INTRINSIC FRACTIONAL TAYLOR FORMULA
FORMULA DI TAYLOR INTRINSECA E FRAZIONARIA

MARIA MANFREDINI

Abstract. We consider a class of non-local ultraparabolic Kolmogorov operators and a suitable fractional Hölder spaces that take into account the intrinsic sub-riemannian geometry induced by the operators. We prove an intrinsic fractional Taylor formula in such spaces with global bounds for the remainder given in terms of the norm naturally associated to the differential operator.

Sunto. Consideriamo una classe di operatori ultraparabolici non locali di tipo Kolmogorov e opportuni spazi frazionari Hölderiani che tengano conto della geometria sub-riemanniana indotta dagli operatori. Dimostriamo una formula di Taylor frazionaria intrinseca in tali spazi con un resto che si esprime in termini della norma naturalmente associata agli operatori.

2010 MSC. Primary 35K70, 35R09, 35R03.
Keywords. non-local Kolmogorov operator, hypoelliptic operator, Hörmander's condition, intrinsic Taylor formula

1. Introduction

We consider a class of Kolmogorov-type hypoelliptic integro-differential operator of the form

$$Lu := \int_{\mathbb{R}^d} (u(t,x,v') - u(t,x,v))K(t,x,v,v')\,dv' + Yu, \qquad (t,x,v) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d, \tag{1}$$

where x, v are the spatial and the velocity variables respectively and Y is the transport term

$$Y = \langle v, \nabla_x\rangle + \partial_t. \tag{2}$$

Bruno Pini Mathematical Analysis Seminar, Vol. 12 (2021) pp. 1–14
Dipartimento di Matematica, Università di Bologna
ISSN 2240-2829.

Note that the integral diffusion term of L acts on the velocity variable only. The regularization effect on the x variable is a consequence of the interaction between the integral diffusion and the transport term. The equation $Lu = 0$ should be understood as a Kolmogorov-type hypoelliptic integrodifferential equation with diffusion term of fractional order. Fractional kinetic operators are used in nuclear- and astro-physics to model the behavior of a collection of particles moving through a plasma. If u is the density function of particles, with t, x, and v being time, space, and velocity respectively, then equation $Lu = 0$ states that these particles move freely through space with their velocities changing in a stochastic manner. If the velocity of a given particle varied according to the process of Wiener type, then u would obey a kinetic Fokker-Planck Equation. However, when the velocity of each particle varies according to a Levy process (without drift), the density function obeys $Lu = 0$. A Levy process allows individual particles to change velocity suddenly and discontinuously, which better approximates the effect of elastic collisions.

The kernel K in (1) belongs to a suitable ellipticity class, whose fundamental example is the homogeneous function

$$K(t, x, v, v') = \frac{1}{|v - v'|^{d+2s}},$$

where $s \in]0, 1[$. The corresponding functional L is the fractional kinetic Fokker-Planck operator

$$L = (-\Delta_v)^s + Y$$

and $(-\Delta_v)^s$ is the classical fractional Laplace operator of order s in the velocity variables.

The limiting case $s = 1$ corresponds to the diffusive operator and the Hölder spaces naturally associated to it were studied by several authors. See, for instance, Manfredini [10], Lunardi [9], Pascucci [13], Di Francesco and Polidoro [4]. In this case the operator has the remarkable property of being invariant with respect to left translations of a group $\left(\mathbb{R} \times \mathbb{R}^{2d}, \circ\right)$ introduced by Lanconelli and Polidoro in [8] studying a class of hypoelliptic ultraparabolic operators including the classical prototype operators of KolmogorovFokker-Planck. In this case the composition of the group coincides with the change of the Galilean variables in the phase space. See also the recent survey by Anceschi and Polidoro [1].

The non-commutative group law $\circ$ is defined by

$$z_1 \circ z_2 = (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)\,, \tag{3}$$

$z_1 = (t_1, x_1, v_1), z_2 = (t_2, x_2, v_2) \in \mathbb{R} \times \mathbb{R}^{2d}$. The identity and the inverse elements are $(0,0,0)$ and $(t,x,v)^{-1} = (-t, tv - x, -v)$ respectively.

Imbert and Silvestre in [7] consider a very general ellipticity class of kernel K. Put

$$K_{(t,x,v)}(w) = K(t,x,v,v+w)$$

and given the order $s \in]0,1[$ and ellipticity constants $0 < \lambda < \Lambda$ then

- $\circ$ $K(w) = K(-w)$.
- $\circ$ For all $r > 0$

$$\int_{B_r} |w|^2 K(w)\, dw \leq \Lambda r^{2-2s},$$

where B_r is a Euclidean ball in $\mathbb{R}^d$ of radius r.

- $\circ$ (Coercivity estimate) For any $r > 0$ and $\phi \in C^2(B_r)$

$$\int_{B_r}\int_{B_r} |\phi(v) - \phi(v')|^2 K(v - v')\, dv dv' \geq \lambda \int_{B_{r/2}}\int_{B_{r/2}} |\phi(v) - \phi(v')|^2 |v - v'|^{-d-2s}\, dv dv'.$$

In case $s <]0,1/2[$, we add the following non-degeneracy assumption

$$\inf_{|e|=1} \int_{B_r} (w \cdot e)_+^2 K(w)\, dw \geq \lambda r^{2-2s}.$$

We remark that for stable-like kernels of the form

$$K(w) = \frac{a(w/|w|)}{|w|^{d+2s}}$$

where a is a positive continuous function, the two conditions in the coercivity estimate are equivalent.

The class of equations $Lu = 0$ are left-invariant with respect the left translation $\circ$ defined in (3), in the sense that if $u = u(z)$ is a solution of $Lu = 0$, then $u_0(z) := u(z_0 \circ z)$ is also a solution of a similar equation with a translated kernel in the same ellipticity class.

We can also define dilations associated to L. Consider the family of vector fields $\partial_{v_1},\dots,\partial_{v_d},Y$ then it satisfies the Hörmander conditon

$$\text{Lie}\{\partial_{v_1},\dots,\partial_{v_d},Y\}(z)=\mathbb{R}\times\mathbb{R}^{2d},\qquad \forall\, z\in\mathbb{R}\times\mathbb{R}^{2d}. \tag{4}$$

In particular the only non null commutators are

$$[\partial_{v_i},Y]=\partial_{x_i},\qquad i=1,\dots,d. \tag{5}$$

We define the dilations $(D(\lambda))_{\lambda>0}$ on $\mathbb{R}\times\mathbb{R}^{2d}$ given by

$$D(\lambda)=\text{diag}\big(\lambda^{2s},\lambda^{2s+1}I_d,\lambda I_d\big), \tag{6}$$

where I_d is the $d\times d$ identity matrix. Then the vector fields Y and ∂_{v_i} are homogeneous of degree $2s$ and 1 with respect to $(D(\lambda))_{\lambda>0}$ respectively.

Moreover, if u solves the equation $Lu=0$ then $u^{(\lambda)}(z)=u(D(\lambda)z)$ solves a similar equation with a scaled kernel in the same ellipticity class.

We remark that the group $\big(\mathbb{R}\times\mathbb{R}^{2d},\circ,D(\lambda)\big)$ is a homogeneous Lie group in the sense of Folland and Stein [5]. Because of this property, it is very natural to define the homogeneous norm on $\mathbb{R}\times\mathbb{R}^{2d}$ as

$$\|(t,x,v)\|=|t|^{\frac{1}{2s}}+|x|^{\frac{1}{2s+1}}+|v|,\qquad (t,x,v)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d,$$

and consider the quasi-distance

$$\left\|z_2^{-1}\circ z_1\right\|,\qquad z_1,z_2\in\mathbb{R}\times\mathbb{R}^{2d}.$$

Actually, a weaker form of triangular inequality holds

$$||\zeta\circ z||\le c\left(||\zeta\circ\eta||+||\eta^{-1}\circ z||\right),\quad z,\zeta,\eta\in\mathbb{R}\times\mathbb{R}^{2d},$$

for a suitable positive constant c.

Imbert and Silvestre in [7] consider a equivalent left invariant quasi-distance

$$d(z_1,z_2)=\min_{w\in\mathbb{R}^d}\Big\{\max\left(|t_1-t_2|^{\frac{1}{2s}},|x_1-x_2-w(t_1-t_2)w|^{\frac{1}{1+2s}},|v_1-w|,|v_2-w|\right)\Big\},$$

$z_1=(t_1,x_1,v_1),z_2=(t_2,x_2,v_2)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$. They prove that if $s\ge 1/2$ then d is a distance in the sense that it satisfies the triangle inequality. When $s<1/2$, the function d^{2s} is a distance.

When the homogeneous structure induced by the dilations is considered, as usual, it is convenient to think of the Lie algebra generated by the vector fields $\{\partial_{v_1}, \dots, \partial_{v_d}, Y\}$ as a graded algebra i.e., it is equipped with the decomposition

$$span\,\{\partial_{v_1}, \dots, \partial_{v_d}\} \oplus span\,\{Y\} \oplus [span\,\{\partial_{v_1}, \dots, \partial_{v_d}\},\, span\,\{Y\}] =$$

$$= span\,\{\partial_{v_1}, \dots, \partial_{v_d}\} \oplus span\,\{Y\} \oplus span\,\{\partial_{x_1}, \dots, \partial_{x_d}\}.$$

Then, we associate a *formal degree* $m_Y = 2s$ to Y and $m_{\partial_{v_i}} = 1$ for $1 \le i \le d$ to the vector fields ∂_{v_i} with respect to the dilations $D(\lambda)$. Any partial derivative with respect to the spatial variable ∂_{x_i} obtained as a commutator of ∂_{v_i} and Y, has to be considered as $D(\lambda)$-homogeneous of degree $2s + 1$.

It is natural define the degree of a monomial m, as the number k so that $m(D(\lambda)z) = \lambda^k m(z)$. In other words, the exponent of the variable t should count times $2s$, every exponent of the variables x_i counts times $1 + 2s$ and every exponent of the variables v_i counts times 1. Note that the degree of a polynomial can be any number in the discrete set $\mathbb{N}_0 + 2s\mathbb{N}_0$.

If $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ will denote a multi-index then as usual $|\beta| := \sum_{j=1}^d \beta_j$ and, for any $x \in \mathbb{R}^d$, $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$.

Let $\alpha \in \mathbb{R}_{>0}$ we say that $p_\alpha = p_\alpha(z) = p_\alpha(t, x, v)$ is a homogeneous polynomial of degree α if p has the form

$$p_\alpha(z) := \sum_{0 \le 2sk + (1+2s)|\gamma| + |\beta| < \alpha} a_{k,\gamma,\beta}\, t^k\, x^\gamma\, v^\beta \tag{7}$$

with coefficients $a_{k,\gamma,\beta} \in \mathbb{R}$ and $(k, \gamma, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^d \times \mathbb{N}_0^d$.

Imbert and Silvestre in [7] have defined a properly scaled version of Hölder spaces. For any $\alpha > 0$, a function $f : D \subset \mathbb{R} \times \mathbb{R}^{2d} \to \mathbb{R}$ is α-Hölderian at a point $z_0 \in \mathbb{R} \times \mathbb{R}^{2d}$ if there exists a polynomial p_α of degree $< \alpha$ and a positive constant C such that such that

$$|f(z) - p_\alpha(z)| \le C\, ||z_0^{-1} \circ z||^\alpha \quad \forall z \in D.$$

When this property holds at every point z_0 in the domain D, with a uniform constant C, we say $f \in C^\alpha(D)$. The semi-norm of f is the smallest value of the constant C so that the inequality above holds for all $z_0, z \in D$.

In the context of Hölderian spaces, Taylor-type formulas constitute a fundamental tool for the development of regularity theory.

In the setting of homogeneous groups, Folland and Stein in [5] proved a classical result about intrinsic Taylor polynomials. Recently, Bonfiglioli in [3] derived an explicit formula for Taylor polynomials on homogeneous groups adapting the classical Taylor formula with integral remainder. In [5] and [3], Taylor polynomials of order n are defined for functions that are differentiable up to order n in the Euclidean sense so that the remainders depend on the norms of the function in the Euclidean Hölder spaces.

Pagliarani, Pascucci and Pignotti in [12] proved a new and more explicit representation of the intrinsic Taylor polynomials for Kolmogorov-type homogeneous groups. They define n-th order Taylor polynomials for functions that are regular in the intrinsic sense and the constants appearing in the error estimates depend only on the norms of the intrinsic derivatives up to order n. A similar result under such intrinsic regularity assumptions only appeared in Arena, Caruso and Causa in [2] in the particular case of Carnot group of step two. The approach in [12] is different from the classic one of Folland and Stein. Indeed, in [5] a classical representation is given as a sum over all possible permutations of the derivatives. In our context since the vector fields $\partial_{v_1}, \dots, \partial_{v_d}$ do not commute with the drift Y, there are different representations for the Taylor polynomials depending on the order of the derivatives. In [12] the authors order the vector fields in a privileged way, in this way they are able to get compact Taylor polynomials with a number of terms increasing linearly with respect to the order of the polynomial itself.

In collaboration with Polidoro and Pagliarani in [11], we extend such formula to the fractional Hölder spaces introduced in [7] associated to the the non-local ultraparabolic Kolmogorov operator L in (1).

2. Hölder spaces and Taylor polynomials

We introduce the notions of intrinsic Hölder regularity and intrinsic Hölder space. We recall that if X is a locally Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^{2d}$. For any $z \in \mathbb{R} \times \mathbb{R}^{2d}$, we denote by $\delta \to e^{\delta X}(z)$ the integral curve of X defined as the unique solution of the problem

$$\begin{cases} \frac{d}{d\delta}e^{\delta X}(z) = X\left(e^{\delta X}(z)\right), & \delta \in \mathbb{R}, \\ e^{\delta X}(z)|_{\delta=0} = z. \end{cases} \tag{8}$$

In particular, a direct calculation shows that the integral curves of ∂_{v_i} and of Y are

$$e^{\delta\partial_{v_i}}(t,x,v) = (t,x,v+\delta e_i), \quad i = 1,\dots,d, \qquad e^{\delta Y}(t,x,v) = (t+\delta, x+\delta v, v), \tag{9}$$

for any $(t,x,v) \in \mathbb{R}\times\mathbb{R}^{2d}$.

Next we recall the general notion of Lie differentiability. Let X be a Lipschitz vector field and u be a real-valued function defined in a neighborhood of $z \in \mathbb{R}^{1+2d}$. We say that u is X-*differentiable* in z if the function $\delta \mapsto u\left(e^{\delta X}(z)\right)$ is differentiable at $\delta = 0$. We will refer to the function $z \mapsto \frac{d}{d\delta}u\left(e^{\delta X}(z)\right)\big|_{\delta=0}$ as X-*Lie derivative of* u.

Definition 2.1. *Let X be a Lipschitz vector field on $\mathbb{R}\times\mathbb{R}^{2d}$ with formal degree $m_X > 0$. For $\alpha \in]0, m_X]$, we say that $u \in C^{\alpha}_X$ if the semi-norm*

$$||u||_{C^{\alpha}_X} := \sup_{z\in\mathbb{R}^{1+2d},\delta\in\mathbb{R}\setminus\{0\}} \frac{\left|u\left(e^{\delta X}(z)\right) - u(z)\right|}{|\delta|^{\frac{\alpha}{m_X}}} < +\infty.$$

Hereafter, let us denote by $(\alpha_n)_{n\in\mathbb{N}_0}$ the sequence given by the ordered elements (with no repetition) of the set

$$\{\alpha > 0 : \alpha = n + 2s\,m, \text{ with } n,m \in \mathbb{N}_0\}. \tag{10}$$

The number α_n denotes the orders at which there is a jump in the regularity and new derivatives along ∂_{v_i} or Y appear.

Now, intuitively it appears that the intrinsic Hölder spaces induced by the formal degrees of the vector fields will be considerably different in the cases $s \in]0,1/2[$ and $s \in [1/2,1[$. In this note we assume that $s \in [\frac{1}{2},1[$, the case $]0,1/2[$ will be considered in [11].

Now we define the intrinsic Hölder spaces on the homogeneous group $\left(\mathbb{R}\times\mathbb{R}^{2d},\circ\right)$, by extending the definitions of Hölder spaces given in [12].

Definition 2.2. *Let $s \in [\frac{1}{2},1[$. Then:*

i) *if* $\alpha \in]0, \alpha_1 = 1]$, *then* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *if* $u \in C^{\alpha}_{Y}(\mathbb{R} \times \mathbb{R}^{2d})$ *and* $u \in C^{\alpha}_{\partial_{v_i}}(\mathbb{R} \times \mathbb{R}^{2d})$ *for any* $i = 1, \dots, d$. *For any* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *we define the semi-norm*

$$\|u\|_{C^{\alpha}} := \|u\|_{C^{\alpha}_{Y}} + \sum_{i=1}^{d} \|u\|_{C^{\alpha}_{\partial_{v_i}}}\,; \tag{11}$$

ii) *if* $\alpha \in]\alpha_1 = 1, \alpha_2 = 2s]$, *then* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *if* $u \in C^{\alpha}_{Y}(\mathbb{R} \times \mathbb{R}^{2d})$ *and* $\partial_{v_i} u \in C^{\alpha-1}(\mathbb{R} \times \mathbb{R}^{2d})$ *for any* $i = 1, \dots, d$. *For any* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *we define the semi-norm*

$$\|u\|_{C^{\alpha}} := \|u\|_{C^{\alpha}_{Y}} + \sum_{i=1}^{d} \|\partial_{v_i} u\|_{C^{\alpha-1}}\,; \tag{12}$$

iii) *if* $\alpha \in]\alpha_n, \alpha_{n+1}]$, *for* $n \in \mathbb{N}$ *with* $n \geq 2$, *then* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *if* $Yu \in C^{\alpha-2s}(\mathbb{R} \times \mathbb{R}^{2d})$ *and* $\partial_{v_i} u \in C^{\alpha-1}(\mathbb{R} \times \mathbb{R}^{2d})$ *for any* $i = 1, \dots, d$. *For any* $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ *we define the semi-norm*

$$\|u\|_{C^{\alpha}} := \|Yu\|_{C^{\alpha-2s}} + \sum_{i=1}^{d} \|\partial_{v_i} u\|_{C^{\alpha-1}}\,. \tag{13}$$

Note that α integer does not imply the existence of derivatives of order α but only the Lipscitzianity of the derivative of lower order, which is explained in Remark 2.3 below.

Remark 2.1. *With the aim of clarifying the definition of Hölder space we consider for example* $s = 3/4$. *In this case*

$$\alpha_1 = 1, \quad \alpha_2 = 2s = 3/2, \quad \alpha_3 = 2, \quad \alpha_4 = 2s + 1 = 5/2, \quad \alpha_5 = 4s = 3, \quad \dots$$

If $u \in C^{\alpha}$ *with* $\alpha = 2 \in]3/2, 2]$ *then by definition the Lie derivative* Yu *exists as well and belongs to* $C^{1/2}$. *Moreover, the first order derivatives* $\partial_{v_1} u, \dots, \partial_{v_d} u \in C^1$. *This doesnt mean that are differentiable functions, but that belong to the Hölder space* C^{α} *with with* $\alpha = 1$.

Otherwise, if $u \in C^{\alpha}$ *with* $\alpha \in]2, 5/2]$, *then* $\alpha - 1 \in]1, 3/2]$ *so that* $\partial_{v_i} u \in C^{\alpha-1}$. *Besides, the second order derivatives* $\partial_{v_i v_j} u$ *exist for* $i, j = 1, \dots, d$. *In particular,* $\partial_{v_i v_j} u \in C^{\alpha-2}$ *and* $\partial_{v_i} u \in C^{\alpha-1}_{Y}$. *Also* $Yu \in C^{\alpha-3/2}$ *since* $\alpha - 3/2 \in]0, 1]$.

Remark 2.2. *It is known that the Euclidean Hölder functions of order greater than one are constant. In our setting a function* $u = u(v, x) \in C^{\alpha}$ *with* $\alpha > 1$ *is a function of* *just the variable x. The α-Hölder regularity is stronger along the fields ∂_{v_i} than along the field Y. In fact, if $\alpha > 1 = \alpha_1$, then a function cannot belong to $C^{\alpha}_{\partial_{v_i}}$ unless it is differentiable and its derivative is constantly null, which implies the function is constant along the directions $\partial_{v_1}, \dots, \partial_{v_d}$. But, it is possible that $u \in C^{\alpha}_{Y}$ with $\alpha \le 3/2 = \alpha_2$, as the Lie derivative along Y only appears when $\alpha > 3/2$.*

Consider the Euclidean Lipschitz continuous function

$$u(t, v, x) = |x|$$

Then $u \in C^{2s}_{loc}(\mathbb{R}^3)$ with $s \ge \frac{1}{2}$. In fact, obviously $\partial_v u \in C^{2s-1}$ and for every $\Omega \subset\subset \mathbb{R}^3$ and $m_Y = 2s$

$$\begin{aligned} ||u||_{C^{2s}_{Y}(\Omega)} &:= \sup_{z\in\Omega} |u(z)| + \sup_{z\in\Omega, \delta\in\mathbb{R}\setminus\{0\}} \frac{\left|u\left(e^{\delta Y}(z)\right) - u(z)\right|}{|\delta|^{\frac{2s}{m_Y}}} \\ &= \sup_{z\in\Omega} |u(z)| + \sup_{z\in\Omega, \delta\in\mathbb{R}\setminus\{0\}} \frac{||x+\delta v| - |x||}{|\delta|} < +\infty. \end{aligned}$$

Then $u \in C^{2s}_{Y,loc}$. Note that $u \notin C^{\alpha}_{loc}$ with $\alpha > 2s$ since u is not differentiable along Y.

Next, consider the function

$$u(t, v, x) = |x|^{1+\frac{1}{2s}},$$

then $u \in C^{1,\frac{1}{2s}}_{Eucl}$. But, the derivative Yu exists (derivative of order $2s$):

$$Yu = (1 + 1/(2s))\, v\, |x|^{\frac{1}{2s}}\, sgn(x) \in C^{1}_{Y,loc}$$

then $Yu \in C^{1}_{loc}$ and $u \in C^{2s+1}_{loc}$.

Theorem 2.1. (Taylor formula) *Let $u \in C^{\alpha}(\mathbb{R} \times \mathbb{R}^{2d})$ then there exist the derivatives*

$$Y^k \partial_\xi^\gamma \partial_\nu^\beta u \in C^{\alpha - 2sk - (1+2s)|\gamma| - |\beta|}(\mathbb{R} \times \mathbb{R}^{2d}), \qquad 0 \le 2sk + (1+2s)|\gamma| + |\beta| < \alpha;$$

and there exists a positive constant c which depends on the dimension d such that

$$|u(z) - T_\alpha u(\zeta, z)| \le c\, \|u\|_{C^\alpha} \|\zeta^{-1} \circ z\|^\alpha, \qquad \forall\, z, \zeta,$$

where $T_\alpha u(\zeta, \cdot)$ is the α-th order intrinsic Taylor polynomial of u around ζ *defined as*

$$T_\alpha u(\zeta, z) := \sum_{0 \le 2sk + (1+2s)|\gamma| + |\beta| < \alpha} \frac{Y^k \partial_\xi^\gamma \partial_\nu^\beta u(\zeta)}{k!\, \gamma!\, \beta!} (t-\tau)^k \big(x - \xi - (t-\tau)\nu\big)^\gamma (v - \nu)^\beta,$$

$z = (t, x, v), \zeta = (\tau, \xi, \nu)$.

Remark 2.3. *There are qualitative differences in the Taylor polynomials of different orders. In fact, the existence of the Euclidean spatial derivatives is not a consequence of definition of Hölder spaces. Such problem arises when defining the Taylor expansion of order* $> 2s+1$*, i.e. when the Euclidean derivatives appear for the first time in the Taylor polynomial. Although, formally we have* $[\partial_{v_i}, Y]u = \partial_{x_i} u$.

Indeed, if we suppose Taylor formula for $\alpha \le 2s+1$ *and we would prove Taylor formula for* $\tilde{\alpha} \in]2s+1, 2s+2]$ *then the polynomial* $T_{\tilde{\alpha}}u$ *contains the derivatives of intrinsic order equal to* $2s+1+\epsilon$ *with* $\epsilon \in]0,1]$*:*

$$Y^k \partial_\xi^\gamma \partial_\nu^\beta u, \qquad 0 \le 2sk + (1+2s)|\gamma| + |\beta| < 2s+1+\epsilon.$$

In particular, contains

$$Y^k \partial_\nu^\beta u, \qquad 0 \le 2sk + |\beta| < 2s+1+\epsilon;$$

whose existence follows by definition of $C^{\tilde{\alpha}}$*. Otherwise,* $T_{\tilde{\alpha}}u$ *also contains the* $2s+1$ *order spatial derivatives* ∂_{ξ_i}*, for* $i = 1, \dots, d$ *whose existence must be proved.*

3. Sketch of the proof

Remark 2.3 suggests that the proof of Theorem 2.1 cannot be carried out by a simple induction on α_n, due to the qualitative differences in the Taylor polynomials of different orders.

We recall that $s \in [1/2, 1[$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ is the sequence given by the ordered elements of the set $\mathbb{N}_0 + 2s\mathbb{N}_0$.

We have the following initial steps:

$$]\alpha_0, \alpha_1] =]0,1], \quad]\alpha_1, \alpha_2] =]1, 2s], \quad]\alpha_2, \alpha_3] =]2s, 2], \quad]\alpha_4, \alpha_5] =]2, 2s+1].$$

If $\alpha \le \alpha_5 = 2s+1$ the derivatives in the Taylor expansion of order α exist by definition of Hölder spaces. For $\alpha \in]\alpha_5, \alpha_6]$, there is a jump in the regularity, indeed the Euclidean derivatives w.r.t. the spatial variables appear for the first time in the Taylor polynomial and then we must also prove the existence of such derivatives. Besides, in the future steps the existence of the Euclidean partial derivatives w.r.t. any variable has already been proved and thus the proof goes smoothly without any further complication.

First, we note that it is not restrictive to prove Taylor formula for t constant:

Remark 3.1. *Suppose $\alpha > 2s$, (for $\alpha \leq 2s$ the proof is easier). Let $z = (t, x, v)$, $\zeta = (\tau, \xi, w)$ be such that with $t \neq \tau$. Consider the integral curve of Y and put*

$$\bar{\zeta} = e^{(t-\tau)Y}(\zeta) = (t, \xi + (t-\tau)w, w).$$

We have

$$\bar{\zeta}^{-1} \circ z = (0, x - \xi - (t-\tau)w, v - w), \quad \zeta^{-1} \circ z = (t-\tau, x - \xi - (t-\tau)w, v - w).$$

Note that, if $u \in C^\alpha$, with $\alpha > 2s$ then $Y^k u \in C^\alpha$ for $0 < 2sk < \alpha$. By the Euclidean mean-value theorem along the vector field Y, for any ζ and $r \in \mathbb{R}$ there exists $\bar{r}$, $|\bar{r}| \leq |r|$ such that

$$u(e^{rY}(\zeta)) - u(\zeta) - \sum_{0<2sk<\alpha} \frac{r^k}{k!} Y^k u(\zeta) = r^{k_1} \left(Y^{k_1} u(e^{\bar{r}Y}(\zeta)) - Y^{k_1} u(\zeta) \right)$$

where $k_1 = max\{k \in \mathbb{N} : 0 < 2sk < \alpha\}$.

By Taylor formula with t constant there exists $c > 0$ such that

$$|u(z) - T_\alpha(\zeta, z)| = |u(z) - T_\alpha(\bar{\zeta}, z)| + |T_\alpha(\bar{\zeta}, z) - T_\alpha(\zeta, z)|$$

$$\leq c||u||_{C^\alpha} ||\zeta^{-1} \circ z||^\alpha + |T_\alpha(\bar{\zeta}, z) - T_\alpha(\zeta, z)|.$$

Rearranging the Taylor polynomials we can write

$$T_\alpha(\bar{\zeta}, z) - T_\alpha(\zeta, z) = \sum_{(1+2s)|\gamma|+|\beta|<\alpha} \frac{1}{\gamma!\,\beta!} \Bigg[\partial_\xi^\gamma \partial_\nu^\beta u(e^{(t-\tau)Y}(\zeta)) -$$

$$- \sum_{2sk<\alpha-(1+2s)|\gamma|-|\beta|} \frac{Y^k \partial_\xi^\gamma \partial_\nu^\beta u(\zeta)(t-\tau)^k}{k!} \Bigg] \big(x - \xi - (t-\tau)w\big)^\gamma (v - w)^\beta.$$

By mean-value theorem, for a suitable k_1

$$|T_\alpha(\bar{\zeta}, z) - T_\alpha(\zeta, z)| \leq$$

$$\leq \sum_{(1+2s)|\gamma|+|\beta|<\alpha} |(t-\tau)^{k_1} \left(Y^{k_1} \partial_\xi^\gamma \partial_\nu^\beta u(e^{\bar{r}Y}(\zeta)) - Y^{k_1} \partial_\xi^\gamma \partial_\nu^\beta u(\zeta) \right)| \, ||\bar{\zeta}^{-1} \circ z||^{(1+2s)|\gamma|+|\beta|}$$

by definition of Hölder spaces

$$\leq c\,||u||_{C^a} \sum_{(1+2s)|\gamma|+|\beta|<\alpha} (t-\tau)^{k_1} \bar{r}^{(\alpha-2sk_1-(1+2s)|\gamma|-|\beta|)/(2s)} ||\bar{\zeta}^{-1}\circ z||^{(1+2s)|\gamma|+|\beta|}$$

$$\leq c\,||u||_{C^a}||\zeta^{-1}\circ z||^{\alpha}.$$

In general, any two points of $\mathbb{R}\times\mathbb{R}^{2d}$ can be connected via integral curves of the vector fields and their commutators but we have no information about the regularity along the directions of the commutators

$$[\partial_{v_i}, Y] = \partial_{x_i}, \qquad i = 1, \ldots, d.$$

Then, the idea is to approximate in the classical way the integral curves of the commutators by using the integral curves of $\pm\partial_{v_i}$ and $\pm Y$.

Remark 3.2. *Let $z = (t, x, v)$ and $\zeta = (t, \xi, w)$. Intuitively, spatial variables and velocity variables belong to different "level", then to connect z and ζ, we progressively correct the velocity variables v and then the spatial variable x.*

We first correct the velocity variables using the integral curve (of the first layer) of $Y_\eta^{(0)} = \sum_{i=1}^{d} \eta_i \partial_{v_i}$ where $\eta \in \mathbb{R}^d$ will be chosen appropriately:

$$\gamma_{\eta,r}^{(0)}(\zeta) := e^{r\,Y_\eta^{(0)}}(\zeta) = (t, \xi, w + r\eta).$$

If $r = ||v-w||$ and $\eta = \frac{v-w}{||v-w||}$ then we have $\gamma_{\eta,r}^{(0)}(\zeta) = (t, \xi, v)$.

Finally, we correct the space variables using the integral curve which approximate the integral curves of the commutators $Y_\eta^{(1)} = [Y_\eta^{(0)}, Y]$ by using a classical technique adapted to the fractional context and by moving along a curve defined as concatenation of integral paths of ∂_{v_i} and Y as follows

$$\gamma_{\eta,r}^{(1)}(\tilde{\zeta}) := e^{-r^{2s}\,Y}\left(\gamma_{\eta,-r}^{(0)}\left(e^{r^{2s}\,Y}\left(\gamma_{\eta,r}^{(0)}(\tilde{\zeta})\right)\right)\right) = (t, \xi + r^{2s+1}\eta, v)$$

where $\tilde{\zeta} = (t, \xi, v)$. Then choosing $r = ||x-\xi||^{\frac{1}{2s+1}}$ and $\eta = \frac{x-\xi}{||x-\xi||}$ we have

$$\gamma_{\eta,r}^{(1)}(t, \xi, v) = (t, x, v).$$

In other words, we can connect two points $z = (t, x, v)$ and $\zeta = (t, \xi, w)$, by only moving along the integral curves $\gamma_{\eta,r}^{(0)}$ and $\gamma_{\eta,r}^{(1)}$.

In these notes we proof only the first steps of the *induction*, the next steps will be proved in [11].

Proof of the Taylor formula: the first step of induction. The proof of the first step $\alpha \in]\alpha_0, \alpha_1] =]0,1]$ contains the idea of proof of the following ones, but technically it is much simpler. In this case Taylor's formula takes the form

$$|u(z) - u(\zeta)| \le c\, ||u||_{C^\alpha}\, ||\zeta^{-1} \circ z||^\alpha,$$

that is u is Hölder continuous according to definition in [14].

Roughly speaking,, we connect any pair of points $z = (t,x,v)$ and $\zeta = (t, \xi, w)$ using the integral curve $\gamma^{(0)}_{\eta,r}$ and $\gamma^{(1)}_{\eta,r}$ to have a control of the increment of u along the connecting path. We write

$$|u(t,x,v) - u(t,\xi,w)| \le |u(t,\xi,v) - u(t,\xi,w)| + |u(t,\xi,v) - u(t,x,v)|.$$

Since $\gamma^{(0)}_{\eta,r}$ is a integral curve of the vector fields ∂_{v_i} and $u \in C^\alpha_{\partial_{v_i}}$ we get

$$|u(t,\xi,v) - u(t,\xi,w)| = |u(\gamma^{(0)}_{\eta,r}(\zeta)) - u(\zeta)| \le c\, ||u||_{C^\alpha}\, |v-w|^\alpha \le c\, ||u||_{C^\alpha}\, ||\zeta^{-1} \circ z||^\alpha.$$

To evaluate the term $|u(t,\xi,v) - u(t,x,v)|$ we write

$$\begin{aligned} |u(t,\xi,v) - u(t,x,v)| =& |u(\tilde\zeta) - u(\gamma^{(1)}_{\eta,r}(\tilde\zeta))| \le \\ \le& |u(\tilde\zeta) - u(\gamma^{(0)}_{\eta,r}(\tilde\zeta))| \\ &+ |u(\gamma^{(0)}_{\eta,r}(\tilde\zeta)) - u(e^{r^{2s}Y}(\gamma^{(0)}_{\eta,r}(\tilde\zeta)))| \\ &+ |u(e^{r^{2s}Y}(\gamma^{(0)}_{\eta,r}(\tilde\zeta))) - u((\gamma^{(0)}_{\eta,-r}(e^{r^{2s}Y}(\gamma^{(0)}_{\eta,r}(\tilde\zeta)))| \\ &+ |u((\gamma^{(0)}_{\eta,-r}(e^{r^{2s}Y}(\gamma^{(0)}_{\eta,r}(\tilde\zeta))) - u(\gamma^{(1)}_{\eta,r}(\tilde\zeta))|. \end{aligned} \tag{14}$$

Then we use the regularity assumption of u along ∂_{v_i} ($u \in C^\alpha_{\partial_{v_i}}$) to estimate the first and the third terms in the right hand side and the regularity assumption of u along Y ($u \in C^\alpha_Y$) to estimate the other terms, obtaining $|u(t,\xi,v) - u(t,x,v)| \le c\, ||u||_{C^\alpha}\; ||\zeta^{-1} \circ z||^\alpha$, for suitable positive constant c.

References

[1] F. Anceschi and S. Polidoro. *A survey on the classical theory for Kolmogorov equation*, preprint.

[2] G. Arena, A.O. Caruso, A. Causa. *Taylor formula on step two Carnot groups.* Rev. Mat. Iberoam. **26(1)** (2010) 239-259.

[3] A. Bonfiglioli. *Taylor formula for homogeneous groups and applications.* Math. Z., **262(2)** (2009) 255279.

[4] M. Di Francesco, S. Polidoro. *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form.* Adv. Differential Equations **11(11)**, (2006) 12611320

[5] G. B. Folland, E. M. Stein. *Hardy spaces on homogeneous groups.* Volume 28 of Mathematical Notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1982).

[6] L. Hörmander. *Hypoelliptic second order differential equations.* Acta Math., **119** (1967) 147171.

[7] C. Imbert, L. Silvestre. *The Schauder estimate for kinetic integral equations.* Anal. PDE, **14(1)** (2021) 171-204.

[8] E. Lanconelli, S. Polidoro. *On a class of hypoelliptic evolution operators.* Rend. Sem. Mat. Univ. Politec. Torino, **52** (1994) 29-63.

[9] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in* $\mathbb{R}$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **(4) 24(1)**, (1997) 133164.

[10] M. Manfredini. *The Dirichlet problem for a class of ultraparabolic equations.* Adv. Differential Equations **2(5)** (1997) 831866.

[11] M. Manfredini, S. Pagliarani, S. Polidoro. *Intrinsic fractional Hölder spaces.* In preparation.

[12] S. Pagliarani, A.Pascucci, M. Pignotti. *Intrinsic Taylor formula for Kolmogorov-type homogeneous groups.* J. Math. Anal. Appl., **435** (2016) 1054-1087.

[13] A. Pascucci. *Hölder regularity for a Kolmogorov equation.* Trans. Amer. Math. Soc. **355(3)**, (2003) 901924.

[14] S. Polidoro. *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type.* Matematiche (Catania), **49** (1995) 53-105.

Dipartimento di Scienze Fisiche, Informatiche e Matematiche Università di Modena e Reggio Emilia, Modena, Italy.

E-mail address: maria.manfredini@unimore.it