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A periodic review policy with quality improvement, setup cost reduction, backorder price discount, and controllable lead time

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ABSTRACT

This paper explores a periodic review inventory model under stochastic demand. The setup (or ordering) cost and the lead time are controllable. The model considers an imperfect production process, whose quality can be improved by means of an investment. A backorder price discount to motivate customers to wait for backorders is included. The demand in the protection interval is first assumed Gaussian; then, the distribution-free approach is adopted. The objective is to determine the review period, the setup cost, the quality level, the backorder price discount, and the length of lead time that minimize the long-run expected total cost per time unit. A solution method for each case is presented. Numerical experiments show that substantial savings can be achieved if the quality level, the setup cost and the lead time are controlled, and if a backorder price discount is applied. A sensitivity analysis is finally carried out.

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Inventory; periodic review; investment; stochastic; optimization

1. Introduction

Inventories can be managed according to two alternative approaches: continuous review or periodic review. In deterministic systems, there is no substantial difference between these models; however, their nature becomes somewhat different in stochastic environments. It is known that, in the past, the number of systems using periodic review was much greater than the number using continuous review (Hadley & Whitin, 1963). In more recent times, it has been observed that periodic review inventory models can often be found in managing inventory cases such as smaller retail stores, drug stores and grocery stores (Taylor, 2015). In addition, periodic review inventory models have gathered over the years great attention from researchers (Braglia, Castellano, & Song, 2017; Sarkar & Mahapatra, 2015; Wensing, 2011).

Traditional inventory literature considers lead time as a prescribed deterministic quantity or a random variable. Hence, under this viewpoint lead time is not controllable (see,

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e.g. Hadley & Whitin, 1963; Zipkin, 2000). However, this may not be realistic. In fact, Tersine (1982) observed that lead time usually consists of several components, such as order preparation, order transit, supplier lead time, delivery time, and setup time. In some practical cases, these components can be shortened at an added crashing cost (Liao & Shyu, 1991). In other words, lead time is controllable. According to the just-in-time (JIT) philosophy, several benefits can be achieved by controlling lead time, e.g. lower investment in inventory, better product quality, higher flexibility, and increased productivity (Glock, 2012; Hariga, 2000; Lin, 2009). The concept of controllable lead time has been widely endorsed in the inventory management literature. The reader is referred to, e.g. Huang (2001), Lin (2009), Glock (2012), Braglia et al. (2017).

Further actions can be tackled to reach JIT goals. One of these initiatives is concerned with setup/ordering cost reduction. This can be achieved in practice by means of various ways, such as procedural changes, specialized equipments acquisition and workers training (Chuang, Ouyang, & Chuang, 2004; Leschke, 1996). Benefits that can be obtained by decreasing the setup/ordering cost include the possibility to improve quality and flexibility, lower investment in inventory, and increase effective capacity (Chuang et al., 2004; Leschke & Weiss, 1997). The importance of investing to reduce setup/ordering cost is also shown by the number of works that include this aspect (see, e.g. Lin, 2009; Priyan & Uthayakumar, 2014; Sarkar & Moon, 2014; Sarkar, Chaudhuri, & Moon, 2015).

In classical inventory models, it is implicitly assumed that the quality level is fixed at an optimal level. That is, all items are assumed to have perfect quality. However, this is not always true in the real production environment. In fact, it can often be observed that there are defective items being produced due to imperfect production processes, which may be related to equipment breakdowns, labour problems, and long-run of machinery systems (Sarkar, Chaudhuri, et al., 2015). The defective items must be rejected, repaired, reworked, or, if they have reached the customer, refunded. In all cases, substantial costs are incurred. Therefore, for the system with an imperfect production process, the decision-maker may consider investing capital on quality improvement, so as to reduce the quality-related costs (Ouyang, Wu, & Ho, 2007). After Porteus (1986) and Rosenblatt and Lee (1986), who are amongst the first to model explicitly the relationship between quality level and lot size, further researchers have considered this aspect. In this regard, it is possible to cite, for example, Ouyang et al. (2007), Sarkar and Moon (2014), Shu and Zhou (2014), Sarkar, Mandal, and Sarkar (2015).

In the real market, many aspects may affect customers' willingness to wait for backorders during the stockout period. Clearly, for some well-famed products or fashionable goods such as brand-name bags, shoes, hi-fi equipment, and clothes, customers may prefer to wait for delivery. Besides, the problem of motivating customers to wait for backorders is worth considering. In other words, we should endeavour to generate high customer loyalty so that the customers is willing to accept backorders. This can be accomplished by offering a price discount to customers (Chuang, Ouyang, & Lin, 2004; Ouyang, Chuang, & Lin, 2007b; Pan & Hsiao, 2001). As Lin (2009) observed, through controlling a price discount, we could generate high customer loyalty. This means that we could reduce cost of lost-sales and reduce holding cost, and then minimize the total relevant cost. The larger the discount, the bigger the advantage to the customers, and hence, a larger number of backorder ratio may result. This argument reveals that, as unsatisfied demands occur during the stockout period, the question of finding an optimal backorder ratio through controlling a price

discount from a retailer to minimize the relevant total cost is a decision-making problem worth discussing. A backorder price discount has been applied in numerous inventory management models (see, e.g. Lee, Wu, & Lei, 2007; Priyan & Uthayakumar, 2014; Sarkar, Mandal, et al., 2015).

In some practical situations, information about the demand distribution may be rather limited. That is, the decision-maker may only know an estimate of the mean and of the variance, but not the specific distribution type. There is a tendency to use the normal distribution under these circumstances (Moon & Gallego, 1994; Sarkar, Mandal, et al., 2015). This approach is called ‘Gaussian approximation’. According to Silver et al. (1998), there are several supporting arguments to adopt the Gaussian approximation, which is appropriate in specific conditions. First, this approach is practical for fast-moving items whose demand is characterized by a relatively small ($<.5$) coefficient of variation. In the case of expensive, slow-moving items this approach is not suitable. Secondly, the Gaussian distribution is generally recommended to model forecast demand: (i) empirically, the normal distribution usually provides a better fit to the data than other distributions; and (ii) the forecast errors in many periods are added together, so we can rely on the central limit theorem to expect a normal distribution (especially over a long time horizon). Lastly, the normal distribution provides analytically tractable results.

However, in some cases the Gaussian approximation may be little practical. In fact, the normal distribution does not offer the best shield against the occurrences of other distributions with the same mean and same variance. This poses a challenge to decision-makers who have to take a decision about what demand distribution should be used. In order to solve this problem, it is reasonable to follow a conservative procedure (Moon & Gallego, 1994). That is, the replenishment policy can be optimized considering the worst non-negative distribution with the given mean and variance. This is called ‘minimax distribution-free approach’. Some applications of this technique can be found in literature, e.g. Chuang et al. (2004), Sarkar, Mandal, et al. (2015), Braglia et al. (2017).

It can be observed that little effort has been done to consider the above aspects jointly. That is, periodic review inventory models that consider investments to reduce setup (or ordering) cost and improve quality, controllable lead time, and backorder price discount seem to be scarce in literature. For this reason, the aim of this paper is to investigate their joint effect in the case of a periodic review policy. See Table 1 for a comparison between this work and others.

The objective is to determine the review period, the setup cost, the quality level, the backorder price discount, and the length of lead time that minimize the long-run expected total cost per time unit. The problem is solved in two different cases: (i) the distribution of the demand in the protection interval is supposed to be Gaussian; and (ii) the minimax distribution-free approach is adopted. Numerical experiments have been carried out to investigate the effect of the main features characterizing the model, and to study the system behaviour when parameter values are made to vary.

The rest of the paper is organized as follows. Section 2 gives notation and assumptions adopted to develop the model; Section 3 deals with the model development and the problem formulation; Section 4 presents the optimization methods; Section 5 concerns numerical experiments; finally, Section 6 concludes the paper and indicates future research.

Table 1. Comparison between this work and others that consider a periodic review policy.

Author(s)	Distribution-free approach	Setup cost reduction	Quality improvement	Backorder price discount	Controllable lead time
Braglia, Castellano, and Frosolini (2016)		✓			✓
Braglia et al. (2017)	✓	✓			✓
Chuang et al. (2004)	✓	✓			✓
Lin (2008)					✓
Ouyang and Chuang (2000)	✓				✓
Ouyang and Chuang (2001)		✓			✓
Lin (2010)				✓	✓
Ouyang, Chuang, and Lin (2007a)	✓				✓
Sarkar and Mahapatra (2015)	✓				✓
Ouyang, Chuang, and Lin (2005)	✓				✓
Annadurai and Uthayakumar (2010)	✓				✓
Kim and Sarkar (2017)			✓	✓	✓
This paper	✓	✓	✓	✓	✓

2. Preliminaries

In this section, we define the preliminary aspects necessary to developing the proposed model and formulating the related optimization problem. In particular, this section introduces notation and assumptions.

2.1. Notation

The main notation adopted in this paper is listed below.

Decision variables

- T Review period [time unit]
- L Length of lead time [time unit]
- A Setup cost [money/setup]
- θ Probability that the production process can go out of control
- π_x Backorder price discount [money/quantity unit]

Parameters

- D Average demand rate [quantity unit/time unit]
- σ Standard deviation of demand rate [quantity unit/time unit]
- R Target level [quantity unit]
- A_0 Initial value of the ordering cost [money/setup]
- θ_0 Initial value of the probability that the production process can go out of control
- β_0 Upper bound of the backorder ratio, i.e. of the fraction of shortage that will be backordered
- h Unit inventory holding cost rate [money/quantity unit/time unit]
- π_0 Marginal profit per unit [money/quantity unit]

Random variables

- X Demand in the protection interval, i.e. within $T + L$

Functions and operators

- $f(\cdot)$ Probability density function (p.d.f.) of X
- $\phi(\cdot)$ Standard normal probability density function
- $\Phi(\cdot)$ Standard normal cumulative distribution function
- $\psi(\cdot)$ Standard normal loss function
- $E[\cdot]$ Mathematical expectation
- x^+ Maximum between 0 and x , i.e. $x^+ \equiv \max\{0, x\}$

Classes

- \mathcal{F} Class of probability density functions with finite mean $D(T + L)$ and finite standard deviation $\sigma\sqrt{T + L}$

2.2. Hypotheses

The main assumptions considered to develop the model are given below:

- (1) A single, fast-moving item is considered.
- (2) The inventory level is reviewed every T time units. A sufficient quantity is ordered up to the target level R and the order arrives after L time units. There is no more than a single order outstanding in a given inventory cycle.
- (3) The demand in the protection interval, X , is a random variable with finite mean $D(T + L)$ and finite standard deviation $\sigma\sqrt{T + L}$.
- (4) The target level is given by $R = D(T + L) + z\sigma\sqrt{T + L}$, where the first addendum is the average demand within the protection interval, while the second one is the safety stock. The safety factor z satisfies $\Pr(X > R) = q$, where q represents the fixed allowable stockout probability during the protection interval.
- (5) Shortages are allowed and partially backordered with ratio β . The backorder ratio β is variable and is proportional to the backorder price discount π_x . In particular, it is assumed that $\beta = \pi_x \frac{\beta_0}{\pi_0}$, with $0 \leq \beta_0 < 1$ and $0 \leq \pi_x \leq \pi_0$ (see, e.g. Lin, 2009; Sarkar, Mandal, et al., 2015).
- (6) The time horizon is infinite.

According to, e.g. Liao and Shyu (1991) and Tersine (1982), the lead time of a generic item can be supposed to be made of several components, such as setup time, process time, and queue time. This observation makes it possible to assume that lead time be negotiable and controllable. That is, each component may be reduced with an additional charge. This approach to controlling lead time was originally proposed by Liao and Shyu (1991) and then endorsed by several authors, e.g. Huang (2001), Chuang et al. (2004), Lin (2009), Glock (2012), Sarkar, Mandal, et al. (2015).

In this paper, the same assumption is made. In particular, it is assumed that the lead time L is made up of M mutually independent, deterministic and constant components. The generic m th component has a minimum duration b_m , a normal duration a_m , and a crashing cost per time unit c_m , with $c_1 \leq c_2 \leq \dots \leq c_M$. Components are crashed one at a time starting with the component of least c_m and so on. If L_m is the length of lead time with components $1, 2, \dots, m$ crashed to their minimum durations, we can write

$$L_m = L_0 - (a_1 - b_1) - (a_2 - b_2) - \dots - (a_m - b_m),$$

where $L_0 \equiv \sum_m s_m$. For $L_n \in [L_m, L_{m-1}]$ with $m = 1, 2, \dots, M$, the lead time crashing cost U can be expressed as follows:

$$U(L) = c_m(L_{m-1} - L) + c_1(a_1 - b_1) + c_2(a_2 - b_2) + \dots + c_{m-1}(a_{m-1} - b_{m-1}). \tag{1}$$

Note that U is a piecewise-linear, decreasing function in the interval $\nexists L_M, L_0 \nexists$, where it is also continuous and convex.

To reflect the industrial practice concerned with controlling setup cost by means of *ad hoc* actions (e.g. worker training, procedural changes, and equipments updating), we assume that A is controllable through a capital investment I , which is a function of A . This investment is required to reduce the setup cost from the original level A_0 to a target level A , with $0 < A \leq A_0$. The function I is the one-time investment cost whose benefits will extend to the long-term into the future. Hence, if τ is the fractional cost of capital investment in the time unit (e.g. interest rate), then τI is the cost of such an investment per time unit. We assume that I follows a logarithmic investment function:

$$I(A) = \frac{1}{\delta_1} \ln\left(\frac{A_0}{A}\right), \quad 0 < A \leq A_0, \tag{2}$$

where δ_1 is the percentage decrease in A per money unit increase in investment. This expression for I was introduced by [Porteus \(1985\)](#) and then has been extensively adopted in literature (see, e.g. [Braglia et al., 2016, 2017](#); [Ouyang, Chen, & Chang, 2002](#); [Sarkar, Chaudhuri, et al., 2015](#); [Sarkar & Majumder, 2013](#)). Note that I is a convex and strictly decreasing function.

We take into account the relationship between lot size and quality outlined by [Porteus \(1986\)](#). During the production process, the process can go ‘out of control’ with a given probability θ each time another unit is produced. The process is assumed to be ‘in control’ when the production of a lot begins. Once ‘out of control’, the process produces defective items and continues to do so until the entire lot is produced.

To improve quality, it is required to control the production process during ‘out of control’ state. In this regard, an investment is needed to reduce the ‘out of control’ state. We denote by K the capital investment to improve process quality, i.e. to reduce the ‘out of control’ probability from the initial value θ_0 to θ , with $0 < \theta \leq \theta_0$. A generally adopted expression for K is the following logarithmic function (see [Porteus, 1986](#)) and more recent works, e.g. [Ouyang et al. \(2002\)](#), [Sarkar and Moon \(2014\)](#), [Sarkar, Mandal, et al. \(2015\)](#), [Sarkar, Chaudhuri, et al. \(2015\)](#):

$$K(\theta) = \frac{1}{\delta_2} \ln\left(\frac{\theta_0}{\theta}\right), \quad 0 < \theta \leq \theta_0, \tag{3}$$

where δ_2 is the percentage decrease in θ per money unit increase in K . We recall that τ is the fractional cost of capital investment in the time unit. Hence, τK is the opportunity cost of quality improvement investment per time unit. Evidently, for $\theta = \theta_0$, there is not any investment for quality improvement.

In real industrial contexts, the ‘out of control’ probability θ takes very small values. This makes it possible to assume $\theta \leq 10^{-5}$ (Sarkar, Mandal, et al., 2015).

The notation and assumptions given in this section will serve to develop the model and the related optimization problem, which are presented in the next section.

3. Model definition and problem formulation

In this section, we first develop the mathematical model according to the assumptions stated in Section 2. Then, the related optimization problem is formulated.

With similar arguments to, e.g. Annadurai and Uthayakumar (2010), the expected inventory holding cost per time unit is $h \left[R - DL - \frac{DT}{2} + (1 - \beta) E \left[(X - R)^+ \right] \right]$, which becomes $h \left[R - DL - \frac{DT}{2} + \left(1 - \pi_x \frac{\beta_0}{\pi_0} \right) E \left[(X - R)^+ \right] \right]$ by assumption 5. Since the expected demand shortage per time unit is $\frac{E \left[(X - R)^+ \right]}{T}$, the expected stockout cost per time unit is $\frac{E \left[(X - R)^+ \right]}{T} \left[\pi_x \beta + \pi_0 (1 - \beta) \right]$ (Lin, 2009), which can be rewritten as $\frac{E \left[(X - R)^+ \right]}{T} \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right)$. Under the hypothesis of small θ , Porteus (1986) proved that the expected number of defective items during the production of a lot made of Q units is approximately $\theta \frac{Q^2}{2}$. If the cost for rework/replace a defective unit is ν , the expected cost for defective items per time unit is $\nu D \theta \frac{Q}{2}$, which becomes $\nu D^2 \theta \frac{T}{2}$ according to the relation $Q = TD$. We finally observe that the ordering cost per time unit is $\frac{A}{T}$ and the lead-time crashing cost per time unit is $\frac{U}{T}$. Hence, taking into account the cost of investments to reduce setup cost (i.e. τI) and to improve quality level (i.e. τK), the long-run expected total cost per time unit is

$$\begin{aligned} \mathfrak{R}_0(T, A, \theta, \pi_x, L) = & \varepsilon_1 \ln \left(\frac{A_0}{A} \right) + \varepsilon_2 \ln \left(\frac{\theta_0}{\theta} \right) + \frac{1}{T} (A + U(L)) \\ & + h \left[R - DL - \frac{DT}{2} + \left(1 - \pi_x \frac{\beta_0}{\pi_0} \right) E \left[(X - R)^+ \right] \right] \\ & + \frac{E \left[(X - R)^+ \right]}{T} \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) + \nu D^2 \theta \frac{T}{2}, \end{aligned} \quad (4)$$

where we have put $\varepsilon_1 \equiv \frac{\tau}{\delta_1}$ and $\varepsilon_2 \equiv \frac{\tau}{\delta_2}$.

The objective is to find the review period, the setup cost, the quality level, the backorder price discount, and the lead time that minimize the long-run expected total cost per time unit. This problem can be formalized as follows:

$$(P) \quad \min_{(T, A, \theta, \pi_x, L)} \mathfrak{R}_0(T, A, \theta, \pi_x, L)$$

$$\text{s.t.} \quad T > 0 \tag{5}$$

$$0 < A \leq A_0 \tag{6}$$

$$0 < \theta \leq \theta_0 \tag{7}$$

$$0 \leq \pi_x \leq \pi_0 \tag{8}$$

$$L \in [L_M, L_0]. \tag{9}$$

The above problem will be approached in two cases depending on the assumption made about the distribution of the demand in the protection interval: (i) Gaussian distribution; and (ii) distribution-free approach. The next section presents the solution methods.

4. Optimization methods

In this section, two optimization methods are presented depending on the assumption made about the distribution of the demand in the protection interval, X . Section 4.1 concerns the case in which X is a Gaussian random variable. Section 4.2 deals with the distribution-free approach.

4.1. Gaussian distribution

We consider the case in which the demand in the protection interval, X , is Gaussian with mean $D(T + L)$ and standard deviation $\sigma\sqrt{T + L}$. In this circumstance, the expected demand shortage per cycle $E[(X - R)^+]$ is given by [Annadurai and Uthayakumar \(2010\)](#)

$$E[(X - R)^+] = \int_R^{+\infty} (x - R) f(x) dx = \sigma\sqrt{T + L}\psi(z), \tag{10}$$

where $\psi(z) = \phi(z) - z(1 - \Phi(z))$ and z is the safety factor (see assumption No. 4). According to Equation (10) and to the expression of the target level $R = D(T + L) + z\sigma\sqrt{T + L}$, Equation (4) becomes

$$\begin{aligned} \mathfrak{K}_N(T, A, \theta, \pi_x, L) = & \varepsilon_1 \ln\left(\frac{A_0}{A}\right) + \varepsilon_2 \ln\left(\frac{\theta_0}{\theta}\right) + \frac{1}{T}(A + U(L)) \\ & + h\left[\frac{DT}{2} + z\sigma\sqrt{T + L} + \left(1 - \pi_x \frac{\beta_0}{\pi_0}\right)\sigma\sqrt{T + L}\psi(z)\right] \\ & + \frac{\sigma\sqrt{T + L}\psi(z)}{T}\left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0\pi_x + \pi_0\right) + \nu D^2 \theta \frac{T}{2}. \end{aligned} \tag{11}$$

The objective is to solve problem (P) under constraints (5)–(9) with \mathfrak{K}_0 replaced by \mathfrak{K}_N . This is a nonlinear program and its solution requires a number of steps that are described below.

By examining the second-order sufficient condition for optimality, it can be verified that \mathfrak{K}_N is not a convex function of (T, A, θ, π_x, L) . However, for fixed (T, A, θ, π_x) , \mathfrak{K}_N is concave in $L \in [L_m, L_{m-1}]$, with $m = 1, 2, \dots, M$, as

$$\begin{aligned} \frac{\partial^2 \mathfrak{K}_N}{\partial L^2} = & -\frac{\sigma}{4(T + L)^{\frac{3}{2}}}\left[hz + h\left(1 - \pi_x \frac{\beta_0}{\pi_0}\right)\psi(z)\right. \\ & \left. + \frac{\psi(z)}{T}\left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0\pi_x + \pi_0\right)\right] < 0. \end{aligned}$$

Hence, for fixed (T, A, θ, π_x) , the minimum of \mathfrak{K}_N in $L \in [L_m, L_{m-1}]$ will occur at one of the endpoints of the interval $[L_m, L_{m-1}]$.

Now, we ignore constraints $0 < \theta \leq \theta_0$, $0 < A \leq A_0$ and $0 \leq \pi_x \leq \pi_0$ and take the first-order partial derivative of \mathfrak{R}_N with respect to T , A , θ , and π_x , respectively:

$$\frac{\partial \mathfrak{R}_N}{\partial T} = -\frac{A + U(L)}{T^2} + h \left[\frac{D}{2} + \frac{z\sigma}{2\sqrt{T+L}} + \left(1 - \pi_x \frac{\beta_0}{\pi_0}\right) \frac{\sigma \psi(z)}{2\sqrt{T+L}} \right] - \sigma \psi(z) \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) \frac{2L + T}{2T^2 \sqrt{T+L}} + \frac{1}{2} v D^2 \theta, \tag{12}$$

$$\frac{\partial \mathfrak{R}_N}{\partial A} = \frac{1}{T} - \frac{\varepsilon_1}{A}, \tag{13}$$

$$\frac{\partial \mathfrak{R}_N}{\partial \theta} = \frac{1}{2} v D^2 T - \frac{\varepsilon_2}{\theta}, \tag{14}$$

$$\frac{\partial \mathfrak{R}_N}{\partial \pi_x} = \sigma \sqrt{T+L} \psi(z) \left[\frac{1}{T} \left(2\pi_x \frac{\beta_0}{\pi_0} - \beta_0 \right) - h \frac{\beta_0}{\pi_0} \right]. \tag{15}$$

If we set Equations (12)–(15) equal to zero, we get the first-order condition for optimality:

$$\frac{A + U(L)}{2\sqrt{T+L}} + h \left[\frac{D}{2} + \frac{z\sigma}{2\sqrt{T+L}} + \left(1 - \pi_x \frac{\beta_0}{\pi_0}\right) \frac{\sigma \psi(z)}{2\sqrt{T+L}} \right] - \sigma \psi(z) \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) \frac{2L + T}{2T^2 \sqrt{T+L}} + \frac{1}{2} v D^2 \theta = 0 \tag{16}$$

$$A = A(T) \tag{17}$$

$$\theta = \theta(T) \tag{18}$$

$$\pi_x = \pi_x(T) \tag{19}$$

where

$$A(T) = \varepsilon_1 T, \tag{20}$$

$$\theta(T) = \frac{2\varepsilon_2}{v D^2 T}, \tag{21}$$

$$\pi_x(T) = \frac{hT + \pi_0}{2}. \tag{22}$$

Inasmuch \mathfrak{R}_N is continuous on the domain identified by constraints (5)–(8), the minimum in (T, A, θ, π_x) , for fixed $L \in [L_m, L_{m-1}]$, lies either on a stationary point, which is obtained by solving the first-order condition, or on the frontier of the domain. We observe that it is hard to determine analytically whether \mathfrak{R}_N is convex in (T, A, θ, π_x) , for fixed $L \in [L_m, L_{m-1}]$, or not. Hence, we have carried out extensive numerical experiments to investigate the properties of \mathfrak{R}_N , for fixed $L \in [L_m, L_{m-1}]$. These tests, which have considered parameter values typically adopted in literature, have shown that \mathfrak{R}_N admits two stationary points. However, only one is located in the region of practically admissible values for decision variables. We can thus assert that, to our practical purposes, the system of Equations (16)–(19) admits a single valid solution in (T, A, θ, π_x) . Let $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ be such admissible solution.

Moreover, the same numerical experiments have shown that this stationary point is a local minimum. If we further observe that $A(T)$, $\theta(T)$, and $\pi_x(T)$ are positive for

Table 2. Overview of all possible cases.

		$\hat{A} \leq A_0$	$\hat{A} > A_0$
$\hat{\theta} \leq \theta_0$	$\hat{\pi}_x \leq \pi_0$	Case 1	Case 2
	$\hat{\pi}_x > \pi_0$	Case 3	Case 4
$\hat{\theta} > \theta_0$	$\hat{\pi}_x \leq \pi_0$	Case 5	Case 6
	$\hat{\pi}_x > \pi_0$	Case 7	Case 8

each $T > 0$, it is clear that $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ gives the solution to our problem, for fixed $L \in [L_m, L_{m-1}]$, when constraints $\theta \leq \theta_0, A \leq A_0$, and $\pi_x \leq \pi_0$ are ignored.

To find $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$, Equations (17)–(19) are put into Equation (16), which is thus solved in T to obtain \hat{T} . The value \hat{T} is then substituted into $A(T), \theta(T)$ and $\pi_x(T)$ to determine $\hat{A}, \hat{\theta}$ and $\hat{\pi}_x$, respectively. Note that to find \hat{T} a numerical method can be used only. In fact, Equation (16) cannot be solved in T analytically.

Once $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ is determined, several cases can be identified depending on which constrains among $\theta \leq \theta_0, A \leq A_0$ and $\pi_x \leq \pi_0$ are satisfied. Table 2 gives an overview of all possible occurrences. In Case 1, the solution $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ results admissible for problem (P) under constraints (5)–(8), for fixed $L \in [L_m, L_{m-1}]$; thus, $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ is the searched solution (for fixed $L \in [L_m, L_{m-1}]$). In any other case, it is necessary to convert those constraints that are not satisfied to equality, and then repeat the calculation to solve the first-order condition. For example, in Case 2 we have to put $A = A_0$ and then solve the system of equations (16), (18) and (19). This procedure must be repeated as long as any of constraints (5)–(8) is unsatisfied. Note that this optimization method based on verifying which constraints are satisfied and then converting to equality those that are unsatisfied is optimal (see, e.g. Sarkar, Mandal, et al., 2015).

The following computational procedure, which is based on studying the cases shown in Table 2, permits us to find the solution $(T^*, A^*, \theta^*, \pi_x^*, L^*)$, and the related cost \mathfrak{R}^* , to problem (P) under constraints (5)–(9):

Algorithm 1. Procedure to approach problem (P) under Gaussian distribution

1. **set** $\mathfrak{R}^* = +\infty$
2. **for** $(m = 0, 1, \dots, M)$ **do**
3. **set** $L \leftarrow L_m$
4. **calculate** $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ by solving Equations (16)–(19)
5. **if** $(\hat{\theta} \leq \theta_0, \hat{\pi}_x \leq \pi_0, \hat{A} \leq A_0)$ **then**
6. **if** $(\mathfrak{R}(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L) \leq \mathfrak{R}^*)$ **then**
7. **set** $\mathfrak{R}^* \leftarrow \mathfrak{R}(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L), (T^*, A^*, \theta^*, \pi_x^*, L^*) \leftarrow (\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L)$
8. **end if**
9. **end if**
10. **if** $(\hat{A} > A_0)$ **then**
11. **set** $\hat{A} \leftarrow A_0, A \leftarrow A_0$
12. **if** $(\hat{\theta} \leq \theta_0)$ **then**
13. **if** $(\hat{\pi}_x \leq \pi_0)$ **then**

```

14.         calculate  $(\hat{T}, \hat{\theta}, \hat{\pi}_x)$  by solving Equations (16), (18) and (19)
15.         go to line 5
16.     else
17.         set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
18.         calculate  $(\hat{T}, \hat{\theta})$  by solving Equations (16) and (18)
19.         go to line 5
20.     end if
21. else
22.     set  $\hat{\theta} \leftarrow \theta_0, \theta \leftarrow \theta_0$ 
23.     if  $(\hat{\pi}_x \leq \pi_0)$  then
24.         calculate  $(\hat{T}, \hat{\pi}_x)$  by solving Equations (16) and (19)
25.         go to line 5
26.     else
27.         set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
28.         calculate  $\hat{T}$  by solving Equation (16)
29.         go to line 5
30.     end if
31. end if
32. else
33.     if  $(\hat{\theta} \leq \theta_0)$  then
34.         if  $(\hat{\pi}_x > \pi_0)$  then
35.             set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
36.             calculate  $(\hat{T}, \hat{A}, \hat{\theta})$  by solving Equations (16), (17) and (18)
37.             go to line 5
38.         end if
39.     else
40.         set  $\hat{\theta} \leftarrow \theta_0, \theta \leftarrow \theta_0$ 
41.         if  $(\hat{\pi}_x \leq \pi_0)$  then
42.             calculate  $(\hat{T}, \hat{A}, \hat{\pi}_x)$  by solving Equations (16), (17) and (19)
43.             go to line 5
44.         else
45.             set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
46.             calculate  $(\hat{T}, \hat{A})$  by solving Equations (16) and (17)
47.             go to line 5
48.         end if
49.     end if
50. end if
51. end for

```

4.2. Distribution-free approach

In Section 4.1, we have developed a model in which the demand in the protection interval, X , is stochastic and follows a normal distribution. However, in some practical situations,

information about the demand distribution may be rather limited. That is, the decision maker may only know an estimate of the mean and of the variance, but not the specific distribution type. In this circumstance, the available information is not sufficient to evaluate the expected demand shortage per cycle $E[(X - R)^+]$, and hence the decision variables cannot be optimized.

To overcome this issue, it is reasonable to follow a conservative approach (Moon & Gallego, 1994), which permits us to optimize the decision variables taking into account the worst non-negative distribution with the given mean and variance. This is called minimax distribution-free approach.

The assumption is made that the p.d.f. f of X belongs to the class \mathcal{F} of probability density functions with finite mean $D(T + L)$ and finite standard deviation $\sigma\sqrt{T + L}$. The minimax principle is to choose f as the most unfavourable p.d.f. in \mathcal{F} for each (T, A, θ, π_x, L) and then minimize over (T, A, θ, π_x, L) .

Although the minimax principle is a conservative approach, several supporting arguments can be raised. First, it can be easily applied in practice, as statistical tables and computer programs used to evaluate distribution functions are not required. In addition, analytically tractable expressions can be obtained (Braglia et al., 2017). Secondly, it is optimal under some conditions (Gallego & Moon, 1993; Moon & Gallego, 1994). Last but not least, the minimax principle has found large consensus in the inventory literature (see, e.g. Kumar & Goswami, 2015; Raza, 2015; Sarkar, Chaudhuri, et al., 2015; Sarkar & Mahapatra, 2015).

According to the minimax principle, problem (P) becomes

$$(\bar{P}) \quad \min_{(T,A,\theta,\pi_x,L)} \max_{f \in \mathcal{F}} \mathfrak{K}_0(T, A, \theta, \pi_x, L).$$

Problem (\bar{P}) can be simplified with the following proposition (Chuang et al., 2004):

Lemma 1: For any $f \in \mathcal{F}$,

$$E[(X - R)^+] \leq \frac{1}{2} \left\{ \sqrt{\sigma^2(T + L) + [R - D(T + L)]^2} - [R - D(T + L)] \right\}. \quad (23)$$

Moreover, the upper bound is tight.

Proposition 1 and the definition of R permit us to consider the following problem

$$(Q) \quad \min_{(T,A,\theta,\pi_x,L)} \mathfrak{K}_D(T, A, \theta, \pi_x, L)$$

in place of problem (\bar{P}) , where \mathfrak{K}_D is the long-run expected total cost per time unit for the distribution-free case, which is expressed as follows:

$$\begin{aligned} &\mathfrak{K}_D(T, A, \theta, \pi_x, L) \\ &= \varepsilon_1 \ln\left(\frac{A_0}{A}\right) + \varepsilon_2 \ln\left(\frac{\theta_0}{\theta}\right) + \frac{1}{T}(A + U(L)) \\ &\quad + h \left[\frac{DT}{2} + z\sigma\sqrt{T + L} + \frac{1}{2}\sigma\sqrt{T + L} \left(1 - \pi_x \frac{\beta_0}{\pi_0}\right) (\sqrt{1 + z^2} - z) \right] \end{aligned}$$

$$+ \frac{1}{2T} \sigma \sqrt{T+L} \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) \left(\sqrt{1+z^2} - z \right) + \nu D^2 \theta \frac{T}{2}.$$

Note that, from inequality (23), \mathfrak{R}_D represents an upper bound to \mathfrak{R}_0 . For the distribution-free case, the objective is thus to solve problem (Q) under constraints (5)–(9).

By similar arguments to the Gaussian distribution case, it is possible to observe that \mathfrak{R}_D is not a convex function of (T, A, θ, π_x, L) . However, for fixed (T, A, θ, π_x) , \mathfrak{R}_D is concave in $L \in [L_m, L_{m-1}]$, with $m = 1, 2, \dots, M$. Hence, for fixed (T, A, θ, π_x) , the minimum of \mathfrak{R}_D in $L \in [L_m, L_{m-1}]$ will occur at one of the endpoints of the interval $[L_m, L_{m-1}]$.

Since \mathfrak{R}_D is continuous on the domain defined by constraints (5)–(8), the minimum in (T, A, θ, π_x) , for fixed $L \in [L_m, L_{m-1}]$, lies either on a stationary point or on the frontier of the domain. Similarly to the case studied in the previous subsection, when constraints $0 < \theta \leq \theta_0$, $0 < A \leq A_0$ and $0 \leq \pi_x \leq \pi_0$ are ignored, it is difficult to verify that \mathfrak{R}_D is a convex function of (T, A, θ, π_x) , for fixed $L \in [L_m, L_{m-1}]$. Extensive numerical experiments have been carried out to investigate the properties of \mathfrak{R}_D in a range of admissible values for parameters and decision variables. Similar conclusions to the Gaussian distribution case have been drawn. In particular, it has been verified that \mathfrak{R}_D admits a single stationary point in the interval of practically admissible values for decision variables. In addition, this stationary point has been observed to be a local minimum.

If this local minimum is denoted by $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$, its components \hat{T} , \hat{A} , $\hat{\theta}$, and $\hat{\pi}_x$ can be determined by solving the first-order condition for optimality. To this aim, we take the first-order partial derivative of \mathfrak{R}_D with respect to T, A, θ , and π_x , respectively:

$$\begin{aligned} \frac{\partial \mathfrak{R}_D}{\partial T} = & -\frac{A+U(L)}{T^2} + h \left[\frac{D}{2} + \frac{z\sigma}{2\sqrt{T+L}} + \sigma \left(1 - \pi_x \frac{\beta_0}{\pi_0} \right) \frac{\sqrt{1+z^2} - z}{4\sqrt{T+L}} \right] \\ & - \sigma \left(\sqrt{1+z^2} - z \right) \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) \frac{2L+T}{4T^2\sqrt{T+L}} + \frac{1}{2} \nu D^2 \theta, \end{aligned} \tag{24}$$

$$\frac{\partial \mathfrak{R}_D}{\partial A} = \frac{1}{T} - \frac{\varepsilon_1}{A}, \tag{25}$$

$$\frac{\partial \mathfrak{R}_D}{\partial \theta} = \frac{2\varepsilon_2}{\nu D^2 T}, \tag{26}$$

$$\frac{\partial \mathfrak{R}_D}{\partial \pi_x} = \frac{hT + \pi_0}{2}. \tag{27}$$

By setting Equations (24)–(27) equal to zero, we get the first-order condition for optimality:

$$\begin{aligned} \frac{A+U(L)}{2\sqrt{T+L}} + h \left[\frac{D}{2} + \frac{z\sigma}{2\sqrt{T+L}} + \sigma \left(1 - \pi_x \frac{\beta_0}{\pi_0} \right) \frac{\sqrt{1+z^2} - z}{4\sqrt{T+L}} \right] \\ - \sigma \left(\sqrt{1+z^2} - z \right) \left(\pi_x^2 \frac{\beta_0}{\pi_0} - \beta_0 \pi_x + \pi_0 \right) \frac{2L+T}{4T^2\sqrt{T+L}} + \frac{1}{2} \nu D^2 \theta = 0, \end{aligned} \tag{28}$$

$$A = A(T), \tag{29}$$

$$\theta = \theta(T), \tag{30}$$

$$\pi_x = \pi_x(T), \tag{31}$$

where $A(T)$, $\theta(T)$, and $\pi_x(T)$ are given by Equations (20)–(22), respectively. Noting that $A(T)$, $\theta(T)$, and $\pi_x(T)$ are positive for each $T > 0$, we can assert that $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ gives the solution to problem (Q), for fixed $L \in [L_m, L_{m-1}]$, when constraints $\theta \leq \theta_0$, $A \leq A_0$, and $\pi_x \leq \pi_0$ are ignored.

To solve the system of equations (28)–(31), the same procedure described in the previous subsection can be adopted. In addition, depending on which constraints among $\theta \leq \theta_0$, $A \leq A_0$ and $\pi_x \leq \pi_0$ are satisfied by the obtained solution, several cases can be identified (see Table 2). We remind the reader that this optimization method based on verifying which constraints are satisfied and then converting to equality those that are unsatisfied is optimal (see, e.g. Sarkar, Mandal, et al., 2015).

With similar arguments to the Gaussian distribution case, the solution $(T^*, A^*, \theta^*, \pi_x^*, L^*)$, and the related cost \mathfrak{R}^* , to problem (Q) under constraints (5)–(9) can be determined with the following computational procedure, which is based on studying the cases shown in Table 2:

Algorithm 2. Procedure to approach problem (Q) under constraints (5)–(9)

1. **set** $\mathfrak{R}^* = +\infty$
2. **for** $(m = 0, 1, \dots, M)$ **do**
3. **set** $L \leftarrow L_m$
4. **calculate** $(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x)$ by solving Equations (28)–(31)
5. **if** $(\hat{\theta} \leq \theta_0, \hat{\pi}_x \leq \pi_0, \hat{A} \leq A_0)$ **then**
6. **if** $(\mathfrak{R}(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L) \leq \mathfrak{R}^*)$ **then**
7. **set** $\mathfrak{R}^* \leftarrow \mathfrak{R}(\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L)$, $(T^*, A^*, \theta^*, \pi_x^*, L^*) \leftarrow (\hat{T}, \hat{A}, \hat{\theta}, \hat{\pi}_x, L)$
8. **end if**
9. **end if**
10. **if** $(\hat{A} > A_0)$ **then**
11. **set** $\hat{A} \leftarrow A_0$, $A \leftarrow A_0$
12. **if** $(\hat{\theta} \leq \theta_0)$ **then**
13. **if** $(\hat{\pi}_x \leq \pi_0)$ **then**
14. **calculate** $(\hat{T}, \hat{\theta}, \hat{\pi}_x)$ by solving Equations (28), (30) and (31)
15. **go to line 5**
16. **else**
17. **set** $\hat{\pi}_x \leftarrow \pi_0$, $\pi_x \leftarrow \pi_0$
18. **calculate** $(\hat{T}, \hat{\theta})$ by solving Equations (28) and (30)
19. **go to line 5**
20. **end if**
21. **else**
22. **set** $\hat{\theta} \leftarrow \theta_0$, $\theta \leftarrow \theta_0$
23. **if** $(\hat{\pi}_x \leq \pi_0)$ **then**
24. **calculate** $(\hat{T}, \hat{\pi}_x)$ by solving Equations (28) and (31)
25. **go to line 5**
26. **else**
27. **set** $\hat{\pi}_x \leftarrow \pi_0$, $\pi_x \leftarrow \pi_0$


```

28.         calculate  $\hat{T}$  by solving Equation (28)
29.         go to line 5
30.     end if
31. end if
32. else
33.     if ( $\hat{\theta} \leq \theta_0$ ) then
34.         if ( $\hat{\pi}_x > \pi_0$ ) then
35.             set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
36.             calculate ( $\hat{T}, \hat{A}, \hat{\theta}$ ) by solving Equations (28), (29) and (30)
37.             go to line 5
38.         end if
39.     else
40.         set  $\hat{\theta} \leftarrow \theta_0, \theta \leftarrow \theta_0$ 
41.         if ( $\hat{\pi}_x \leq \pi_0$ ) then
42.             calculate ( $\hat{T}, \hat{A}, \hat{\pi}_x$ ) by solving Equations (28), (29) and (31)
43.             go to line 5
44.         else
45.             set  $\hat{\pi}_x \leftarrow \pi_0, \pi_x \leftarrow \pi_0$ 
46.             calculate ( $\hat{T}, \hat{A}$ ) by solving Equations (28) and (29)
47.             go to line 5
48.         end if
49.     end if
50. end if
51. end for

```

5. Numerical experiments

The first part of this section is concerned with investigating the solution procedures and examining the effect of the main features (i.e. investments to reduce setup cost and improve quality, and backorder price discount) characterizing the inventory models presented in the previous sections. The second part deals with the sensitivity analysis to study the system behaviour when changes in parameter values occur.

5.1. Analysis on the impact of the main features

For these experiments, we consider the following parameter values (Sarkar & Moon, 2014): $D = 600$ units/year; $\sigma = 7$ units/week; $h = \$20$ /unit/year; $\pi_0 = \$150$ /unit; $\theta_0 = .0002$; $A_0 = \$200$ /setup; $v = \$75$ /defective unit; $\tau = .1$ \$/year. In addition, we assume $q = .2$ (Ouyang & Chuang, 2000). The lead time has three components whose data are shown in Table 3 (Sarkar, Mandal, et al., 2015). It is supposed 1 year = 52 weeks. In the experiments below, β_0 takes values in the set $\{0, 0.3, 0.6, 0.9, 0.95, 1\}$, which are similar to those typically adopted in literature (see, e.g. Chuang et al., 2004). The values assigned to δ_1 and δ_2 are in the range adopted in similar works, e.g. Sarkar and Moon (2014). However, δ_1 also takes smaller values than the typical ones to make evident the effect of the related investment.

Table 3. Lead time data.

Lead time component, m	Normal duration, a_m [days]	Minimum duration, b_m [days]	Unit crashing cost, c_m [\$/day]
1	20	6	.4
2	20	6	1.2
3	16	9	5.0

Table 4. Numerical results for Example 1.

δ_1	T^* [years]	A^* [\$/setup]	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
1/12,000	.1387	166.4	56	2637	.4
1/10,000	.1158	115.8	56	2565	3.1
1/8000	.0930	74.4	56	2413	8.9
1/4000	.0481	19.2	56	1777	32.9
	.1520	(No investment)	56	2648	

Table 5. Numerical results for Example 2.

δ_1	T^* [years]	A^* [\$/setup]	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
1/12,000	.1396	167.5	56	2648	.4
1/10,000	.1169	116.9	56	2578	3.0
1/8000	.0944	75.5	56	2428	8.7
1/4000	.0505	20.2	56	1803	32.2
	.1524	(No investment)	56	2658	

5.1.1. Example 1

Here, we assume that the demand within the protection interval follows a Gaussian distribution. In this circumstance, from the table of the standard normal we find $z = .845$. We consider the case with investments to reduce setup cost; while investments to improve quality and backorder price discount are not included in this example (i.e. we put $\theta = \theta_0$ and $\pi_x = \pi_0$). We obtain the solution for $\beta_0 = .95$, $\delta_2 = 1/400$, and $\delta_1 = 1/12,000$, $1/10,000$, $1/8000$, and $1/4000$. Applying Algorithm 1, the results of the solution procedure are summarized in Table 4. Furthermore, the results for the no-investment in setup cost reduction case are shown too. From Table 4, we can note that the greater the parameter about the percentage decrease in A per money unit increase in investment (i.e. δ_1), the higher the savings. In particular, savings range from .4% to 32.9%. In addition, it is possible to note that, as δ_1 increases, T^* and A^* become smaller, while no change is observed in L^* .

5.1.2. Example 2

We use the same data and assumptions as in Example 1, except that the distribution of the demand within the protection interval is supposed to be unknown, and therefore the minimax distribution-free approach is adopted. To obtain the value of z that satisfies the condition $q = .2$, the maximal distribution is considered (Zipkin, 2000). The quantile corresponding to $q = .2$ is $z = .750$. Algorithm 2 is used to obtain the results shown in Table 5. Even in this example, the effect of investing to reduce setup cost is evident: savings range from .4 to 32.2%. Similarly to the previous example, as δ_1 increases, T^* and A^* decrease, while no change is evident in L^* .

Table 6. Numerical results for Example 3.

δ_2	T^* [years]	θ^*	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
1/4000	.1527	1.9×10^{-4}	56	2648	.4
1/3000	.1597	1.4×10^{-4}	56	2626	1.2
1/2000	.1671	8.9×10^{-5}	56	2571	3.3
1/500	.1789	2.1×10^{-5}	56	2363	11.1
	.1524	(No investment)	56	2658	

Table 7. Numerical results for Example 4.

δ_2	T^* [years]	θ^*	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
1/4000	.1532	1.9×10^{-4}	56	2655	.1
1/3000	.1602	1.4×10^{-4}	56	2638	.8
1/2000	.1676	8.8×10^{-5}	56	2580	2.9
1/500	.1794	2.1×10^{-5}	56	2372	10.8
	.1524	(No investment)	56	2658	

5.1.3. Example 3

In this example, the demand within the protection interval is supposed to be Gaussian. We consider the case with investments to improve quality; while investments to reduce setup cost and backorder price discount are not included (i.e. we put $A = A_0$ and $\pi_x = \pi_0$). The solution for $\beta_0 = .95$, $\delta_1 = 1/8000$, and $\delta_2 = 1/4000, 1/3000, 1/2000$, and $1/500$ is determined. Applying Algorithm 1, the results of the solution procedure are given in Table 6, where the solution for the no-investment case is also shown. The results in Table 6 highlight that .4–11.1% of savings can be achieved if investments to improve quality are carried out. We can also observe that T^* increases while θ^* decreases with an increment in δ_2 . No change is apparent in L^* .

5.1.4. Example 4

The same assumptions and data of Example 3 are considered here, but the distribution within the protection interval is unknown. The results, obtained by means of Algorithm 2, are shown in Table 7. When information about the demand distribution are incomplete and the distribution-free approach is adopted, it is advisable to invest to improve quality. From Table 7, we note that savings are between .1 and 10.8%. Similarly to the outcomes in the previous example, T^* increases while θ^* decreases with an increment in δ_2 , and no change is evident in L^* .

5.1.5. Example 5

This example analyses the case in which backorder price discount is considered, while investments to reduce setup cost and improve quality are not included (i.e. we put $A = A_0$ and $\theta = \theta_0$). The demand within the protection interval is Gaussian, and Algorithm 1 is thus executed to find the solution. The results are given in Table 8, and have been obtained for $\delta_1 = 1/8000$, $\delta_2 = 1/2000$, and $\beta_0 = 0.0, 0.3, 0.6, 0.9$, and 1.0 . From Table 8, we observe that applying backorder price discount makes it possible to reach savings, which are however quite limited probably because of the specific parameter values adopted in the example. Note that, for larger values of β_0 , T^* decreases slowly, while no change is evident in π_x^* and L^* .

Table 8. Numerical results for Example 5.

β_0	T^* [years]	π_x^* [\$/unit]	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
.0	.1520	76.5	56	2648	$\approx .0$
.3	.1519	76.5	56	2647	< .1
.6	.1519	76.5	56	2647	< .1
.9	.1519	76.5	56	2646	< .1
1.0	.1519	76.5	56	2646	< .1
	.1520	(No discount)	56	2648	

Table 9. Numerical results for Example 6.

β_0	T^* [years]	π_x^* [\$/unit]	L^* [days]	\mathcal{R}^* [\$/year]	Savings (%)
.0	.1524	75.0	56	2658	$\approx .0$
.3	.1523	76.5	56	2656	< .1
.6	.1523	76.5	56	2655	.1
.9	.1522	76.5	56	2654	.2
1.0	.1522	76.5	56	2653	.2
	.1524	(No discount)	56	2658	

5.1.6. Example 6

The same assumptions and data of Example 5 are considered here, but the distribution within the protection interval is unknown. The results, obtained by means of Algorithm 2, are shown in Table 9. We can observe that, when limited information about the demand distribution are available and the distribution-free approach is adopted, applying backorder price discount is recommended. In fact, savings can be reached, although these cannot be fully appreciated probably because of the specific parameter values adopted in the example. Also note that the percentage of average annual cost reduction appears to be greater in the case of distribution-free approach than in the Gaussian case. In this example, a change in β_0 produces a larger effect in T^* and π_x^* than in the previous one. We can observe that T^* decreases more rapidly and π_x^* increases as β_0 becomes larger, while changes are still not present in L^* .

5.1.7. Example 7

This example studies the case in which backorder price discount is considered, in addition to investments to reduce setup cost and improve quality. The demand within the protection interval is Gaussian. The solution is obtained with Algorithm 1 for $\delta_1 = 1/8000$, $\delta_2 = 1/2000$, and $\beta_0 = 0.0, 0.3, 0.6, 0.9$, and 1.0, and is shown in Table 10. The results make evident the benefits from carrying out investments to reduce setup cost and improve quality, along with considering backorder price discount. In particular, although for different β_0 savings change little, it is noteworthy that the joint effect of these improvement actions permits to achieve larger benefits than if considered individually (see Tables 4, 6, and 8). The results also show that T^* and A^* decrease while π_x^* increases with an increment in β_0 . No change can be appreciated in θ^* and L^* .

5.1.8. Example 8

In this example, we use the same assumptions and data as in Example 7. The only difference is that the distribution of the demand within the protection interval is unknown and hence the distribution-free approach is used. The solution for the considered problems, obtained

Table 10. Numerical results for Example 7.

β_0	T^* [years]	A^* [\$/setup]	θ^*	π_x^* [\$/unit]	L^* [days]	\mathfrak{R}^* [\$/year]	Savings (%)
.0	.1014	82.2	1.5×10^{-4}	75.0	56	2406	9.1
.3	.1013	81.1	1.5×10^{-4}	76.0	56	2405	9.2
.6	.1012	81.0	1.5×10^{-4}	76.0	56	2404	9.2
.9	.1011	81.0	1.5×10^{-4}	76.0	56	2403	9.3
1.0	.1011	81.0	1.5×10^{-4}	76.0	56	2403	9.3
	.1520		(No investment)	(No discount)	56	2648	

Table 11. Numerical results for Example 8.

β_0	T^* [years]	A^* [\$/setup]	θ^*	π_x^* [\$/unit]	L^* [days]	\mathfrak{R}^* [\$/year]	Savings (%)
.0	.1195	95.6	6.2×10^{-5}	75.0	56	2348	11.7%
.3	.1193	95.4	6.2×10^{-5}	76.2	56	2346	11.7%
.6	.1191	95.3	6.2×10^{-5}	76.2	56	2345	11.8%
.9	.1189	95.1	6.2×10^{-5}	76.2	56	2343	11.9%
1.0	.1188	95.0	6.2×10^{-5}	76.2	56	2342	11.9%
	.1524		(No investment)	(No discount)	56	2658	

with Algorithm 2, is shown in Table 11. Similar conclusions to the previous example can be drawn here. That is, when information about the demand distribution are limited, investing to reduce setup cost and improve quality, in addition to applying backorder price discount, permits to achieve savings. Moreover, although for different β_0 they change little, these savings are larger than if improvement actions are performed individually (see Tables 5, 7, and 9). Similarly to the previous example, T^* and A^* decrease while π_x^* increases with an increment in β_0 , and no change can be noted in θ^* and L^* .

5.2. Sensitivity analysis

In this section, we examine the effect of changes in the system parameters D , h , and $c_v \equiv \frac{\sigma}{D}$ on $(T^*, A^*, \theta^*, \pi_x^*, L^*)$ and \mathfrak{R}^* for both cases Gaussian distribution and distribution-free approach. This analysis is carried out by changing each of the considered parameters by +50%, +25%, -25%, and -50%, taking one parameter at a time and keeping the value of the remaining parameters unchanged. Experiments are done for $\delta_1 = 1/8000$, $\delta_2 = 1/2000$, and $\beta_0 = .95$. The results are shown in Table 12 for the Gaussian distribution case, and in Table 13 for the distribution-free approach case.

From the results in Table 12, the following managerial insights can be obtained for the Gaussian case:

- T^* , A^* , θ^* and π_x^* decrease, while \mathfrak{R}^* increases with an increase in D . Moreover, T^* , A^* and θ^* appear to be highly sensitive to a change in D , while \mathfrak{R}^* is moderately sensitive and π_x^* is slightly sensitive.
- T^* and A^* decrease, while θ^* and \mathfrak{R}^* increase with an increase in h . Moreover, T^* , A^* , θ^* and \mathfrak{R}^* appear to be highly sensitive to a change in h , while π_x^* is almost insensitive.
- T^* , A^* and \mathfrak{R}^* increase, while θ^* decreases with an increase in c_v . Note that T^* , A^* , \mathfrak{R}^* and θ^* seem to be slightly sensitive to a change in c_v , while π_x^* is almost insensitive.

Table 12. Results of the sensitivity analysis for the Gaussian distribution case.

Parameter	% of change	% of change in the solution					
		T^*	A^*	θ^*	π_x^*	L^*	\mathcal{R}^*
D	+50	-33.0	-33.0	-2.8	-4	0	+17.0
	+25	-19.8	-19.8	+16.9	-3	0	+9.4
	-25	+33.0	+33.0	+95.9	+4	0	-12.0
	-50	+116.1	+116.1	+10.0	+1.5	0	-28.5
h	+50	-31.6	-31.6	+100.0	≈ 0	0	+10.3
	+25	-19.8	-19.8	+82.7	≈ 0	0	+5.7
	-25	+33.0	+33.0	+10.2	≈ 0	0	-7.3
	-50	+99.0	+99.0	-26.4	≈ 0	0	-17.5
c_v	+50	+5	+5	+45.8	≈ 0	0	+2
	+25	+3	+3	+46.1	≈ 0	0	+1
	-25	-3	-3	+46.9	≈ 0	0	-1
	-50	-5	-5	+47.3	≈ 0	0	-2

Table 13. Results of the sensitivity analysis for the distribution-free approach case.

Parameter	% of change	% of change in the solution					
		T^*	A^*	θ^*	π_x^*	L^*	\mathcal{R}^*
D	+50	-32.6	-32.5	-4.8	-4	0	+17.1
	+25	-19.5	-19.5	+15.0	-3	0	+9.4
	-25	+32.5	+32.6	+93.8	+4	0	-12.1
	-50	+114.3	+114.3	+100.0	+1.5	0	-28.6
h	+50	-31.5	-31.5	+100.0	≈ 0	0	+10.4
	+25	-19.5	-19.5	+79.6	≈ 0	0	+5.7
	-25	+32.6	+32.6	+9.0	≈ 0	0	-7.3
	-50	+97.7	+97.7	-26.9	≈ 0	0	-17.6
c_v	+50	+1.2	+1.2	+42.9	≈ 0	0	+4
	+25	+6	+6	+43.7	≈ 0	0	+2
	-25	-6	-6	+45.4	≈ 0	0	-2
	-50	-1.2	-1.2	+46.3	≈ 0	0	-4

- Changes in L^* are not reported for any variation in the model parameters D , h and c_v .

If we consider the distribution-free approach, Table 13 permits us to observe that the solution behaves similarly to the Gaussian case, in terms of response direction (i.e. positive or negative), to changes in the model parameters D , h and c_v . However, we can note that the absolute magnitude of changes is somewhat different:

- The sensitivity of \mathcal{R}^* and θ^* is larger than in the Gaussian case for changes in any parameter D , h and c_v .
- The sensitivity of T^* and A^* is larger than in the Gaussian case for changes in c_v while is smaller for changes in D and h .
- The sensitivity of π_x^* is identical to the Gaussian case.
- Similarly to the Gaussian case, changes in L^* are not reported for any variation in D , h and c_v .

6. Discussion

The issues of quality improvement, setup cost reduction, backorder price discount, and controllable lead time discussed in the previous sections are consistent with the approach called ‘changing the givens’ as suggested by Silver (1992). The numerical experiments carried out in Section 5 permit us to observe the benefits that can be reached taking into account the main features that this paper introduces. In particular, the following findings can be highlighted:

- (1) The larger the parameter δ_1 about the percentage decrease in A per money unit increase in investment, the greater the savings. A similar conclusion can be drawn concerning the parameter δ_2 about the percentage decrease in θ per money unit increase in investment.
- (2) The application of a backorder price discount permits to reach savings, which are even greater if investments to reduce setup cost and improve quality are considered as well.
- (3) Investing to reduce setup cost or to improve quality gives greater savings in the Gaussian case than when the distribution-free approach is used. The contrary happens when a backorder price discount is applied.

The observed effects of applying a backorder price discount and carrying out investments to reduce setup cost and improve quality are congruous with the results of similar studies (see, e.g. Ouyang et al., 2002; Sarkar and Moon, 2014; Sarkar, Mandal, et al., 2015). Moreover, although the observed findings were obtained with a single dataset, it is likely that the behaviour of the model does not change if other parameter values are used. This can be deduced noting that our results are consistent with those of other researchers, which used a different dataset.

7. Conclusions

This paper investigated a single-item periodic review inventory model with investments to reduce setup/ordering cost and improve quality, backorder price discount, and controllable lead time. The objective was to determine the review period, the setup/ordering cost, the quality level, the backorder price discount, and the length of lead time that minimize the long-run expected total cost per time unit. The problem was solved in two different cases: (i) the distribution of the demand within the protection interval is assumed to be Gaussian; and (ii) the minimax distribution-free approach is adopted.

Numerical experiments served to demonstrate the importance of investing to reduce setup/ordering cost and to improve quality, and applying a backorder price discount in a periodic review inventory model. In fact, we observed that considerable savings can be achieved if these actions are carried out. Further tests were performed to analyse the system behaviour when parameter values are made to vary.

Future researches may be directed to introduce fuzziness into the developed model, or to include investment to reduce lost-sale rate.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

- Annadurai, K., & Uthayakumar, R. (2010). Reducing lost-sales rate in (t, r, l) inventory model with controllable lead time. *Applied Mathematical Modelling*, 34(11), 3465–3477.
- Braglia, M., Castellano, D., & Frosolini, M. (2016). Joint-replenishment problem under stochastic demands with backorders-lost sales mixtures, controllable lead times, and investment to reduce the major ordering cost. *Journal of the Operational Research Society*, 67(8), 1108–1120.
- Braglia, M., Castellano, D., & Song, D. (2017). Distribution-free approach for stochastic joint-replenishment problem with backorders-lost sales mixtures, and controllable major ordering cost and lead times. *Computers & Operations Research*, 79, 161–173.
- Chuang, B.-R., Ouyang, L.-Y., & Chuang, K.-W. (2004). A note on periodic review inventory model with controllable setup cost and lead time. *Computers and Operations Research*, 31(4), 549–561.
- Chuang, B.-R., Ouyang, L.-Y., & Lin, Y.-J. (2004). A minimax distribution free procedure for mixed inventory model with backorder discounts and variable lead time. *Journal of Statistics and Management Systems*, 7(1), 65–76.
- Gallego, G., & Moon, I. (1993). The distribution free newsboy problem: Review and extensions. *Journal of the Operational Research Society*, 44(8), 825–834.
- Glock, C. (2012). Lead time reduction strategies in a single-vendor-single-buyer integrated inventory model with lot size-dependent lead times and stochastic demand. *International Journal of Production Economics*, 136(1), 37–44.
- Hadley, G., & Whitin, T. (1963). *Analysis of inventory systems*. Englewood Cliffs, NJ: Prentice-Hall.
- Hariga, M. (2000). Setup cost reduction in (q, r) policy with lot size, setup time and lead-time interactions. *Journal of the Operational Research Society*, 51(11), 1340–1345.
- Huang, C. (2001). Inventory model involving lead time as a decision variable with a mixture of lost sales and backorders. *Journal of Statistics & Management Systems*, 4(3), 227–237.
- Kim, M.-S., & Sarkar, B. (2017). Multi-stage cleaner production process with quality improvement and lead time dependent ordering cost. *Journal of Cleaner Production*, 144, 572–590.
- Kumar, R., & Goswami, A. (2015). A continuous review production-inventory system in fuzzy random environment: Minmax distribution free procedure. *Computers and Industrial Engineering*, 79, 65–75.
- Lee, W.-C., Wu, J.-W., & Lei, C.-L. (2007). Computational algorithmic procedure for optimal inventory policy involving ordering cost reduction and back-order discounts when lead time demand is controllable. *Applied Mathematics and Computation*, 189(1), 186–200.
- Leschke, J. (1996). An empirical study of the setup-reduction process. *Production and Operations Management*, 5(2), 121–131.
- Leschke, J., & Weiss, E. (1997). The multi-item setup-reduction investment-allocation problem with continuous investment-cost functions. *Management Science*, 43(6), 890–894.
- Liao, C.-J., & Shyu, C.-H. (1991). An analytical determination of lead time with normal demand. *International Journal of Operations & Production Management*, 11(9), 72–78.
- Lin, Y.-J. (2008). A periodic review inventory model involving fuzzy expected demand short and fuzzy backorder rate. *Computers and Industrial Engineering*, 54(3), 666–676.
- Lin, Y.-J. (2009). An integrated vendor-buyer inventory model with backorder price discount and effective investment to reduce ordering cost. *Computers and Industrial Engineering*, 56(4), 1597–1606.
- Lin, Y. J. (2010). A stochastic periodic review integrated inventory model involving defective items, backorder price discount, and variable lead time. *4OR*, 8(3), 281–297.
- Moon, I., & Gallego, G. (1994). Distribution free procedures for some inventory models. *Journal of the Operational Research Society*, 45(6), 651–658.
- Ouyang, L.-Y., Chen, C.-K., & Chang, H.-C. (2002). Quality improvement, setup cost and lead-time reductions in lot size reorder point models with an imperfect production process. *Computers and Operations Research*, 29(12), 1701–1717.
- Ouyang, L.-Y., & Chuang, B.-R. (2000). A periodic review inventory model involving variable leadtime with a service level constraint. *International Journal of Systems Science*, 31(10), 1209–1215.

- Ouyang, L.-Y., & Chuang, B.-R. (2001). A periodic review inventory-control system with variable lead time. *International Journal of Information and Management Sciences*, 12(1), 1–13.
- Ouyang, L.-Y., Chuang, B.-R., & Lin, Y.-J. (2005). Periodic review inventory models with controllable lead time and lost sales rate reduction. *Journal of the Chinese Institute of Industrial Engineers*, 22(5), 355–368.
- Ouyang, L.-Y., Chuang, B.-R., & Lin, Y.-J. (2007a). Effective investment to reduce lost-sales rate in a periodic review inventory model. *OR Spectrum*, 29(4), 681–697.
- Ouyang, L.-Y., Chuang, B.-R., & Lin, Y.-J. (2007b). The inter-dependent reductions of lead time and ordering cost in periodic review inventory model with backorder price discount. *International Journal of Information and Management Sciences*, 18(3), 195–208.
- Ouyang, L.-Y., Wu, K.-S., & Ho, C.-H. (2007). An integrated vendor-buyer inventory model with quality improvement and lead time reduction. *International Journal of Production Economics*, 108(1–2), 349–358.
- Pan, J.-H., & Hsiao, Y.-C. (2001). Inventory models with back-order discounts and variable lead time. *International Journal of Systems Science*, 32(7), 925–929.
- Porteus, E. (1985). Investing in reduced setups in the eoq model. *Management Science*, 31(8), 998–1010.
- Porteus, E. (1986). Optimal lot sizing, process quality improvement and setup cost reduction. *Operations Research*, 34(1), 137–144.
- Priyan, S., & Uthayakumar, R. (2014). Trade credit financing in the vendor-buyer inventory system with ordering cost reduction, transportation cost and backorder price discount when the received quantity is uncertain. *Journal of Manufacturing Systems*, 33(4), 654–674.
- Raza, S. (2015). An integrated approach to price differentiation and inventory decisions with demand leakage. *International Journal of Production Economics*, 164, 105–117.
- Rosenblatt, M., & Lee, H. (1986). Economic production cycles with imperfect production processes. *IIE Transactions (Institute of Industrial Engineers)*, 18(1), 48–55.
- Sarkar, B., Chaudhuri, K., & Moon, I. (2015). Manufacturing setup cost reduction and quality improvement for the distribution free continuous-review inventory model with a service level constraint. *Journal of Manufacturing Systems*, 34(C), 74–82.
- Sarkar, B., & Mahapatra, A. (2015). Periodic review fuzzy inventory model with variable lead time and fuzzy demand. *International Transactions in Operational Research*.
- Sarkar, B., & Majumder, A. (2013). Integrated vendor-buyer supply chain model with vendor's setup cost reduction. *Applied Mathematics and Computation*, 224, 362–371.
- Sarkar, B., Mandal, B., & Sarkar, S. (2015). Quality improvement and backorder price discount under controllable lead time in an inventory model. *Journal of Manufacturing Systems*, 35, 26–36.
- Sarkar, B., & Moon, I. (2014). Improved quality, setup cost reduction, and variable backorder costs in an imperfect production process. *International Journal of Production Economics*, 155, 204–213.
- Shu, H., & Zhou, X. (2014). An optimal policy for a single-vendor and a single-buyer integrated system with setup cost reduction and process-quality improvement. *International Journal of Systems Science*, 45(5), 1242–1252.
- Silver, E. (1992). Changing the givens in modelling inventory problems: the example of just-in-time systems. *International Journal of Production Economics*, 26(1–3), 347–351.
- Silver, E., Pyke, D., & Peterson, R. (1998). *Inventory management and production planning and scheduling* (3rd ed.). New York, NY: John Wiley & Sons.
- Taylor, B. I. (2015). *Introduction to management science* (12th ed.). London: Pearson.
- Tersine, R. (1982). *Principles of inventory and materials management* (4th ed.). New York, NY: North Holland.
- Wensing, T. (2011). *Periodic review inventory systems*. Lecture notes in economics and mathematical systems: Springer-Verlag, Berlin, Germany.
- Zipkin, P. (2000). *Foundations of inventory management*. New York, NY: McGraw-Hill/Irwin.