|I| = r$, according to (5.4). In total, there are

$$2^k \binom{r}{k} \tag{5.6}$$

Q-funtions Q_I at level k. At the last two levels the Q-functions split according to (5.3) and (5.4) such that (5.6) remains valid for $1 \leqslant k \leqslant r-2$ and $2 \cdot 2^{r-1}$ spinorial Q-functions $S_{\vec{\alpha}}$ distinguished by $\prod_{i=1}^r \alpha_i = \pm 1$ are assigned to $(r-1)$'th and r'th spinor node, respectively.

Let S_I and S_J denote two Q-functions labelled by some acceptable sets I and J verifying $|I \cap J| = r-2$, i.e.

$$I = \{i_1, \ldots, i_{r-2}, i_{r-1}, i_r\} \qquad \text{and} \qquad J = \{i_1, \ldots, i_{r-2}, i'_{r-1}, i'_r\}\,. \tag{5.7}$$

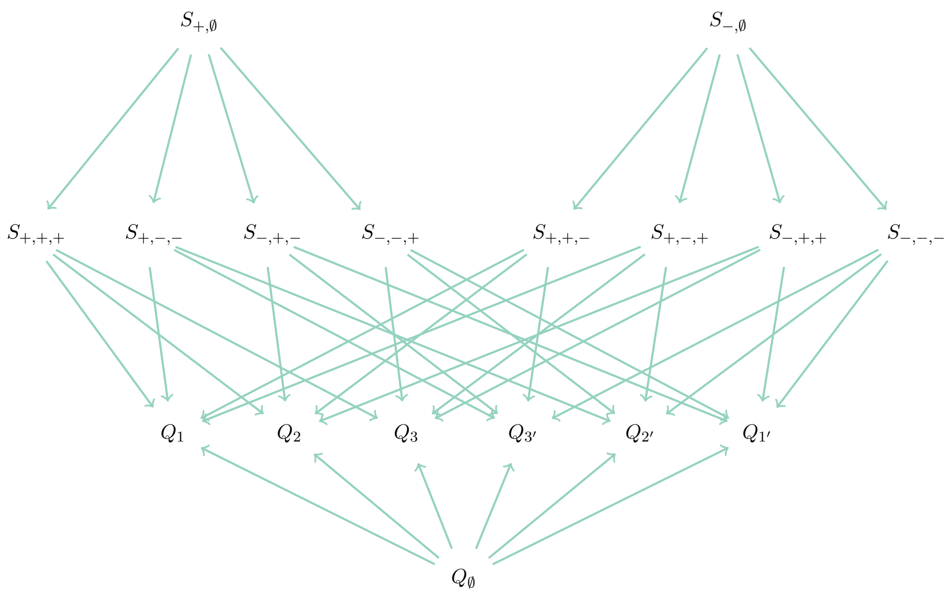

Figure 4. Hasse diagram of mixed orientation for D_3. In a particular gauge, the functions at the first and last level nodes can be chosen as in (5.17).

It follows that they must belong to the same node of the Dynkin diagram. Among them we have the QQ-relations

$$S_I^{[+1]}S_J^{[-1]} - S_I^{[-1]}S_J^{[+1]} = \frac{\tau_{i_{r-1}}\tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}}\tau_{i_r}}} Q_{I\cap J} \tag{5.8}$$

which relate the spinorial Q-functions to the last Q-functions on the tail of the Dynkin diagram, i.e. at the $r-2$'th node. Notice that for each level $r-2$ Q-function there are two ways to obtain them from spinorial Q-functions, e.g.: when $r=4$, $I\cap J = \{1,3\}$ can come from $I=\{1,3,2,4\}$ and $J=\{1,3,7,5\}$ or from $I=\{1,3,2,5\}$ and $J=\{1,3,7,4\}$. This relation allows us to resolve the last two levels in the D_r Hasse diagram, cf. figure 4 and figure 5 for the cases D_3 and D_4, respectively. A more detailed explanation of the elements of the Hasse diagram can be found in appendix E. Let us note that the D_3 Hasse diagram of figure 4 is (up to a gauge transformation setting $Q_\emptyset$ to 1) the same as the A_3 one, this is not surprising since the two algebras are isomorphic. The D_4 Hasse diagram, on the other hand, is new and gives a clear idea of the higher rank picture. Here we used the directions of the arrows in the Hasse diagram to distinguish from the QQ-relations (5.1) used for the last nodes as depicted in figure 3.

Using the QQ-relations in (5.1) we can express all Q-functions Q_I in terms of Casoratian determinants of single-index Q-functions. We find

$$Q_{\{i_1,\ldots,i_k\}} = \frac{\left(\sqrt{\tau_{i_1}\cdots\tau_{i_k}}\right)^{k-1}}{\prod_{1\leqslant a<b\leqslant k}(\tau_{i_a}-\tau_{i_b})} \frac{\left|Q_{\{i_a\}}^{[k+1-2b]}\right|_k}{\prod_{l=1}^{k-1} Q_\emptyset^{[k-2l]}} \tag{5.9}$$

with $i_a \neq i_b$, $i_a \neq i'_b$ and $\tau_i = \tau_{i'}^{-1}$ for $i > r$. Similar formulas exist with spinorial Q-functions: if I is an acceptable set of order $k \leqslant r-2$ and $i_{k+1},\ldots,i_r$ are such that

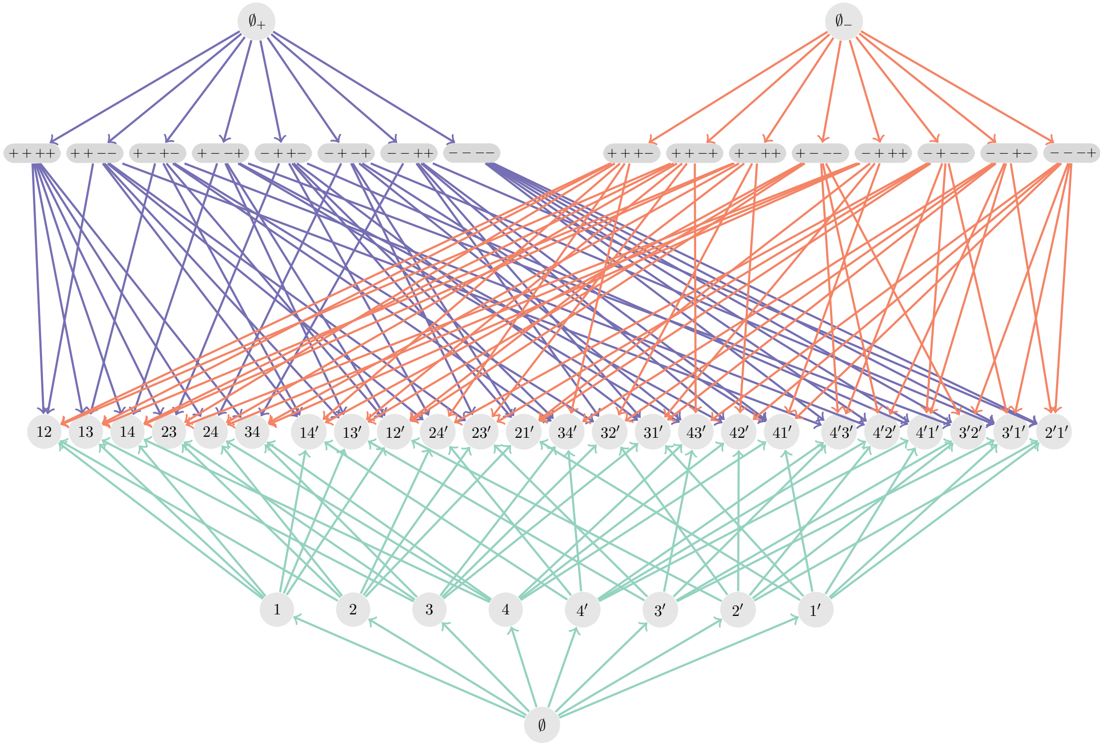

Figure 5. Hasse diagram of mixed orientation for D_4. Here, the level 1 and level 2 Q-operators Q_I are abbreviated by their index set I. The third level contains the spinorial Q-operators $S_{\vec{\alpha}}$ which are abbreviated by $\vec{\alpha}$. Finally, we have $Q_\emptyset$ (denoted by $\emptyset$) at the lowest level and $S_{\pm,\emptyset}$ (denoted by $\emptyset_\pm$) at the highest level. These are proportional to the identity and can be fixed via (5.17).

$I_r = I \cup \{i_{k+1}, \ldots, i_r\}$ is acceptable of order r then one has

$$Q_I = \frac{\left(\sqrt{\tau_{i_{k+1}} \cdots \tau_{i_r}}\right)^{r-k-1}}{\prod_{k+1 \leqslant a < b \leqslant r} (\tau_{i_b} - \tau_{i_a})} \frac{\begin{vmatrix} S^{[r-k-1]}_{I\cup\{i'_{k+1}, i_{k+2}, \ldots, i_r\}} & S^{[r-k-3]}_{I\cup\{i'_{k+1}, i_{k+2}, \ldots, i_r\}} & \cdots & S^{[1+k-r]}_{I\cup\{i'_{k+1}, i_{k+2}, \ldots, i_r\}} \\ S^{[r-k-1]}_{I\cup\{i_{k+1}, i'_{k+2}, \ldots, i_r\}} & S^{[r-k-3]}_{I\cup\{i_{k+1}, i'_{k+2}, \ldots, i_r\}} & \cdots & S^{[1+k-r]}_{I\cup\{i_{k+1}, i'_{k+2}, \ldots, i_r\}} \\ \vdots & \vdots & \ddots & \vdots \\ S^{[r-k-1]}_{I\cup\{i_{k+1}, \ldots, i_{r-1}, i'_r\}} & S^{[r-k-3]}_{I\cup\{i_{k+1}, \ldots, i_{r-1}, i'_r\}} & \cdots & S^{[1+k-r]}_{I\cup\{i_{k+1}, \ldots, i_{r-1}, i'_r\}} \end{vmatrix}}{\prod_{l=1}^{r-k-2} S^{[r-k-1-2l]}_{I_r}} . \tag{5.10}$$

Gauge transformation. The QQ-system as written above corresponds to a particular choice of gauge. In order to describe this gauge freedom, we draw inspiration from the $r = 3$ case, see appendix D. One needs to introduce two new Q-functions $S_{\pm,\emptyset}$, (5.1) and (5.3) remain unchanged while (5.4) and (5.8) become

$$Q_I = S^{[+1]}_I S^{[-1]}_I S_{-\epsilon(I),\emptyset} \tag{5.11}$$

and

$$S^{[+1]}_I S^{[-1]}_J - S^{[-1]}_I S^{[+1]}_J = \frac{\tau_{i_{r-1}} \tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}} \tau_{i_r}}} Q_{I\cap J}\, S_{\epsilon(I),\emptyset} \tag{5.12}$$

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where $I = \{i_1, \ldots, i_r\}$ and $J = \{i_1, \ldots, i_{r-2}, i'_{r-1}, i'_r\}$ are acceptable sets of order r and we define $\epsilon(I) = \prod_{i=1}^r \alpha_i = \epsilon(\vec{\alpha})$ with $\vec{\alpha}$ associated to I according to (5.5). These QQ-relations remain unchanged if one applies the gauge transformation, depending on three arbitrary functions g, g_+ and g_-, given by

$$S_{+,\emptyset} \mapsto \frac{g_+^{[+3]} g_-^{[-1]}}{g_+^{[+1]} g_-^{[-3]}} S_{+,\emptyset}\,, \qquad S_{-,\emptyset} \mapsto \frac{g_-^{[+3]} g_+^{[-1]}}{g_-^{[+1]} g_+^{[-3]}} S_{-,\emptyset}\,, \tag{5.13}$$

$$S_{\vec{\alpha}} \mapsto \frac{g_+^{[+2]} g_-}{g_+ g_-^{[-2]}} g S_{\vec{\alpha}}\,, \quad \text{if} \quad \epsilon(\vec{\alpha}) = +\,, \tag{5.14}$$

$$S_{\vec{\alpha}} \mapsto \frac{g_-^{[+2]} g_+}{g_- g_+^{[-2]}} g S_{\vec{\alpha}}\,, \quad \text{if} \quad \epsilon(\vec{\alpha}) = -\,, \tag{5.15}$$

$$Q_I \mapsto \frac{g_+^{[|I|+3-r]} g_-^{[|I|+3-r]}}{g_+^{[r-3-|I|]} g_-^{[r-3-|I|]}} g^{[r-1-|I|]} g^{[|I|+1-r]} Q_I \tag{5.16}$$

for I acceptable. In this paper we work in the "spin chain" gauge

$$Q_\emptyset(x) = x^N\,, \quad S_{\pm,\emptyset}(x) = 1 \tag{5.17}$$

and the Q-functions are polynomials in the spectral parameter up to twist-dependent exponential prefactors.

6 Transfer matrix in terms of fundamental Q's

In section 2.2 we gave the transfer matrix in terms of one single Q-function for each nesting level. We can use the Casoratian formula (5.9) to express the transfer matrix only in terms of $Q_\emptyset$ and a half the number of fundamental Q-functions $Q_{\{i\}}$. We will show in this section that the transfer matrix is then given by

$$T = Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]} \frac{|Q_{\{i_a\}}^{[r+2-2b-2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[r+2-2b]}|_r} + Q_\emptyset^{[1-r]} Q_\emptyset^{[r-3]} \frac{|Q_{\{i_a\}}^{[2b-r-2+2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[2b-r-2]}|_r} \tag{6.1}$$

with $i_a \neq i_b$ and $i_a \neq i'_b$ for all $a \neq b$.

This formula fulfills, at least for the fundamental T-function, one of the main purposes of our paper – to derive the Weyl-type expressions for the transfer matrices of spin chains based on D_r algebra, "quantizing" in this way the classical Weyl character determinant formula. The latter can be restored in the classical limit $x \to \infty$. In that limit $Q_{\{j\}}(x) \underset{x\to\infty}{\sim} \tau_j^x x^{J_j+N}$ while the fundamental T behaves as $x^{2N} \sum_{j=1}^r (\tau_j + \frac{1}{\tau_j})$.

6.1 Induction

We can prove the formula (6.1) by expressing the transfer matrix in terms of the first r fundamental Q-functions, as in (5.9), and inserting it into (2.17). We obtain

$$T_\pm = Q_\emptyset^{[\pm(r-1)]} Q_\emptyset^{[\pm(3-r)]} \sum_{k=1}^r \frac{\left|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}\right|_{k-1}}{\left|Q_{\{i\}}^{[\pm(2k-r-2j)]}\right|_{k-1}} \frac{\left|Q_{\{i\}}^{[\pm(2k-r-2j)]}\right|_k}{\left|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}\right|_k} \tag{6.2}$$

The desired expression (6.1) for the transfer matrix (in the case $i_a = a$) follows from (6.2) using the identity

$$\sum_{k=1}^{r} \frac{\left|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}\right|_{k-1}}{\left|Q_{\{i\}}^{[\pm(2k-r-2j)]}\right|_{k-1}} \frac{\left|Q_{\{i\}}^{[\pm(2k-r-2j)]}\right|_{k}}{\left|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}\right|_{k}} = \frac{\left|Q_{\{i\}}^{[\pm(r+2-2j-2\delta_{j,r})]}\right|_{r}}{\left|Q_{\{i\}}^{[\pm(r+2-2j)]}\right|_{r}} \tag{6.3}$$

which can be shown by induction on r. It obviously holds true for $r = 1$. It remains to show that

$$\frac{\left|Q_{\{i\}}^{[\pm(r+3-2j-2\delta_{j,r})]}\right|_{r+1}}{\left|Q_{\{i\}}^{[\pm(r+3-2j)]}\right|_{r+1}} = \frac{\left|Q_{\{i\}}^{[\pm(r+1-2j-2\delta_{j,r})]}\right|_{r}}{\left|Q_{\{i\}}^{[\pm(r+1-2j)]}\right|_{r}} + \frac{\left|Q_{\{i\}}^{[\pm(r-2j+3)]}\right|_{r}}{\left|Q_{\{i\}}^{[\pm(r-2j+1)]}\right|_{r}} \frac{\left|Q_{\{i\}}^{[\pm(r-2j+1)]}\right|_{r+1}}{\left|Q_{\{i\}}^{[\pm(r-2j+3)]}\right|_{r+1}}, \tag{6.4}$$

or equivalently (assuming the determinants are non-vanishing)

$$\begin{aligned}\left|Q_{\{i\}}^{[\mp(2j-1+2\delta_{j,r+1})]}\right|_{r+1} \left|Q_{\{i\}}^{[\mp(2j+1)]}\right|_{r} = & \left|Q_{\{i\}}^{[\mp(2j-1)]}\right|_{r+1} \left|Q_{\{i\}}^{[\mp(2j+1+2\delta_{j,r})]}\right|_{r} \\ & + \left|Q_{\{i\}}^{[\mp(2j-1)]}\right|_{r} \left|Q_{\{i\}}^{[\mp(2j+1)]}\right|_{r+1}.\end{aligned} \tag{6.5}$$

The latter identity can be proven as follows: one first expands each of the $(r+1) \times (r+1)$ determinants with respect to the row involving $Q_{\{r+1\}}$. Both sides become linear combination of $Q_{\{r+1\}}^{[\mp(2j-1)]}$ for $1 \leqslant j \leqslant r+2$ and one just has to check that the coefficients on each side are the same. For $j \in \{1, r+1, r+2\}$ this is completely trivial whereas for $j \in \{2, \ldots, r\}$ this becomes[3]

$$\begin{aligned}&|C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_r, C_{r+2}||C_2, \ldots, C_{r+1}| \\ &= |C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_r, C_{r+1}||C_2, \ldots, C_r, C_{r+2}| \\ &\quad - |C_1, \ldots, C_r||C_2, \ldots, C_{j-1}, C_{j+1}, \ldots, C_{r+2}|\end{aligned} \tag{6.6}$$

where C_j is the transpose of the row vector $\left(Q_{\{1\}}^{[\mp(2j-1)]}, \ldots, Q_{\{r\}}^{[\mp(2j-1)]}\right)$. This last equality is a particular case of a Plücker identity (or Sylvester's lemma): if M and N are two matrices of the same size with columns $M_1, \ldots, M_r$ and $N_1, \ldots, N_r$ respectively then the following identity holds for any $k \in \{1, \ldots, r\}$,

$$\det M \det N = \sum_{l=1}^{r} |M_1, \ldots, M_{k-1}, N_l, M_{k+1}, \ldots, M_r||N_1, \ldots, N_{l-1}, M_k, N_{l+1}, \ldots, N_r|. \tag{6.7}$$

In our case $M = (C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_r, C_{r+2})$ and $N = (C_2, \ldots, C_{r+1})$ have many columns in common so that if we decide to exchange $M_r = C_{r+2}$ only two terms survive in the sum (when $l = j-1$ or $l = r$) and they give exactly what we want.

[3] We use here another notation for determinants: if M is a $p \times p$ matrix with columns $M_1, \ldots, M_p$, we write $\det M = |M_1, \ldots, M_p|$.

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6.2 Reshuffling Q-functions in the transfer matrix

Here we show that the expression for the transfer matrix (6.1) in terms of r fundamental Q-functions is invariant under the replacement $Q_{i_a} \mapsto Q_{i'_a}$ for any a. By obvious symmetry with respect to permutations of the functions Q_i, $i \in \{1, 2, \dots, r\}$ it suffices to show that the transfer matrix is invariant under $Q_{i_r} \mapsto Q_{i'_r}$. This is the case if

$$\frac{Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]}}{Q_\emptyset^{[r-3]} Q_\emptyset^{[1-r]}} = -\frac{\check{T}_-^{\{i_1,\dots,i_r\}} - \check{T}_-^{\{i_1,\dots,i'_r\}}}{\check{T}_+^{\{i_1,\dots,i_r\}} - \check{T}_+^{\{i_1,\dots,i'_r\}}} \tag{6.8}$$

where we defined

$$\check{T}_\pm^{\{a_1,\dots,a_r\}} = \frac{\left| Q_{\{a_i\}}^{[\mp(2j-r-2+2\delta_{j,r})]} \right|_r}{\left| Q_{\{a_i\}}^{[\mp(2j-r-2)]} \right|_r} . \tag{6.9}$$

Using the Jacobi identity on determinants, one can rewrite the numerator and the denominator in the previous condition as

$$\check{T}_-^{\{i_1,\dots,i_r\}} - \check{T}_-^{\{i_1,\dots,i'_r\}} = (-1)^{1+\lfloor \frac{r}{2} \rfloor} \frac{W_{i_1,\dots,i_{r-1}}^{[-2]} W_{i'_r,i_1,\dots,i_{r-1},i_r}}{W_{i_1,\dots,i_{r-1},i_r}^{[-1]} W_{i_1,\dots,i_{r-1},i'_r}^{[-1]}} \tag{6.10}$$

and

$$\check{T}_+^{\{i_1,\dots,i_r\}} - \check{T}_+^{\{i_1,\dots,i'_r\}} = (-1)^{1+\lfloor \frac{r-1}{2} \rfloor + r} \frac{W_{i_1,\dots,i_{r-1}}^{[-2]} W_{i'_r,i_1,\dots,i_{r-1},i_r}}{W_{i_1,\dots,i_{r-1},i_r}^{[+1]} W_{i_1,\dots,i_{r-1},i'_r}^{[+1]}} \tag{6.11}$$

with

$$W_{i_1,\dots,i_k} := \left| Q_{\{i_a\}}^{[k+1-2b]} \right|_k . \tag{6.12}$$

The condition (6.8) then reads

$$\frac{Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]}}{Q_\emptyset^{[r-3]} Q_\emptyset^{[1-r]}} = \frac{W_{i_1,\dots,i_{r-1}}^{[-2]} W_{i_1,\dots,i_{r-1},i_r}^{[+1]} W_{i_1,\dots,i_{r-1},i'_r}^{[+1]}}{W_{i_1,\dots,i_{r-1}}^{[+2]} W_{i_1,\dots,i_{r-1},i_r}^{[-1]} W_{i_1,\dots,i_{r-1},i'_r}^{[-1]}} \tag{6.13}$$

which is indeed satisfied due to the trivial relation

$$\frac{Q_{\{i_1,\dots,i_{r-1},i_r\}} Q_{\{i_1,\dots,i_{r-1},i'_r\}}}{Q_{\{i_1,\dots,i_{r-1}\}}^{[+1]} Q_{\{i_1,\dots,i_{r-1}\}}^{[-1]}} = 1 , \tag{6.14}$$

following immediately from the factorisation properties of the Q-functions (5.3) and (5.4).

7 Bethe ansatz equations of Wronskian type

We propose here a Wronskian relation on $r+1$ Q-functions which could serve for finding the Bethe roots and, eventually, the energy of the state. We call it the Wronskian BAE, in analogy to the very useful Wronskian BAE for the A_r Heisenberg XXX spin chain which has the form

$$|Q_j(x + r - 2k + 2)|_{r+1} = x^N \prod_{1 \leqslant i < j \leqslant r+1} (\tau_i - \tau_j) \tag{7.1}$$

where Q-functions have the form (1.2). Solving the Wronskian relation above is often more efficient than solving the Bethe equations. This alternative method of finding Bethe roots was proposed in [36] and further extended in [40, 70, 71].

A similar relation for D-type spin chain is not as simple. In the following, we propose to use for this purpose the equation (5.10) when $I = \emptyset$:

$$\begin{vmatrix} S^{[r-1]}_{\{i'_1,i_2,\dots,i_r\}} & S^{[r-3]}_{\{i'_1,i_2,\dots,i_r\}} & \cdots & S^{[1-r]}_{\{i'_1,i_2,\dots,i_r\}} \\ S^{[r-1]}_{\{i_1,i'_2,\dots,i_r\}} & S^{[r-3]}_{\{i_1,i'_2,\dots,i_r\}} & \cdots & S^{[1-r]}_{\{i_1,i'_2,\dots,i_r\}} \\ \vdots & \vdots & \ddots & \vdots \\ S^{[r-1]}_{\{i_1,\dots,i_{r-1},i'_r\}} & S^{[r-3]}_{\{i_1,\dots,i_{r-1},i'_r\}} & \cdots & S^{[1-r]}_{\{i_1,\dots,i_{r-1},i'_r\}} \end{vmatrix} = \frac{\prod_{1\leqslant a<b\leqslant r}(\tau_{i_b}-\tau_{i_a})}{\left(\sqrt{\tau_{i_1}\cdots\tau_{i_r}}\right)^{r-1}} Q_\emptyset \prod_{l=1}^{r-2} S^{[r-1-2l]}_{I_r} \tag{7.2}$$

where we recall that $I_r = \{i_1,\dots,i_r\} \subset \{1,\dots,2r\}$ is such that $\{i_a,i'_a\} \cap \{i_b,i'_b\} = \emptyset$ for all $a \neq b$. The spinorial Q-functions are polynomials up to a twist-dependent exponential prefactor, their leading asymptotic behaviour is completely determined by the global charges $J_i = -J_{i'} = \sum_{k=1}^N J^{(k)}_{ii}$, it is given by

$$S_{\{i_1,\dots,i_r\}}(x) \underset{x\to\infty}{\sim} \left(\prod_{a=1}^r \tau_{i_a}\right)^{\frac{x}{2}} x^{\frac{1}{2}\left(\sum_{a=1}^r J_{i_a}+N\right)}. \tag{7.3}$$

The hope would be that, once the global charges are fixed, it suffices to solve equation (7.2) for the unknowns that are the coefficients of the polynomial parts of the spinorial Q-functions. In the A_r case, we thus get exactly all the eigenstates with such a weight. However, this does not seem to be the case here. First of all, one should notice that there are 2^r equations of the type (7.2) (as many as there are spinorial Q-functions S_{I_r}). For a given choice of I_r, the number of unknown coefficients can be easily computed to be $N + \frac{r-1}{2}\left(\sum_{a=1}^r J_{i_a} + N\right)$, it thus seems natural to chose I_r such that $\sum_{a=1}^r J_{i_a}$ is minimal. Nonetheless, since the degree of the polynomials on each side of the equation is $N + \frac{r-2}{2}\left(\sum_{a=1}^r J_{i_a} + N\right)$, as soon as $\sum_{a=1}^r J_{i_a} > -N$, there does not seem to be enough equations to fix all the coefficients. This is understandable if one looks at the case $r = 3$: the proposed equation does not coincide with (7.1), it is instead the expression of Q-functions with three indices in terms of single-index Q-functions. A possible way to resolve this issue would be to solve (7.2) for different choices of I_r and to look for common sets of solutions.

Once the Wronskian BAE has been solved, we have enough spinorial Q-functions to recover r single-index Q-functions using (5.10). Any of them can be used to compute the energy of the state through (2.23).

A Wronskian BAE for single-index Q-functions would be (6.14), which also reads

$$Q^{[r-2]}_\emptyset Q^{[2-r]}_\emptyset \left|Q^{[r+1-2b]}_{\{i_a\}}\right|_{r-1} \left|Q^{[r-1-2b]}_{\{i_a\}}\right|_{r-1} = \prod_{a=1}^{r-1} \frac{\tau_{i_a}}{(\tau_{i_a}-\tau_{i_r})(\tau_{i_a}-\tau_{i'_r})} \left|Q^{[r+1-2b]}_{\{i_a\}}\right|_r \left|Q^{[r+1-2b]}_{\{j_a\}}\right|_r \tag{7.4}$$

where $j_a = i_a$ for $1 \leqslant a \leqslant r-1$ and $j_r = i'_r$ and the asymptotic behaviour of the relevant functions is given by

$$Q_{\{i\}}(x) \underset{x\to\infty}{\sim} \tau_i^x x^{N+J_i}. \tag{7.5}$$

Once again there are 2^r equations of this type but they are of higher order than (7.2).

8 QQ′-type formulas for T-functions

In this section we first present QQ′-formulas for the symmetric and spinorial T-operators. The reasoning behind our rather heuristic derivation is in analogy to [29] where the BGG resolution [72] was used. Here we give arguments on the level of characters, see also [33], which we take as hints to obtain the actual BGG-type relation for the fundamental and spinorial transfer matrices. The final formulas have been checked in several examples for small finite lengths. Further we provide a consistency check. Namely, we recover the Weyl-type expression for the fundamental transfer matrix (6.1) by reducing there the number of used single-index Q-functions from $2r$ to r. In the final subsection we introduce the Hirota equation [73] and solve it using the QQ′-formulas for symmetric T-operators. This yields QQ′-type formulas for any rectangular transfer matrix $T_{a,s}$.

8.1 Symmetric transfer matrices

In [31] it was argued that the product of Lax matrices can be brought to the form

$$L_i^{(1)}(x+x_i)L_{i'}^{(2)}(x-x_i) = S_i\mathfrak{L}_i^{+,(1)}(x)G_i^{(2)}S_i^{-1} \tag{8.1}$$

where the Lax operator L_i are defined via $L_i(x) = \tilde{B}_{1,i}\,L(x)\,\tilde{B}_{1,i}$ for $i = 1,\dots,r$ and $L_i(x) = \mathrm{J}L_{i'}(x)\mathrm{J}$ for $i = r+1,\dots,2r$, see section 4.1. The superscripts $(1,2)$ indicate two different families of oscillators. The letter S_i denotes a similarity transformation in the oscillators space and G_i a dummy matrix that does not depend on the spectral parameter and commutes with the Lax matrix $\mathfrak{L}_i^+(x)$. Their precise form is given in [31]. We identify the Lax matrix $\mathfrak{L}_i^+(x)$ as a realisation of (2.5). The parameter x_i then plays the role of the representation label. We stress that the term linear in the spectral parameter is given by the generators J_{ij}, cf. (2.5). In the case (8.1), the representation of $\mathfrak{so}(2r)$ is infinite-dimensional in the oscillators space and becomes reducible for certain values of the parameter x_i. The infinite-dimensional representation of $\mathfrak{so}(2r)$ is characterised by its character. For example for $i=1$ the Cartan elements are of the form

$$J_{1,1} = 1 - r + 2x_1 - \sum_{k=2}^{2r-1}\bar{\mathbf{a}}_k\mathbf{a}_k\,, \tag{8.2}$$

$$J_{i,i} = \bar{\mathbf{a}}_i\mathbf{a}_i - \bar{\mathbf{a}}_{i'}\mathbf{a}_{i'}\,, \qquad 2 \leqslant i \leqslant r\,. \tag{8.3}$$

The character can then be computed via

$$\chi_1^+(x_1) \equiv \operatorname{tr}\prod_{i=1}^{r}\tau_i^{J_{ii}} = \tau_1^{2x_1}\prod_{k=2}^{r}\frac{\tau_1}{(\tau_1-\tau_k)(\tau_1-\tau_{k'})}\,. \tag{8.4}$$

We find similar formulas for the product of Lax matrices $L_i(x+x_i)L_{i'}(x-x_i)$ by exchanging $\tau_1 \leftrightarrow \tau_i$ and $x_1 \to x_i$ for $1 \leqslant i \leqslant r$ and $\tau_j \to \tau_j^{-1}$, $x_i \to x_{i'}$ for $i > r$, cf. section 4.1. The twist dependent prefactor is invariant under $\tau_i \to \tau_i^{-1}$. We find

$$\chi_i^+(x_i) = \begin{cases} \tau_i^{+2x_i}\prod_{k\neq i}\frac{\tau_i}{(\tau_i-\tau_k)(\tau_i-\tau_{k'})}, & 1 \leqslant i \leqslant r \\ \tau_{i'}^{-2x_i}\prod_{k\neq i'}^{r}\frac{\tau_{i'}}{(\tau_{i'}-\tau_k)(\tau_{i'}-\tau_{k'})}, & r < i \leqslant 2r \end{cases}\,. \tag{8.5}$$

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The finite dimensional characters are related to the one above by the sum formula

$$\chi_s = \sum_{i=1}^{2r} \chi_i^+ \left(\frac{s+r-1}{2}\right) = \sum_{i=1}^{r} \left[\prod_{j\neq i} \frac{\tau_i}{(\tau_i - \tau_{j'})(\tau_i - \tau_j)}\right] \left(\tau_i^{s+r-1} + \tau_{i'}^{s+r-1}\right) . \tag{8.6}$$

From our results for finite length and the discussion above we find that the formula can be lifted to transfer matrices and Q-operators. It reads

$$T_{1,s}(x) = \sum_{i=1}^{r} \left[\prod_{j\neq i} \frac{\tau_i}{(\tau_i - \tau_{j'})(\tau_i - \tau_j)}\right] \left(Q_{\{i\}}^{[s+r-1]} Q_{\{i'\}}^{[1-r-s]} + Q_{\{i\}}^{[1-r-s]} Q_{\{i'\}}^{[s+r-1]}\right) . \tag{8.7}$$

Notice that in the limit $x \to \infty$ (8.7) becomes (8.6), as it should be.

8.2 Spinorial transfer matries

A similar factorisation formula as (8.1) exists for the spinorial Lax matrices (4.11). It reads

$$\check{L}_{\vec{\alpha}}^{(1)}(x + x_{\vec{\alpha}}) \check{L}_{-\vec{\alpha}}^{(2)}(x - x_{\vec{\alpha}} - \kappa) = \check{S}_{\vec{\alpha}} \check{\mathfrak{L}}_{\vec{\alpha}}^{+,(1)}(x) \check{G}_{\vec{\alpha}}^{(2)} \check{S}_{\vec{\alpha}}^{-1} . \tag{8.8}$$

Here we defined $\check{L}_{\vec{\alpha}}(x) = B(\vec{\alpha})\check{L}(x)B(\vec{\alpha})$ and use a notation similar to that in (8.1). The similarity transformation $\check{S}_{\vec{\alpha}}$ only depends on the oscillators and $\check{G}_{\vec{\alpha}}$ is a matrix that is independent of the spectral parameter and commutes with the Lax matrix $\check{\mathfrak{L}}_{\vec{\alpha}}^{+}$. The latter denotes an infinite-dimensional realisation of the spinorial Lax matrix

$$\check{\mathfrak{L}}(x) = z\mathrm{I} + J_{ij} \otimes E_{ji} , \tag{8.9}$$

where J_{ij} denote the generators of a spinorial representation. Again the parameter $x_{\vec{\alpha}}$ in (8.8) has the role of the representation label. As before we compute the character of the oscillator representation. In the case $\vec{\alpha} = (+, \ldots, +)$ we find

$$\chi_{(+,\ldots,+)} \equiv \operatorname{tr} \prod_{i=1}^{r} \tau_i^{J_{ii}} = \prod_{i=1}^{r} \tau_i^{x_{(+,\ldots,+)}} \prod_{1\leqslant j<k\leqslant r} \frac{\tau_j \tau_k}{\tau_j \tau_k - 1} , \tag{8.10}$$

where

$$J_{ii} = x_{(+,\ldots,+)} - \sum_{j=i+1}^{r} \bar{\mathbf{a}}_{i,j'} \mathbf{a}_{j',i} - \sum_{j=1}^{i-1} \bar{\mathbf{a}}_{j,i'} \mathbf{a}_{i',j} , \qquad 1 \leqslant i \leqslant r . \tag{8.11}$$

The general formula can be obtained using the relations among the spinorial Lax matrices as presented in section 4.2. We get

$$\chi_{\vec{\alpha}}^{+}(x_{\vec{\alpha}}) = \prod_{i=1}^{r} \tau_i^{\alpha_i x_{\vec{\alpha}}} \prod_{1\leqslant j<k\leqslant r} \frac{\tau_j^{\alpha_j} \tau_k^{\alpha_k}}{\tau_j^{\alpha_j} \tau_k^{\alpha_k} - 1} , \tag{8.12}$$

The characters of the finite-dimensional spinor representations with $f = (s/2, \ldots, s/2, \pm s/2)$ can then be written as

$$\chi_{\pm,s} = \sum_{\{\alpha_i\}_\pm} \chi_{\vec{\alpha}}^{+}\left(\frac{s}{2}\right) = \sum_{\{\alpha_i\}_\pm} \prod_{i=1}^{r} \tau_i^{\frac{s}{2}\alpha_i} \prod_{1\leqslant j<k\leqslant r} \frac{\tau_j^{\alpha_j} \tau_k^{\alpha_k}}{\tau_j^{\alpha_j} \tau_k^{\alpha_k} - 1} . \tag{8.13}$$

Here the sum is taken over all configurations $\{\vec{\alpha}\}_\pm$ such that $\prod_i \alpha = \pm 1$.

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On the level of monodromies, we propose the formula

$$T_{\pm,s} = \sum_{\{\alpha_i\}_\pm} \prod_{1\leqslant j<k\leqslant r} \frac{\tau_j^{\alpha_j}\tau_k^{\alpha_k}}{\tau_j^{\alpha_j}\tau_k^{\alpha_k}-1} \prod_{i=1}^{r} \tau_i^{-\frac{\kappa}{2}\alpha_i} S_{\vec{\alpha}}^{[r+s-1]} S_{-\vec{\alpha}}^{[1-s-r]}\,. \tag{8.14}$$

This formula has been verified for small finite lengths by comparing to the transfer matrices directly constructed within the quantum inverse scattering method using the Lax matrices in (8.9) for finite-dimensional spinor representations.

8.3 Derivation of Weyl-type formula for $T_{1,1}$ from QQ′-relations

Let us write (8.7) as

$$T_{1,s} = \sum_{i=1}^{2r} h_i\, Q_{\{i\}}^{[s+r-1]} Q_{\{i'\}}^{[1-r-s]} \tag{8.15}$$

where $h_i = \prod_{j(\neq i,i')}^{r}(u_i-u_j)^{-1}$ and $u_j = \tau_j + 1/\tau_j$. We further assume that when $s \in \{1-r,\dots,0\}$ the identity is still verified if one sets

$$T_{1,0} = Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} \quad \text{and} \quad T_{1,s} = 0 \quad \text{for} \quad 1-r \leqslant s \leqslant -1\,. \tag{8.16}$$

We show here that the conditions (8.15) and (8.16) are enough to recover the expression (6.1) giving $T_{1,1}$ in terms of only r of the single-index Q-functions, and so are consistent with it. We also show in appendix F.1 how to retrieve the Wronskian equation (7.4) from these conditions.

One simply has to notice that (8.15) implies that there exist some Q-dependent coefficients $C_{j,k',k}$ (defined for $0 \leqslant k' \leqslant k \leqslant r$ and $0 \leqslant j \leqslant k-k'$) such that

$$\sum_{k'=0}^{k}\sum_{j=0}^{k-k'} C_{j,k',k} T_{1,k'+1-r}^{[2j+k'-k]} = \sum_{i=1}^{r} h_i \begin{vmatrix} Q_1^{[-k]} & Q_1^{[-k+2]} & \cdots & Q_1^{[k]} \\ \vdots & \vdots & & \vdots \\ Q_k^{[-k]} & Q_k^{[-k+2]} & \cdots & Q_k^{[k]} \\ Q_i^{[-k]} & Q_i^{[-k+2]} & \cdots & Q_i^{[k]} \end{vmatrix} \begin{vmatrix} Q_1^{[-k]} & Q_1^{[-k+2]} & \cdots & Q_1^{[k]} \\ \vdots & \vdots & & \vdots \\ Q_k^{[-k]} & Q_k^{[-k+2]} & \cdots & Q_k^{[k]} \\ Q_{i'}^{[-k]} & Q_{i'}^{[-k+2]} & \cdots & Q_{i'}^{[k]} \end{vmatrix} \tag{8.17}$$

It suffices indeed to expand the determinants with respect to their last row and perform the sum over i. One has for instance

$$C_{0,k,k} = (-1)^k \begin{vmatrix} Q_1^{[-k]} & \cdots & Q_1^{[k-2]} \\ \vdots & & \vdots \\ Q_k^{[-k]} & \cdots & Q_k^{[k-2]} \end{vmatrix} \begin{vmatrix} Q_1^{[-k+2]} & \cdots & Q_1^{[k]} \\ \vdots & & \vdots \\ Q_k^{[-k+2]} & \cdots & Q_k^{[k]} \end{vmatrix}. \tag{8.18}$$

In particular, plugging the constraints (8.16) in the previous relation when $k=r$ gives us

$$C_{0,r-1,r} Q_\emptyset^{[r-3]} Q_\emptyset^{[1-r]} + C_{1,r-1,r} Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]} + C_{0,r,r} T_{1,1} = 0\,. \tag{8.19}$$

Since

$$\begin{aligned} C_{0,r-1,r} &= (-1)^{r+1} \left|Q_i^{[-r+2j]}\right|_r \times \left|Q_i^{[-r+2j-2+2\delta_{j,r}]}\right|_r \\ \text{and} \quad C_{1,r-1,r} &= (-1)^{r+1} \left|Q_i^{[-r+2j-2]}\right|_r \times \left|Q_i^{[-r+2j-2\delta_{j,1}]}\right|_r \end{aligned} \tag{8.20}$$

we recover (6.1) in the case $i_a = a$. Notice that with this derivation, the symmetry under $Q_{\{i\}} \leftrightarrow Q_{\{i'\}}$ is immediate because the equations we started from were already symmetric.

8.4 Transfer matrices for general rectangular representations

In this section we propose relatively simple formulas for T-functions in rectangular representations in terms of bi-linear expressions involving Wronskians of both types of single-index Q-functions, Q_i and $Q_{i'}$, where $i = 1, 2, \ldots, r$. These formulas follow from (8.7) when solving the Hirota equations [73] satisfied by the T-functions. These equations read as follows ($s \in \mathbb{N}^*$):

$$T_{a,s}^{[+1]}\, T_{a,s}^{[-1]} = T_{a,s+1}\, T_{a,s-1} + T_{a-1,s}\, T_{a+1,s} \tag{8.21}$$

for $1 \leqslant a \leqslant r-3$,

$$T_{r-2,s}^{[+1]}\, T_{r-2,s}^{[-1]} = T_{r-2,s+1}\, T_{r-2,s-1} + T_{r-3,s}\, T_{+,s}\, T_{-,s}\,, \tag{8.22}$$

which can be written in the same form as the previous equation if one sets $T_{r-1,s} = T_{+,s}T_{-,s}$, and

$$T_{\pm,s}^{[+1]}\, T_{\pm,s}^{[-1]} = T_{\pm,s+1}\, T_{\pm,s-1} + T_{r-2,s}\,. \tag{8.23}$$

The boundary conditions are ($0 \leqslant a \leqslant r-2$, $s \in \mathbb{N}$)

$$T_{a,0} = Q_{\emptyset}^{[r-a-1]} Q_{\emptyset}^{[a+1-r]}\,, \qquad T_{0,s} = Q_{\emptyset}^{[r+s-1]} Q_{\emptyset}^{[1-r-s]}\,, \tag{8.24}$$

and

$$T_{\pm,0}(x) = Q_{\emptyset}(x)\,. \tag{8.25}$$

We shall determine here the QQ′-type relations for $T_{a,s}$ for $1 \leqslant a \leqslant r-1$, but not for $T_{\pm,s}$. For these spinorial transfer matrices, the spinorial Q-functions seem more suitable, see equation (8.14). We start from

$$T_{1,s}^{[+1]} T_{1,s}^{[-1]} - T_{1,s-1} T_{1,s+1} = \sum_{1 \leqslant i_1 < i_2 \leqslant 2r} h_{i_1} h_{i_2} \begin{vmatrix} Q_{\{i_1\}}^{[s+r]} & Q_{\{i_1\}}^{[s+r-2]} \\ Q_{\{i_2\}}^{[s+r]} & Q_{\{i_2\}}^{[s+r-2]} \end{vmatrix} \begin{vmatrix} Q_{\{i_1'\}}^{[2-s-r]} & Q_{\{i_1'\}}^{[-s-r]} \\ Q_{\{i_2'\}}^{[2-s-r]} & Q_{\{i_2'\}}^{[-s-r]} \end{vmatrix} \tag{8.26}$$

which can be also written, if the transfer matrices satisfy the Hirota equation (8.21) with boundary conditions (8.24), as follows

$$T_{1,s}^{[+1]}\, T_{1,s}^{[-1]} - T_{1,s-1}\, T_{1,s+1} = T_{0,s}\, T_{2,s} = Q_{\emptyset}^{[r+s-1]} Q_{\emptyset}^{[1-r-s]} T_{2,s}. \tag{8.27}$$

Putting the two expressions together yields the following expression for the second row of transfer matrices:

$$T_{2,s} = \frac{1}{Q_{\emptyset}^{[r+s-1]} Q_{\emptyset}^{[1-r-s]}} \sum_{1 \leqslant i_1 < i_2 \leqslant 2r} h_{i_1} h_{i_2} \begin{vmatrix} Q_{\{i_1\}}^{[s+r]} & Q_{\{i_1\}}^{[s+r-2]} \\ Q_{\{i_2\}}^{[s+r]} & Q_{\{i_2\}}^{[s+r-2]} \end{vmatrix} \begin{vmatrix} Q_{\{i_1'\}}^{[2-s-r]} & Q_{\{i_1'\}}^{[-s-r]} \\ Q_{\{i_2'\}}^{[2-s-r]} & Q_{\{i_2'\}}^{[-s-r]} \end{vmatrix}\,. \tag{8.28}$$

This procedure can be continued for $1 \leqslant a \leqslant r-1$, it yields

$$T_{a,s} = \frac{1}{\prod_{k=1}^{a-1} Q_{\emptyset}^{[r+s+2k-a-1]} Q_{\emptyset}^{[1+a-r-s-2k]}} \sum_{1 \leqslant i_1 < \ldots < i_a \leqslant 2r} h_{i_1} \cdots h_{i_a} W_{i_1,\ldots,i_a}^{[s+r-1]} W_{i_1',\ldots,i_a'}^{[1-s-r]}\,, \tag{8.29}$$

where we recall that $W_{i_1,\ldots,i_k} = \left|Q^{[k+1-2b]}_{\{i_a\}}\right|_k$. The proof of this formula, which we present in appendix F.2, boils down to verifying the relation

$$\frac{1}{2}\sum_{\substack{1\leqslant i_1<\cdots<i_a\leqslant 2r\\1\leqslant j_1<\cdots<j_a\leqslant 2r}}\begin{vmatrix}W^{[s+r]}_{i_1,\ldots,i_a} & W^{[s+r-2]}_{i_1,\ldots,i_a}\\W^{[s+r]}_{j_1,\ldots,j_a} & W^{[s+r-2]}_{j_1,\ldots,j_a}\end{vmatrix}\begin{vmatrix}W^{[2-s-r]}_{i'_1,\ldots,i'_a} & W^{[-s-r]}_{i'_1,\ldots,i'_a}\\W^{[2-s-r]}_{j'_1,\ldots,j'_a} & W^{[-s-r]}_{j'_1,\ldots,j'_a}\end{vmatrix}$$
$$=\left(\sum_{1\leqslant i_1<\cdots<i_{a-1}\leqslant 2r}W^{[s+r-1]}_{i_1,\ldots,i_{a-1}}W^{[1-s-r]}_{i'_1,\ldots,i'_{a-1}}\right)\left(\sum_{1\leqslant i_1<\cdots<i_{a+1}\leqslant 2r}W^{[s+r-1]}_{i_1,\ldots,i_{a+1}}W^{[1-s-r]}_{i'_1,\ldots,i'_{a+1}}\right). \quad (8.30)$$

9 Weyl-type formulas for T-functions from tableaux representations

The tableaux sum formulas of [46] give expressions for the transfer matrices of any rectangular representation $T_{a,s}$ through the single terms in the sum of the transfer matrix (2.15) as given in (2.17). In total there are $2r$ different terms (boxes), r for T_+ and r for T_-. Instead of using the summands in the form (2.17) involving Q's of different levels, we shall express them either in terms of $Q_\emptyset$ and r single-index Q-functions as in (6.2) or in terms of $r+1$ spinorial Q-functions. This will yield new expressions for totally symmetric $T_{1,s}$ and totally antisymmetric $T_{a,1}$ T-functions.

We start from the expressions (6.2) for $T_\pm$ such that

$$T_{1,1} = T_+ + T_- = \sum_{k=1}^{2r} b_{k,r} \quad (9.1)$$

where $b_{k,r}$ denotes a box as given in [46] for D_r with index k. The expression above, in the character limit $x\to\infty$, allows to identify $b_{k,r}$ in terms of the single-index Q-functions. We get

$$b_{k,r} = Q^{[r-1]}_\emptyset Q^{[3-r]}_\emptyset \frac{\left|Q^{[2k-r-2j+2]}_{\{i\}}\right|_{k-1}}{\left|Q^{[2k-r-2j]}_{\{i\}}\right|_{k-1}}\frac{\left|Q^{[2k-r-2j]}_{\{i\}}\right|_{k}}{\left|Q^{[2k-r-2j+2]}_{\{i\}}\right|_{k}} \quad (9.2)$$

for $1\leqslant k\leqslant r$ and

$$b_{k,r} = Q^{[1-r]}_\emptyset Q^{[r-3]}_\emptyset \frac{\left|Q^{[r-2j-2]}_{\{i\}}\right|_{k'-1}}{\left|Q^{[r-2j]}_{\{i\}}\right|_{k'-1}}\frac{\left|Q^{[r+2-2j]}_{\{i\}}\right|_{k'}}{\left|Q^{[r-2j]}_{\{i\}}\right|_{k'}} \quad (9.3)$$

for $r+1\leqslant k\leqslant 2r$, and we recall that $k'=2r-k+1$.

The simplest examples of the tableaux sum formulas beyond $T_{1,1}$ are for $T_{1,2}$ and $T_{2,1}$. They arise when writing

$$T^-_{1,1}T^+_{1,1} = \left[\left(\sum_{1\leqslant i\leqslant j\leqslant 2r} b^{[-1]}_{i,r}b^{[+1]}_{j,r}\right) - b^{[-1]}_{r,r}b^{[+1]}_{r',r}\right] + \left[\left(\sum_{1\leqslant i<j\leqslant 2r} b^{[+1]}_{i,r}b^{[-1]}_{j,r}\right) + b^{[+1]}_{r',r}b^{[-1]}_{r,r}\right] \quad (9.4)$$

and identifying the terms in the brackets with $T_{1,0}T_{1,2}$ and $T_{0,1}T_{2,1}$ from the Hirota equation (8.21), so that

$$T_{1,2}\, Q^{[r-2]}_\emptyset Q^{[2-r]}_\emptyset = \sum_{1\leqslant i\leqslant j\leqslant 2r} b^{[-1]}_{i,r}b^{[+1]}_{j,r} - b^{[-1]}_{r,r}b^{[+1]}_{r',r}\,, \quad (9.5)$$

and

$$T_{2,1}\, Q_\emptyset^{[+r]} Q_\emptyset^{[-r]} = \sum_{1\leqslant i<j\leqslant 2r} b_{i,r}^{[+1]} b_{j,r}^{[-1]} + b_{r',r}^{[+1]} b_{r,r}^{[-1]}\,. \tag{9.6}$$

As we see, it is independent of the actual representation of the box terms $b_{k,r}$. As we will see in the following, substituting (9.2) and (9.3) will yield new expressions for the transfer matrices that only depend on r single-index Q-functions and $Q_\emptyset$.

9.1 Symmetric representations

The transfer matrices for generic symmetric representations are given by [46]

$$T_{1,s} = \frac{1}{\prod_{k=1}^{s-1} Q_\emptyset^{[r-s-2+2k]} Q_\emptyset^{[-(r-s-2+2k)]}} \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_s\leqslant 2r}{}' \; b_{i_1,r}^{[1-s]} \cdots b_{i_s,r}^{[s-1]} \tag{9.7}$$

where the symbol $\sum'$ stands for a sum in which we do not allow for r and $r+1$ to appear at the same time. The denominator appears as a consequence of our boundary conditions for the Hirota equation.

9.1.1 General symmetric sum

Let us define

$$\tilde{b}_k = \frac{\left|Q_{\{i\}}^{[2k-2j+2]}\right|_{k-1}}{\left|Q_{\{i\}}^{[2k-2j]}\right|_{k-1}} \frac{\left|Q_{\{i\}}^{[2k-2j]}\right|_{k}}{\left|Q_{\{i\}}^{[2k-2j+2]}\right|_{k}} \tag{9.8}$$

for $1 \leqslant k \leqslant r$ and

$$\tilde{b}_k = \frac{\left|Q_{\{i\}}^{[-(2k'-2j+2)]}\right|_{k'-1}}{\left|Q_{\{i\}}^{[-(2k'-2j)]}\right|_{k'-1}} \frac{\left|Q_{\{i\}}^{[-(2k'-2j)]}\right|_{k'}}{\left|Q_{\{i\}}^{[-(2k'-2j+2)]}\right|_{k'}} \tag{9.9}$$

for $r+1 \leqslant k \leqslant 2r$ such that

$$b_{k,r} = Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]} \tilde{b}_k^{[-r]} \quad \text{if} \quad k\leqslant r \quad \text{and} \quad b_{k,r} = Q_\emptyset^{[1-r]} Q_\emptyset^{[r-3]} \tilde{b}_k^{[r]} \quad \text{if} \quad r+1\leqslant k\,. \tag{9.10}$$

For $l \geqslant 1$, one has

$$\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_l\leqslant r} \tilde{b}_{i_1}^{[-2l+1]} \cdots \tilde{b}_{i_l}^{[-1]} = \frac{\left|Q_{\{i\}}^{[2r+1-2j-2l\delta_{j,r}]}\right|_r}{\left|Q_{\{i\}}^{[2r+1-2j]}\right|_r} \tag{9.11}$$

and

$$\sum_{r+1\leqslant i_1\leqslant\ldots\leqslant i_l\leqslant 2r} \tilde{b}_{i_1}^{[1]} \cdots \tilde{b}_{i_l}^{[2l-1]} = \frac{\left|Q_{\{i\}}^{[-(2r+1-2j-2l\delta_{j,r})]}\right|_r}{\left|Q_{\{i\}}^{[-(2r+1-2j)]}\right|_r}\,. \tag{9.12}$$

The two identities are equivalent, so it is enough to prove the first one. We do it by induction in r. It is trivial when $r=1$. If it is true for some $r_0 \geqslant 1$ then let us show by induction on l that it is also true for r_0+1: the case $l=1$ has been proven earlier in section 6.1 so we assume that the identity holds for some $l_0 \geqslant 1$. We then write

$$\begin{aligned}\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_{l_0+1}\leqslant r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]} \cdots \tilde{b}_{i_{l_0+1}}^{[-1]} &= \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_{l_0+1}\leqslant r_0} \tilde{b}_{i_1}^{[-2l_0-1]} \cdots \tilde{b}_{i_{l_0+1}}^{[-1]} \\ &\quad + \tilde{b}_{r_0+1}^{[-1]} \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_{l_0}\leqslant r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]} \cdots \tilde{b}_{i_{l_0}}^{[-3]}\,.\end{aligned} \tag{9.13}$$

Since we have assumed that the identity holds for r_0 and any l, for (r_0+1, l_0) we can write

$$\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_{l_0+1}\leqslant r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]}\cdots\tilde{b}_{i_{l+1}}^{[-1]} = \frac{\left|Q_{\{i\}}^{[2r_0+1-2j-2(l_0+1)\delta_{j,r_0}]}\right|_{r_0}}{\left|Q_{\{i\}}^{[2r_0+1-2j]}\right|_{r_0}} + \frac{\left|Q_{\{i\}}^{[2r_0-2j+3]}\right|_{r_0}}{\left|Q_{\{i\}}^{[2r_0-2j+1]}\right|_{r_0}} \frac{\left|Q_{\{i\}}^{[2r_0+1-2j-2l_0\delta_{j,r_0+1}]}\right|_{r_0+1}}{\left|Q_{\{i\}}^{[2r_0-2j+3]}\right|_{r_0+1}}. \tag{9.14}$$

Consequently, for (9.11) to hold for (r_0+1, l_0+1), one only has to show that

$$\left|Q_{\{i\}}^{[-2j]}\right|_{r_0} \left|Q_{\{i\}}^{[2-2j-2(l_0+1)\delta_{j,r_0+1}]}\right|_{r_0+1} = \left|Q_{\{i\}}^{[2-2j]}\right|_{r_0+1} \left|Q_{\{i\}}^{[-2j-2(l_0+1)\delta_{j,r_0}]}\right|_{r_0} + \left|Q_{\{i\}}^{[2-2j]}\right|_{r_0} \left|Q_{\{i\}}^{[-2j-2l_0\delta_{j,r_0+1}]}\right|_{r_0+1}. \tag{9.15}$$

This last relation can be proven in much the same way as (6.5) which itself corresponds to the case $l_0 = 0$.

9.1.2 Application to the computation of transfer matrices

In order to apply the summation formulas (9.11) and (9.12) we first rewrite equation (9.7) as

$$\begin{aligned} T_{1,s} = &\sum_{l=0}^{s} Q_\emptyset^{[2j+r-s-2]} Q_\emptyset^{[2+2j-r-s]} \sum_{\substack{1\leqslant i_1\leqslant\ldots\leqslant i_l\leqslant r \\ r+1\leqslant i_{l+1}\leqslant\ldots\leqslant i_s\leqslant 2r}} \left(\tilde{b}_{i_1}^{[1-s-r]}\cdots\tilde{b}_{i_l}^{[2l-s-r-1]}\right) \\ &\times \left(\tilde{b}_{i_{l+1}}^{[2l-s+r+1]}\cdots\tilde{b}_{i_s}^{[s+r-1]}\right) - \sum_{l=1}^{s-1} Q_\emptyset^{[2l+r-s-2]} Q_\emptyset^{[2+2l-r-s]} \tilde{b}_r^{[2l-s-r-1]} \tilde{b}_{r+1}^{[2l-s+r+1]} \\ &\times \sum_{\substack{1\leqslant i_1\leqslant\ldots\leqslant i_{l-1}\leqslant r \\ r+1\leqslant i_{l+2}\leqslant\ldots\leqslant i_s\leqslant 2r}} \left(\tilde{b}_{i_1}^{[1-s-r]}\cdots\tilde{b}_{i_{l-1}}^{[2l-s-r-3]}\right)\left(\tilde{b}_{i_{l+2}}^{[2l-s+r+3]}\cdots\tilde{b}_{i_s}^{[s+r-1]}\right). \end{aligned} \tag{9.16}$$

In virtue of (9.11) and (9.12), this gives

$$\begin{aligned} T_{1,s} = &\sum_{l=0}^{s} Q_\emptyset^{[2l+r-s-2]} Q_\emptyset^{[2+2l-r-s]} \frac{\left|Q_{\{i\}}^{[2l+r-s+1-2j-2l\delta_{j,r}]}\right|_r}{\left|Q_{\{i\}}^{[2l+r-s+1-2j]}\right|_r} \frac{\left|Q_{\{i\}}^{[-(r+s+1-2l-2j-2(s-l)\delta_{j,r})]}\right|_r}{\left|Q_{\{i\}}^{[-(r+s+1-2l-2j)]}\right|_r} \\ &- \sum_{l=1}^{s-1} Q_\emptyset^{[2l+r-s-2]} Q_\emptyset^{[2+2l-r-s]} \frac{\left|Q_{\{i\}}^{[2l+r-s-1-2j-2(l-1)\delta_{j,r}]}\right|_r}{\left|Q_{\{i\}}^{[2l+r-s+1-2j]}\right|_r} \frac{\left|Q_{\{i\}}^{[-(r+s-1-2l-2j-2(s-1-l)\delta_{j,r})]}\right|_r}{\left|Q_{\{i\}}^{[-(r+s+1-2l-2j)]}\right|_r}. \end{aligned} \tag{9.17}$$

The terms for $1 \leqslant l \leqslant s-1$ of each sum can be combined thanks to a Plücker identity, to give an explicit and concise Weyl-type representation of symmetric T-functions for D_r algebra

$$T_{1,s} = \sum_{l=0}^{s} Q_\emptyset^{[2l+r-s-2]} Q_\emptyset^{[2+2l-r-s]} \frac{\left|Q_{\{i\}}^{[2l+r-s+1-2j+2(s-l)\delta_{j,1}-2l\delta_{j,r}]}\right|_r}{\left|Q_{\{i\}}^{[2l+r-s+1-2j]}\right|_r}. \tag{9.18}$$

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Once again there are 2^r formulas of this type depending on which set of r single-index Q-functions we use in the right-hand side. Finally, let us mention that we also checked that one can get the same formula starting from the conditions (8.15) and (8.16) using a method similar to that of section 8.3 (or more directly for low ranks, see appendix G.1 for the case $r = 2$).

9.2 Antisymmetric representations

The transfer matrices for generic antisymmetric representations are given by [46]

$$T_{a,1} = \frac{1}{\prod_{k=1}^{a-1} Q_\emptyset^{[r-a+2k]} Q_\emptyset^{[-(r-a+2k)]}} \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant r \\ r+1\leqslant j_1<\cdots<j_l\leqslant 2r \\ a-k-l\in 2\mathbb{N}}} b_{i_1,r}^{[a-1]} \cdots b_{i_k,r}^{[a+1-2k]} \\ \times b_{r+1,r}^{[a-1-2k]} b_{r,r}^{[a-3-2k]} \cdots b_{r+1,r}^{[2l+3-a]} b_{r,r}^{[2l+1-a]} b_{j_1,r}^{[2l-a-1]} \cdots b_{j_l,r}^{[1-a]} . \tag{9.19}$$

As it happens, in order to obtain nice expressions for these transfer matrices involving a reduced number of Q-functions, it is more convenient to turn to spinorial Q-functions. If, in order to shorten the notations, we write

$$S_{I_r} = S_{\{1,\ldots,r\}}\,, \quad S_i = S_{\{1,\ldots,r-i,r+i,r-i+2,\ldots,r\}} \quad \text{for} \quad i \in \{1,\ldots,r\} \tag{9.20}$$

then, according to (5.10), the boxes are also given by

$$b_{k,r} = Q_\emptyset^{[r-1]} Q_\emptyset^{[1-r]} \frac{S_{I_r}^{[-2]}}{S_{I_r}} \frac{\left|S_i^{[-2j]}\right|_{r-k}}{\left|S_i^{[2-2j]}\right|_{r-k}} \frac{\left|S_i^{[4-2j]}\right|_{r+1-k}}{\left|S_i^{[2-2j]}\right|_{r+1-k}} \tag{9.21}$$

for $1 \leqslant k \leqslant r$ and

$$b_{k,r} = Q_\emptyset^{[r-1]} Q_\emptyset^{[1-r]} \frac{S_{I_r}^{[+2]}}{S_{I_r}} \frac{\left|S_i^{[2j]}\right|_{k-r-1}}{\left|S_i^{[2j-2]}\right|_{k-r-1}} \frac{\left|S_i^{[2j-4]}\right|_{k-r}}{\left|S_i^{[2j-2]}\right|_{k-r}} \tag{9.22}$$

for $r+1 \leqslant k \leqslant 2r$. The relevant summation formulas read

$$\sum_{1\leqslant i_1<\ldots<i_l\leqslant r} b_{i_1,r}^{[-1]} \cdots b_{i_l,r}^{[1-2l]} = \frac{S_{I_r}^{[-1-2l]}}{S_{I_r}^{[-1]}} \left(\prod_{a=1}^{l} Q_\emptyset^{[r-2l-2+2a]} Q_\emptyset^{[-r-2l+2a]} \right) \frac{\left|S_i^{[1-2j+2\theta(l-j))]}\right|_r}{\left|S_i^{[1-2j]}\right|_r} \tag{9.23}$$

and

$$\sum_{r+1\leqslant i_1<\ldots<i_l\leqslant 2r} b_{i_1,r}^{[2l-1]} \cdots b_{i_l,r}^{[1]} = \frac{S_{I_r}^{[2l+1]}}{S_{I_r}^{[+1]}} \left(\prod_{a=1}^{l} Q_\emptyset^{[r-2+2a]} Q_\emptyset^{[-r+2a]} \right) \frac{\left|S_i^{[2r-1-2j+2\theta(r-l-j))]}\right|_r}{\left|S_i^{[2r+1-2j]}\right|_r} \tag{9.24}$$

where we used the Heaviside function θ that is 0 for negative arguments and 1 for non-negative ones. These formulas can be proven in much the same way as (9.11) and (9.12).

This permits us to write

$$T_{a,1}=\frac{Q_\emptyset^{[r-a]}Q_\emptyset^{[a-r]}}{S_{I_r}^{[a-1]}S_{I_r}^{[1-a]}}\sum_{\substack{0\leqslant k,l\leqslant a\\ a-k-l\in 2\mathbb{N}}}S_{I_r}^{[a+1-2k]}S_{I_r}^{[2l-1-a]}\frac{\left|S_i^{[a+1-2j+2\theta(k-j)]}\right|_r}{\left|S_i^{[a+1-2j]}\right|_r}$$
$$\times\frac{\left|S_i^{[2r-a-1-2j+2\theta(r-l-j)]}\right|_r}{\left|S_i^{[2r-a+1-2j]}\right|_r}\,. \tag{9.25}$$

This equation should be compared with the much more complicated expression given in appendix G.2 for the same quantity but in terms of single-index Q-functions. Similarly, the expression for $T_{1,s}$ in terms of spinorial Q-functions is not as simple as (9.18).

In the particular case $a=1$ the previous expression reads

$$T_{1,1}=\frac{Q_\emptyset^{[r-1]}Q_\emptyset^{[1-r]}}{S_{I_r}}\left(S_{I_r}^{[-2]}\frac{\left|S_i^{[2-2j+2\delta_{1,j}]}\right|_r}{\left|S_i^{[2-2j]}\right|_r}+S_{I_r}^{[+2]}\frac{\left|S_i^{[2r-2j-2\delta_{j,r}]}\right|_r}{\left|S_i^{[2r-2j]}\right|_r}\right)\,. \tag{9.26}$$

It should be compared with (6.1). When $a=r-1$, as expected from the fact that $T_{r-1,1}=T_{+,1}T_{-,1}$, there is a factorization:

$$T_{r-1,1}=\frac{1}{S_{I_r}^{[r-2]}S_{I_r}^{[2-r]}}\frac{Q_\emptyset^{[+1]}Q_\emptyset^{[-1]}}{\left|S_i^{[r+2-2j]}\right|_r\left|S_i^{[r-2j]}\right|_r}\left(\sum_{\substack{k=0\\ k\text{ even}}}^{r}S_{I_r}^{[r-2k]}\left|S_i^{[r-2j+2\theta(k-j)]}\right|_r\right)$$
$$\times\left(\sum_{\substack{k=0\\ k\text{ odd}}}^{r}S_{I_r}^{[r-2k]}\left|S_i^{[r-2j+2\theta(k-j)]}\right|_r\right)\,. \tag{9.27}$$

9.3 Spinorial representations

Following [42] we express the spinorial T-functions $T_{\pm,1}$ in terms of the Q-functions along a nesting path, cf. section 2.2. One finds

$$T_{\pm,1}=\sum_{|\alpha|=\pm1}Q_\emptyset^{[-\alpha_1]}\left(\frac{S_{(+,\dots,+)}^{[\rho_+(\vec\alpha)+1]}}{S_{(+,\dots,+)}^{[\rho_+(\vec\alpha)-1]}}\right)^{\frac{\alpha_{r-1}+\alpha_r}{2}}\left(\frac{S_{(+,\dots,+,-)}^{[\rho_-(\vec\alpha)+1]}}{S_{(+,\dots,+,-)}^{[\rho_-(\vec\alpha)-1]}}\right)^{\frac{\alpha_{r-1}-\alpha_r}{2}}\prod_{k=1}^{r-2}\left(\frac{Q_{\{1,\dots,k\}}^{[\rho_k(\vec\alpha)+1]}}{Q_{\{1,\dots,k\}}^{[\rho_k(\vec\alpha)-1]}}\right)^{\frac{\alpha_k-\alpha_{k+1}}{2}} \tag{9.28}$$

where $Q_\emptyset=x^N$ and the shifts are determined via

$$\begin{aligned}\rho_k(\vec\alpha)&=\alpha_1+\ldots+\alpha_{k-1}+\frac{\alpha_k-\alpha_{k+1}}{2}\qquad\text{for}\qquad 1\le k\le r-2\,,\\ \rho_\pm(\vec\alpha)&=\alpha_1+\ldots+\alpha_{r-2}+\frac{\alpha_{r-1}\pm\alpha_r}{2}\,.\end{aligned} \tag{9.29}$$

Expressing all Q-functions in terms of spinorial ones using (5.10) we obtain determinant formulas for $T_{\pm,1}$. We find

$$T_{+,1}=\frac{\left(\sqrt{\tau_{i_1}\cdots\tau_{i_r}}\right)^{r-1}}{\prod_{1\leqslant a<b\leqslant r}(\tau_{i_b}-\tau_{i_a})}\frac{1}{\prod_{l=1}^{r-1}S_{I_r}^{[r-2l]}}\sum_{\substack{k=0\\ k\text{ even}}}^{r}S_{I_r}^{[r-2k]}\left|S_i^{[2j-r-2\theta(r-k-j)]}\right|_r \tag{9.30}$$

and

$$T_{-,1} = \frac{\left(\sqrt{\tau_{i_1} \cdots \tau_{i_r}}\right)^{r-1}}{\prod_{1\leqslant a<b\leqslant r} (\tau_{i_b} - \tau_{i_a})} \frac{1}{\prod_{l=1}^{r-1} S_{I_r}^{[r-2l]}} \sum_{\substack{k=0 \\ k\,\mathrm{odd}}}^{r} S_{I_r}^{[r-2k]} \left| S_i^{[2j-r-2\theta(r-k-j))]} \right|_r \tag{9.31}$$

These expressions have been verified for $r = 3, 4, 5$ for a particular choice of I_r and we are missing a generic proof. However, the formulas are consistent with the factorisation $T_{r-1,1} = T_{+,1}T_{-,1}$ in (9.27) expected from the Hirota relations. In principle, one can now generate, from (9.25), (9.30) and (9.31) above, all transfer matrices of rectangular representations using Cherednik-Bazhanov-Reshetikhin type formulas written for D_r symmetry in [42].

10 Discussion

In this work, we proposed the full system of Baxter Q-functions – the QQ-system – for the spin chains with SO($2r$) symmetry. This QQ-system is described by a novel type of Hasse diagram presented for various ranks on figures 4, 5 and 9. We found Weyl-type formulas for transfer matrices (T-functions) of symmetric and antisymmetric representations in terms of sums of ratios of determinants of r basic Q-functions. We proposed QQ′-type formulas expressing the T-functions through $2r$ basic single-index Q-functions. These could be a powerful tool for the study of spin chains and sigma models with D_r symmetry. We also reformulated the Bethe ansatz equations in the form of a single Wronskian relation on $r+1$ basic Q-functions. It is the analogue of a similar Wronskian relation for spin chains with A_r symmetry. However, apart from the Bethe roots our equation contain extra solutions whose role has yet to be clarified.

Our main assumptions in this article are the Plücker QQ-relations (5.1) and (5.8), as well as the QQ′-relations (8.7) and (8.14). The QQ-relations are motivated by the asymptotics of the Q-operators and the QQ′-type relations by the factorisation formulas for the Lax matrices for Q-operators and the corresponding character formulas, as discussed in section 8.1 and 8.2. Both relations remain to be proven but we have tested them explicitly for several examples of small finite length T and Q-operators. The QQ-relations (5.1) allow to express the fundamental transfer matrix, for which an expression in terms of one Q-function of each level is known from the algebraic Bethe ansatz, in terms of r single-index Q-functions and $Q_\emptyset$, cf. (6.1). This Weyl-type expression has been independently obtained from the QQ′-type relations (8.7), see section 8.3. We take this as a consistency check. The new formulas for $T_{a,s}$ are obtained from the Hirota equation and the tableaux formulas of [46].

Unlike the well understood QQ-system of A-type, in the D-type QQ-system there are still many questions left and issues to be clarified. The questions exists already on the operator level: the Yangian for SO($2r$) spin chains is constructed only for "rectangular" representations and the R-matrix – the main building block for Q and T-functions is known only for the symmetric and spinorial representations [31, 63, 74]. A full classification of Lax matrices including the ones for the Q-operators was recently given in [75] for A-type. This may shed some light upon the transfer matrices for general rectangular representations and beyond. It would be interesting, using the tableaux formulas (which look quite involved [76]) to find the Weyl-type determinant formulas for the arbitrary rectangular representations, generalizing our formulas for symmetric and antisymmetric representations. These could also be interesting for the study of the Q-system and its relation to cluster algebras, see e.g. [77, 78]. Moreover, a solid proof of our QQ-system and our Wronskian formulas for T's in symmetric and antisymmetric representations is yet to be found on both the analytic and operator level. It may be possible using the BGG resolution or the analogue of coderivative method proposed in [32] and used for this purpose in [33], see also [79] for a review. Unfortunately, we do not know yet a suitable analogue of Baxter TQ equations (quantum spectral determinant) which appeared to be so useful for the spin chains with A_r symmetry [1, 2], see [40] for the modern description in terms of forms as well as [80] in terms of the quantum determinant. It is possible that the QQ′-type formulas for transfer matrices proposed in this work can replace the Baxter equations for D_r algebra.

It would be interesting to generalize our approach to the study of spin chains with open boundary conditions and to the non-compact, highest weight and principal series representations of D algebras. For a much better studied case of these aspects in A-type integrable system we refer the reader to [50, 81–84]. One encounters non-compact representations in sigma models [85] and spin chains [86] with principal series representations of the d-dimensional conformal group $\mathrm{SO}(2, d)$ [86]. They recently appeared in the study of d-dimensional fishnet CFT [70, 87, 88] and the associated planar graphs (of the shape of regular 2-dimensional lattice) [89]. While for the lowest, 4-dimensional conformal $\mathrm{SO}(2, 4)$ symmetry one can use its isomorphism to the A type $\mathrm{SU}(2, 2)$ group to construct the suitable QQ-system and Baxter TQ system for the efficient study of Fishnet CFT [90, 91], for $d > 4$ we have to find an alternative approach which can be based on the D-type QQ-system constructed in this work. The structure of the QQ-system does not depend on the choice of real section of the orthogonal group but the Wronskian, Weyl-type formulas for T do depend. So one could try to construct the quantum spectral curve (QSC) formalism for $d > 4$ fishnet CFT in analogy to the $d = 4$ case [92].

Finally, we hope that our methods can be generalized to B, C and exceptional types of algebras and their deformations, as well as to superalgebras such as $\mathfrak{osp}(m|2n)$ where the QQ-system and T-functions are yet to be constructed. This includes the case relevant for AdS_4/CFT_3 for which the QSC has recently been studied in [93–95]. A first step could be the evaluation of the oscillators type Lax matrices for Q-operators using the results of [96, 97] as done in [98] for type A.

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A D_r Kirillov-Reshetikhin modules and characters

The Kirillov-Reshetikhin modules form a family $\{\mathrm{W}_{a,s}(x)|a \in \{1,\dots,r-2,+,-\}, s \in \mathbb{N}^*, x \in \mathbb{C}\}$ of modules of the Yangian $Y(D_r)$ that were first introduced in [59]. When restricted to $D_r \subset Y(D_r)$ they decompose into irreducible representations of D_r according to

$$\mathrm{W}_{a,s}(x) \simeq \bigoplus_{\substack{n_i\in\mathbb{N}\\ n_1+n_3+\cdots+n_a=s}} \mathrm{V}(n_1\omega_1 + n_3\omega_3 + \cdots + n_a\omega_a) \tag{A.1}$$

for odd $a \leqslant r-2$,

$$\mathrm{W}_{a,s}(x) \simeq \bigoplus_{\substack{n_i\in\mathbb{N}\\ n_0+n_2+\cdots+n_a=s}} \mathrm{V}(n_0\omega_0 + n_2\omega_2 + \cdots + n_a\omega_a) \tag{A.2}$$

for even $a \leqslant r-2$,

$$\mathrm{W}_{+,s}(x) \simeq \mathrm{V}(s\,\omega_{r-1}) \quad \text{and} \quad \mathrm{W}_{-,s}(x) \simeq \mathrm{V}(s\,\omega_r)\,. \tag{A.3}$$

Here $\omega_0 = 0$, while $\omega_1,\dots,\omega_r$ are the fundamental weights of D_r, $\mathrm{V}(f)$ denotes the irreducible D_r-module with highest weight f. Notice that the previous decompositions are independent of the spectral parameter x. The characters are expressed in terms of weights $f_1,\dots,f_r$ that are related to the non-negative integers $n_1,\dots,n_r$ (Dynkin labels) via

$$f_a = n_a + \cdots + n_{r-2} + \frac{1}{2}(n_{r-1}+n_r) \tag{A.4}$$

for $1 \leqslant a \leqslant r-2$ and

$$f_{r-1} = \frac{1}{2}(n_{r-1}+n_r)\,, \qquad f_r = \frac{1}{2}(n_{r-1}-n_r)\,. \tag{A.5}$$

The finite-dimensional irreducible representations of $\mathrm{SO}(2r)$ are in one-to-one correspondence with $(f_1,\dots,f_r)$ such that

$$f_1 \geqslant \cdots \geqslant |f_r| \geqslant 0 \quad \text{and} \quad \begin{cases} \forall i \in \{1,\dots,r\},\ f_i \in \mathbb{Z} \\ \text{or} \quad \forall i \in \{1,\dots,r\},\ f_i \in \frac{1}{2}+\mathbb{Z} \end{cases}. \tag{A.6}$$

The characters of these irreducible representations are given by ($\ell_j = f_j + r - j$)

$$\chi_f^{\mathrm{SO}(2r)}(\tau) = \frac{|\tau_i^{\ell_j}+\tau_i^{-\ell_j}|_r + |\tau_i^{\ell_j}-\tau_i^{-\ell_j}|_r}{|\tau_i^{r-j}+\tau_i^{-r+j}|_r} = \frac{|\tau_i^{\ell_j}+\tau_i^{-\ell_j}|_r + |\tau_i^{\ell_j}-\tau_i^{-\ell_j}|_r}{2\prod_{1\leqslant i<j\leqslant r}(\tau_i+\tau_i^{-1}-\tau_i-\tau_j^{-1})}\,. \tag{A.7}$$

One should notice that, when $f_r = 0$, the second determinant in the numerator is 0 because its last column vanishes.

Since the Kirillov-Reshetikhin modules for the symmetric representations $(f_1,\dots,f_r) = (s,0,\dots,0)$ coincide with the usual irreducible D_r modules, so do the characters. They are given by

$$\chi_s(\tau) = h_s(\tau_1,\dots,\tau_r,\tau_1^{-1},\dots,\tau_r^{-1}) - h_{s-2}(\tau_1,\dots,\tau_r,\tau_1^{-1},\dots,\tau_r^{-1}) \tag{A.8}$$

where $h_{-2}=h_{-1}=0$ and h_s for $s \geqslant 0$ is the homogeneous symmetric polynomial defined by

$$h_s(x_1, \dots, x_p) = \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_s \leqslant p} x_{i_1} \cdots x_{i_s} \,. \tag{A.9}$$

We also have the following generating series:

$$\sum_{s=0}^{+\infty} t^s h_s(x_1, \dots, x_p) = \frac{1}{\prod_{k=1}^{p}(1 - t x_k)} \,, \quad \sum_{s=0}^{+\infty} t^s \chi_s(\tau) = \frac{1 - t^2}{\prod_{k=1}^{r}(1 - t\tau_k)(1 - t\tau_k^{-1})} \,. \tag{A.10}$$

B Q-function example: one site

For $N = 1$ the Q-operators are diagonal and we can read off the Q-functions. For $Q_1(x)$ we find

$$\begin{aligned}
(Q_1(x))_{11} = \tau_1^x \Bigg[& x^2 - x \sum_{k=2}^{r} \left(1 + \frac{\tau_k}{\tau_1 - \tau_k} + \frac{\tau_k^{-1}}{\tau_1 - \tau_k^{-1}} \right) \\
& + \sum_{k=2}^{r} \left[\frac{1}{(\tau_1 - \tau_k)(\tau_1 - \tau_k^{-1})} + \frac{\tau_k}{2(\tau_1 - \tau_k)} + \frac{\tau_k^{-1}}{2(\tau_1 - \tau_k^{-1})} \right] + \frac{2r-3}{4} \Bigg] \,,
\end{aligned} \tag{B.1}$$

$$(Q_1(x))_{ii} = \tau_1^x \left[x - \frac{1}{2} + \frac{\tau_1^{-1}}{\tau_1^{-1} - \tau_i} \right] , \qquad 1 < i \leq r \,, \tag{B.2}$$

$$(Q_1(x))_{ii} = \tau_1^x \left[x + \frac{1}{2} - \frac{\tau_{i'}}{\tau_1 - \tau_{i'}} \right] , \qquad r < i \leq 2r - 1 \,, \tag{B.3}$$

$$(Q_1(x))_{2r2r} = \tau_1^x \,. \tag{B.4}$$

C Crossing relations

C.1 Crossing symmetry of transfer matrix

The transfer matrix (2.11) satisfies the crossing relations

$$T_{1,s}(x) = T_{1,s}^t(-x)\Big|_{\tau_i \to \tau_i^{-1}} \tag{C.1}$$

We further note that when defining reflection matrix

$$\mathrm{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix} \,. \tag{C.2}$$

the twist parameters of the transfer matrix exchange: $\mathrm{J} T_{1,s}(x) \mathrm{J} = T_{1,s}(x)|_{\tau_i \to \tau_i^{-1}}$. It thus follows that

$$T_{1,s}(x) = \mathrm{J} T_{1,s}^t(-x) \mathrm{J} = T_{1,s}'(-x) \,. \tag{C.3}$$

C.2 Crossing symmetry of single-index Q-operators

In this appendix we discuss the derivation of the crossing relation for the single-index Q-operators. The corresponding Lax matrices satisfy

$$L^t(-z-1)|_{\text{p.h.}} = L(z)G\,. \tag{C.4}$$

here t denotes the transpose in the matrix space and "p.h." denotes the particle hole transformation

$$(\mathbf{a}_i, \bar{\mathbf{a}}_i)|_{\text{p.h.}} = (-\bar{\mathbf{a}}_i, \mathbf{a}_i) \tag{C.5}$$

and G is the diagonal matrix

$$G = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & -\mathrm{I} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\,. \tag{C.6}$$

Using the symmetries of the twist in the Q-operator

$$D|_{\tau_i\to\tau_i^{-1}} = \left(\tau_1^{-2}\right)^r D|_{\text{p.h.}} \tag{C.7}$$

we find that the normalised trace is independent of particle hole transformation. The extra factor above drops. We obtain

$$Q_1^t(-x)\Big|_{\tau_i\to\tau_i^{-1}} = Q_1(x)\,(G\otimes\ldots\otimes G) \tag{C.8}$$

Such equation holds for any Q_i. On the level of eigenvalues the transformation $G\otimes\ldots\otimes G$ only yields a possible sign, depending on the magnon number.

It further follows that

$$\begin{aligned} Q'_1(-x) &= (\mathrm{J}\otimes\ldots\otimes\mathrm{J})\,Q_1^t(-x)\,(\mathrm{J}\otimes\ldots\otimes\mathrm{J}) \\ &= (\mathrm{J}\otimes\ldots\otimes\mathrm{J})\,Q_1(x)|_{\tau_j\to\tau_j^{-1}}\,(\mathrm{J}\otimes\ldots\otimes\mathrm{J})\,(G\otimes\ldots\otimes G) \\ &= Q_{1'}(x)\,(G\otimes\ldots\otimes G)\,. \end{aligned} \tag{C.9}$$

D QQ-system of $A_3 \simeq D_3$

We show in this appendix that, as is expected from the isomorphism $A_3 \simeq D_3$, the known QQ-system for A_3 can be interpreted as the QQ-system for D_3, albeit in a particular gauge. We start with a reminder of the QQ-system for A_3: in order to avoid confusion, we shall denote $\mathcal{Q}_I$ for $I \subset \{1,2,3,4\}$ the Q-functions for A_3 and the SL(4) twists will be z_1, z_2, z_3 and z_4 such that $z_1z_2z_3z_4 = 1$. The following relations hold (neither i nor j belongs to I):

$$\mathcal{Q}_{I\cup\{i\}}^{[+1]}\mathcal{Q}_{I\cup\{j\}}^{[-1]} - \mathcal{Q}_{I\cup\{i\}}^{[-1]}\mathcal{Q}_{I\cup\{j\}}^{[+1]} = \frac{z_i - z_j}{\sqrt{z_iz_j}}\mathcal{Q}_I\mathcal{Q}_{I\cup\{i,j\}}\,. \tag{D.1}$$

From these relations, one can also deduce that

$$\begin{vmatrix} \mathcal{Q}_{\{i,j\}}^{[-2]} & \mathcal{Q}_{\{i,j\}} & \mathcal{Q}_{\{i,j\}}^{[+2]} \\ \mathcal{Q}_{\{i,k\}}^{[-2]} & \mathcal{Q}_{\{i,k\}} & \mathcal{Q}_{\{i,k\}}^{[+2]} \\ \mathcal{Q}_{\{i,l\}}^{[-2]} & \mathcal{Q}_{\{i,l\}} & \mathcal{Q}_{\{i,l\}}^{[+2]} \end{vmatrix} = \frac{(z_j-z_k)(z_j-z_l)(z_k-z_l)}{z_jz_kz_l}\mathcal{Q}_{\{i\}}^{[-1]}\mathcal{Q}_{\{i\}}^{[+1]}\mathcal{Q}_{\{1,2,3,4\}} \tag{D.2}$$

and

$$\begin{vmatrix} \mathcal{Q}^{[-2]}_{\{i,j\}} & \mathcal{Q}_{\{i,j\}} & \mathcal{Q}^{[+2]}_{\{i,j\}} \\ \mathcal{Q}^{[-2]}_{\{j,k\}} & \mathcal{Q}_{\{j,k\}} & \mathcal{Q}^{[+2]}_{\{j,k\}} \\ \mathcal{Q}^{[-2]}_{\{i,k\}} & \mathcal{Q}_{\{i,k\}} & \mathcal{Q}^{[+2]}_{\{i,k\}} \end{vmatrix} = \frac{(z_j - z_i)(z_j - z_k)(z_i - z_k)}{z_i z_j z_k} \mathcal{Q}^{[-1]}_{\{i,j,k\}} \mathcal{Q}^{[+1]}_{\{i,j,k\}} \mathcal{Q}_{\emptyset} \,. \tag{D.3}$$

Both of these equations are identified with equation (5.11). More generally, both QQ-systems are the same if one makes the following identification between the two sets of Q-functions:

$$Q_{\{1\}} = \mathcal{Q}_{\{1,2\}} \,, \qquad Q_{\{2\}} = \mathcal{Q}_{\{1,3\}} \,, \qquad Q_{\{3\}} = \mathcal{Q}_{\{1,4\}} \,, \tag{D.4}$$

$$Q_{\{1'\}} = \mathcal{Q}_{\overline{\{1,2\}}} = \mathcal{Q}_{\{3,4\}} \,, \qquad Q_{\{2'\}} = \mathcal{Q}_{\{2,4\}} \,, \qquad Q_{\{3'\}} = \mathcal{Q}_{\{2,3\}} \,, \tag{D.5}$$

$$S_{(+,+,+)} = \mathcal{Q}_{\{1\}} \,, \quad S_{(+,-,-)} = \mathcal{Q}_{\{2\}} \,, \quad S_{(-,+,-)} = \mathcal{Q}_{\{3\}} \,, \quad S_{(-,-,+)} = \mathcal{Q}_{\{4\}} \,, \tag{D.6}$$

$$S_{(-,-,-)} = \mathcal{Q}_{\{2,3,4\}} \,, \quad S_{(+,+,-)} = \mathcal{Q}_{\{1,2,3\}} \,, \quad S_{(+,-,+)} = \mathcal{Q}_{\{1,2,4\}} \,, \quad S_{(-,+,+)} = \mathcal{Q}_{\{1,3,4\}} \,. \tag{D.7}$$

The twists are related via

$$\tau_1 = z_1 z_2 = \frac{1}{z_3 z_4} \,, \quad \tau_2 = z_1 z_3 = \frac{1}{z_2 z_4} \,, \quad \tau_3 = z_1 z_4 = \frac{1}{z_2 z_3} \tag{D.8}$$

while the remaining Q-functions are

$$Q_{\emptyset} = 1 \,, \quad S_{+,\emptyset} = \mathcal{Q}_{\emptyset} \quad \text{and} \quad S_{-,\emptyset} = \mathcal{Q}_{\{1,2,3,4\}} \,. \tag{D.9}$$

The previous equation shows that in identifying the two QQ-systems we had to partly fix the gauge for D_3. This explains why in the A_3 QQ-system there are only two gauge degrees of freedom [40] while there are three of them for the D_3 one.

E Elements of the Hasse diagram

In this appendix we give a more detailed explanation of the Hasse diagrams in figure 4 and figure 5.

QQ-relations along the tail. Along the tail of the Dynkin diagram the QQ-relations are given in (5.1). They are depicted by the plaquette in figure 6.

QQ-relations at the spinor nodes. The QQ-relations for the spinorial nodes were introduced in (5.8) and (5.12). We depict them as the QQ-relations along the tail by a plaquette, see figure 7. Here we chose the opposite orientation of the arrows to avoid confusion with plaquettes at the fork, cf. figure 8. Further depending on the spinor node, we choose a blue or red color for the arrows.

$$
\begin{tikzcd}
& Q_{I\cup\{i\}\cup\{j\}} & \\
Q_{I\cup\{i\}} \arrow[ur] & & Q_{I\cup\{j\}} \arrow[ul] \\
& Q_I \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 6. Plaquette for QQ-relations along the tail depicted in green.

$$
\begin{tikzcd}
& S_{+,\emptyset} \arrow[dl] \arrow[dr] & & & & S_{-,\emptyset} \arrow[dl] \arrow[dr] & \\
S_I \arrow[dr] & & S_J \arrow[dl] & & S_I \arrow[dr] & & S_J \arrow[dl] \\
& Q_{I\cap J} & & & & Q_{I\cap J} &
\end{tikzcd}
$$

Figure 7. Spinorial QQ-relations for spinor nodes $\pm$ depicted in blue and red respectively.

QQ-relations at the fork. At the fork we have the QQ-relations (5.1) with $|J| = r-3$. In this case the Q-function at level $r-1$ factorises into two spinorial Q-functions, see (5.3). For $J = \{j_1, \dots, j_{r-3}\}$ we can write the QQ-relations as

$$
Q^{[+1]}_{J\cup\{j_{r-2}\}} Q^{[-1]}_{J\cup\{j_{r-1}\}} - Q^{[-1]}_{J\cup\{j_{r-2}\}} Q^{[+1]}_{J\cup\{j_{r-1}\}} = \frac{\tau_{j_{r-2}} - \tau_{j_{r-1}}}{\sqrt{\tau_{j_{r-2}}\tau_{j_{r-1}}}} Q_J S_{\{j_1,\dots,j_{r-1},j_r\}} S_{\{j_1,\dots,j_{r-1},j'_r\}}\,,
$$

where $S_{\{j_1,\dots,j_{r-1},j_r\}}$ and $S_{\{j_1,\dots,j_{r-1},j'_r\}}$ belong to different spinor nodes. We denote these relations by the "cat" shaped diagram in figure 8. To avoid confusion with the plaquettes in figure 6 and figure 7 we chose all arrows to be ingoing at level $r-2$.

F Details for the computations of section 8

F.1 Wronskian condition from QQ′-type constraints

Plugging the constraints (8.16) into equation (8.17) for $k = r-1$ we get

$$
C_{0,r-1,r-1} Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} = h_r \begin{vmatrix} Q_1^{[-r+1]} & Q_1^{[-r+3]} & \cdots & Q_1^{[r-1]} \\ \vdots & \vdots & & \vdots \\ Q_{r-1}^{[-r+1]} & Q_{r-1}^{[-r+3]} & \cdots & Q_{r-1}^{[r-1]} \\ Q_r^{[-r+1]} & Q_r^{[-r+3]} & \cdots & Q_r^{[r-1]} \end{vmatrix} \begin{vmatrix} Q_1^{[-r+1]} & Q_1^{[-r+3]} & \cdots & Q_1^{[r-1]} \\ \vdots & \vdots & & \vdots \\ Q_{r-1}^{[-r+1]} & Q_{r-1}^{[-r+3]} & \cdots & Q_{r-1}^{[r-1]} \\ Q_{r'}^{[-r+1]} & Q_{r'}^{[-r+3]} & \cdots & Q_{r'}^{[r-1]} \end{vmatrix}. \tag{F.1}
$$

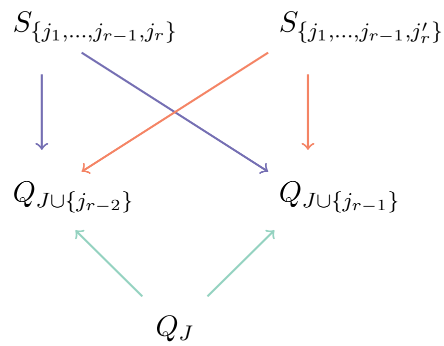

Figure 8. QQ-relations at the fork. To avoid confusion with the plaquette we introduce a different direction for the arrows pointing away from spinorial nodes.

Using the explicit expression of $C_{0,r-1,r-1}$ gives

$$W^{[-]}_{1,\dots,r-1}W^{[+]}_{1,\dots,r-1}Q^{[r-2]}_{\emptyset}Q^{[2-r]}_{\emptyset}=\frac{1}{\prod_{j=1}^{r-1}(u_j-u_r)}W_{1,\dots,r-1,r}W_{1,\dots,r-1,r'} \tag{F.2}$$

where we used the notation $W_{i_1,\dots,i_k}=\left|Q^{[k+1-2b]}_{\{i_a\}}\right|_k$. The derivation makes it clear that the previous identity still holds if one exchanges some $Q_{\{i\}}$ with $Q_{\{i'\}}$ so that one may actually write

$$W^{[-]}_{i_1,\dots,i_{r-1}}W^{[+]}_{i_1,\dots,i_{r-1}}Q^{[r-2]}_{\emptyset}Q^{[2-r]}_{\emptyset}=\frac{1}{\prod_{j\neq i_r,i'_r}^{r}(u_j-u_r)}W_{i_1,\dots,i_{r-1},i_r}W_{i_1,\dots,i_{r-1},i'_r}\,. \tag{F.3}$$

where we only assume that for all $1\leqslant a\neq b\leqslant r$ one has $\{i_a,i'_a\}\cap\{i_b,i'_b\}=\emptyset$. This is exactly equation (7.4).

F.2 Proof of equation (8.29)

We prove here the following claim: if $T_{a,s}$ satisfy the Hirota equations and $T_{1,s}$ is given by equation (8.15) then $T_{a,s}$ for $a\leqslant r-1$ is given by equation (8.29).

The proof is made by induction: the claim is true for $a=1$ by assumption and we have also shown, in the main text, that it is true for $a=2$. For higher a, the claim is clearly equivalent to equation (8.30) which is itself a particular case of the following identity:

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{1\leqslant i_1<\dots<i_a\leqslant 2r\\ 1\leqslant j_1<\dots<j_a\leqslant 2r}}\begin{vmatrix} W^{[+1]}_{i_1,\dots,i_a} & W^{[-1]}_{i_1,\dots,i_a}\\ W^{[+1]}_{j_1,\dots,j_a} & W^{[-1]}_{j_1,\dots,j_a}\end{vmatrix}\begin{vmatrix} \widetilde{W}^{[+1]}_{i_1,\dots,i_a} & \widetilde{W}^{[-1]}_{i_1,\dots,i_a}\\ \widetilde{W}^{[+1]}_{j_1,\dots,j_a} & \widetilde{W}^{[-1]}_{j_1,\dots,j_a}\end{vmatrix}\\
&=\left(\sum_{1\leqslant i_1<\dots<i_{a-1}\leqslant 2r}W_{i_1,\dots,i_{a-1}}\widetilde{W}_{i_1,\dots,i_{a-1}}\right)\left(\sum_{1\leqslant i_1<\dots<i_{a+1}\leqslant 2r}W_{i_1,\dots,i_{a+1}}\widetilde{W}_{i_1,\dots,i_{a+1}}\right)\,.
\end{aligned} \tag{F.4}$$

where $W_{i_1,\dots,i_a}=\left|Q^{[a+1-2k]}_{i_j}\right|_a$ and $\widetilde{W}_{i_1,\dots,i_a}=\left|P^{[a+1-2k]}_{i_j}\right|_a$ for $\{Q_i\}_{1\leqslant i\leqslant 2r}$ and $\{P_i\}_{1\leqslant i\leqslant 2r}$ two sets of arbitrary functions. In this appendix, most of the summation indices run from 1 to $2r$ so we will not write these bounds under each summation symbols in the following.

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The only indices for which it will be different will be called m, n, $\tilde{m}$ or $\tilde{n}$, the values they may take will be indicated each time.

We shall now prove (F.4). Let us start from the left-hand side, we expand each of the determinants W and $\widetilde{W}$ with respect to the columns with shifts $\pm a$, for instance: $W^{[+1]}_{i_1,\ldots,i_a} = \sum_{m=1}^{a}(-1)^{m+1}Q^{[a]}_{i_m}W_{i_1,\ldots,\widehat{i_m},\ldots i_a}$ and $W^{[-1]}_{j_1,\ldots,j_a} = \sum_{n=1}^{a}(-1)^{n+a}Q^{[-a]}_{j_n}W_{j_1,\ldots,\widehat{j_n},\ldots j_a}$ where the hat over an index means that we omit it. We thus obtain

$$\begin{aligned}
&\sum_{\substack{i_1<\ldots<i_a\\ j_1<\ldots<j_a}} \begin{vmatrix} W^{[+1]}_{i_1,\ldots,i_a} & W^{[-1]}_{i_1,\ldots,i_a} \\ W^{[+1]}_{j_1,\ldots,j_a} & W^{[-1]}_{j_1,\ldots,j_a} \end{vmatrix} \begin{vmatrix} \widetilde{W}^{[+1]}_{i_1,\ldots,i_a} & \widetilde{W}^{[-1]}_{i_1,\ldots,i_a} \\ \widetilde{W}^{[+1]}_{j_1,\ldots,j_a} & \widetilde{W}^{[-1]}_{j_1,\ldots,j_a} \end{vmatrix} \\
&= \frac{1}{(a!)^2} \sum_{\substack{i_1,\ldots,i_a\\ j_1,\ldots,j_a}} \begin{vmatrix} W^{[+1]}_{i_1,\ldots,i_a} & W^{[-1]}_{i_1,\ldots,i_a} \\ W^{[+1]}_{j_1,\ldots,j_a} & W^{[-1]}_{j_1,\ldots,j_a} \end{vmatrix} \begin{vmatrix} \widetilde{W}^{[+1]}_{i_1,\ldots,i_a} & \widetilde{W}^{[-1]}_{i_1,\ldots,i_a} \\ \widetilde{W}^{[+1]}_{j_1,\ldots,j_a} & \widetilde{W}^{[-1]}_{j_1,\ldots,j_a} \end{vmatrix} \\
&= \frac{1}{(a!)^2} \sum_{\substack{i_1,\ldots,i_a\\ j_1,\ldots,j_a\\ 1\leqslant m,n,\tilde{m},\tilde{n}\leqslant a}} (-1)^{m+n+\tilde{m}+\tilde{n}} \begin{vmatrix} Q^{[a]}_{i_m} & Q^{[-a]}_{i_m} \\ Q^{[a]}_{j_n} & Q^{[-a]}_{j_n} \end{vmatrix} \begin{vmatrix} P^{[a]}_{i_{\tilde{m}}} & P^{[-a]}_{i_{\tilde{m}}} \\ P^{[a]}_{j_{\tilde{n}}} & P^{[-a]}_{j_{\tilde{n}}} \end{vmatrix} \\
&\quad \times W_{i_1,\ldots,\widehat{i_m},\ldots i_a} W_{j_1,\ldots,\widehat{j_n},\ldots j_a} \widetilde{W}_{i_1,\ldots,\widehat{i_{\tilde{m}}},\ldots i_a} \widetilde{W}_{j_1,\ldots,\widehat{j_{\tilde{n}}},\ldots j_a} = L_1 + L_2 + L_3
\end{aligned} \tag{F.5}$$

where we have split the sums over m, n, $\tilde{m}$ and $\tilde{n}$ into three contributions L_1, L_2 and L_3. L_1 contains all the terms with $m = \tilde{m}$ and $n = \tilde{n}$, L_2 all the terms with $m = \tilde{m}$ and $n \neq \tilde{n}$ or $m \neq \tilde{m}$ and $n = \tilde{n}$ while L_3 contains all the terms with $m \neq \tilde{m}$ and $n \neq \tilde{n}$. In each of the three cases the remaining sums (over i's and j's) do not depend on the actual values of m, n, $\tilde{m}$ and $\tilde{n}$ anymore so that we can perform the sums over these latter indices. We thus get

$$L_1 = \left(\frac{1}{(a-1)!} \sum_{i_1,\ldots,i_{a-1}} W_{i_1,\ldots,i_{a-1}} \widetilde{W}_{i_1,\ldots,i_{a-1}} \right)^2 \sum_{i,j} \begin{vmatrix} Q^{[a]}_{i} & Q^{[-a]}_{i} \\ Q^{[a]}_{j} & Q^{[-a]}_{j} \end{vmatrix} \begin{vmatrix} P^{[a]}_{i} & P^{[-a]}_{i} \\ P^{[a]}_{j} & P^{[-a]}_{j} \end{vmatrix}, \tag{F.6}$$

$$\begin{aligned}
L_2 &= \frac{2}{(a-2)!} \left(\frac{1}{(a-1)!} \sum_{i_1,\ldots,i_{a-1}} W_{i_1,\ldots,i_{a-1}} \widetilde{W}_{i_1,\ldots,i_{a-1}} \right) \\
&\quad \times \left(- \sum_{i,j,k,i_1,\ldots,i_{a-2}} \begin{vmatrix} Q^{[a]}_{i} & Q^{[-a]}_{i} \\ Q^{[a]}_{j} & Q^{[-a]}_{j} \end{vmatrix} \begin{vmatrix} P^{[a]}_{i} & P^{[-a]}_{i} \\ P^{[a]}_{k} & P^{[-a]}_{k} \end{vmatrix} W_{k,i_1,\ldots,i_{a-2}} \widetilde{W}_{j,i_1,\ldots,i_{a-2}} \right),
\end{aligned} \tag{F.7}$$

$$\begin{aligned}
L_3 &= \frac{1}{((a-2)!)^2} \sum_{\substack{i_1,\ldots,i_{a-2}\\ j_1,\ldots,j_{a-2}\\ i,j,k,l}} \begin{vmatrix} Q^{[a]}_{i} & Q^{[-a]}_{i} \\ Q^{[a]}_{j} & Q^{[-a]}_{j} \end{vmatrix} \begin{vmatrix} P^{[a]}_{k} & P^{[-a]}_{k} \\ P^{[a]}_{l} & P^{[-a]}_{l} \end{vmatrix} \\
&\quad \times W_{k,i_1,\ldots,i_{a-2}} W_{l,j_1,\ldots,j_{a-2}} \widetilde{W}_{i,i_1,\ldots,i_{a-2}} \widetilde{W}_{j,j_1,\ldots,j_{a-2}}\,.
\end{aligned} \tag{F.8}$$

One can rewrite L_3 using the Plücker identity (6.7). We first use it to write

$$W_{k,i_1,\ldots,i_{a-2}} W_{l,j_1,\ldots,j_{a-1}} = W_{l,i_1,\ldots,i_{a-2}} W_{k,j_1,\ldots,j_{a-2}} + \sum_{p=1}^{a-2} (-1)^{p-1} W_{k,l,i_1,\ldots,\widehat{i_p},\ldots,i_{a-2}} W_{i_p,j_1,\ldots,j_{a-2}} \tag{F.9}$$

which we then plug in the expression for L_3, after some renaming of the indices this yields

$$L_3 = -\,L_3 + \frac{1}{(a-3)!(a-2)!} \sum_{\substack{i_1,\dots,i_{a-3}\\ j_1,\dots,j_{a-1}\\ i,j,k,l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \times W_{k,l,i_1,\dots,i_{a-3}} W_{j_1,\dots,j_{a-1}} \widetilde{W}_{i,j_1,i_1,\dots,i_{a-3}} \widetilde{W}_{j,j_2,\dots,j_{a-1}}\,. \tag{F.10}$$

This means that

$$L_3 = \frac{1}{2(a-3)!(a-2)!} \sum_{\substack{i_1,\dots,i_{a-3}\\ j_1,\dots,j_{a-1}\\ i,j,k,l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \times W_{k,l,i_1,\dots,i_{a-3}} W_{j_1,\dots,j_{a-1}} \widetilde{W}_{i,j_1,i_1,\dots,i_{a-3}} \widetilde{W}_{j,j_2,\dots,j_{a-1}}\,. \tag{F.11}$$

We now apply again the Plücker identity:

$$\begin{aligned}&\widetilde{W}_{i,j_1,i_1,\dots,i_{a-3}} \widetilde{W}_{j,j_2,\dots,j_{a-1}} \\ &= \widetilde{W}_{i,j,i_1,\dots,i_{a-3}} \widetilde{W}_{j_1,j_2,\dots,j_{a-1}} + \sum_{p=2}^{a-1} (-1)^p \widetilde{W}_{i,j_p,i_1,\dots,i_{a-3}} \widetilde{W}_{j,j_1,j_2,\dots,\widehat{j_p},\dots,j_{a-1}}\end{aligned} \tag{F.12}$$

so that

$$L_3 = \frac{1}{2(a-3)!(a-2)!} \sum_{\substack{i_1,\dots,i_{a-3}\\ j_1,\dots,j_{a-1}\\ i,j,k,l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \times W_{k,l,i_1,\dots,i_{a-3}} W_{j_1,\dots,j_{a-1}} \widetilde{W}_{i,j,i_1,\dots,i_{a-3}} \widetilde{W}_{j_1,\dots,j_{a-1}} - (a-2)L_3\,. \tag{F.13}$$

Finally, we arrive at the following expression:

$$L_3 = \frac{1}{2(a-3)!} \left(\frac{1}{(a-1)!} \sum_{i_1,\dots,i_{a-1}} W_{i_1,\dots,i_{a-1}} \widetilde{W}_{i_1,\dots,i_{a-1}} \right) \times \left(\sum_{i,j,k,l,i_1,\dots,i_{a-3}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} W_{k,l,i_1,\dots,i_{a-3}} \widetilde{W}_{i,j,i_1,\dots,i_{a-3}} \right)\,. \tag{F.14}$$

In order to prove (F.4) we need to show that

$$\frac{L_1+L_2+L_3}{2} = \left(\frac{1}{(a-1)!} \sum_{i_1,\dots,i_{a-1}} W_{i_1,\dots,i_{a-1}} \widetilde{W}_{i_1,\dots,i_{a-1}} \right) \times \left(\frac{1}{(a+1)!} \sum_{i_1,\dots,i_{a+1}} W_{i_1,\dots,i_{a+1}} \widetilde{W}_{i_1,\dots,i_{a+1}} \right)\,. \tag{F.15}$$

From expressions (F.6), (F.7) and (F.14) this is equivalent to showing that

$$\begin{aligned}
&\frac{(a+1)a}{2}\sum_{i,j,i_1,\ldots,i_{a-1}}\begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]}\end{vmatrix}\begin{vmatrix} P_i^{[a]} & P_i^{[-a]} \\ P_j^{[a]} & P_j^{[-a]}\end{vmatrix} W_{i_1,\ldots,i_{a-1}}\widetilde{W}_{i_1,\ldots,i_{a-1}} \\
&-(a+1)a(a-1)\sum_{i,j,k,i_1,\ldots,i_{a-2}}\begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]}\end{vmatrix}\begin{vmatrix} P_i^{[a]} & P_i^{[-a]} \\ P_k^{[a]} & P_k^{[-a]}\end{vmatrix} W_{k,i_1,\ldots,i_{a-2}}\widetilde{W}_{j,i_1,\ldots,i_{a-2}} \\
&+\frac{(a+1)a(a-1)(a-2)}{4}\sum_{i,j,k,l,i_1,\ldots,i_{a-3}}\begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]}\end{vmatrix}\begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]}\end{vmatrix} W_{k,l,i_1,\ldots,i_{a-3}}\widetilde{W}_{i,j,i_1,\ldots,i_{a-3}} \\
&=\sum_{i_1,\ldots,i_{a+1}} W_{i_1,\ldots,i_{a+1}}\widetilde{W}_{i_1,\ldots,i_{a+1}}\,.
\end{aligned} \tag{F.16}$$

This last identity is proven by expanding the determinants in the right-hand side with respect to their first and last columns:

$$W_{i_1,\ldots,i_{a+1}} = \sum_{1\leqslant m<n\leqslant a+1}(-1)^{m+n+a}\begin{vmatrix} Q_{i_m}^{[a]} & Q_{i_m}^{[-a]} \\ Q_{i_n}^{[a]} & Q_{i_n}^{[-a]}\end{vmatrix} W_{i_1,\ldots,\widehat{i_m},\ldots,\widehat{i_n},\ldots,i_{a+1}} \tag{F.17}$$

and

$$\widetilde{W}_{i_1,\ldots,i_{a+1}} = \sum_{1\leqslant \tilde{m}<\tilde{n}\leqslant a+1}(-1)^{\tilde{m}+\tilde{n}+a}\begin{vmatrix} P_{i_{\tilde{m}}}^{[a]} & P_{i_{\tilde{m}}}^{[-a]} \\ P_{i_{\tilde{n}}}^{[a]} & P_{i_{\tilde{n}}}^{[-a]}\end{vmatrix} \widetilde{W}_{i_1,\ldots,\widehat{i_{\tilde{m}}},\ldots,\widehat{i_{\tilde{n}}},\ldots,i_{a+1}}\,. \tag{F.18}$$

We then once again group the terms depending on the values of m, n, $\tilde{m}$ and $\tilde{n}$ and recover exactly the identity (F.16). There are indeed $\frac{(a+1)a}{2}$ terms with $(m,n)=(\tilde{m},\tilde{n})$, $(a+1)a(a-1)$ terms with $m=\tilde{m}$ and $n\neq\tilde{n}$ or $m\neq\tilde{m}$ and $n=\tilde{n}$, and $\frac{(a+1)a(a-1)(a-2)}{4}$ terms with $m\neq\tilde{m}$ and $n\neq\tilde{n}$.

G More on Weyl-type formulas

G.1 From QQ′-relations to Weyl-type formulas for $D_2\simeq A_1\oplus A_1$

Here we demonstrate for the examples of D_2 spin chains how to use the QQ′-relations to recover the Weyl-type formulas for T-functions. The Hasse diagram is depicted in figure 9.

From the two constraints

$$Q_1Q_{1'}+Q_2Q_{2'}=0\,, \tag{G.1}$$

and

$$Q_1^{[1]}Q_{1'}^{[-1]}+Q_1^{[-1]}Q_{1'}^{[1]}+Q_2^{[1]}Q_{2'}^{[-1]}+Q_2^{[-1]}Q_{2'}^{[1]}=Q_\emptyset^2\,, \tag{G.2}$$

cf. (8.16), we obtain

$$\left(\frac{Q_{1'}^{[1]}}{Q_2^{[1]}}-\frac{Q_{1'}^{[-1]}}{Q_2^{[-1]}}\right)=\frac{Q_\emptyset^2}{W_1} \tag{G.3}$$

where $W_n=Q_1^{[n]}Q_2^{[-n]}-Q_1^{[-n]}Q_2^{[n]}$.

Further on, excluding $Q_{2'}$ from

$$T_{1,s}=Q_1^{[s+1]}Q_{1'}^{[-s-1]}+Q_1^{[-s-1]}Q_{1'}^{[s+1]}+Q_2^{[s+1]}Q_{2'}^{[-s-1]}+Q_2^{[-s-1]}Q_{2'}^{[s+1]} \tag{G.4}$$

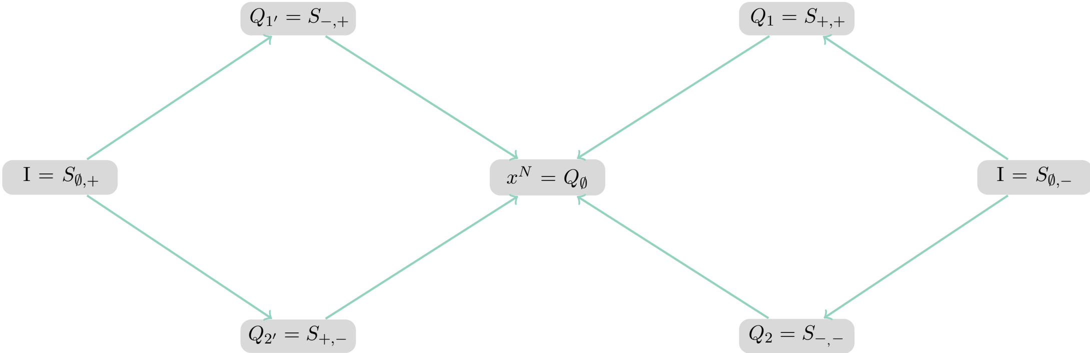

Figure 9. Hasse diagram for $D_2 \simeq A_1 \oplus A_1$.

we get

$$T_{1,s} = \left(\frac{Q_{1'}^{[s+1]}}{Q_2^{[s+1]}} - \frac{Q_{1'}^{[-s-1]}}{Q_2^{[-s-1]}} \right) W_{s+1} \,. \tag{G.5}$$

Excluding the difference in the first bracket in the r.h.s. using (G.3) we arrive at

$$T_{1,s} = W_{s+1} \sum_{l=0}^{s} \frac{(Q_\emptyset^{[2l-s]})^2}{W_1^{[2l-s]}} \,. \tag{G.6}$$

This coincides with the $r = 2$ case of the determinant formula (9.18).

G.2 Additional formulas in the general case

For the sake of completeness, we give here the Weyl-type formulas complementary to those given in section 9, i.e. for $T_{1,s}$ in terms of spinorial Q-functions:

$$\begin{aligned} T_{1,s} = {} & Q_\emptyset^{[r+s-2]} Q_\emptyset^{[2-r-s]} \\ & \times \left(\sum_{l=0}^{s} \frac{S_{I_r}^{[-s-1]} S_{I_r}^{[s+1]}}{S_{I_r}^{[2l-s-1]} S_{I_r}^{[2l-s+1]}} \frac{\left| S_i^{[-(2r+s-1-2j-2l\delta_{j,r})]} \right|_r}{\left| S_i^{[-(2r+s-1-2j)]} \right|_r} \frac{\left| S_i^{[2r+s-1-2j-2(s-l)\delta_{j,r}]} \right|_r}{\left| S_i^{[2r+s-1-2j]} \right|_r} \right. \\ & \left. - \sum_{l=1}^{s-1} \frac{S_{I_r}^{[-s-1]} S_{I_r}^{[s+1]}}{S_{I_r}^{[2l-s-1]} S_{I_r}^{[2l-s+1]}} \frac{\left| S_i^{[-(2r+s-1-2j-2(l-1)\delta_{j,r})]} \right|_r}{\left| S_i^{[-(2r+s-1-2j)]} \right|_r} \frac{\left| S_i^{[2r+s-1-2j-2(s-1-l)\delta_{j,r}]} \right|_r}{\left| S_i^{[2r+s-1-2j]} \right|_r} \right) \end{aligned} \tag{G.7}$$

and for $T_{a,1}$ in terms of single-index Q-functions:

$$\begin{aligned} T_{a,1} = &\frac{1}{\prod_{k=1}^{a-1} Q_{\emptyset}^{[r-a+2k]} Q_{\emptyset}^{[-(r-a+2k)]}} \sum_{\substack{0\leqslant k,l\leqslant a \\ a-k-l\in 2\mathbb{N}}} \left(\prod_{m=1}^{k} Q_{\emptyset}^{[r+a-2m]} Q_{\emptyset}^{[4+a-r-2m]} \right) \\ &\times \frac{\left| Q_{\{i\}}^{[r+a+3-2k-2j-2\theta(j+k-r-1)]} \right|_r}{\left| Q_{\{i\}}^{[r+a+3-2k-2j]} \right|_r} \prod_{m=1}^{\frac{a-k-l}{2}} \left(Q_{\emptyset}^{[r+a-2k-4m]} Q_{\emptyset}^{[4+a-r-2k-4m]} \right)^2 \\ &\times \frac{\left| Q_{\{i\}}^{[r+a+1-2k-2j]} \right|_r \left| Q_{\{i\}}^{[r+a-3-2k-2j]} \right|_r^2 \cdots \left| Q_{\{i\}}^{[2l+r+5-a-2j]} \right|_r^2 \left| Q_{\{i\}}^{[2l+r+1-a-2j]} \right|_r}{\left| Q_{\{i\}}^{[r+a-1-2k-2j]} \right|_r^2 \cdots \left| Q_{\{i\}}^{[2l+r+3-a-2j]} \right|_r^2} \\ &\times \left(\prod_{m=1}^{l} Q_{\emptyset}^{[2+2l-r-a-2m]} Q_{\emptyset}^{[2l+r-a-2m-2]} \right) \frac{\left| Q_{\{i\}}^{[-(r+a+3-2l-2j-2\theta(j+l-r-1))]} \right|_r}{\left| Q_{\{i\}}^{[-(r+a+3-2l-2j)]} \right|_r} . \end{aligned} \tag{G.8}$$

Notice that the first formula, expressing $T_{1,s}$ in terms of spinorial Q-functions, is much more complicated than the expression (9.18) in terms of fundamental Q-functions, whereas the second formula expressing $T_{a,1}$ in terms of fundamental Q-functions, is more complicated than (9.25) expressing it in terms of spinorial Q-functions.

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