Università degli Studi di Modena e Reggio Emilia

Dipartimento di Scienze Fisiche Informatiche e Matematiche

Physics and Nanosciences - School of Graduate Studies

THE STRING-INSPIRED FORMALISM MEETS THE PERTURBINER METHOD: COMPUTATIONS OF SCATTERING AMPLITUDES IN GAUGE THEORIES AND GRAVITY

Tutor:

Prof. Olindo Corradini

PhD candidate:

Filippo Maria Balli

Cycle XXXV

The unexamined life is not worth living.

— Socrates

ABSTRACT

In quantum field theory, arguably the most important class of observables which can be studied are scattering amplitudes, i.e. probability amplitudes for processes of scattering of particles (or strings) off each other. Scattering amplitudes are computed in perturbation theory as a sum of Feynman diagrams, mathematical quantities that depend on the nature of particles involved in the scattering process. In gauge theories, the individual Feynman diagrams can be factorized into a color part, depending on the structure of the gauge group, and a kinematic part, depending on the momenta and polarizations of the external particles. In the recent years, the color-kinematics duality by Bern, Carrasco, and Johansson (BCJ) has been discovered for gauge theories. It represents a duality where, diagram-by diagram, the kinematic factors are in a representation such that they exhibit the same algebraic structures as their color counterparts. When organized in such a representation, the color factors can be replaced by another copy of kinematic factors. This procedure is known as double-copy and the resulting scattering amplitudes are amplitudes for gravitational theories. The great advantage of the double-copy construction is that, once a suitable representation in gauge theory has been achieved, gravitational amplitudes are computed automatically, and the complexity in the diagrammatic computation directly from Einstein Lagrangian is overcome. In this thesis, we detail recent results addressing BCJ duality and double-copy in the context of the worldline formalism. The worldline formalism represents an equivalent but independent way to study relativistic quantum mechanics with respect to the canonical quantum field theory. Essentially, in the worldline approach the scattering amplitudes are described no more through path integrals over fields but through path integrals over particle coordinates, i.e. integrals over space-time paths (worldline). In the recent years, the worldline approach in flat and curved space has had promising developments and it is now used as a powerful tool for the computation of amplitudes at tree- and loop-level also in the presence of gravity. Various interesting avenues of applications have been recently considered, such as the use of perturbiner and double copy techniques within the worldline formalism as will be discussed shortly, as well as the development of the so-called *worldline quantum field theory* formalism, whereby classical scatterings of relativistic bodies are studied.

In the first part of this manuscript, we review the way the worldline formalism is used in the computation of dressed propagators. In particular, tree-level scattering amplitudes for a scalar particle coupled to an arbitrary number of photons and a single graviton are computed using worldline techniques. Specifically, we consider the case of a scalar propagator dressed with two photons and one graviton, and, as the amplitude is fully off-shell, we use it to sew together the two external photons and to construct one-loop radiative corrections to the scalar-scalar-graviton vertex. We test our construction by verifying the on-shell gauge and Ward identities. In the second part of the thesis, we develop a novel procedure to construct the so-called Berends-Giele (BG) currents using the worldline formalism for one-loop gluon amplitudes (Bern-Kosower formalism). The Berends-Giele currents are a set of auxiliary fields introduced in the study of multiparticle scattering amplitudes in non-abelian gauge theory and are interpreted as tree-level amplitudes with one leg off-shell. These currents are often used fundamental building blocks for on-shell amplitudes: applying the so-called pinch procedure of the BK formalism to a suitable special case, the currents are naturally obtained in terms of multiparticle fields in a color-kinematic-dual representation. Using the same construction from the worldline Bern-Dunbar-Shimada formalism for one-loop gravity amplitudes, we naturally obtain gravity multiparticle polarization tensors as tensor product of multiparticle fields in a color-kinematic-dual representation. This allows us to formulate a revised prescription for double-copy gravity BG currents, and to obtain both the color-dressed Yang-Mills BG currents in a color-kinematic-dual representation and the gravitational BG currents explicitly.

In teoria quantistica dei campi, le ampiezze di scattering sono una delle categorie di osservabili più importanti da studiare. Esse descrivono la probabilità per processi di interazione tra particelle (o stringhe). Le ampiezze di scattering sono calcolate in teoria perturbativa come somma di diagrammi di Feynman, oggetti matematici con proprietà determinate dalle particelle coinvolte nel processo di scattering. In teorie di gauge, i diagrammi di Feynman possono essere fattorizzati individualmente in un termine di *colore*, legato alla struttura del gruppo di gauge, e un termine *cinematico*, legato ai momenti e alle polarizzazioni delle particelle esterne. Recentemente, Bern, Carrasco e Johansson (BCJ) hanno scoperto, per le teorie di gauge, una dualità che permette di convertire i fattori cinematici in una rappresentazione che mostra le stesse strutture algebriche dei termini di colore. In questa rappresentazione, i fattori di colore possono essere sostituiti da un'altra copia di fattori cinematici. Questa procedura è nota come *double copy* e permette di ottenere ampiezze per teorie di gravità. Il vantaggio di questa procedura è che, una volta ottenuta una rappresentazione appropriata nella teoria di gauge, le ampiezze gravitazionali possono essere calcolate automaticamente, evitando le complessità del calcolo diagrammatico direttamente dalla Lagrangiana di Einstein. In questa tesi, verranno esaminati e discussi recenti sviluppi riguardo la dualità BCJ e la procedura di *double copy* all'interno del contesto del formalismo worldline. Il formalismo worldline è un metodo alternativo per lo studio della meccanica quantistica relativistica che si basa sull'utilizzo di integrali di cammino su coordinate spazio-temporali invece che su campi. Questo approccio ha dimostrato di avere un grande potenziale per la realizzazione di calcoli sia in spazi piatti che curvi, e si è rivelato efficace per il calcolo delle ampiezze di scattering sia ad *albero* che a *loop*, anche in presenza di effetti gravitazionali. Negli ultimi anni, sono state sviluppate diverse applicazioni interessanti del formalismo worldline, tra cui l'utilizzo di tecniche come *perturbiner* e *double copy*, e la creazione di una nuova teoria chiamata *worldline quantum field theory*, che si occupa dello studio delle ampiezze classiche di particelle relativistiche.

Nella prima parte della tesi, esploriamo l'utilizzo del formalismo worldline per calcolare propagatori vestiti. In particolare, calcoliamo le ampiezze di scattering ad albero per una particella scalare accoppiata a un numero variabile di fotoni e un singolo gravitone. In dettaglio, consideriamo il caso di un propagatore scalare vestito con due fotoni e un gravitone e, poiché l'ampiezza è completamente off-shell, la utilizziamo per unire i due fotoni esterni e quindi costruire correzioni a un loop per il vertice scalare-scalare-gravitone. Verifichiamo la validità della nostra procedura testando la trasversalità e le identità di Ward. Nella seconda parte, proponiamo un nuovo metodo per generare le correnti di Berends-Giele (BG) utilizzando il formalismo worldline per le ampiezze di gluoni a un loop (formalismo di Bern-Kosower). Le correnti di Berends-Giele sono un insieme di campi ausiliari utilizzati nello studio delle ampiezze di scattering multi-particellari in teorie di gauge non-abeliane e vengono interpretate come ampiezze a livello albero con una gamba off-shell. Questi campi vengono spesso utilizzati come blocchi fondamentali per costruire ampiezze on-shell: applicando la procedura di "pinching" del formalismo BK, le correnti vengono naturalmente ottenute in termini di campi multiparticellari nella rappresentazione BCJ. Successivamente utilizziamo una procedura simile, basata sul formalismo worldline di Bern-Dunbar-Shimada per le ampiezze di gravità a un loop, per ottenere tensori di polarizzazione multiparticellare come prodotto tensoriale di campi multiparticellari nella rappresentazione BCJ. Ciò ci consente di formulare una nuova prescrizione *double copy* per le correnti BG gravitazionali e di ottenere esplicitamente sia le correnti BG in Yang-Mills nella rappresentazione BCJ che le correnti BG gravitazionali.

ACKNOWLEDGMENTS

There are a multitude of people who should be thanked for helping me reach this point. To begin, I am grateful to my advisor, Olindo. I first met you during my master and I have been fortunate enough to continue working with you during my Ph.D. studies. I have deeply appreciated your scientific mentorship, your support and your friendship, which have never failed me over the years. It has been a privilege to learn from you, and I hope to keep some of your lessons with me as I embark on my journey beyond Modena.

After Olindo, I would like to express my gratitude to my collaborators Christian, Naser, Cristhiam and Alexander, whose expertise and dedication have not only granted me precious insights during my studies but have also been essential in the realization of our project.

I feel fortunate to have had the opportunity to spend a significant part of my Ph.D. at Uppsala University, which has been a truly life-changing experience for me. There are many people who have contributed to this journey and I am grateful to all of them. It goes without saying that Oliver, my second advisor in this experience, deserves a special mention. I am grateful for the priceless knowledge I gained from the hours spent together in front of a blackboard, as well as for the time we spent outside of the university in front of a beer or at your amazing s***** place. Working together with you has been, and continues to be, a source of pride for me.

Next up is Alex. In a nutshell, the best flatmate I could have asked for. You are a brilliant physicist, a Mathematica guru, an excellent chef and, last but not least, a good friend. The number of times I have asked for your help is countless, and I cannot think of a better place than this to express my gratitude.

In conclusion, I would like to mention a few other members of the *hep-th* group in Uppsala who have made my time there truly wonderful. A big thank you to Carlos: your peculiar enthusiasm, not just for physics, has been contagious since my (literally) first step in Sweden. Thank you to Paolo, your life lessons are a baggage that I will carry with me forever. Finally, thank you to Maor, Lucile, Simon, Lucia, Paul, Andrea, Martijn (and anyone else I may have omitted). Whether it was enjoying a *fika*, a game of frisbee, or simply a chat in the corridor, these moments were truly precious and will remain with me for the rest of my life.

Questa è la prima volta che mi trovo a scrivere dei ringraziamenti, non perché non abbia mai considerato importanti i passaggi precedenti della mia vita universitaria, ma perché sono sempre stato convinto che il mio percorso sarebbe proseguito. Tuttavia, ora che la mia vita da studente sta per finire (almeno formalmente), è giunto il momento di guardare indietro e ringraziare coloro che mi hanno permesso di arrivare fin qui.

Per primi, vorrei ringraziare quelli che hanno fatto parte della mia vita universitaria a Modena. Quindi grazie a Elena, Giulia, Alice, Matteo e Giacomo, amici preziosi che porterò con me. Grazie poi a Samuele. In questi anni, forse più di ogni altro, hai sopportato ogni mia stranezza, crisi o lamentela, aiutandomi sempre con parole mai banali. Infine, grazie a Frank, molto più di un compagno di ufficio sempre all'attacco. Con te questi anni sono stati più allegri, nonostante tutto.

Si sa, la vita di paese ti porta a conoscere tutti. Eppure mi sento fortunato a poter considerare amici tanti di loro (ma mai troppi). Ringraziarvi uno ad uno sarebbe dovuto, ma mi servirebbe molto più di una singola pagina. Quindi semplicemente grazie a chi c'è stato, c'è e ci sarà, siete tutti un tassello di un colore diverso in questo viaggio. Un grazie va comunque ai fratelli (e sorelle), non di sangue ma certamente di fatto. La vita cambia sapore lontano da voi. Com'era? Ci siamo r... Diciamo che ci siamo divertiti ecco.

A questo punto due nomi devo comunque farli, e un grazie speciale va a Ludovica e Stefano. Siete le mie rocce. Non so cosa ci riserverà la vita, ma qualsiasi cosa sarà insieme, e non ho bisogno di altro.

Grazie a chi non c'è più, ma avrebbe tanto voluto esserci. Nonno Vittorio e nonna Franca, siete sempre al mio fianco. Carlo, questo traguardo è anche e soprattutto dedicato a te.

A conclusione l'ultimo e più sentito grazie va alla mia famiglia. Grazie in particolare a Roma e Alessandro, nonni per diritto ma secondi genitori per scelta. Siete, oggi come sempre, la mia certezza. Grazie a Carolina, autentica forza della natura ed esempio costante. Le mia testa è sempre più libera quando parlo con te, di qualsiasi cosa si tratti. Grazie infine a Cesare ed Elisabetta, o più semplicemente mamma e papà. Non ho mai parlato abbastanza, ma il vostro supporto

puro ed incondizionato ad ogni mia scelta mi ha dato il coraggio e la serenità per arrivare fino a qui. Vi voglio bene.

CONTENTS

GLOSSARY OF ACRONYMS

BCJ	Bern-Carrasco-Johansson
BK	Bern-Kosower
KK	Kleiss–Kuijf
BDS	Bern-Dunbar-Shimada
BCFW	Britto-Cachazo-Feng-Witten
KLT	Kawai-Lewellen-Tye
YM	Yang-Mills
BG	Berends-Giele
GJI	Generalized Jacobi Identities
QED	Quantum Electroynamics
QFT	Quantum Field Theory
IBP	Integration By Parts
1PI	One-Particle Irreducible
MHV	Maximally Helicity Violating

1 INTRODUCTION

1.1 AN INVITATION TO SCATTERING AMPLITUDES

Scattering amplitudes are a fundamental concept in quantum physics, used to describe the probability amplitudes for the possible outcomes of processes of scattering of fundamental particles off each other. These amplitudes are essential for understanding the behavior of subatomic particles and for testing the mathematical consistency of our physical models, as well as for making concrete predictions about the possible results in quantum scattering experiments.

The study of scattering phenomena has a long history, dating back to the pioneering experiment that led to Rutherford's discovery of the atomic nucleus in the early 20th century. More recently, the discovery of the Higgs boson at the Large Hadron Collider (LHC) [1] in 2012, possible due to the precise measurements of scattering amplitudes, was a major breakthrough in our understanding of the laws of Nature. The need for accurate theoretical predictions for current and upcoming experiments, as well as the desire of theoretical physicists to rigorously test their models, has led to the development of highly efficient methods for not only calculating amplitudes but also extracting physical quantities from them. The use of these advanced techniques has revealed striking connections between theories relevant to particle scattering at the LHC, and General Relativity. These connections between scattering amplitudes play a crucial role in giving a new perspective on black holes and the physics of gravitational waves, which have been recently detected by the LIGO and VIRGO collaborations [2]. As a result of this new wave of breakthroughs, scattering amplitudes are now established as a major new field in theoretical high-energy physics.

The recent advancements and the new techniques developed for the calculation of scattering amplitudes have a common thread: they reveal that scattering amplitudes are much simpler than what one naively expects from their construction in terms of Feynman diagrams. Historically, diagrammatic calculation of scattering amplitudes has been extremely challenging in various physical models. This is due to the fact that the number of relevant Feynman diagrams increases rapidly with both the number of particles involved in the scattering process and the number of loops. Despite the complexity of the intermediate expressions in the diagrammatic calculation, it is remarkable that the finals results are often surprisingly simple. One of the most iconic examples is the famous Parke-Taylor n-gluon tree amplitude, which illustrates the remarkable simplicity that can be achieved in these calculations. By making use of the spinor helicity formalism and focusing on color-ordered partial amplitudes in Yang-Mills theory, the Maximally Helicity Violating (MHV) gluon amplitude at any number of points is expressed as [3]:

$$A_n^{MHV}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = ig^{n-2}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \quad (1.1.1)$$

There is no need to understand the meaning of the symbols in the expression above (which originates from spinor helicity formalism) to appreciate the simplicity of the result, which is effectively independent of the number of gluons. On the other hand, Feynman diagrams fail to account for it: the number of Feynman diagrams necessary for carrying out this computation is 4 at $n = 4$, 25 at $n = 5$, at $n = 6$ is already 220, and the number is exponentially growing. The take-home message from this example is that scattering amplitudes have intrinsically a hidden simplicity that cannot be fully captured by Feynman diagrams. Therefore, it is crucial to continue searching for more efficient methods to compute amplitudes, as this may lead to the discovery of new symmetries that can explain this simplicity.

Over the last decades a number of powerful techniques for the on-shell matrix elements calculation have emerged. Among the others, here we mention the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [4, 5], where tree-level amplitudes of increasing multiplicity are computed by using on-shell amplitudes from lower-level trees, and generalized unitarity methods [6, 7], which allow for the systematic construction of complex loop-level predictions using compact on-shell tree-level data. In an independent development, the perturbiner expansion method was introduced by Rosly and Selivanov [8, 9] as an efficient method for obtaining tree-level scattering amplitudes for a generic massless quantum field theory. In the case of Yang-Mills, it can also be used to compute multiparticle trees with one particle off-shell, which are referred to as Berends-Giele currents [10].

Recent studies have revealed a deeper understanding of the mathematical structure of amplitudes in gauge theory through the development of the Bern-Carrasco-Johansson (BCJ) duality [11]. This duality investigates the connection between the color and kinematic factors of the gauge theory amplitudes. When this duality is satisfied, the color and kinematic parts of the contributing diagrams obey the same algebraic relations, leading to a more organized and simplified representation of the gauge theory amplitudes. This also implies that the kinematic numerators of the amplitude are not independent, and that by making use these relations, the number of required calculations to obtain an n-point amplitude can be significantly reduced.

In addition to their applications in particle physics, scattering amplitudes also play a crucial role in string theory, a theoretical framework that attempts to unify all known forces of nature, including gravity. In string theory, point-like particles of particle physics are replaced by one-dimensional objects, called strings, which interact with each other as they propagate through space. Scattering amplitudes are thus used to calculate the probabilities of different outcomes in these string interactions. Because of mathematical richness in the structure of string theory, the study of scattering amplitudes has led to the development of new and powerful mathematical tools. One such example are the so-called Kawai-Lewellen-Tye (KLT) relations between open string amplitudes and closed string amplitudes [12]. These relations, which come from open-closed duality, state that the n-point tree-level closed string scattering amplitudes can be expressed as a sum of products of n-point open string partial amplitudes at the perturbative string level.

This result has severe implications on scattering amplitudes in quantum field theories. Indeed, string theory is thought to be the ultraviolet completion of (super)gravity theory. In the limit of infinite tension, where α' goes to zero, the strings become point particles, and Einstein and Yang-Mills (YM) theories are recovered. In particular, in the particle limit $\alpha' \to 0$, KLT leads to relations between tree-level graviton amplitudes and tree-level gluon amplitudes in Yang-Mills theories, which are often summarized as

$$\text{Gravity} = (\text{Gauge Theory})^2. \tag{1.1.2}$$

Such duality holds even though the structures of the non-abelian Yang-Mills and the Einstein-Hilbert Lagrangians are rather different: the former contains only up to four-point interactions while the latter contains infinitely many vertices. Therefore the validity of the above duality in the field theory limit has been a major puzzle for many years. The connection between gravity and gauge theory starts already at three points:

$$\mathcal{M}_3(1,2,3) = A_3(1,2,3)\tilde{A}_3(1,2,3), \tag{1.1.3}$$

where $\mathcal{M}_3$ and A_3 are the three-point gravity and gauge theory amplitudes accordingly, and complex momenta are used to avoid the vanishing of the three-point amplitudes for real on-shell momenta. For the four- and five-point amplitudes the relations keep still very simple:

$$\mathcal{M}_4^{\text{tree}}(1,2,3,4) = -\, s_{12} A_4(1,2,3,4) A_4(1,2,4,3) \tag{1.1.4}$$
$$\mathcal{M}_5^{\text{tree}}(1,2,3,4,5) = s_{23} s_{45} A_5(1,2,3,4,5) A_5(1,3,2,5,4) + (3 \leftrightarrow 4), \tag{1.1.5}$$

where $s_{ij} = (k_i + k_j)^2 = 2\, k_i \cdot k_j$.

This remarkable simplicity in the construction of gravity amplitudes has been also captured by Bern, Carrasco and Johansson, who recently discovered an alternative direct way of constructing gravity amplitudes from gauge theory ones. In detail, this construction, called *double copy*

[13], allows for tree-level gravity amplitudes to be easily obtained by replacing the color factors of the gauge theory amplitude with another copy of the associated numerators, as long as the amplitude numerators satisfy the color-kinematics duality. Specifically, at tree-level, color dressed scattering amplitudes in Yang-Mills theories can be written in the following form

$$\mathcal{A}_n^{\text{tree}} \sim \sum \frac{c_j n_j}{\prod_{i_j} s_{i_j}}. \tag{1.1.6}$$

where the color factors c_i are formed by combining the structure constants of the gauge Lie algebra, the kinematic numerators n_i are determined by the momenta and polarizations of the external particles, and the s_{i_j} represent the propagators in different diagrams. Color-kinematics duality states that the relations satisfied by the color factors (as determined by Jacobi identities) are mirrored by the corresponding kinematic numerators:

$$c_i + c_j + c_k = 0 \qquad \Leftrightarrow \qquad n_i + n_j + n_k = 0. \tag{1.1.7}$$

The great advantage of having the amplitude in a color-kinematics dual representation is that the calculation of the associated gravity amplitude is automatic. The n-point gravity amplitudes are obtained in terms of the gauge theory information simply by replacing the color factors by another copy of the kinematic numerators:

$$\mathcal{M}_n^{\text{tree}} \sim \sum_{j \in \text{ trivalent}} \frac{\tilde{n}_j n_j}{\prod_{i_j} s_{i_j}}. \tag{1.1.8}$$

At tree level, proofs exist [14–16] that the color-kinematics duality and the double copy hold. However, at loop level, less is known but explicit examples have shown that the duality between color and kinematics and the double copy hold for a broad range of cases [17–21]. Among the applications of the double copy prescription, we also mention the computation of classical solutions for gravity [22–25], and relations to gravitational-wave physics [26, 27].

In this brief overview, we have only presented a selection of the methods and techniques developed in recent years for computing scattering amplitudes. Many more exist —see [28] for an extensive review. However, our hope is that this introduction has convinced the reader of its main point: scattering amplitudes are relatively simple, or at least more so than what was previously believed by the theoretical physics community until few decades ago. The greatest challenge facing the field of scattering amplitudes in the future is understanding the underlying reasons for this simplicity. Eventually, we will uncover more fundamental principles, that will shed light on this crucial problem, and that will lead us to a deeper understanding of the quantum laws of Nature.

1.2 A GLIMPSE INTO THE WORLDLINE FORMALISM

The operatorial approach of non-relativistic quantum mechanics was developed in the late twenties by a number of physicists, including Paul Dirac and David Hilbert. This approach allowed for a unification of the earlier, separate descriptions of quantum mechanics based on wave mechanics (Schrödinger) and matrix mechanics (Heisenberg). In 1948, Richard Feynman proposed a new method [29] for understanding non-relativistic quantum mechanics that was based on the concept of path integrals. This approach, while mathematically equivalent to the operatorial approach, was distinct, as it was derived from the principle of least action, rather than Hamiltonian dynamics. The path integral formulation provides an intuitive way to understand quantum mechanics by viewing the system as the sum of all possible paths between two points at different times, each path weighted by a complex phase factor. Two years later, Feynman started publishing his work that laid the foundation for relativistic Quantum Field Theory (QFT) and specifically Quantum Electroynamics (QED) through the use of Feynman diagrams. However, at the same time he also developed a different formalism, later called *worldline* formalism. Here the QED scattering amplitudes are described no more through path integrals over fields but through relativistic particle path integrals, i.e. path integrals over particle coordinates. Essentially, both the worldline formalism and (second quantized) quantum field theory are ways to study relativistic quantum mechanics, but they are independent of each other. In the worldline formalism, both time and space coordinates are treated as operators and the proper time is used as a parameter, while in quantum field theory, they are treated as parameters, that label the different kinds of fields. Thus, in the path integral approach, the worldline formalism becomes an integral over spacetime paths, worldlines —see figure 1.1.

Initially, the worldline approach was not widely recognized as a promising method and was not used extensively in research for several years after its introduction. The potential of the worldline formalism to improve on standard field theory methods was recognized few decades later in the early nineties, which coincides with a period of rapid development of string theory. This was not a coincidence. Indeed, similarly to the worldline formalism, string theory is a first quantized approach, i.e. the coordinates of the string in space are promoted to operators and the canonical quantization is realized by imposing commutation relations among them. The study of string theory is driven in part by its property to converge with quantum field theory as the size of the string shrinks to zero. In the 1970s-1980s, many advancements in field theory have been made by examining this point-particle limit of strings. As examples, here we mention the calculation made in 1982 by Green, Schwarz and Brink [30], who obtained the one-loop four-gluon amplitude in $\mathcal{N} = 4$ Super Yang-Mills theory from the low energy limit of superstring theory, and the computation in 1988 of the one-loop β-function for Yang-Mills [31–33]. A systematic investigation of the infinite string tension limit was undertaken in the following years by Bern and Kosower [34, 35]. They succeeded in deriving a novel type of parameter integral representation for the on-shell n-gluon amplitude in Yang-Mills theory, at the tree- and one-loop level. Moreover, they established a set of rules which allows one to construct this parameter integral, for any number of gluons and choice of helicities, without referring to string theory any more. The efficiency of these rules has been demonstrated by the first complete calculation of the one–loop five–gluon amplitude. Motivated by this wave of new results, and by the challenges of computations in string theory, this set of questions naturally arose:

Is it possible to rederive these results completely inside particle theory? If so, what form should this theory take? Would it be more efficient than the standard field theory and Feynman diagrams, at least for specific calculations?

The reader should not be surprised by a positive answer. The worldline formalism, as mentioned, represents a first-quantized formulation of quantum field theory (instead of the usual second-quantized approach), and can be tought as *string-inspired* in the sense that it has a natural interpretation in terms of a one dimensional field theory in a first-quantized formulation. Thus, it's ideal for adapting string theory techniques for specific calculations in field theory.

With this in mind, Strassler, in the early nineties, began investigating seriously the worldline formalism [36]. In particular, he showed that ordinary perturbation theory can be obtained in the first quantized approach, simply by mimicking string perturbation theory. Let us demonstrate the one-loop scalar QED effective action as an example. This can be represented through a worldline path integral as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A\right)}. \tag{1.2.1}$$

By rewriting the Maxwell field $A^\mu(x)$ as a sum of plane waves,

$$A_\mu(x) = \sum_{i=1}^{n} \varepsilon_{i\mu}\, e^{ik_i\cdot x}, \tag{1.2.2}$$

and performing some algebraic manipulations, one is able to obtain a master formula that encodes a closed expression for the one-loop n-photon amplitude in scalar QED. At this stage, we are not interested in its mathematical formulation (which will be covered later in the thesis), but there are some key points we want to highlight here:

- The formula does not involve loop momentum, which reduces the number of kinematic invariants from the beginning, and allows for an efficient use of the spinor helicity method.

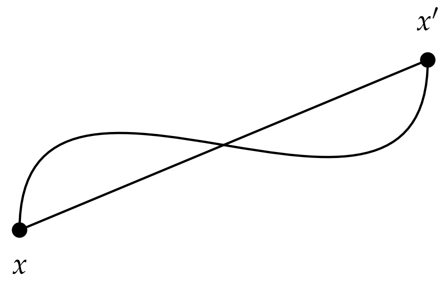


Fig. 1.1: In the worldline approach, the path integrals are performed over spacetime coordinates, and no more over fields, as usually happens in standard (second quantized) field theory.

- The formula results in a set of parameter integrals that define the ordering in which the external photon legs merge into the loop. However the complete integral does not represent any particular Feynman diagram, with a fixed ordering of the external legs, but the sum of them. In other words, we are not referring to Feynman diagrams anymore, and our description has a superior organization of gauge invariance.

- The integral representation of the amplitude is valid off-shell. Thus, we can sew together a pair of legs, and obtain a parameter integral representing the complete two-loop $(n-2)$-photon amplitude.

Those are all important steps in the quest for simplicity, which is crucial for the current research in scattering amplitudes. As a drawback, the description of interactions among particles appears to be less intuitive compared to the second-quantized approach and thus, innovative methods had to be derived to overcome this challenge.

Since the work by Strassler, who rederived the Bern-Kosower master formulas directly from point particle path integrals (for a comprehensive review see [37]), many extensions and applications of the worldline formalism have been considered: multiloop computations [38], the numerical worldline approach to the Casimir effect [39], the worldline formalism in curved spacetime [40–42], photon-graviton mixing at one loop [43], as well as applications to noncommutative QFT [44, 45], Standard Model physics [46] and its grand-unified extensions [47]. More recently, a classical double-copy relation was developed [48] in the context of a worldline QFT description [49] of the classical gravitational scattering of massive bodies.

In our work, we try to exploit recent develpements in worldline techniques. Our analysis will initially focus on the calculation of dressed propagators in curved spacetime in an efficient manner. Then, we will integrate the worldline approach with recent developments in scattering amplitudes, such as color-kinematics duality and double copy. The goal of this manuscript is to convince the reader that the worldline formalism not only provides a concrete alternative to standard field theory, but also has a concrete role in recent developments and techniques in scattering amplitudes.

1.3 OUTLINE

This work is divided into four main parts.

- The first part provides an overview of the string-based formalism, together with recent developments in scattering amplitudes such as color-kinematics duality and double copy, and Berends-Giele currents calculation method using the perturbiner method. Chapter 2 offers a brief summary of the string-inspired or worldline formalism in perturbative quantum field theory and its use as an alternative computation tool for effective actions and dressed propagators, similar to string perturbation theory, without the use of common mathematical tools and structures in standard field theory. The chapter also includes examples of one-loop amplitudes and tree-amplitudes calculation in QED and gravity using the Bern-Kosower formalism, which will be a key element in this manuscript. Chapter 3 reviews the basics of the color-kinematics duality and double copy as recent and significant developments in scattering amplitudes. This includes reviewing algebraic tools for dealing with the color structure of Yang-Mills theory, rederiving relations among partial amplitudes (Kleiss-Kuijf and BCJ) and introducing the main ideas behind the color-kinematics duality and the double copy prescription. Chapter 4 reviews the Berends-Giele recursion relations and their application in computing tree-level scattering amplitudes in Yang-Mills theory, connecting it to the perturbiner technique that can be used to obtain generating functions for all tree-level scattering amplitudes in a given theory.

- In the second part, we present recent results on scattering amplitudes in scalar QED within a curved background, i.e. we include the integration of gravity inside dressed propagators from the worldline approach. In detail, in chapter 5, we compute the tree-level scattering amplitudes of a scalar particle interacting with n photons and a single graviton. The worldline formalism is applied to calculate the irreducible part of the amplitude where all photons and gravitons are attached to the scalar line. Also, a tree replacement rule is introduced to construct the reducible parts of the amplitude. Chapter 6 covers the calculation of off-shell one-loop QED correction to the graviton-scalar vertex in any covariant gauge using the formalism developed in the previous chapter. The process includes re-deriving the off-shell amplitude for two-photon and one-graviton and applying the sewing procedure to compute the radiative correction.

- In the third part, we use standard worldline techniques to develop new methods for computing Berends-Giele currents, which reveal interesting connections to color-kinematics duality and the double copy. In chapter 7, we introduce a novel procedure for constructing Berends-Giele currents using the Bern-Kosower formalism for one-loop gluon amplitudes by applying the pinch procedure of that formalism to a specific case. The resulting currents are expressed in terms of multiparticle fields and obey the color-kinematics duality. Chapter 8 generalizes the previous result by providing a new method for computing the gravity polarization tensor as a product of multiparticle fields using the Bern-Dunbar-Shimada formalism for one-loop gravity amplitudes. We also introduce a revised prescription for double-copy for gravity Berends-Giele currents and show how to obtain them explicitly in the BCJ gauge, i.e. in a representation such that color-kinematics duality is obeyed.

- The thesis concludes with a final chapter, where we summarize the main results and suggest potential avenues for future research, as outlined in chapter 9. Additionally, an appendix is provided, containing supplementary information: the detailed calculation of a scalar propagator dressed with two photons and one graviton in appendix A, a proof of transversality on the graviton line in scalar amplitudes with two or fewer photons and one graviton in appendix B, a list of Feynman integrals used for the computation of radiative corrections to the graviton-scalar vertex in QED, along with reduction formulae for specific integrals in appendix C, the color-dressed Berends-Giele polarization current in the BCJ gauge with the corresponding double copy version for gravity in appendix D.

Part I

STATE OF THE ART

2

THE STRING-INSPIRED FORMALISM

In this chapter we give a short review of the basics for the application of the so-called *string-inspired*, or *worldline*, formalism to perturbative quantum field theory. The formalism offers the possibility of computing effective actions and dressed propagators in a way which is similar in spirit to string perturbation theory, and avoids the use of many of the mathematical tools and structures that are commonly used in standard second-quantized field theory. Here we present an overview of the fundamentals in the calculation of string scattering amplitudes and we also provide an intuitive explanation of how this relates to the worldline formalism. Later we will present a considerable number of sample calculations: effective actions for the perturbative calculation of one-loop amplitude with n external particles (gluons or gravitons), and dressed propagators for the computation of tree-amplitudes, focusing on QED and gravity.

2.1 STRING SCATTERING AMPLITUDES IN A NUTSHELL

The basic tool for the calculation of string scattering amplitudes is the Polyakov path integral. In the simplest case, the closed bosonic string propagating in flat spacetime, this integral is of the form

$$\langle V_1 \cdots V_n \rangle \sim \sum_{\text{top}} \int \mathcal{D}h \int \mathcal{D}X(\sigma,\tau) V_1 \cdots V_n e^{-S[X,h]}, \tag{2.1.1}$$

where $X(\sigma,\tau)$ are the coordinates of the string. This path integral corresponds to first quantization in the sense that it is performed over the spacetime coordinates of a single string that propagates in a Minkowski background. The worldsheet action in the path integral is the *Polyakov action*, which is given by

$$S[X,h] = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.1.2}$$

where $h \equiv \det h_{\alpha\beta}$ and $1/2\pi\alpha'$ is the string tension. The Polyakov action corresponds to a two-dimensional conformal field theory describing the worldsheet of a string in string theory. The parameters σ, τ in (2.1.1) parametrize the worldsheet surface swept out by the string in its motion —see figure 2.1, and the integral $\int \mathcal{D}X(\sigma,\tau)$ has to be performed over the space of all embeddings of the string worldsheet with a fixed topology into spacetime. The tensor $h_{\alpha\beta}$ determines a specific worldsheet metric and the Polyakov path integral, through $\int \mathcal{D}h$, instruct us to sum over all the possible values of it. At the same

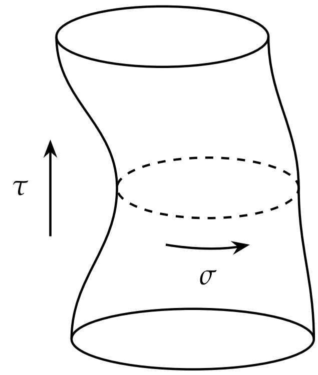


FIG. 2.1: Worldsheet swept out by a string in its motion in Minkowski space. The worldsheet is locally parametrized by the coordinates (τ, σ), where τ is time-like while σ is space-like.

time, worldsheets can have different topologies, and we have to sum over these according to $\sum_{\text{top}}$. This sum gives rise to a perturbative expansion in string theory —we will see later that this corresponds to a loop expansion in quantum field theory in the particle limit of string theory. The weight of the different surfaces in the perturbative expansion is understood once we make explicit the dependence over the string coupling constant g_s inside (2.1.1). In particular, the power that accompanies each g_s depends on the topology of a specific worldsheet and on whether we are dealing with open or closed strings. For closed strings, the worldsheets have no boundary and each topology is weighted by a factor

$$(g_s)^{2(g-1)} \,. \tag{2.1.3}$$

Here the parameter g describes the *genus* of a given surface. After a conformal map, the worldsheets involved in the sum inside (2.1.1) are transformed into Riemann surfaces with specific genus. For example, the tree-level scattering of closed strings corresponds to a worldsheet with the topology of a sphere: the genus is $g = 0$ and the amplitudes are proportional to $1/g_s^2$. One-loop scattering corresponds to toroidal worldsheets, where $g = 1$, and have no power of g_s —although, obviously, these are suppressed by g_s^2 relative to tree-level processes. Higher terms in the expansion are associated to values $g > 1$ and correspond to higher powers of g_s. Thus, the perturbative expansion in string theory becomes a sum over Riemann surfaces of increasing genus, as depicted in the first line of figure 2.2 for the closed string case.

On the other hand, open string worldsheets have topologies with

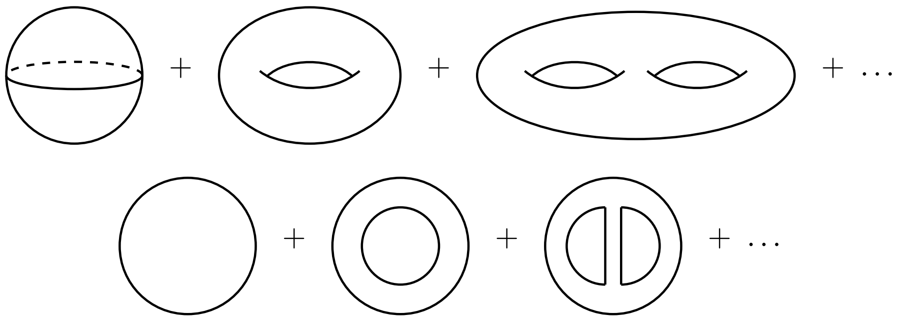


FIG. 2.2: The loop expansion in perturbation theory for closed (top) and open (bottom) strings.

boundaries and the string coupling constant has to be redefined according to[1]

$$g_{\text{open}}^2 = g_s. \tag{2.1.4}$$

The expansion for open string scattering becomes an expansion in the number of boundaries added to the worldsheet. Adding a boundary corresponds to the addition of a strip to the worldsheet, i.e. the emission and absorption of a virtual open string. The disc is weighted by $1/g_s$; the annulus has no factor of g_s and so on.

The last element in (2.1.1) that has not been analyzed yet are the terms $V_1 \dots V_n$: these are string vertex operators and represent the external scattering states of the string. The external states of the string represents points in the correlation function taken to infinity: $x_i \to \infty$, and are assigned to some spacetime momentum p_i —see figure 2.3. Using the state-operator map, it is known that each of these states at infinity is equivalent to the insertion of an appropriate vertex operator V_i on the worldsheet. Therefore, to compute the scattering amplitude we use a conformal transformation to bring each of these infinite legs to a finite distance. The end result is a worldsheet punctured with vertex operators where the legs used to be. In the case of the open string, which is the more relevant one for our discussion, the vertex operators are inserted on the boundaries. For instance, for the open string at one-loop level the worldsheet is an annulus as shown in the second line of figure 2.2, and a vertex operator may be integrated along either one of the two boundary components. On the other hand, at tree level the worldsheet is a disk and there is only one boundary where the vertices can be integrated —see figure 2.4. An important observation is that the constraint of Weyl invariance imposes that vertex operators are necessarily **on shell**. This has important consequences, as this is the reason why we can only compute on shell

1 The string coupling constant appears by adding to the Polyakov action an extra topological invariant term. The presence (or absence) of boundaries determines how the string coupling constant is computed —for example, see [50] for more details.

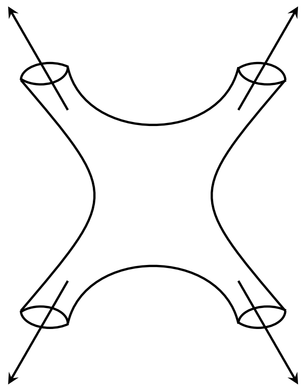


FIG. 2.3: In a string scattering process, the external legs are taken to infinity to suppress the redundancy of the gauge transformations.

correlation functions in string theory. This constraint is lifted in the (particle-derived) worldline formalism, as we shall see later.

The vertex operators most relevant for open string calculations are of the form

$$V^{\phi}[k] = \int d\tau \, e^{ik\cdot X(\tau)} \tag{2.1.5}$$

$$V^{A}[k,\varepsilon,a] = \int d\tau \, T^{a}\varepsilon\cdot\dot{X}(\tau)e^{ik\cdot X(\tau)}. \tag{2.1.6}$$

They represent a tachyon and a gauge boson particle with definite momentum k and polarization vector ε. T^{a} is a generator of the gauge group in a specific representation. The integration variable τ parametrizes the boundary in question. Since the action is Gaussian (note that he Polyakov action in (2.1.2) is quadratic in the coordinate x), the path integral $\int \mathcal{D}x$ can be performed using Wick contractions of type

$$\langle X^{\mu}(\tau_1)X^{\nu}(\tau_2)\rangle = G(\tau_1,\tau_2)\eta^{\mu\nu}, \tag{2.1.7}$$

where G denotes the Green's function for the Laplacian on the worldsheet, restricted to its boundary, and $\eta^{\mu\nu}$ the Lorentz metric. By exploiting the conformal invariance of the path integral, the remaining integration over the infinite dimensional space of the worldsheet metrics h can be reduced to space of conformal equivalent classes, which is finite dimensional. The actual integration domain, called *moduli space*, is somewhat smaller, since a further discrete symmetry group has to be taken into account, i.e. modded out.

The uses of the Polyakov path integral are not restricted to the calculation of scattering amplitudes. It has been shown [51] that it is equally useful for the calculation of string effective actions. For example, an open string propagating in the background of a Yang-Mills field A_{μ}

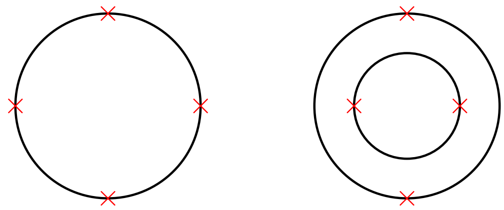


FIG. 2.4: Vertex operators inserted on the single boundary of the disk (left) and on the two boundaries of the annulus (right).

would generate an effective action for this background field given by the following modification of the Polyakov path integral,

$$\Gamma[A] \sim \sum_{\text{top}} \int \mathcal{D}h \int \mathcal{D}X(\sigma,\tau) e^{-S_0 - S_I}, \tag{2.1.8}$$

where

$$S_I = \int d\tau \, ie \, \dot{X}^\mu A_\mu(X(\tau)), \tag{2.1.9}$$

and S_0 is the standard Polyakov action introduced in 2.1.2. For simplicity, we have written the interaction term for the abelian case; in the non-abelian case, a color trace and path ordering are required, as it will be discussed later.

2.2 WORLDLINE AS A STRING-INSPIRED APPROACH

One of the main motivations for the study of string theory, and string theory amplitudes in particular, is the fact that it reduces to quantum field theory in the low energy limit, i.e. in the $\alpha' \to 0$ limit where the size of the string shrinks to zero and the tension becomes infinite [52–55]. In this limit all the massive modes of the string are suppressed, and only the massless modes survive. These modes are easily identified as the standard massless particles in field theory, such as gluons, gravitons and massless spin-$\frac{1}{2}$ particles, and their interactions take place in a way that is consistent with standard field theory.

Despite the complexity of computing scattering amplitudes in string theory, the analysis of their field theory limit is still meaningful: the techniques available in string theory are extremely different and this brings to interesting insights on field theory amplitudes. In ordinary quantum field theory perturbative calculations are usually performed using second quantization, and Feynman diagrams. This is not the case in string theory, where amplitudes are computed in first quantization using the Polyakov path integral (2.1.1), which describes the propagation of a single string in a given background through the spacetime coordinate $X^\mu(\sigma,\tau)$. In the low energy limit, thus we expect to obtain a description of the scattering amplitudes in the first quantization scheme. The major focus of this section is to show that this is indeed the case, and some hints will be given in order to understand

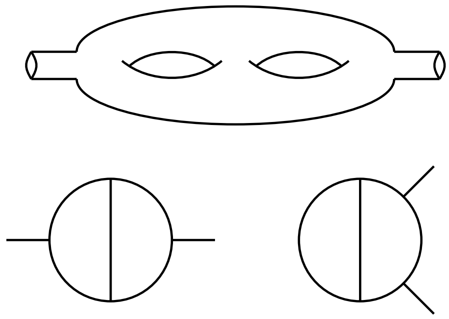


Fig. 2.5: Infinite string tension limit of a string diagram.

how and when first quantization can help in improving standard field theory techniques.

As seen in the previous section, string scattering amplitudes are computed using integrals of the form (2.1.1). After fixing the topology of the worldsheet and utilizing conformal symmetry, the amplitude is calculated by integrating over the moduli space punctured with the external vertices. At this stage, the amplitude is in a form suitable for performing the $\alpha' \to 0$ limit. In this limit, only certain corners of the moduli space contribute and the amplitude splits into a number of explicit parameter integrals that eliminate all contributions due to propagating massive modes, and represent the corresponding field theory amplitude. Visually, as α' approaches 0, the Riemann surfaces get squeezed to specific Feynman graphs. However, this process is complicated: several Feynman diagrams with different topologies can be generated from a single Riemann surface, and this proliferation gets worse at higher orders —see figure 2.5. As an additional fact, in gauge theory or gravity quartic and higher order vertices are involved, and this leads to many more possible diagrams. On the other hand, the generating string theories have a much smaller number of topologies then the limiting field theories. This is one of the major motivation for the introduction of string-inspired techniques for the computation of field theory quantities, as it will be clear from some insights presented in this manuscript.

The mathematical details of the computation of the field theory limit $\alpha' \to 0$ in string amplitudes go beyond the scope of the thesis. Here we just mention some relevant results that will be helpful in the proceedings of the calculations. In particular, it is crucial for our purposes the analysis of the one-loop n-gluon amplitude carried out by Bern and Kosower in [35]. By an explicit analysis of the infinite string tension limit[2], they derived a novel type of parameter integral representation for the on-shell n-gluon amplitude in Yang-Mills theory at one-loop level. In particular, they established a set of rules which allows one to construct the amplitude for any number of gluons without referring to string theory any more. Once this set of rules has been defined, it was natural to compare them with the corresponding field theoretic Feynman rules, and some advantages were immediately clear: absence of loop momentum, that reduces the number of kinematic variables and allows for an early use of the spinor helicity formalism, and nice combination with spacetime supersymmetry, that smoothly relates amplitudes with different particles running into the loop. Gauge invariance is also guaranteed in a superior way. The efficiency of the so called *Bern-Kosower formalism* has been demonstrated by the first complete calculation of the one–loop five–gluon amplitude in [56]. We will give a comprehensive overview of the Bern-Kosower rules in section 2.3.3. The formalism has been extended to gravity from the field theory limit of closed strings and a similar set of rules is available for the computation of graviton scattering [57]. Those have been used for the first calculation of the complete one–loop four–graviton amplitude in quantum gravity [58]. This extension to gravity, known as the *Bern-Dunbar-Shimada formalism*, will be reviewed in section 2.3.5.

As it has been pointed out earlier in this section, the Bern-Kosower rules do not refer to string theory anymore, so it is natural to wonder if an independent re-derivation can be carried out starting directly from a particle point of view. For this purpose,obviously we need a first-quantization description of standard field theory. This is not a big issue, as already in 1950 [29] Richard Feynman presented such a formalism for the case of scalar quantum electrodynamics. Here the scattering amplitude for a charged scalar particle that moves under the influence of the external potential A_μ, from point x_μ to x'_μ in Minkowski space is given by[3]

$$\left\langle \phi(x')\bar{\phi}(x) \right\rangle_A = \int_0^\infty ds\, e^{-\frac{1}{2}ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x(\tau)\, e^{-\int_0^s d\tau\left(\frac{i}{2}\dot{x}^2 - ie\,\dot{x}\cdot A(x(\tau))\right)}. \tag{2.2.1}$$

We will give a formal derivation of this formula later in section 2.4.1. In the equation above, the parameter s can be identified with a Schwinger proper time, that is used to construct a path integral representation of the amplitude. The path integral is performed over all the possible

2 They have used a specific heterotic string model containing $SU(N)$ Yang-Mills theory in the α' limit. This allows for a consistent reduction to four dimensions, but the representation of the amplitude is more elaborate.

3 In the original Feynman's formula, an additional interaction term was present, describing an arbitrary number of virtual photons emitted and re-absorbed along the trajectory of the particle. Here we omit this term, as its contribution won't play any role in the results showed in the manuscript.

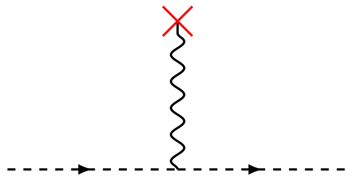


FIG. 2.6: Feynman diagram representing the interaction of a scalar particle with a Maxwell background field in the path integral (2.2.1).

trajectories running from the point x to x' in the fixed proper time s. The path integral action contains a familiar kinetic term, plus an interaction term, that corresponds to the coupling with an external gauge field. In second quantization, this corresponds to the diagrammatic representation in figure 2.6.

With a small effort, we can extend the Feynman's formula (2.2.1) to path integrals for closed loops, and obtain a representation of the one-loop effective action for the Maxwell field:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}. \quad (2.2.2)$$

Here the Schwinger proper time has been rescaled and Wick rotated according to $s \to -i2T$, for consistency with formulas that will be used later in the manuscript. Worthy of note is the path integrals performed now over a closed trajectory, i.e. the limits of the trajectory coincide $x^\mu(0) = x^\mu(T)$. This formula will be better investigated in section 2.3.1, where it will be derived from standard field theory results. It is natural now to compare (2.2.2) with the string theory formula (2.1.8): the former is clearly the infinite string tension limit of the latter. We have achieved our first glimpse of the origin of the string-inspired formalism in field theory.

The path integral in (2.2.2) has been promptly generalized to various cases, like spinor quantum electrodynamics and supersymmetric extensions [59–64], and this first quantized approach has gained a wide use in literature: among the others, we mention attempts to nonperturbative computations [65–68] and applications to the calculations of anomalies. In fact the use of particle path integrals to compute anomalies has already been known since the seminal work of Alvarez-Gaume and Witten [69, 70]. However, we want to keep the correspondence with string theory going on and we show how the first quantized formalism can turn into an extremely helpful tool for perturbative calculations in scattering amplitudes and effective actions. The basic idea is simple: we evaluate the path integrals in precisely the same way as one calculates the Polyakov path integral in string theory, i.e. in a one-dimensional perturbation theory. As an example, we consider the scalar loop path integral defined in (2.2.2)[4] and focus on the interaction term. We can expand it according to:

$$e^{-\int_0^T d\tau\, ie\dot{x}^\mu(\tau)A_\mu(x(\tau))} = \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} \prod_{i=1}^{n} \int_0^T d\tau\, ie\dot{x}^\mu(\tau_i)A_\mu(x(\tau_i)). \quad (2.2.3)$$

We can interpret each term of the sum as a Feynman diagram describing a fixed number of interactions of the scalar loop with the external field —see figure 2.7. Now we specialize the background field $A_\mu(x)$ to a sum of plane waves with definite polarizations

$$A_\mu(x) = \sum_{i=1}^{n} \varepsilon_{i\mu}\, e^{ik_i\cdot x}, \quad (2.2.4)$$

and we pick out the term containing every ε_i once. This procedure is by no means trivial: according to standard field theory, the path integral now corresponds exactly to the one-loop n-photon correlator, where the external photon lines are already truncated [71]. Furthermore, we can define a photon vertex operator

$$V_A[k,\varepsilon] = -ie\int_0^T d\tau\, \varepsilon\cdot\dot{x}(\tau)e^{ik\cdot x(\tau)}. \quad (2.2.5)$$

This corresponds exactly to one introduced for string amplitudes in (2.1.6), where we take $T^a \sim 1$ for abelian theories. The one-loop n-photon correlator now reads

$$\Gamma[n] = \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int \mathcal{D}x(\tau) \left(V^A[k_1,\varepsilon_1]\cdots V^A[k_n,\varepsilon_n]\right) e^{-\frac{1}{4}\int_0^T d\tau\, \dot{x}^2}. \quad (2.2.6)$$

This amplitude is interpreted as the field theory limit of the corresponding string amplitude, where the photon vertex operators are inserted on a circle instead than on the boundary of the annulus (see the annulus in figure 2.4). However, there is an important difference that is worth mentioning: the path integral 2.2.2 is a field-theoretic quantity and no on-shellness conditions have been imposed to obtain it, as it will be sketched in section 2.3.1. This is not true in string theory, where vertices for the external states of type (2.1.6) have necessary to be on-shell in order to guarantee Weyl invariance. This is a huge distinction, as (2.2.2) describes an effective action for photons that, in principle, are off-shell. We will exploit some advantages of this property later in the manuscript. Also, we mention that the path integral in (2.2.2) has now become Gaussian, and can be solved with little effort taking into account the appropriate boundary conditions. We will detail the calculation later in section 2.3. At this stage, the take-home message is that the first quantized approach for the calculation of

4 An identical procedure can be carried out if we consider the scalar propagator instead of the loop.

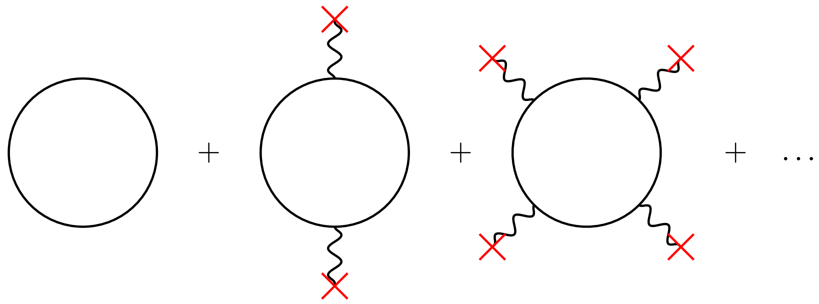


Fig. 2.7: Expansion of the path integral in powers of the background field.

scattering amplitudes is really string-inspired, in the sense that we obtain formulae that precisely mimic the structure of the corresponding string amplitudes. On the other hand, now we are dealing with path integral performed on the line ($1+0$ dimension) and no more on a worldsheet ($1+1$ dimensions), and the resulting calculations will be less complex, as it will be clear in the following.

2.3 EFFECTIVE ACTIONS FROM THE WORLDLINE

In this section we are going to review the fundamentals of the worldline description of effective actions. In 2.3.1 we will formally re-derive worldline formulae for a scalar particle coupled with a set of different external fields, focusing our attention on background gauge fields. In 2.3.2 we will specialize our study to the coupling with a background Maxwell field and we will see how the worldline description can be related to standard perturbation theory by computing a master formula for the one-loop n-photon amplitude. In 2.3.3 and 2.3.4 we will review the basics of the Bern-Kosower formalism, a string-inspired method that is used to compute the full one-loop n-gluon amplitude and that will have a crucial importance later in the manuscript. In 2.3.5 a similar formalism, called Bern-Dunbar-Shimada formalism, is presented for the computation of the full one-loop n-graviton amplitude.

2.3.1 *Worldline Formulation of Effective Actions*

Consider the simplest case of a real massive scalar field ϕ with a self-interaction potential $U(\phi)$. The (Wick-rotated) action thus reads

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{m^2}{2} \phi^2 + U(\phi) \right) . \qquad (2.3.1)$$

According to standard quantum field theory (see, e.g., [72, 73]), the one-loop effective action for this field theory can be written as[5]

$$\Gamma[\phi] = -\mathrm{Tr}\log\left[\frac{-\Box + m^2 + U''(\phi)}{-\Box + m^2}\right], \tag{2.3.2}$$

where $U''(\phi) = \frac{\delta^2 U(\phi)}{\delta\phi^2}$ and ϕ is the background field. Now we make use of the Schwinger formula

$$-\mathrm{Tr}\log\left(\frac{A}{B}\right) = \int_0^\infty \frac{dT}{T}\mathrm{Tr}\left(e^{-AT} - e^{-BT}\right), \tag{2.3.3}$$

valid for positive definite operators A and B. In (2.3.2) the denominator does not involve the self-interaction potential $U(\phi)$ and operates as a regulator in the representation (2.3.3). Neglecting the irrelevant ϕ-independent terms, and performing the functional trace in the D-dimensional space of coordinates x, we express the one-loop effective action as

$$\Gamma[\phi] = \int_0^\infty \frac{dT}{T}\int d^D x\, \langle x|\, e^{-T\left(-\Box + m^2 + U''(\phi(x))\right)}|x\rangle, \tag{2.3.4}$$

where we identify T as the Schwinger proper time. We can directly compare the integrand above with the standard Feynman's path integral representation for the transition amplitude in non-relativistic quantum mechanics (e.g. see [74] for more details). This reads as

$$\langle x''|e^{-i(t''-t')\hat{H}}|x'\rangle = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\, e^{i\int_{t'}^{t''} dt\,\mathcal{L}}, \tag{2.3.5}$$

where $\hat{H}$ is the Hamiltonian operator for the system:

$$\hat{H} = -\frac{\nabla^2}{2\tilde{m}} + \tilde{V}(x). \tag{2.3.6}$$

Here $\tilde{m}$ is the mass of the particle and $\tilde{V}(x)$ is a generic time-independent potential. From (2.3.6) we can extract the expression for the Hamiltonian operator in the momentum space ($\hat{p} = -i\nabla$),

$$\hat{H} = \frac{\hat{p}^2}{2\tilde{m}} + \tilde{V}(x), \tag{2.3.7}$$

and we compute the Lagrangian $\mathcal{L}$ that appears in the right hand side of (2.3.5) simply as the Legendre transform of this Hamiltonian:

$$\mathcal{L} = \frac{\tilde{m}}{2}\dot{x}^2 - \tilde{V}(x). \tag{2.3.8}$$

5 In quantum field theory it is common knowledge that one-loop effective actions can generally be expressed in terms of the determinant of the kinetic operator. Using the relation $\log(\det) = \mathrm{Tr}(\log)$, we obtain the standard definition in (2.3.2).

We interpret now the operator in (2.3.6) as the Hamiltonian for a fictitious particle moving in D dimensions, where we identify:

$$\Box = \nabla^2 \qquad \tilde{V}(x) = m^2 + U''(\phi(x)), \qquad \tilde{m} = \frac{1}{2}, \qquad i(t'' - t') = T. \tag{2.3.9}$$

Using the path-integral prescription in (2.3.5), we can immediately write

$$\langle x | \, e^{-T\left(-\Box + m^2 + U''(\phi(x))\right)} | x \rangle = \int_{x(0)=x}^{x(T)=x} \mathcal{D}x(t) \, e^{-\int_0^T dt \left(\frac{1}{4}\dot{x}^2 + m^2 + U''(\phi(x(\tau)))\right)}, \tag{2.3.10}$$

where we have identified $\tau = it$. Taking into account that

$$\int d^D x \int_{x(0)=x(T)=x} \mathcal{D}x(\tau) = \int_{x(0)=x(T)} \mathcal{D}x(\tau), \tag{2.3.11}$$

we obtain the following representation for the effective action

$$\Gamma[\phi] = \int_0^\infty \frac{dT}{T} \, e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x(\tau) \, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + U''(\phi(x(\tau)))\right)}. \tag{2.3.12}$$

We have shown a formal procedure to obtain a path integral representation of the one-loop effective action of a self-interacting scalar field theory. The formulation in (2.3.12) is truly a **worldline** representation, in the sense that the effective action is truly computed through a particle path integral performed over spacetime trajectories described by the coordinate $x(\tau)$. We have chosen a simple theory, i.e. a self-interacting scalar field theory, to show the general procedure to obtain the worldline representation of the effective action. However, in the present manuscript we are mostly interested to the coupling with an external gauge field, so a generalization of (2.3.12) is needed. This is not a problem at all, as we can simply recur to quantum mechanics to include the coupling of the massive scalars to a background Maxwell field. In particular, the contribution of an external gauge field is added by generalizing the standard derivative in (2.3.2) to the gauge covariant derivative (e.g. see [73]), that is

$$\partial_\mu \quad \longrightarrow \quad \partial_\mu - ieA_\mu. \tag{2.3.13}$$

Here A_μ is the gauge field and e is the coupling constant. The field theory kinetic operator now reads

$$-\left(\partial - ieA\right)^2 + m^2, \tag{2.3.14}$$

and the one-loop effective action is generalized to

$$\Gamma[A] = -\mathrm{Tr}\log\left[\frac{-(\partial - ieA)^2 + m^2}{-\Box + m^2}\right]. \tag{2.3.15}$$

Note that the self-interacting potential in the kinetic operator of (2.3.2) has been neglected, as it won't enter in the calculations in the remainder of the manuscript. Now we can follow the exact same procedure proposed for the self-interacting scalar theory. The fictitious Hamiltonian here translates to

$$H = \frac{(p + eA)^2}{2\tilde{m}} + m^2. \tag{2.3.16}$$

and, using the Feynman's path integral representation for the transition amplitude, we end up with

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}. \tag{2.3.17}$$

The expression describes the one-loop effective action of a scalar particle coupled to an external (abelian) gauge field, as already anticipated earlier in (2.2.2). We can use this formula to describe the contribution to the photon scattering due to a scalar loop. However, this is not our final goal, as we will be interested in the scattering of gluons, i.e. we need to introduce a external non-abelian gauge field inside the scalar loop effective action. We can retrieve the procedure used in the abelian case, with two major changes:

1. The trace now includes a global color trace.
2. The corresponding quantum mechanical Hamilton operators at different times do not commute any more, so that the exponential must be taken path-ordered. Note that this guarantees the gauge invariance of Γ.

We have thus

$$\Gamma[A^a] = \mathrm{Tr} \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, \mathcal{P}\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}, \tag{2.3.18}$$

where now $A_\mu = A_\mu^a T^a$, $\mathcal{P}$ denotes the path ordering operator, and Tr the color trace. The effective action in (2.3.18) is enough for the calculations that will be presented later in the manuscript. However, it is worth mentioning that the worldline description of effective actions goes beyond scalar theories: in literature one can find various representations for the worldline path integral of spin-$\frac{1}{2}$ [59–63] and spin-1 [64] particles, i.e. we can introduce the spinor and gluon loop contributions inside the effective action. More details will be mentioned in section 2.3.3 where the Bern-Kosower formalism for the one-loop n-gluon full amplitude will be introduced.

2.3.2 *Perturbative Calculations from Effective Actions*

At the end of section 2.2, we have shown that the path integral (2.2.2), re-derived in (2.3.17), can be expanded around the background Maxwell field $A_\mu(x)$ to generate an amplitude in perturbation theory.

In particular, we have seen that, by rewriting the external field as a sum of plane waves

$$A_\mu(x) = \sum_{i=1}^{n} \varepsilon_{i\mu}\, e^{ik_i\cdot x}, \tag{2.3.19}$$

we obtain the path integral in (2.2.6), that describes the scalar-loop n-photon amplitude in a first-quantized shape. Our next goal is to demonstrate how the path integral can be solved to obtain a more practical formula in terms of parameter integrals only. As we have already mentioned, the path integral written in form (2.2.6) is Gaussian: it can be evaluated through Wick contraction of the expression

$$\left\langle \dot{x}_1^{\mu_1} e^{ik_1\cdot x_1} \cdots \dot{x}_n^{\mu_n} e^{ik_n\cdot x_n} \right\rangle. \tag{2.3.20}$$

The Green's function to be used to solve the Wick contractions is now simply the one for the second-derivative operator, that acts on periodic functions. To derive this Green's function, we first observe that $\int \mathcal{D}x(\tau)$ contains the constant functions: we must get rid of it to obtain a well-defined inverse. Let us therefore introduce the *center of mass* position x_0^μ of the loop,

$$x_0^\mu = \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau), \tag{2.3.21}$$

and restrict our integral over the space of all loops by fixing a specific center of mass. This corresponds to redefine the spacetime coordinate $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ [6] with the condition $\int_0^T y^\mu(\tau) = 0$, which allows to invert the kinetic operator. In scattering amplitude calculations, the D-dimensional integral over x_0 just gives momentum conservation $(2\pi)^D\delta(\sum k_i)$. The reduced path integral $\int \mathcal{D}y(\tau)$ has an invertible kinetic operator

$$2\langle\tau_i|\left(\frac{d}{d\tau}\right)^{-2}|\tau_j\rangle = G(\tau_i,\tau_j), \tag{2.3.22}$$

where the worldline Green's function $G(\tau_i,\tau_j)$, up to an irrelevant constant, is given by (see [37])

$$G(\tau_i,\tau_j) = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}. \tag{2.3.23}$$

Note that the choice of the zero mode as center of mass leaves the worldline Poincaré invariance unbroken, i.e. $G(\tau_i,\tau_j) = G(\tau_i-\tau_j)$. For the executions of the Wick contractions, it is convenient to formally re-exponentiate all the n photon vertex operators, and write the $\dot{x}_i$'s included in (2.2.5) as

$$\varepsilon_i\cdot\dot{x}_i e^{ik_i\cdot x_i} = e^{\varepsilon_i\cdot\dot{x}_i + ik_i\cdot x_i}\big|_{\mathrm{lin}(\varepsilon_i)}. \tag{2.3.24}$$

6 Note that this is a linear transformation, so no Jacobi determinant has to be included in the path integral.

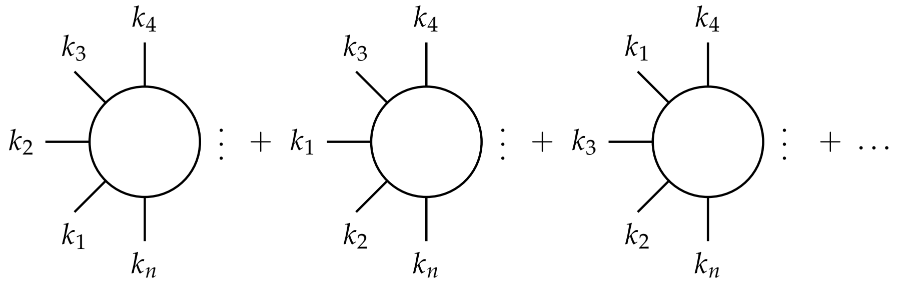


FIG. 2.8: Sum of one-loop diagrams with permuted legs.

Now we just have to complete the square and we arrive at the following closed expression for the one-loop n-photon amplitude in scalar QED:

$$\Gamma(k_1,\varepsilon_1;\dots;k_n,\varepsilon_n) = (-ie)^n (2\pi)^D \delta\left(\sum k_i\right) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} \mathrm{e}^{-m^2 T}$$
$$\prod_{i=1}^n \int_0^T d\tau_i \exp\left\{ \sum_{i,j=1}^n \left(\frac{1}{2} G_{ij} k_i \cdot k_j - i \dot{G}_{ij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{ij} \varepsilon_i \cdot \varepsilon_j \right) \right\} \Big|_{\varepsilon_1 \dots \varepsilon_n} . \tag{2.3.25}$$

Note that, besides the Green's function $G_{ij} \equiv G(\tau_i, \tau_j)$, also its first and second derivative appear:

$$\dot{G}(\tau_i, \tau_j) = \mathrm{sgn}(\tau_i - \tau_j) - 2\frac{(\tau_i - \tau_j)}{T} \tag{2.3.26}$$
$$\ddot{G}(\tau_i, \tau_j) = 2\delta(\tau_i - \tau_j) - \frac{2}{T}. \tag{2.3.27}$$

In our notation, the derivative is always applied on the first variable of the Green's function, such that

$$\dot{G}(\tau_i, \tau_j) \equiv \frac{\partial}{\partial \tau_i} G(\tau_i, \tau_j), \qquad \ddot{G}(\tau_i, \tau_j) \equiv \frac{\partial^2}{\partial \tau_i^2} G(\tau_i, \tau_j). \tag{2.3.28}$$

For a given n, the notation $|_{\varepsilon_1 \dots \varepsilon_n}$ implies that we have to expand the exponential in (2.3.25) keeping only the terms linear in each of the polarization vectors $\varepsilon_1, \dots, \varepsilon_n$. The factor $(4\pi T)^{-D/2}$ represents the free Gaussian path integral determinant factor:

$$\int \mathcal{D}y(\tau)\, e^{-\int_0^T d\tau \frac{1}{4} \dot{y}^2} = (4\pi T)^{-D/2}. \tag{2.3.29}$$

It can be easily calculated starting again from quantum mechanics results and performing the same formal substitutions of (2.3.9) —e.g. see [71].

At this stage, it is important to stress the significance of (2.3.25): we have at hand here a single unifying generating functional for the one-loop n-photon amplitude, that already contains the contribution from **all** possible disposition of the n photons. This is not true in second-quantized QED, where the contribution from each diagram (that is, order and interaction within vertices) is calculated separately. However, we still can establish a link between the parameter integrals in (2.3.25) and the integrals appearing in an ordinary Feynman parameter calculation of this amplitude. Note that in the formula every photon leg is integrated around the loop independently, so, once we restrict the integration domain to a fixed ordering $\tau_{i_1} > \tau_{i_2} > \ldots > \tau_{i_n}$, it is not difficult to identify the integrand with the corresponding Feynman parameter integral —see figure 2.8. In particular, there is an exact correspondence between the δ-function appearing in $\ddot{G}$, and the seagull–vertex of scalar QED.

Another important property of (2.3.25), already pointed out in its original representation (2.2.2), is that it valid **off-shell**. This opens for a wide range of possible calculations: for example, we can use this formula to sew together a pair of legs and obtain a parameter integral representing the complete two-loop $(n-2)$-photon amplitude. This is interesting, as we have at hand a single integral formula that describes the sum of many diagrams of different topologies. On our side, we will exploit the off-shellness of similar formulae later in the manuscript to carry out calculations on specific amplitudes.

2.3.3 *The One-Loop n-Gluon Amplitude*

In the previous section we have found out how the worldline description of the effective action can be exploited to obtain a master formula for the scalar-loop n-photon amplitude. Now we extend our analysis to a perturbative description of the interaction with an external non-abelian gauge field, i.e. we want to treat scattering of gluons. Before proceeding with out analysis, it is worth mentioning the additional difficulties that have to be taken into account in dealing with gluon scattering. Firstly, the Lie-algebra of the generators of the gauge group (we will introduce our conventions in section 3.1) imposes carefulness in the derivation of a worldline formula for the effective action for scalar particles: we have already seen in (2.3.18) that this can be handled correctly with the introduction of a global color trace and of a path-ordering operator for the correct application of the Feynman's path integral formulation. Moreover, non-abelian gauge theories allow for self-interactions among gluons: three- and four-gluon vertices have to be considered in our study, as they will give rise to one-particle reducible contributions inside the scattering amplitude, i.e. diagrams that can be divided into two subdiagrams by cutting a gluon line. In our analysis so far, we have treated the external fields as background fields coupled to a particle running into a loop. In particular, a kinetic term for the external fields is not present in (2.3.18), so new techniques for the description of their dynamics have to be introduced. This difficulty has been efficiently treated within the so-called **Bern-Kosower formalism**, a method developed during the 90s and obtained by the authors directly from the low energy limit of specific string amplitudes. The central object in the Bern-Kosower (BK) formalism is the color-ordered one-loop n-gluon correlator with a massive scalar loop. The One-Particle Irreducible (1PI) of the amplitude is encoded in the following master formula,

$$\Gamma(k_1,\varepsilon_1;\ldots;k_n,\varepsilon_n) = (-ig)^n \mathrm{Tr}(T^{a_1}\cdots T^{a_n}) \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}} \mathrm{e}^{-m^2 T} \int_0^T d\tau_1$$
$$\cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \exp\left\{ \sum_{i,j=1}^{n} \left(\frac{1}{2} G_{ij} k_i \cdot k_j - i\dot{G}_{ij}\varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i \cdot \varepsilon_j \right) \right\} \Big|_{\varepsilon_1 \ldots \varepsilon_n}, \tag{2.3.30}$$

meaning that the associated diagrams cannot be divided into two self-standing contributions by cutting an internal line. In the formula above G_{ij}, $\dot{G}_{ij}$ and $\ddot{G}_{ij}$ have the forms specified in (2.3.23) and (2.3.26)-(2.3.27) respectively. Note the presence of the term $\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$, typical of a color-ordered amplitude. Again, the notation $|_{\varepsilon_1\ldots\varepsilon_n}$ implies that we have to expand the exponential in (2.3.30) keeping only the terms linear in each polarization. The resulting integrand is of the form

$$\exp\left\{ \cdot \right\} |_{\varepsilon_1 \ldots \varepsilon_n} \equiv (-i)^n P_n \left(\dot{G}_{ij}, \ddot{G}_{ij} \right) e^{\frac{1}{2} \sum_{i,j=1}^{n} \frac{1}{2} G_{ij} k_i \cdot k_j}, \tag{2.3.31}$$

where P_n is a polynomial dependent on the derivatives of the Green's function $\dot{G}_{ij}$ and $\ddot{G}_{ij}$ as well as on kinematic variables ($\varepsilon' s$ and k's).

According to the Bern-Kosower formalism, the reducible contributions are constructed using the following *pinching procedure*:

1. Expand the kinematic expression, and perform integrations by parts, till all double derivatives $\ddot{G}_{ij}$ are removed (ignore boundary terms). This step leads to the replacement $P_n \left(\dot{G}_{ij}, \ddot{G}_{ij} \right) \to Q_n \left(\dot{G}_{ij} \right)$. The integration by parts is carried out using the *symmetric partial integration* algorithm that will be described in section 2.3.4.

2. Draw all possible one-loop diagrams D_i with cubic vertices[7] and with n legs, labeled $1,\ldots,n$ and following the ordering of the color trace.

3. The pinching rule amounts to the replacement
$$\dot{G}_{ij} \longrightarrow \frac{2}{s_{ij}} = \frac{2}{(k_i + k_j)^2}, \tag{2.3.32}$$
removing the vertex and transferring the label i to the ingoing leg (see figure 2.9). The integration over τ_j is omitted and the index j replaced by i in the remaining G_{kl} and $\dot{G}_{kl}$.

7 The contribution of the quartic self-interactions among the gluons is automatically included within the pinching procedure.

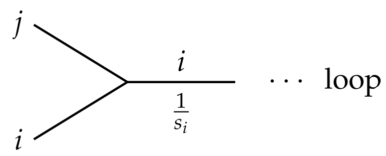


FIG. 2.9: Pinching of a vertex according to the Bern-Kosower rules.

4. The previous replacement can only occur iff Q_n contains $\dot{G}_{ij}$ linearly. This is a trivial consequence of the replacement $j \to i$ in the previous step and of the antisymmetry of the $\dot{G}_{ij}$ function. Moreover, a diagram will contribute iff each vertex except the ones attached directly to the loop corresponds to a possible pinch, i.e. it must be coherent with the color-ordering of the amplitude. If more than two legs are attached to the same external tree in a given diagram, the pinching procedure starts with the outermost vertices and recursively removes the trees attached to the loop.

The pinching procedure can be improved with the introduction of a suitable **pinch operator**, that will be presented later in the manuscript, and the implementation of the perturbiner multi-particle techniques that help to have a better understanding of the mathematical structure of the trees attached to the loop.

As a further benefit of the Bern-Kosower formalism, the contributions of the spinor and gluon loop to the n-gluon amplitudes can be constructed at the integrand level using a set of *loop replacement rules* —see [35, 37, 75] for more details.

2.3.4 *Symmetric Partial Integration*

In this section we explain a partial integration algorithm that allows one to remove all the double derivatives $\ddot{G}_{ij}$ contained in the numerator polynomial P_n in (2.3.31) of the n-gluon color ordered amplitude. The algorithm is known as **symmetric partial integration** algorithm, as it solves for the double derivatives in P_n in such a way that the permutation symmetry in the n gluons is preserved. This property plays a crucial role, as it will be demonstrated in future sections of the manuscript.

The symmetric partial integration algorithm can be defined in the following way:

1. In every step, we partially integrate away all the $\ddot{G}_{ij}$'s appearing in the terms under inspection *simultaneously*. This is possible since different $\ddot{G}_{ij}$'s do not share variables, since, as it is evident from (2.3.30), $\ddot{G}_{ij}$is hooked to $\varepsilon_i \cdot \varepsilon_j$ which ought to appear only once, and this property is preserved by all partial integrations. New $\ddot{G}_{ij}$'s may be created in this step.

2. In the first step, for every $\ddot{G}_{ij}$ we partially integrate both over τ_i and τ_j, and we take the average of the two results. Note that boundary terms are ignored in this algorithm, we systematically trash them away.

3. At every following step, clearly any $\ddot{G}_{ij}$ appearing must have been created in the previous step. Therefore either both i and j were used in the previous step, or just one of them. If both, the rule is to again use both variables in the actual step for partial integration, and take the average of the results. If only one variable was used in the previous step, then the other one should be used in the actual step.

EXAMPLE: We consider the $n = 4$ one-loop gluon scattering and we look at the term $\ddot{G}_{12}\ddot{G}_{34}$ appearing in the polynomial P_4, obtained after the multilinear expansion of the exponential in (2.3.30) has been performed. We can solve for both the double derivatives following the procedure depicted above. In the first step we get

$$\begin{aligned} \ddot{G}_{12}\ddot{G}_{34} \;\rightarrow\; & \frac{1}{4}\dot{G}_{12}\dot{G}_{34}\Big[\left(\dot{G}_{1i}k_1\cdot k_i - \dot{G}_{2i}k_2\cdot k_i\right)\left(\dot{G}_{3j}k_3\cdot k_j - \dot{G}_{4j}k_4\cdot k_j\right) \\ & - \ddot{G}_{13}k_1\cdot k_3 + \ddot{G}_{14}k_1\cdot k_4 + \ddot{G}_{23}k_2\cdot k_3 - \ddot{G}_{24}k_2\cdot k_4\Big] . \end{aligned} \tag{2.3.33}$$

In the second line we have obtained terms that still show double derivatives, so we apply the algorithm a second time. Considering just the first double derivative $\ddot{G}_{13}$ (the other terms can be treated in an identical way), we apply partial integrations over both variables τ_1 and τ_3, as both were involved in the previous step. We obtain:

$$\begin{aligned} -\dot{G}_{12}\dot{G}_{34}\ddot{G}_{13} \;\rightarrow\; & \frac{1}{2}\dot{G}_{12}\dot{G}_{34}\dot{G}_{13}\left(\dot{G}_{1i}k_1\cdot k_i - \dot{G}_{3i}k_3\cdot k_i\right) \\ & + \frac{1}{2}\dot{G}_{13}\left(\ddot{G}_{12}\dot{G}_{34} - \dot{G}_{12}\ddot{G}_{34}\right) . \end{aligned} \tag{2.3.34}$$

Again double derivatives appear. We consider only the term involving $\ddot{G}_{12}$ in the second line. This time, only the variable τ_1 has been used in the previous step, therefore only τ_2 has to be involved now. As final result, we get:

$$\dot{G}_{13}\ddot{G}_{12}\dot{G}_{34} \;\rightarrow\; \dot{G}_{13}\dot{G}_{12}\dot{G}_{34}\dot{G}_{2i}k_2\cdot k_i \,. \tag{2.3.35}$$

All the variables involved in the algorithm are treated on the same footing, so the final result has to be permutation-symmetric. The non-trivial fact is that the process terminates after a finite number of steps, and does not become cyclic. This is not difficult to derive: first of all note that partial integration is *not* always symmetric —if this was the case, in (2.3.35) the process would have become cyclic. Also, we point out that, inside the generic P_n, the indices appearing in the $\ddot{G}_{ij}$'s and the first indices of the $\dot{G}_{ij}$'s are associated to the polarization vectors, and thus must all take different values. In other words, it is not possible to have terms of type $\ddot{G}_{12}\dot{G}_{12}$, that would become cyclic inside the algorithm.

The symmetric partial integration described above is applied to polynomials $P_n\left(\dot{G}_{ij}, \ddot{G}_{ij}\right)$ and transforms them into polynomials $Q_n\left(\dot{G}_{ij}\right)$, that do not depend on double derivatives of the Green's function. Because of the special structure of the symmetric partial integration algorithm, the polynomials Q_n show two important properties:

- Unlike P_n, they are **homogeneous** not only in the polarizations, but also in the momenta.
- They are manifestly **permutation invariant** under to switch of any leg.

These properties allow one to rewrite the polynomials Q_n in a compact way via a decomposition into cycles and tails. A cycle of length k is defined by

$$\dot{G}(i_1, i_2, \cdots, i_k) \equiv \dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_k i_1} Z_k(i_1, i_2, \ldots, i_k), \tag{2.3.36}$$

where

$$Z_k(i_1, i_2, \ldots, i_k) \equiv \left(\frac{1}{2}\right)^{\delta_{k2}} \mathrm{Tr}(f_{i_1}\cdots f_{i_k}). \tag{2.3.37}$$

Here $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$ is the (abelian part of the) gluon field strength tensor. The tails are the left-overs after factorizing out all possible cycles. The k-tail $T(i_1, i_2, \ldots, i_k)$ involves k polarization vectors that have not yet been absorbed into field strength tensors.

For example, the cycle decomposition of Q_3 reads as

$$Q_3 = Q_3^3 + Q_3^2, \tag{2.3.38}$$

where

$$Q_3^3 = \dot{G}(1,2,3), \quad Q_3^2 = \dot{G}(1,2)T(3) + \text{perms}. \tag{2.3.39}$$

The superscripts on the left-hand side of the equations above indicate the cycle-content of each term.

Similarly, we show the cyclic decomposition of Q_4:

$$Q_4 = Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22}, \tag{2.3.40}$$

where

$$\begin{aligned}
Q_4^4 =&\dot{G}(1,2,3,4)+\dot{G}(1,2,4,3)+\dot{G}(1,3,2,4)\\
Q_4^3 =&\dot{G}(1,2,3)T(4)+\dot{G}(2,3,4)T(1)\\
&+\dot{G}(3,4,1)T(2)+\dot{G}(4,1,2)T(3)\\
Q_4^2 =&\dot{G}(1,2)T(3,4)+\dot{G}(1,3)T(2,4)+\dot{G}(1,4)T(2,3)\\
&+\dot{G}(2,3)T(1,4)+\dot{G}(2,4)T(1,3)+\dot{G}(3,4)T(1,2)\\
Q_4^{22} =&\dot{G}(1,2)\dot{G}(3,4)+\dot{G}(1,3)\dot{G}(2,4)+\dot{G}(1,4)\dot{G}(2,3).
\end{aligned} \tag{2.3.41}$$

In (2.3.39)-(2.3.41) the one- and two-tails appear,

$$T(a)\equiv\sum_r \dot{G}_{ar}\varepsilon_a\cdot k_r \tag{2.3.42}$$

$$\begin{aligned}
T(a,b)\equiv&\sum_{\substack{r,s\\(r,s)\neq(b,a)}}\dot{G}_{ar}\varepsilon_a\cdot k_r\dot{G}_s\varepsilon_b\cdot k_s\\
&+\frac{1}{2}\dot{G}_{ab}\varepsilon_a\cdot\varepsilon_b\Big[\sum_{r\neq b}\dot{G}_{ar}k_a\cdot k_r-\sum_{s\neq a}\dot{G}_s k_b\cdot k_s\Big].
\end{aligned} \tag{2.3.43}$$

Note that the cycle decomposition of Q_n involves the tails of length up to $n-2$. Up to length 4, the tails are given in [37], while the five-tail is computed in [76]. In principle, one is able to compute tails at any order simply by applying iteratively the algorithm showed in the current section.

2.3.5 *The One-Loop n-Graviton Amplitude*

The gluon master formula (2.3.30) was generalized to amplitudes with gravitons by Bern, Dunbar and Shimada [57] and later refined by Dunbar and Norridge [58]. Using again string theory as a guiding principle, we focus on closed strings to extract the particle limit and study graviton scattering. The master formula for the irreducible one-loop n-graviton amplitudes with a massless scalar loop is constructed as

$$\begin{aligned}
\Gamma(k_1,h_1;\cdots)=&-(-\frac{\kappa}{4})^n\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\int_0^T d\tau_1\cdots\int_0^T d\tau_n\\
&\times\exp\Bigg\{\sum_{i,j=1}^n\Big[\frac{1}{2}G_{ij}k_i\cdot k_j-i(\dot{G}_{ij}\varepsilon_i+\dot{\bar{G}}_{ij}\bar{\varepsilon}_i)\cdot k_j\\
&+\frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j+\frac{1}{2}\ddot{\bar{G}}_{ij}\bar{\varepsilon}_i\cdot\bar{\varepsilon}_j+\frac{1}{2}H_{ij}(\varepsilon_i\cdot\bar{\varepsilon}_j+\varepsilon_j\cdot\bar{\varepsilon}_i)\Big]\Bigg\}\Big|_{\varepsilon_1\ldots\bar{\varepsilon}_1\ldots},
\end{aligned} \tag{2.3.44}$$

where the on-shell conditions allow to reconstruct the graviton polarization tensor as $\epsilon^{\mu\nu}=\varepsilon_i^\mu\bar{\varepsilon}_i^\nu$. The equation above differs from the Bern-Kosower master formula in (2.3.25) as we are incorporating both the left-moving variables τ_i and the right-moving variables $\bar{\tau}_i$.

This originates from the closed string periodicity, that allows the string worldsheet variables to be separated into left-moving and right-moving components. The function $G_{ij} = G(\tau_i, \tau_j)$ is the worldline Green's function introduced in (2.3.23), $\dot{G}_{ij}$ and $\ddot{G}_{ij}$ are derivatives of this worldline Green's function with respect to left-moving variables τ_i, and $\dot{\bar{G}}_{ij}$ and $\ddot{\bar{G}}_{ij}$ are derivatives with respect to right-movers $\bar{\tau}_i$ —their expressions both match with (2.3.26)-(2.3.27). The term $H_{ij} = H(\tau_i, \tau_j)$ is the derivative of G_{ij} with respect to one left-mover and one right-mover variable, and its value inside the parameter integrals is given by the constant value

$$H_{ij} = \frac{2}{T}. \tag{2.3.45}$$

In (2.3.44) the terms involving H_{ij}, which were not present in the Bern-Kosower formula, are now included as they reflect the coupling of the left- and right-movers through the zero mode of the string. Once we extract the particle limit $\alpha' \to 0$ to compute the master formula (2.3.44), the left- and right-moving variables τ_i and $\bar{\tau}_i$ are no longer independent. In detail, we can identify $\tau_i = \bar{\tau}_i$, and only one integration variable is needed in (2.3.44).

The full amplitude is obtained by identifying the multi-linear terms in ε_i and $\bar{\varepsilon}_i$ from the exponential expansion in (2.3.44). The symmetric Integration By Parts (IBP) algorithm is used to eliminate all double derivatives $\ddot{G}_{ij}$ and $\ddot{\bar{G}}_{ij}$, and the pinching rules can then be applied to the resulting integrand in order to construct the reducible contributions to the amplitude. However, differently from the gluon case, it is now generally not possible to remove all of the $\ddot{G}_{ij}$, $\ddot{\bar{G}}_{ij}$ using partial integrations in the single variables τ_i. Instead, one has to return to the string level and invoke the fact that, before taking the infinite string tension limit, the left-and right movers depend on independent variables τ_i and $\bar{\tau}_i$. This allows one to treat $\dot{G}_{ij}$, $\ddot{G}_{ij}$ and $\dot{\bar{G}}_{ij}$, $\ddot{\bar{G}}_{ij}$ independently in the partial integration procedure. Additionally, the following rules must be used in the computation of derivatives,

$$\frac{\partial}{\partial \bar{\tau}_k}\dot{G}_{ij} = \frac{1}{2}(\delta_{ki}H_{ij} - \delta_{kj}H_{ij}) \tag{2.3.46}$$

$$\frac{\partial}{\partial \tau_k}\dot{\bar{G}}_{ij} = \frac{1}{2}(\delta_{ki}H_{ij} - \delta_{kj}H_{ij}) \tag{2.3.47}$$

$$\frac{\partial}{\partial \bar{\tau}_k}\ddot{G}_{ij} = 0 \tag{2.3.48}$$

$$\frac{\partial}{\partial \tau_k}\ddot{\bar{G}}_{ij} = 0. \tag{2.3.49}$$

Essentially, the symmetric partial integration rules introduced in section 2.3.4 can be extended in graviton amplitudes calculations simply by treating the variables τ_i and $\bar{\tau}_i$, entering in $\dot{G}_{ij}$ and $\dot{\bar{G}}_{ij}$ respectively, on an independent footing. The addition of the derivative rules in (2.3.46) generates extra terms involving H_{ij}, that are treated as constants in the partial integration procedure. The integrand can thus be ordered according to the powers of H_{ij}, where the terms in the prefactor polynomial not containing any H_{ij} can be factorized into $Q_n(\dot{G})Q_n(\dot{\bar{G}})$ and terms with m factors of H_{ij} appear with $(n-m)$ factors of $\dot{G}$ and $\dot{\bar{G}}$ each.

After the removal of all of the $\ddot{G}_{ij}, \ddot{\bar{G}}_{ij}$ using the symmetric partial integration procedure, the computation of the reducible contributions of the graviton amplitude can be achieved by a pinching procedure that mimics the one for the gluon case introduced in section 2.3.3. The most important difference lies in the fact that the pinching of a vertex with labels i and j now is possible iff the integrand contains both $\dot{G}_{ij}$ and $\dot{\bar{G}}_{ij}$ linearly. In addition, the replacement in (2.3.32) has to be modified to

$$\dot{G}_{ij}\dot{\bar{G}}_{ij} \to \frac{4}{s_{ij}}. \tag{2.3.50}$$

After the recursive removal of all trees attached to the loop, one obtains a parameter integral representation for the full on-shell n-graviton amplitude with a massive scalar loop. Representations for other spins in the loop (Weyl fermion, vector, gravitino, graviton) can again be obtained from this by precise loop replacement rules that are essentially independent applications of the rules mentioned at the end of section 2.3.3 to the left- and right-mover parts, with additional substitutions rules for the cross terms involving H_{ij} —see [57, 58] for more details.

In conclusion of this section, it's worth mentioning that the calculations presented in this thesis will not aim to build the full n-graviton amplitude through the Bern-Dunbar-Shimada (BDS) formalism. In chapter 8, we will focus on the part of the integrand in (2.3.44) that is independent on H_{ij}. In this case, the expansion of the exponent leads to a prefactor polynomial that simply factorizes into two copies of the one of the gluonic case in (2.3.31),

$$\exp\left\{\cdot\right\}\Big|_{\varepsilon_1\ldots\varepsilon_n\bar{\varepsilon}_1\ldots\bar{\varepsilon}_n} = P_n(\dot{\bar{G}}_{ij}, \ddot{\bar{G}}_{ij})P_n(\dot{G}_{ij}, \ddot{G}_{ij})\mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{n} G_{ij}k_i\cdot k_j}. \tag{2.3.51}$$

One of the main goals of the thesis is the study of the algebraic structure of the gravitational trees attached to the loop, and, in this sense, (2.3.51) will be enough. However, as we emphasized above, the contributions that come from the H_{ij} terms cannot be neglected if one aims to compute the full one-loop amplitude.

2.4 PROPAGATORS FROM THE WORLDLINE

This section will examine the basic concepts of dressed propagators in worldline representation. Following the guiding principles seen in section 2.3, in 2.4.1 we will briefly re-derive worldline representations for a scalar propagator dressed with a set of different external fields. In particular, we will focus our attention on the coupling with background Maxwell fields and, in 2.4.2, we will compute a Bern-Kosower-like formula for the n-photon scalar propagator. The procedure will follow the step used for one-loop calculations, but substantial differences will be pointed out.

2.4.1 *Worldline Formulation of Propagators*

Previously, in section 2.3.1 we have seen how one-loop effective actions can be reformulated in the worldline language by means of standard quantum field theory techniques: using the Schwinger proper-time representation, one obtains an integral over the space of all closed trajectories of a quantum mechanical particle moving in spacetime. Generally, this procedure can be extended from closed loop path integrals to path integrals on open lines, representing the field theory propagators of specific particles in a background field. The path integral description of propagators involves the same worldline Lagrangian as the one-loop case, with some boundary terms added if necessary. The path integral is to be performed over the space of trajectories connecting two fixed points in spacetime, with appropriate boundary conditions. As a first example, we consider the path integral representation of the scalar propagator. The starting point is the standard definition of the propagator as the inverse of the kinetic operator (e.g. see [73]):

$$\begin{aligned}\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle &= \langle x'|\frac{1}{-\Box+m^2+U''(\phi)}|x\rangle \\ &= \int_0^\infty \frac{dT}{T}\,\langle x'|\,e^{-T\left(-\Box+m^2+U''(\phi)\right)}|x\rangle,\end{aligned} \tag{2.4.1}$$

Here we have included inside the kinetic operator the contribution $U''(\phi)$, that takes origin from the self-interacting potential $U(\phi)$ and that depends on the background field ϕ. Note that in the second line of (2.4.1) we have made use of the Schwinger parametrization

$$\frac{1}{A} = \int_0^\infty dT\, e^{-TA}. \tag{2.4.2}$$

Here the procedure follows step by step the method in (2.3.5)-(2.3.9) and we easily end up with the following representation for the scalar propagator:

$$\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle = \int_0^\infty dT\, e^{-Tm^2}\int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+U''(\phi(x(\tau)))\right)}. \tag{2.4.3}$$

Similarly to the one-loop representation, we can compute the propagator for a massive scalar coupled to a background Maxwell field. The kinetic operator is given by (2.3.14) and the propagator is given by

$$\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_A = \langle x'|\frac{1}{-(\partial-ieA)^2+m^2}|x\rangle, \tag{2.4.4}$$

that transforms to[8]

$$\left\langle \phi(x')\bar{\phi}(x) \right\rangle_A = \int_0^\infty dT\, e^{-Tm^2} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}. \tag{2.4.5}$$

Again, we can in principle extend this result to non-abelian gauge theories and describe a scalar propagator dressed with external gluons. Here the generalization is not as immediate as (2.3.18), because of the color degrees of the external scalar that have to be taken into account and because of the realization of the path ordering on the line. In literature, various solutions have been proposed that include the presence of auxiliary Grassmann variables that, once quantized, allow to efficiently describe the path ordering and the color of the external particles [77–79]. We won't give further details about this, as it won't be relevant for the results that will be presented later in the manuscript.

2.4.2 *Perturbative Calculations from Propagators*

Following the guiding principles of section 2.3.2, we can complete perturbative calculations on the line by expanding the exponential in (2.4.5). In particular, the n-photon scalar propagator, i.e. the scalar propagator with the insertion of n photons, can be obtained using analogous recipes to the one-loop case. Firstly, we write the external field as a sum of n plane waves

$$A_\mu(x) = \sum_{i=1}^n \varepsilon_{i\mu}\, e^{ik_i\cdot x}, \tag{2.4.6}$$

then extract from (2.4.5) the multi-linear part in the various polarizations ε_l. This is a representation of the propagator in x-space, but it can be Fourier transformed on the two external scalar lines in order to get a result more familiar in standard QFT. This leads to

$$\begin{aligned} D^{(n)}(p,p';\varepsilon_1,k_1;\ldots;\varepsilon_n,k_n) &= (-ie)^n \int_0^\infty dTe^{-m^2T} \int d^4x \int d^4x' \\ &\times e^{i(p\cdot x+p'\cdot x')} \int_{x(0)=x}^{x(T)=x'} Dx\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \prod_{l=1}^n \int_0^T d\tau_l\, \varepsilon_l\cdot\dot{x}(\tau_l) e^{ik_l\cdot x(\tau_l)}. \end{aligned} \tag{2.4.7}$$

It is thus convenient to split the particle path in terms of a background $\bar{x}^\mu(\tau) = x^\mu + (x'^\mu - x^\mu)\frac{\tau}{T}$, that satisfies the boundary conditions $\bar{x}^\mu(0) = x$ and $\bar{x}^\mu(T) = x'$, and fluctuations $y^\mu(\tau)$ with vanishing boundary conditions:

$$x^\mu(\tau) = x^\mu + (x'^\mu - x^\mu)\tfrac{\tau}{T} + y^\mu(\tau). \tag{2.4.8}$$

8 This result corresponds exactly to the first Feynman proposal in (2.2.1) once the proper time s has been rescaled and Wick rotated, $s \to -i2T$.

One thus gets

$$D^{(n)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n) = (-ie)^n \int_0^\infty dT e^{-m^2 T} \int d^4x \int d^4x'$$
$$\times\, e^{i(p\cdot x+p'\cdot x')-\frac{1}{4T}(x-x')^2+\sum_l \left(ik_l\cdot x+\frac{\varepsilon_l}{T}\cdot(x'-x)\right)} \int_{y(0)=0}^{y(T)=0} Dy\, e^{-\int_0^T d\tau \frac{1}{4}\dot{y}^2}$$
$$\times \prod_{l=1}^{n} \int_0^T d\tau_l\, e^{ik_l\cdot\left((x'-x)\frac{\tau_l}{T}+y(\tau_l)\right)+\varepsilon_l\cdot\dot{y}(\tau_l)}\Big|_{\varepsilon_1\dots\varepsilon_n}, \tag{2.4.9}$$

where again $|_{\varepsilon_1\dots\varepsilon_n}$ indicates that we are only meant to pick out the multilinear part in all the polarizations. The latter path integral thus provides the correlation function of the product of n photon vertex operators. This, with our choice of coordinates, takes now the form

$$V_A[k,\varepsilon] = e^{ik\cdot x+\frac{\varepsilon}{T}\cdot(x'-x)} \int_0^T d\tau\, e^{ik\cdot\left((x-x')\frac{\tau}{T}+y(\tau)\right)+\varepsilon\cdot\dot{y}(\tau)}\Big|_{\text{lin}}, \tag{2.4.10}$$

with respect to the Gaussian measure $\int \mathcal{D}y(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{y}^2}$, which has normalization $(4\pi T)^{-D/2}$ —compare with (2.3.29). This yields the Green's functions

$$-\frac{1}{2}\langle y^\mu(\tau) y^\nu(\tau')\rangle = \delta^{\mu\nu}\Delta(\tau,\tau'), \tag{2.4.11}$$
$$\Delta(\tau,\tau') = \frac{\tau\tau'}{T} + \frac{1}{2}|\tau-\tau'| - \frac{1}{2}(\tau+\tau'). \tag{2.4.12}$$

Note the difference in the definition of $\Delta(\tau,\tau')$ with respect to the Green's function $G(\tau,\tau')$ defined in (2.3.23) for the one-loop case: because of the different configuration of the worldline, line and loop respectively, and accordingly the different boundary conditions, the kinetic operator has a different Green's function in the two cases. Going back to the integral in (2.4.9), after some straightforward algebra one finds the Bern-Kosower-like[9] master formula originally obtained by Daikouji et al. [80] and later in the worldline formalism in [81], i.e.

$$D^{(n)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n) = (-ie)^n \int_0^\infty dT e^{-T(m^2+p'^2)} \prod_{l=1}^{n}\int_0^T d\tau_l$$
$$\times\, e^{(p'-p)\cdot\sum_{l=1}^n(-k_l\tau_l+i\varepsilon_l)+\sum_{l,l'=1}^n\left(k_l\cdot k_{l'}\Delta_{l-l'}-2i\varepsilon_l\cdot k_{l'}\dot{\Delta}_{l-l'}+\varepsilon_l\cdot\varepsilon_{l'}\ddot{\Delta}_{l-l'}\right)}\Big|_{\varepsilon_1\dots\varepsilon_n}. \tag{2.4.13}$$

Here

$$\Delta_{l-l'} := \frac{1}{2}|\tau_l-\tau_{l'}|, \tag{2.4.14}$$

is the translation-invariant part of (2.4.12). Above we have also stripped off the overall momentum-conservation delta function. In particular,

9 Compare with the one-loop formula in (2.3.25).

note that Fourier transforming x, x' into p, p', i.e. writing $D^{(n)}$ in full momentum space, removes the (UV divergent) factor $(4\pi T)^{D/2}$. The Feynman amplitude for the tree-level scattering of two scalars and n photons can thus be obtained from (2.4.13) by truncating the external scalar lines, i.e. multiplying by $(p^2+m^2)(p'^2+m^2)$,

$$\begin{aligned}\mathcal{D}^{(n)}&(p, p'; \varepsilon_1, k_1; \ldots; \varepsilon_n, k_n)\\ &= (p^2+m^2)(p'^2+m^2) D^{(n)}(p, p'; \varepsilon_1, k_1; \ldots; \varepsilon_n, k_n).\end{aligned} \quad (2.4.15)$$

Note that, as already mentioned about one-loop calculations, this expression holds off the mass-shell of the external particles. This property, peculiar of the worldline formalism, is extremely relevant and will be properly investigated later in the manuscript. Moreover, going on-shell leads to transversality in all the photon lines, i.e. the amplitude is guaranteed to vanish upon the replacement $\varepsilon_l(k_l) \to k_l$ as expected. This will also be reviewed later.

3 COLOR-KINEMATICS DUALITY AND DOUBLE COPY

The present chapter reviews the basics of the color-kinematics duality and double copy as recent and significant developments in scattering amplitudes. The color-kinematics duality states that the kinematic factors of an amplitude in gauge theory can be given in a representation such that they satisfy the very same algebraic relations of the corresponding color factors. The double copy principle, on the other hand, states that the scattering amplitudes of a gauge theory can be related to those of a gravity theory through a simple transformation of the color factors into another copy of the kinematic factors. The double copy prescription requires color-kinematics duality to be satisfied satisfied. In this chapter we firstly review the algebraic tools for dealing with the color structure of Yang-Mills theory, then we rederive relations among partial amplitudes (Kleiss-Kuijf and BCJ) that naturally lead to color-kinematics duality. We later introduce the main ideas behind the double copy prescription. We conclude the chapter with final remarks about the related KLT relations.

3.1 THE COLOR-STRUCTURE OF YANG-MILLS THEORY

The simplest example of theory exhibiting color-kinematics duality is Yang-Mills theory, a special example of gauge theory with a non-abelian symmetry group. Yang-Mills Lagrangian describes the propagation and self-interactions of gluons through the Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \tag{3.1.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}\left[A_\mu, A_\nu\right]$ and $A_\mu = A^a_\mu T^a$. For definiteness, we consider the semisimple gauge group $G = SU(N)$; the gluon fields are in the adjoint representation, so the *color* indices run over $a, b, \ldots = 1, 2, \ldots, N^2 - 1$. The generators of the gauge group T^a are chosen to be hermitian, and are normalized as

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}, \tag{3.1.2}$$

and the totally antisymmetric group-theory structure constants $\tilde{f}^{abc}$ are defined through the relation

$$\left[T^a, T^b\right] = i\tilde{f}^{abc}T^c. \tag{3.1.3}$$

By combining the two relations above we express the color factor $\tilde{f}^{abc}$ as

$$i\tilde{f}^{abc} = \mathrm{Tr}\left(\left[T^a, T^b\right]T^c\right). \tag{3.1.4}$$

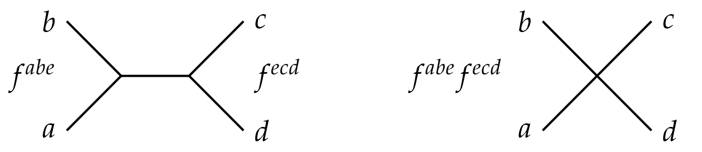
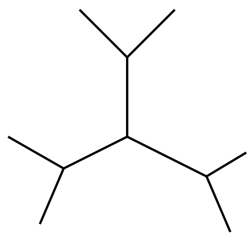


FIG. 3.1: The structure constants in a four point amplitude due to the Feynman rules for three point vertices in the gauge field. The resulting color factor is identical to the one associated to a quartic vertex.

Once gauge redundancy is fixed[1], the Lagrangian in (3.1.1) can be used to extract standard Feynman rules. In the Feynman gauge, the gluon propagator is given by $\delta^{ab}\frac{\eta^{\mu\nu}}{p^2}$, while the 3- and 4-gluon vertices involve $\tilde{f}^{abc}$ and $\tilde{f}^{abx}\tilde{f}^{xcd}$ (+perms.), respectively, each dressed up with kinematic factors (see figure 3.1). These rules are useful to construct scattering amplitudes, where the different group theory structures are dressed with a suitable factor depending on momenta and polarizations.

In the remainder of the chapter, the discussion will be mainly focused on scattering amplitudes at *tree-level*. Most of the properties of amplitudes that will be exploited in the following, can be efficiently extended at *loop-level* and final remarks about this generalization will be done later in this chapter.

The full color-dressed n-point tree amplitude of Yang-Mills theory can be conveniently rewritten in terms of diagrams with only cubic vertices (as described below), such as



$$, \tag{3.1.5}$$

and the amplitude takes the form

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{j \in \text{ trivalent}} \frac{c_j n_j}{\prod_{i_j} s_{i_j}}. \tag{3.1.6}$$

The sum runs over the set of distinct n-point graphs (labeled by j) with only three-point vertices. The factors $1/s_{i_j}$ are ordinary scalar Feynman propagators, where i_j runs over the propagators for diagram j. The numerators factorize into a group-theoretic color-part c_j, which is a polynomial of structure constants f^{abc}, and a purely kinematic part n_i, which is a polynomial of Lorentz-invariant contractions of

1 We ignore ghosts, since the main focus here is on tree-level amplitudes.

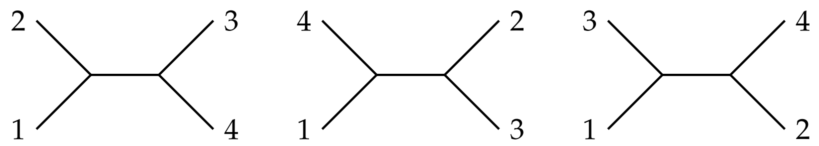


FIG. 3.2: The three diagrams with cubic vertices describing a four-point tree amplitude.

polarization vectors ε_i and momenta p_i. The contribution of quartic vertices in Yang-Mills theory to (3.1.6) certainly contains less propagators and can be absorbed into the n_i by multiplying and dividing by appropriate propagators $1 = s_{i_j}/s_{i_j}$ for compatibility with the pole structure.

As an example, we consider the 4-point amplitude for Yang-Mills theory. The amplitude is given by the sum of the three cubic diagrams in figure 3.2, and takes the form

$$\mathcal{A}_4^{tree} = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}. \tag{3.1.7}$$

The color factors of the s-, t- and u-channel diagram are

$$c_s \equiv \tilde{f}^{a_1 a_2 x} \tilde{f}^{x a_3 a_4}, \quad c_t \equiv \tilde{f}^{a_1 a_3 x} \tilde{f}^{x a_4 a_2}, \quad c_u \equiv \tilde{f}^{a_1 a_4 x} \tilde{f}^{x a_2 a_3}. \tag{3.1.8}$$

The three color factors are not independent but they obey Jacobi relations that are inherited from the Lie algebra structure. In particular, the following relation holds:

$$c_s + c_t + c_u = 0. \tag{3.1.9}$$

So there are only two independent color-structures for the tree-level 4-gluon amplitude. As pointed out above, the Yang-Mills 4-point contact terms can be absorbed into s-, t- or u-channel 3-vertex pole diagrams and only the three diagrams in figure 3.2 contribute to the full tree amplitude. This is trivially achieved by multiplying the distinct 4-point vertices by $1 = s/s = u/u = t/t$. However, since $c_s + c_t + c_u = 0$, there is not a unique prescription for how to assign a given 4-point vertex into the cubic diagrams, and, generalizing this procedure to a generic n-point amplitude, it is manifest that the numerators in (3.1.6) are not uniquely defined.

There are several ways in which one can deform the numerators n_i without changing the full amplitude. First of all, one can make use of the gauge invariance of the amplitude and exploit the transversality propriety. This property comes directly from *Ward-Takahashi identities* in QFT, and states that if in the expression of the amplitude is invariant if one substitutes $\varepsilon_i(p_i) \to \varepsilon_i(p_i) + p_i$; this shift modifies the expression of the numerators n_i, but gauge invariance guarantees that the full amplitude is untouched.

A more non-trivial deformation involves a shift in the distinct numerators n_i of type (consider arbitrary functions Δ_i)

$$n_i = n_i' + \Delta_i, \tag{3.1.10}$$

subject to the constraint

$$\sum_i \frac{c_i \Delta_i}{s_i} = 0. \tag{3.1.11}$$

For example, in the four point case we can take the shifts $n_s \to n_s + s\Delta$, $n_t \to n_t + t\Delta$ and $n_u \to n_u + u\Delta$. The amplitude is left invariant, as the net deformation is proportional to $c_s + c_t + c_u$, that is vanishing because of the color factor Jacobi identity (3.1.9).

In general, inside the full amplitude in (3.1.6), we can pinpoint sets of three trivalent diagrams (see figure 3.3) whose color factors are related through a Jacobi identity of type

$$c_i + c_j + c_k = 0. \tag{3.1.12}$$

The following transformation on the numerators leaves the amplitude invariant:

$$n_i \to n_i + s_i\Delta, \quad n_j \to n_j + s_t\Delta, \quad n_k \to n_k + s_k\Delta, \tag{3.1.13}$$

where $1/s_i$, $1/s_j$ and $1/s_k$ are the unique propagators that are not shared among the three diagrams, as shown in figure 3.3. The function Δ can be thought of as *generalized gauge functions*, that produces transformations over the numerator factors n_i and that drop out of the amplitude. In particular, the freedom (3.1.12)-(3.1.13) is often called **generalized gauge transformation** [11]. The gauge choice can only affect the numerators n_i by adding terms that cancel one of the s_{i_j} poles[2].

3.2 PARTIAL AMPLITUDES AND KLEISS-KUIJF RELATIONS

From the considerations in the previous section we have learned that the single numerators n_i are not unique nor separately gauge invariant, but there is nothing wrong with this: the individual Feynman diagrams are not physical observables, the relevant property is the gauge invariance of the full amplitude. For practical purposes, it is often useful to work with gauge invariant quantities. Note that, if we are able to arrange the color factors in a basis that is independent under Jacobi identities, the coefficient in front of every terms has to be gauge invariant. These coefficients are referred to as *partial amplitudes*,

2 Note that the ambiguity in the decomposition of quartic-vertices into cubic diagrams has the same effect of a generalized gauge transformation.

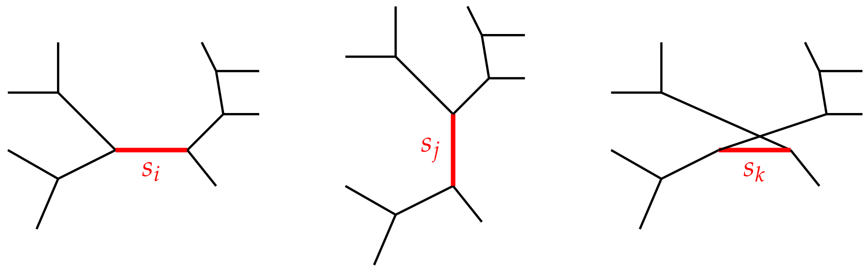


FIG. 3.3: A triplet of diagrams where the sum over the associated color factors vanishes due to the Jacobi relation.

that combine to construct the full amplitude and by construction are fully gauge invariant.

We show now alternative ways to obtain such partial amplitudes starting from the full amplitude (3.1.6). First of all we can use the definitions (3.1.2)-(3.1.4) to reshuffle the contraction of two structure constants as

$$\tilde{f}^{abe}\tilde{f}^{ecd} = -\mathrm{Tr}\left(\left[T^a, T^b\right]\left[T^c, T^d\right]\right). \tag{3.2.1}$$

By iterating identities like (3.1.4) and the one above, all the structure constants f^{abc} in the color factors of the amplitude can be replaced by traces of generators T^d. Hence, we can reformulate the full amplitude in (3.1.6) as

$$\mathcal{A}_n^{\mathrm{tree}} = g^{n-2} \sum_{\rho \in S_{n-1}} \mathrm{Tr}\left(T^{a_1} T^{a_{\rho(2)}} \dots T^{a_{\rho(n)}}\right) A_n\left(1, \rho(2, \dots, n)\right), \tag{3.2.2}$$

where S_{n-1} denotes the group of permutations in $(2, 3, \dots, n)$. The partial amplitudes $A_n\left(1, \rho(2, \dots, n)\right)$ in (3.2.2) are called **color-ordered amplitudes**. The cyclic nature of the trace has been exploited to obtain a trace-basis of $(n-1)!$ elements. The most important property of the color-ordered amplitudes is that they must be separately gauge invariant [3], as opposed to individual Feynman diagrams. In (3.2.2) we have accomplished one of our goals, i.e. we have obtained a decomposition of the full amplitude $\mathcal{A}_n^{tree}$ into smaller gauge invariant pieces.

The color-ordered amplitudes have a number of properties worth noting [83]:

1. *Cyclic*: As mentioned above, the cyclic symmetry of the color-ordered amplitudes follows directly from the trace-structure

$$A_n\left(1, 2, \dots, n\right) = A_n\left(2, \dots, n, 1\right) = \dots \tag{3.2.3}$$

2. *Reflection*:

$$A_n\left(1, 2, \dots, n\right) = (-1)^n A_n\left(n, \dots, 2, 1\right) \tag{3.2.4}$$

3 This follows from a partial orthogonality property of the single-traces [82].

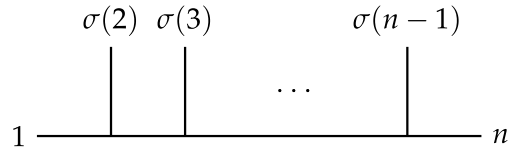


FIG. 3.4: Graphic representation of a diagram where the color factors have been converted in the multi-peripheral form.

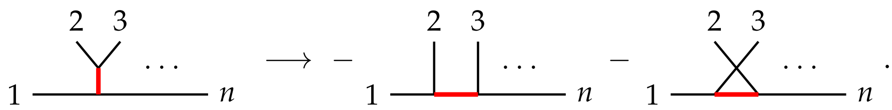
3. *$U(1)$ decoupling identity*:

$$A_n(1,2,\ldots,n) + A_n(2,1\ldots,n) + A_n(2,3,1\ldots,n) + \ldots + A_n(2,3,\ldots,1,n) = 0. \quad (3.2.5)$$

   This relation among $n-1$ color-ordered amplitudes is also known as photon decoupling identity. It is obtained by taking the generator $T^{a_1} \propto 1$ in (3.2.2): after this substitution, the particle behaves like a non-interacting *fictitious photon* and any scattering amplitude involving it must be zero.

The existence of relations among partial amplitudes is not a coincidence: the trace-basis (3.2.2) is overcomplete and this forces the presence of relations of type (3.2.3)-(3.2.5).

To better understand why the trace-basis in (3.2.2) is not the minimal basis, we consider now a different procedure [84] to organize the full amplitude in (3.1.6). The main idea is to iteratively use the Jacobi identity to disentangle the color factors of individual Feynman diagrams. In particular, we can make repeated use of the identity (3.1.12) to convert each diagram into a sum of color factors in *multi-peripheral form*, i.e. color factors where the position of two legs (1 and n by convention) is fixed, and the remaining $n-2$ legs are permuted (see figure 3.4). As an example, we show how an external tree with legs 2 and 3 attached extending from the baseline can be converted to a linear combination of two diagrams in multi-peripheral form:



(3.2.6)

The procedure depicted above can be iterated unlimited times to convert any trivalent diagram into a linear combination of diagrams in multi-peripheral form, i.e. diagrams in same shape of figure 3.4. The huge advantage of this representation is that the color factors in multi-peripheral form are not related by any Jacobi identities, so there is a total of $(n-2)!$ independent color factors. We write the full amplitude in the multi-peripheral basis as

$$\mathcal{A}_n^{tree} = \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma(2)} b_1} \tilde{f}^{b_1 a_{\sigma(3)} b_2} \cdots \tilde{f}^{b_{n-3} a_{\sigma(n-1)} a_n} \tilde{A}_n \left(1, \sigma(2, \ldots, n-1), n\right), \tag{3.2.7}$$

where S_{n-2} indicates the group of permutations in $(2, 3, \ldots, n-1)$. As the color structures are independent, the coefficient that comes with each color factor must be a gauge invariant quantity.

Now we can compare the representation in the multi-peripheral basis in (3.2.7) and the one in the trace basis in (3.2.2) obtained above. The important thing to notice is that in the trace basis representation, there are $(n-1)!$ distinct traces, while the independent color factors are only $(n-2)!$. This implies that the trace basis is overcomplete and the color-ordered partial amplitudes have to satisfy special linear relations among them. These linear relations are known as the **Kleiss-Kuijf relations** [85], and are exactly the relations that reduce the number of independent color ordered amplitudes from $(n-1)!$ to $(n-2)!$. An example of one of those relations has already been carried out in (3.2.5). A compact representation of the Kleiss-Kuijf relations is given by

$$A_n \left(1, \alpha_1, \ldots, \alpha_s, n, \beta_1, \ldots, \beta_r\right) = (-1)^r \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^T} A_n \left(1, \sigma, n\right), \tag{3.2.8}$$

where $\{\beta^T\}$ denotes the reverse ordering of the labels $\{\beta\}$ and $\sqcup\!\sqcup$ denotes the sum ordered permutations, namely permutations of the labels in the joined set $\{\alpha\} \cup \{\beta^T\}$ such that the ordering within $\{\alpha\}$ and $\{\beta^T\}$ is preserved. The $\sqcup\!\sqcup$ operator is commonly referred to as the **shuffle product**, and section 4.1 will provide a more comprehensive explanation of it. The proof of (3.2.8) is carried out in [84] and based on group-theoretic properties only such as the Jacobi identity and the behavior of color traces. As pointed out above, the effect of the relations (3.2.8) is to reduce the number of independent color-ordered amplitudes that appear in (3.2.2): the $(n-2)!$ partial amplitudes $\tilde{A}_n$ in the representation (3.2.7) are exactly the color-ordered partial amplitudes that are independent under the Kleiss-Kuijf relations. As a final remark for the present section, note that the partial amplitudes $\tilde{A}_n$ are not unique: the choice of legs 1 and n as reference legs has been completely arbitrary and other legs could have been preferred.

3.3 DUALITY BETWEEN COLOR AND KINEMATICS

In section 3.1 we have learned that the kinematic numerators n_i in the amplitude (3.1.6) are building blocks for colored amplitudes in Yang-Mills theory. These numerators are not unique: they can be modified using suitable sets of transformations. The non-trivial statement

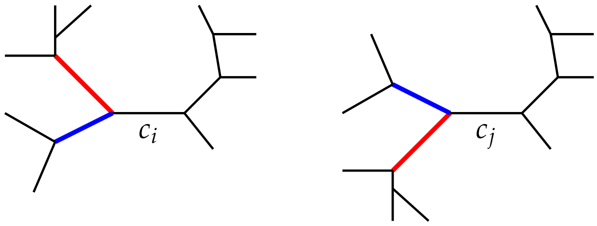


FIG. 3.5: Two trivalent diagrams obtained by twisting two lines. The corresponding color factors c_i and c_j are linked by antisymmetry.

is that these transformations can be used to parametrize the numerators n_i in such a way that they obey the same algebraic relations as the corresponding color factors c_i, although they appear completely unrelated at first glance. This property is known as **color-kinematics duality** and was first proposed by Bern, Carrasco and Johansson (BCJ) in 2008 [11].

More precisely, the main proposal of the duality is that one can always find a representation of the numerators n_i in (3.1.6) such that

$$c_i + c_j + c_k = 0 \qquad \Leftrightarrow \qquad n_i + n_j + n_k = 0 \tag{3.3.1}$$

$$c_i = -c_j \qquad \Leftrightarrow \qquad n_i = -n_j. \tag{3.3.2}$$

In (3.3.1), the labels i, j and k refer to three graphs which are identical except for one internal propagator (see figure 3.3). The relation indicates that the numerator factors must satisfy exactly the same Jacobi relations as their associated color factors, i.e. there exists a representation for the numerators such that $n_i + n_j + n_k = 0$. The identity in (3.3.2) is referred to the switching of two lines on a three-point vertex (see figure 3.5). The color factors are related by a minus sign[4]: $c_i = -c_j$, so the color-kinematics duality indicates that there is a representation where the numerator factors of the two diagrams share the same antisymmetry property: $n_i = -n_j$.

The entire set of identities that are obtained by exploiting the relations (3.3.1)-(3.3.2) inside the full amplitude (3.1.6) is known as Generalized Jacobi Identities (GJI) [86]. A consistent way to construct it will be provided later in the manuscript by investigating the Lie algebra generated by n colored particles.

3.4 BCJ RELATIONS

The existence of the color-kinematics duality presented in (3.3.1)-(3.3.2) has interesting consequences on the search for a minimal basis of color

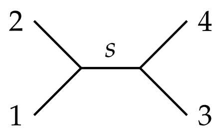
4 This property comes out naturally from the definition of the group-theory structure constants $\tilde{f}^{abc}$ through a commutator, i.e. $\left[T^a, T^b\right] = i\tilde{f}^{abc}T^c$.

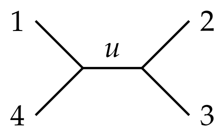
ordered field theory amplitudes showed in section 3.2. The representation of the full Yang-Mills amplitude in the multi-peripheral basis (3.2.7) allowed to identify $(n-2)!$ independent partial amplitudes, but this is not the end of the story: the color ladder of the partial amplitudes can be further reduced to a basis of $(n-3)!$ by means of the so-called **BCJ relations** , found by Bern, Carrasco and Johansson in 2008 [11].

The identities among partial amplitudes emerging from the color structure of the gauge group have been fully exploited to obtain the multi-peripheral representation in (3.2.7), i.e. Jacobi identities can not be used further to make progress in the computation of a minimal basis for partial amplitudes. Thus, the only way we have to go beyond the multi-peripheral representation and find new identities among partial amplitudes is by means of a massive use of the generalized Jacobi identities introduced in section 3.1. This is indeed the idea behind the BCJ relations. Before introducing these relations in a closed form, we briefly sketch how the scheme works in the four-point case. Consider the expression of the full four-point amplitude in Yang-Mills:

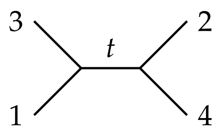
$$\mathcal{A}_4^{tree} = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}. \tag{3.4.1}$$

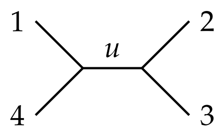
Now we decompose the expression above in terms of color-ordered amplitudes using the trace-basis representation in (3.2.2). Note that only the three partial amplitudes $A_4(1,2,3,4)$, $A_4(1,3,2,4)$ and $A_4(1,3,4,2)$ are independent under reflection symmetry. They are parametrized by kinematic numerators n_s, n_t and n_u numerators along with the $s-$, $t-$ and $u-$channel poles. Only two out of three channels are compatible with the individual color orderings:

$A_4(1,2,3,4) = -$ 

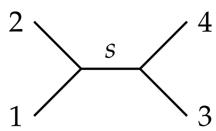
$+$ 

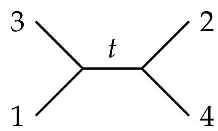
$= -\frac{n_s}{s} + \frac{n_u}{u}$

$A_4(1,3,2,4) = +$ 

$-$ 

$= +\frac{n_t}{t} - \frac{n_u}{u}$

$A_4(1,3,4,2) = +$ 

$-$ 

$= +\frac{n_s}{s} - \frac{n_t}{t}$

(3.4.2)

At this point we can make use of the generalized gauge transformations introduced in (3.1.13) to modify the numerators n_s, n_t and n_u and ensure that they satisfy color-kinematics duality, i.e. they obey the relations (3.3.1)-(3.3.2). However, this does not exploit the full potentialities of the generalized gauge transformations: these can be extended to *non-local* transformations. In particular, the full amplitude in (3.4.1) is unchanged if we add an extra term

$$\chi(s,t,u)(c_s + c_t + c_u), \tag{3.4.3}$$

where the generalized gauge parameter χ is now non-local. Note that this extra term transforms the numerators to a new set $(n_s, n_t, n_u) \to (\hat{n}_s, \hat{n}_t, \hat{n}_u)$, but the overall effect on the full amplitude is irrelevant because of the Jacobi identity among the color factors inside (3.4.3). Now we can use the non-local choice

$$\chi = -\frac{n_u}{u} \tag{3.4.4}$$

to force the transformed numerator $\hat{n}_u$ to vanish:

$$(n_s, n_t, n_u) \quad \longrightarrow \quad (\hat{n}_s, \hat{n}_t, \hat{n}_u) = \left(n_s - \frac{s}{u} n_u,\, n_t - \frac{t}{u} n_u,\, 0 \right). \tag{3.4.5}$$

Note that the Jacobi identity among the transformed $(\hat{n}_s, \hat{n}_t, \hat{n}_u)$ still holds, but it just leaves one independent numerator $\hat{n}_s = -\hat{n}_t$, as $\hat{n}_u = 0$. Ultimately, using the transformed numerators inside the colored-amplitudes (3.4.2) we straightforwardly obtain

$$A_4(1,2,3,4) = \frac{t}{s} A_4(1,3,2,4) = \frac{t}{u} A_4(1,3,4,2). \tag{3.4.6}$$

This is a non-trivial set of relations among color-ordered amplitudes and it is the first example of the BCJ relations.

This procedure can be generalized at higher points to obtain new relations among color-ordered amplitudes. By refining the scheme introduced above, the BCJ relations at n-point can be obtained with the following steps:

- express all the numerators in terms of $(n-2)!$ independent ones using Jacobi identities according to (3.3.1)-(3.3.2);
- solve for $(n-3)!$ basis numerators in terms of a color-ordered amplitude to which they contribute;
- force the remaining $(n-2)! - (n-3)!$ to vanish by means of non-local gauge transformations.

The remaining $(n-2)! - (n-3)!$ partial amplitudes which did not get involved in the second step are then naturally expressed in terms of a $(n-3)!$ basis from their $\sum_i \frac{n_i}{\prod_{\alpha_i} s_{\alpha_i}}$ parametrization. The resulting system of equations among partial amplitudes can be brought in the form

$$\begin{aligned} s_{12} A_n(2,1,3,\ldots,n) + (s_{12}+s_{13}) A_n(2,3,1,4\ldots,n) + \ldots + \\ (s_{12}+s_{13}+\ldots+s_{1,n-1}) A_n(2,3,\ldots,n-1,1,n) = 0. \end{aligned} \tag{3.4.7}$$

As a final remark, we emphasize that the existence of a generalized gauge where all the Jacobi triplets sum to zero ($n_{i_j} + n_{i_k} + n_{i_l} = 0$) is just a tool in [11] to perform the basis reduction to $(n-3)!$ partial amplitudes. The BCJ relations are stand-alone objects and must hold independently on the choice of the numerators n_i. In other words, duality-satisfying numerators make identities in (3.4.7) valid on the spot, but are not the unique solutions to the BCJ relations (see [87, 88] for more details).

3.5 DOUBLE COPY

The existence of a gauge where color-kinematics duality is satisfied was crucial in the computation of the BCJ relations and in the resulting computation of a minimal basis of $(n-3)!$ partial amplitudes. On the other hand, this is not the full story about color-kinematics duality: a second remarkable consequence is that, once we have obtained numerators n_i that obey the same algebraic relations as the color factors c_i according to (3.3.1)-(3.3.2), we can replace

$$c_i \;\to\; n_i \tag{3.5.1}$$

inside the Yang-Mills amplitude (3.1.6). This simple substitution plays a crucial role: it allows us to get gravity from Yang-Mills theory on the spot! The expression

$$\mathcal{M}_n^{\text{tree}} = \left(\frac{\kappa}{2}\right)^{n-2} \sum_{j \in \text{trivalent}} \frac{n_j^2}{\prod_{i_j} s_{i_j}} \tag{3.5.2}$$

indeed correctly reproduces gravitational amplitudes. In the expression above, $\kappa^2 = 32\pi G$ with G Newton's constant, and n_j are the kinematic numerator factors of the gauge-theory amplitude. This relation is called the **BCJ double-copy relation** [13, 89]. First of all, note that the formula (3.5.2) manifestly reproduces all possible poles that should appear in the gravity amplitude. Indeed, in color-ordered amplitudes the allowed physical poles are only those that involve adjacent momenta, i.e. $1/s_{i,i+1,\ldots,j-1,j}$, as expected from color-ordered Feynman rules. On the other hand, gravity amplitudes do not require a color-ordering and the poles can involve any combination of external momenta. This is in agreement with (3.5.2), as the sum is performed over all the possible cubic diagrams with n external legs and all the possible poles are thus explored in the sum. The double-copy construction in (3.5.2) has another important property (it will not be exploited in the present manuscript): the squaring relation can be generalized to

$$\mathcal{M}_n^{\text{tree}} = \left(\frac{\kappa}{2}\right)^{n-2} \sum_{j \in \text{trivalent}} \frac{\tilde{n}_j n_j}{\prod_{i_j} s_{i_j}}, \tag{3.5.3}$$

where $\tilde{n}_j$ and n_j are two kinematic factors of two *distinct* Yang-Mills theories. Only one set of the two numerators has to satisfy the color-kinematics duality, while the other copy can be in an arbitrary representation. This property is extremely useful once one recalls that the spectrum of many supergravity theories can be obtained through the tensor product of two different Yang-Mills theories. For example, the spectrum of pure $\mathcal{N} = 4$ supergravity is given by (e.g. see [90, 91])

$$\mathcal{N} = 4 \text{ supergravity} = (\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = 0 \text{ YM}) \,. \tag{3.5.4}$$

Thus the main insight of (3.5.3) is that BCJ double-copy prescription can be used to construct the supergravity scattering amplitude by using the numerators of two distinct (S)YM theories, and only one copy of the numerators needs to satisfy the duality.

3.6 KLT RELATIONS

In the previous section we have clarified that gravity and gauge theory amplitudes are not independent, but a strong link between the two exists, as gravity amplitudes can be obtained from gauge theory ones using the double-copy prescription. However, the relation in (3.5.2) is not unique, and more general relations are available in literature to understand the link between gravity and gauge theory amplitudes. The first such example are the so-called **KLT relations**, derived by Kawai, Lewellen and Tye in 1985 [12]. The KLT relations have been initially derived in string theory and they state that the n-point tree-level closed string scattering amplitude is related to a sum over products of n-point open string string partial amplitudes. The coefficients of the sum depend on the string tension $1/(2\pi\alpha')$ as well as on kinematic variables. The KLT relations are derived by monodromy arguments on the worldsheet, and their existence reflects the fact that the Hilbert space of closed string states is simply the tensor product of two open string states. The non-triviality of the KLT relations is that the factorization into open string amplitudes survives the integrals over the insertion points of the vertex operators, i.e. the closed string integrand over the moduli space factorizes into left- and right-movers without any correlation function among them. The KLT relations at n-points can be expressed in the following form:

$$\begin{aligned}\mathcal{M}^{\text{tree}}_{\text{closed},n} = \sum_{\rho,\tau \in S_{n-3}} & A^{\text{tree}}_{\text{open},n}(1, \rho(2, \ldots, n-2), n-1, n) \\ & S_{\alpha'}(\rho|\tau) \overline{A^{\text{tree}}_{\text{open},n}(1, \tau(2, \ldots, n-2), n, n-1)}.\end{aligned} \tag{3.6.1}$$

In the expression above, S_{n-3} denotes the group of permutations in $(2, 3, \ldots, n-2)$, and $S_{\alpha'}$, called *KLT matrix*, is a $(n-3) \times (n-3)$ matrix that depends on α' and on kinematic variables and that pairs the different permutations. Also, $A^{\text{tree}}_{\text{open},n}$ describes a color-ordered open string amplitude at n-point. As an example, we can consider the easiest case $n = 4$, where the KLT relation reads as

$$\mathcal{M}^{\text{tree}}_{\text{closed},4} = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} A^{\text{tree}}_{\text{open},4}(1,2,3,4) \overline{A^{\text{tree}}_{\text{open},4}(1,2,4,3)}. \tag{3.6.2}$$

In the expression above, the external momenta are rescaled by a factor α' to guarantee the correct dimensions.

String scattering amplitudes are fascinating objects per se. However they have interesting implications on scattering amplitudes in quantum field theories. Indeed, it is believed that string theories form the UV completions of (super)gravity theories and, in the infinite tension limit $\alpha' \to 0$, the strings shrink to point particles and Einstein and Yang-Mills theories are recovered. In particular, in this limit the closed string amplitudes with massless spin-2 string external states become the regular graviton scattering amplitudes $\mathcal{M}_n$ discussed in section 3.5 and the open-string partial amplitudes with external massless spin-1 states become the color-ordered gluon amplitudes A_n. Thus, in the limit $\alpha' \to 0$, KLT offers a relationship between tree-level $\mathcal{M}_n$ and A_n for each n. We show now how the field theory limit is extracted in the case $n = 4$. The relation (3.6.2) can be reformulated as

$$\begin{aligned}
\mathcal{M}^{\text{tree}}_{\text{closed},4} =& -\frac{\pi s_{12}}{\Gamma(1+s_{12})\Gamma(1-s_{12})} A_4(1,2,3,4)\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\
&\times \overline{A_4(1,2,4,3)}\frac{\Gamma(1+s_{12})\Gamma(1+s_{13})}{\Gamma(1+s_{12}+s_{13})} \\
=&\mathcal{M}^{\text{tree}}_4 \frac{\Gamma(1+s_{12})\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})},
\end{aligned} \tag{3.6.3}$$

where in the first line we have used

$$\frac{\sin(\pi s_{12})}{2\pi\alpha'} = \frac{\pi s_{12}}{\Gamma(1+s_{12})\Gamma(1-s_{12})}, \tag{3.6.4}$$

and

$$A^{\text{tree}}_{\text{open},4}(1,2,3,4) = A_4(1,2,3,4)\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}. \tag{3.6.5}$$

The last relation in (3.6.5) is an exact relation in string scattering amplitudes: essentially, it states that the field theory limit of the amplitude, i.e. a color-ordered amplitude in Yang-Mills, is completely disentangled from the integral over the moduli space. It is worth to point out further details about (3.6.5). In particular, we can make explicit the dependence on α' of the kinematic variables, i.e. rescale $s_{ij} \to \alpha' s_{ij}$, and perform a series expansion in α':

$$A^{\text{tree}}_{\text{open},4}(1,2,3,4) = A_4(1,2,3,4)\left(1 - \frac{1}{6}\alpha'^2\left(\pi^2 s_{12}s_{23}\right) + O\left(\alpha'^3\right)\right). \tag{3.6.6}$$

At order zero in α', the amplitude is exactly the a color-ordered amplitude in Yang-Mills as expected. The extra-terms in α'^2 and higher orders are interpreted in the field theory language as effective vertices obtained by integrating out massive modes in string theory. Similar considerations can be carried out in the last line of (3.6.3), where the order zero in α' gives exactly the four point (super)gravity amplitude. In particular, we can formulate the field theory limit of the KLT relation in (3.6.3):

$$\mathcal{M}_4^{\text{tree}}(1,2,3,4) = -s_{12}A_4(1,2,3,4)A_4(1,2,4,3). \tag{3.6.7}$$

The idea behind this expression is similar to the double copy construction in section 3.5, i.e. tree level scattering amplitudes in gravity are expressed as squares of scattering amplitudes in Yang-Mills theory[5]. Note the presence of the prefactor s_{12}, it is crucial to ensure that the locality structure of the resulting gravity amplitude is correct, that is, double poles are avoided. Going back to the KLT relations at n points in (3.6.1), we formulate the field theory limit $\alpha' \to 0$ of KLT relations as

$$\begin{aligned}\mathcal{M}_n^{\text{tree}} = \sum_{\rho,\tau\in S_{n-3}} & A_n(1,\rho(2,\dots,n-2),n-1,n) \\ & S_0(\rho|\tau)\overline{A_n(1,\tau(2,\dots,n-2),n,n-1)}.\end{aligned} \tag{3.6.8}$$

The entries of the $(n-3)\times(n-3)$ KLT matrix are of type s_{ij}, and it is widely known among the theoretical physics community that their task is to guarantee that no double poles appear in the product of the two Yang-Mills amplitude. In other words, the KLT formula guarantees that all the possible poles inside a gravity amplitude are reconstructed. However, there is still a lack of a rigorous mathematical proof for this property —e.g. see [93] for more details. For $n=4,5$ the field theory KLT relations read

$$\begin{aligned}\mathcal{M}_5^{\text{tree}}(1,2,3,4,5) =& s_{23}s_{45}A_5(1,2,3,4,5)A_5(1,3,2,5,4) + (3\leftrightarrow 4)\\ \mathcal{M}_6^{\text{tree}}(1,2,3,4,5,6) =& -s_{12}s_{45}A_6(1,2,3,4,5,6)\Big(s_{35}A_6(1,5,3,4,6,2)+\\ &+(s_{34}+s_{35})A_6(1,5,4,3,6,2)\Big) + \text{perms.}(2,3,4).\end{aligned} \tag{3.6.9}$$

The KLT relations have a deep origin in string theory and the link to the double copy prescription in (3.5.2) is manifest. Currently, KLT relations have been established for tree-level scattering amplitudes. However, in a recent work by Stieberger [94], an extension to loop-level, known as the *one-loop KLT relation*, has been presented in the literature. The amplitudes community had foreseen the existence of this extension based on several recent results, including the explicit one-loop KLT formula in field theory that was derived in [95] using forward limits of tree amplitudes (CHY representations), and the extension of this result to one-loop matrix elements of effective operators from string tree-level, which was discussed in [96].

The connection between the KLT formula and the double copy prescription provides further indications of the feasibility of uplifting

5 The correspondence of double copy and KLT formulae is well established in literature, e.g. see [92].

the former to loop level. Indeed, the color-kinematics duality and double copy construction discussed in this chapter are believed to hold true when extending scattering amplitudes from tree-level to loop-level. Analogous to the tree level case (3.1.6), an L-loop m-point gauge theory scattering amplitude can be organized as,

$$\mathcal{A}_m^{(L)} = g^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{3.6.10}$$

where the sum runs over the distinct L-loop m-point diagrams with only cubic vertices. Each such diagram corresponds to a unique color factor c_i. It also has an associated denominator corresponding to the product of the denominators of the Feynman propagators $\sim 1/s_{\alpha_i}$ of each internal line of the diagram. As for tree level, the representation of the amplitude in terms of cubic diagrams is trivial. The non-trivial part is to find representations of the amplitude where the duality holds so that the integrand kinematic numerators n_i satisfy the duality in (3.3.1)-(3.3.2). Whether this can be done in general at loop level remains a conjecture, although there is considerable evidence in literature [17–20]. If the amplitude in (3.6.10) manifests the color-kinematics duality, we may now replace the color factors of the first amplitude with the duality-satisfying numerators $\tilde{n}_i$ of the second one. This gives the loop-level double copy formula for gravitational scattering amplitudes [11, 13]:

$$\mathcal{M}_m^{(L)} = \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}. \tag{3.6.11}$$

A formal proof of the constructions in (3.6.10)-(3.6.11) is not present yet, even if many results at loop level have already been obtained in literature and the general idea in the amplitudes community is that this is not just a conjecture. In this case, strong links with string theory are expected.

4 BERENDS-GIELE CURRENTS AND PERTURBINERS

In this chapter, we review the basic ideas of Berends–Giele recursion relations and their applications to the computation of tree-level scattering amplitudes in Yang-Mills theory. The Berends–Giele recursion shows a close connection to the more recent perturbiner technique. This technique originates from a systematic solution of the classical field equations in massless quantum field theories and allows to obtain a generating function for all tree-level scattering amplitudes in a given theory. We will show the basics of this technique focusing on pure Yang-Mills theory.

4.1 BERENDS–GIELE RECURSIONS

In 1987 Berends and Giele have introduced an efficient approach to determine the tensor structure of tree amplitudes in arbitrary D dimensions in pure Yang-Mills theory [10]. The key idea of the method is to recursively combine all color-ordered Feynman diagrams involving multiple external on-shell legs and a single off-shell leg, which are now referred to as *Berends-Giele currents*. This recursion is implemented via currents $J^{\mu}_{12\dots p}$ that depend on the polarization vectors ε^{μ}_{i} and momenta k^{μ}_{i} of the external particles $i = 1, 2, \dots, p$. These are subject to the following on-shell constraints

$$\varepsilon_i \cdot k_i = k_i \cdot k_i = 0 \qquad\qquad i = 1, 2, \dots \,. \tag{4.1.1}$$

The Berends-Giele recursion is stated through the following combination of currents of arbitrary multiplicity:

$$J^{\mu}_{i} = \varepsilon^{\mu}_{i}, \qquad\qquad s_P J^{\mu}_{P} = \sum_{XY=P} \left[J_X, J_Y\right]^{\mu} + \sum_{XYZ=P} \left\{J_X, J_Y, J_Z\right\}^{\mu}, \tag{4.1.2}$$

where

$$\left[J_X, J_Y\right]^{\mu} = (k_Y \cdot J_X)\, J^{\mu}_{Y} - (k_X \cdot J_Y)\, J^{\mu}_{X} + \frac{1}{2}\left(k^{\mu}_{X} - k^{\mu}_{Y}\right)(J_X \cdot J_Y) \tag{4.1.3}$$

$$\left\{J_X, J_Y, J_Z\right\}^{\mu} = (J_X \cdot J_Z)\, J^{\mu}_{Y} - \frac{1}{2}(J_X \cdot J_Y)\, J^{\mu}_{Z} - \frac{1}{2}(J_Y \cdot J_Z)\, J^{\mu}_{X}. \tag{4.1.4}$$

The external states have been grouped into multiparticle labels (or *words*, as commonly are defined) $P = 12\dots p$, usually represented with capital letters. The summation over $XY = P$ on the right-hand side of (4.1.2) tells us to deconcatenate P into non-empty words $X = 12\dots j$ and $Y = j+1\dots p$ with $j = 1, 2, \dots, p-1$, generating in this way

$|P|-1$ terms. Note that a deconcatenation includes the notion of ordering, no labels have to be swapped in this process: for instance, the summation over $XY = P$ with $P = 1234$ of length four incorporates the pairs $(X,Y) = (123,4)$, $(12,34)$ and $(1,234)$. In a similar fashion, the sum over $XYZ = P$ involves deconcatenations into non-empty words $X = 12\ldots j$, $Y = j+1\ldots l$ and $Z = l+1\ldots p$ with $1 \leq j < l \leq p-1$. This presentation of the Berends-Giele setup using words was motivated by recent literature reviews, such as the ones found in [97, 98].

In (4.1.2) s_P represents a Mandelstam invariant, defined through the multiparticle momentum k_P as

$$k_P^\mu = k_1^\mu + k_2^\mu + \ldots + k_p^\mu, \qquad s_P = \frac{1}{2}k_P^2. \tag{4.1.5}$$

The brackets in (4.1.3)-(4.1.4) capture the cubic and quartic Feynman vertices of pure Yang-Mills theory in Lorenz gauge. In figure 4.1 we depict the mechanism of the Berends-Giele recursion: by deconcatenating $XY = P$ and $XYZ = P$, we are effectively connecting lower-rank currents J_X^μ, J_Y^ν and J_Z^λ using cubic and quartic interactions in all possible ways such that the color ordering of the on-shell legs in the word $P = 12\ldots p$ is preserved.

From this construction, it is clear that Berends-Giele currents must be connected somehow to color-ordered on-shell amplitudes. In particular, the $n = p+1$ amplitude is computed by taking the off-shell leg in the current J_P^μ on shell: we can implement this by contracting with the polarization vector $J_n^\mu = \varepsilon_n^\mu$ of the last leg and by truncating the propagator $s_{12\ldots p}^{-1}$ contained in J_P^μ. This is necessary in order to remove the divergence coming out from the momentum conservation $k_{12\ldots p}^2 = (-k_n)^2 = 0$. Finally, the compact expression for the $n = p+1$ color-ordered amplitude is given by

$$A_n(1,2,\ldots,n) = s_{12\ldots n-1} J_{12\ldots n-1}^\mu J_{n,\mu}. \tag{4.1.6}$$

As an example, we consider the rank-two Berends-Giele current J_{12}^μ: using the recursion relation in (4.1.2) with the value of the letters $X = 1$ and $Y = 2$, we obtain

$$s_{12} J_{12}^\mu = (k_2 \cdot \varepsilon_1)\varepsilon_2^\mu - (k_1 \cdot \varepsilon_2)\varepsilon_1^\mu + \frac{1}{2}\varepsilon_1 \cdot \varepsilon_2 (k_1^\mu - k_2^\mu). \tag{4.1.7}$$

With this current, we are can use (4.1.6) and compute the three-point Yang-Mills color-ordered amplitude as

$$\begin{aligned} A_3(1,2,3) &= s_{12} J_{12}^\mu J_{3\,\mu} \\ &= (k_2 \cdot \varepsilon_1)(\varepsilon_2 \cdot \varepsilon_3) - (k_1 \cdot \varepsilon_2)(\varepsilon_1 \cdot \varepsilon_3) + (\varepsilon_1 \cdot \varepsilon_2)\varepsilon_3 \cdot k_1, \end{aligned} \tag{4.1.8}$$

where in the second line we have exploited momentum conservation and on-shellness condition to write $\varepsilon_3 \cdot k_2 = -\varepsilon_3 \cdot k_1$.

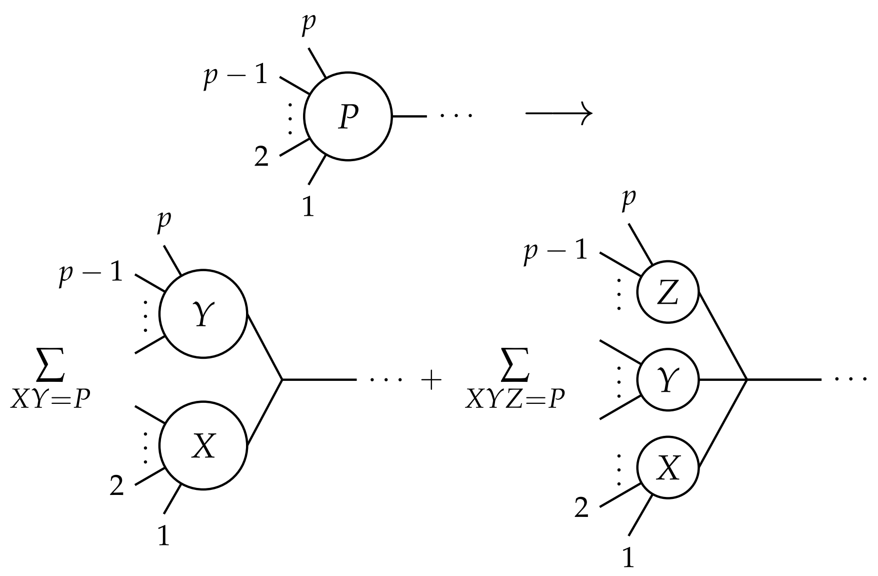


FIG. 4.1: Berends-Giele currents $J^{\mu}_{12\ldots p}$ expressed as combinations of lower-weighted currents connected with cubic and quartic Feynman vertices in such a a way that the color order is preserved.

In order to further investigate the properties of the Berends-Giele currents in (4.1.2), it is worth now to introduce the **shuffle product**. The shuffle product $P \sqcup\!\sqcup Q$ of two words $P = p_1 \ldots p_{|P|}$ and $Q = q_1 \ldots q_{|Q|}$ is defined as a sum over all permutations of $P \cup Q$ that preserve orderings of both words P and Q. It is recursively defined by

$$P \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup P = P, \qquad P \sqcup\!\sqcup Q = p_1(p_2 \ldots p_{|P|} \sqcup\!\sqcup Q) + q_1(q_2 \ldots q_{|Q|} \sqcup\!\sqcup P). \tag{4.1.9}$$

Here we assume linearity property once we unfold the shuffle product and sums of words appear in a subscript, e.g. $J^{\mu}_{1\sqcup\!\sqcup 2} = J^{\mu}_{12+21} = J^{\mu}_{12} + J^{\mu}_{21}$.

Now we point out that the symmetry properties $[J_X, J_Y] = -[J_Y, J_X]$ and $\{J_X, J_Y, J_Z\} + \text{cyc}(X, Y, Z) = 0$ of the brackets in (4.1.3) and (4.1.4) imply that the currents in (4.1.2) obey shuffle symmetry

$$J^{\mu}_{P \sqcup\!\sqcup Q} = 0, \qquad\qquad P, Q \neq 0. \tag{4.1.10}$$

While [99] had previously confirmed this property at lower multiplicities, a rigorous proof for it has been presented recently in the appendix of [100]. Making use of (4.1.10), it is now easy to see that the amplitude formula (4.1.6) propagates the shuffle symmetry of the currents and we end up with the relations

$$A_n((P \sqcup\!\sqcup Q), n) = 0, \qquad\qquad P, Q \neq 0, \tag{4.1.11}$$

where the words P and Q involve external state labels $1, 2, \ldots, n-1$. It should be noted that the aforementioned relations offer an alternative expression of the Kleiss–Kuijf (KK) relations discussed in section 3.2. While this equivalence is not immediately apparent and was not demonstrated in the original work of Berends and Giele, it has been proven in more recent works, as described in [97, 101].

4.2 PERTURBINER METHODS

The Berends–Giele construction of the previous section shows a close connection to the more recent **perturbiner** technique. This technique originates from the common statement that tree-level scattering amplitudes are all encoded in the solutions of the classical field equations in massless quantum field theories. Taking inspiration from this idea, Rosly and Selivanov [8, 9, 102–105] introduced an ansatz for such solutions as an infinite expansion in terms of plane-wave states. Using this ansatz, called **perturbiner expansion**, one actually obtains a generating function for all tree-level scattering amplitudes in a given theory. This formalism is closely related to the Berends-Giele recursion method outlined in the previous section, as was initially highlighted in [100]. Since then, numerous applications of the perturbiner formalism have been documented in literature.

In this section we introduce two different types of perturbiner expansions [106]: color-stripped and color-dressed. These can be used to construct recursion relations for various theories with and without color degrees of freedom, but we will mainly focus our attention on pure Yang-Mills theory.

4.2.1 *Color-Stripped Perturbiners*

Color-stripped perturbiners can be used to construct Berends–Giele currents and partial amplitudes for theories with colors. We consider here the case of $U(N)$ Yang-Mills theory —see [106, 107] for a similar setup. Here the gauge-theory Lagrangian, already seen in (3.1.1),

$$\mathcal{L}_{YM} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{4.2.1}$$

leads to the following non-linear equation of motion

$$\frac{\partial \mathcal{L}_{YM}}{\partial A_\lambda} = \left[\nabla_\mu, F^{\lambda\mu}\right] = 0. \tag{4.2.2}$$

Here the Lie-algebra valued gluon field is set as $A_\mu = A^a_\mu T^a$, and the group theory generators and structure constants have the same conventions used in section 3.1. The connection ∇_μ and the corresponding field strength $F_{\mu\nu}$ are defined as

$$\nabla_\mu = \partial_\mu - A_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \tag{4.2.3}$$

For convenience in the calculation, the coupling constant has been set to $g = -i\sqrt{2}$. Lorenz gauge $\partial_\mu A^\mu = 0$ simplifies the equations of motion in (4.2.2) to the equation

$$\begin{aligned}\Box A^\lambda &= [A^\mu, \partial_\mu A^\lambda] + [A_\mu, F^{\mu\lambda}] \\ &= 2[A^\mu, \partial_\mu A^\lambda] + [\partial^\lambda A^\mu, A_\mu] + \Big[[A^\mu, A^\lambda], A_\mu\Big]. \end{aligned} \tag{4.2.4}$$

We can derive formal solutions to (4.2.4) by means of the perturbiner ansatz

$$A^\mu = \sum_P J_P^\mu T^P e^{k_P \cdot x} = \sum_i J_i^\mu T^{a_i} e^{k_i \cdot x} + \sum_{ij} J_{ij}^\mu T^{a_i} T^{a_j} e^{k_{ij}\cdot x} + \dots, \tag{4.2.5}$$

Here P represents a non-empty word $P = 12\dots m$, such that

$$T^P = T^{a_1} T^{a_2} \cdots T^{a_m}, \tag{4.2.6}$$

and $k_P^\mu = k_1^\mu + k_2^\mu + \cdots k_m^\mu$, using the same notation introduced in (4.1.5). At this stage, we take the momenta k_i^μ to be imaginary for later convenience, however this won't cause problems in tree-level computations. Similarly, we can write down the perturbiner expansion for the field strength:

$$F^{\mu\nu}(x) = \sum_P F_P^{\mu\nu} T^P e^{k_P \cdot x}. \tag{4.2.7}$$

The reason we refer to the above perturbiner expansions as *color-stripped* is because the coefficients A_P^μ and $F_P^{\mu\nu}$ appearing in the expansion do not have any color degrees of freedom. The equation of motion in (4.2.4) can be solved recursively for the coefficients J_P^μ by collecting terms of the same order, i.e., with the same number of Lie algebra generators T^{a_i} on both sides. The choice of the letter J_P^μ was not a coincidence: these coefficients represent exactly the Berends-Giele currents encountered in section 4.1, as it will be clear in the following.

At the linear order in the generators T^{a_i}, we have

$$\sum_i k_i^2 J_i^\mu T^{a_i} e^{k_i \cdot x} = 0, \tag{4.2.8}$$

and, using the definition of the field strength in (4.2.3),

$$\sum_i F_i^{\mu\nu} T^{a_i} e^{k_i \cdot x} = \sum_i \left(k_i^\mu J_i^\nu - k_i^\nu J_i^\mu\right) T^{a_i} e^{k_i \cdot x}. \tag{4.2.9}$$

These relations are equivalent to imposing the momenta to be lightlike, $k_i^2 = 0$, and the field strength to be $F_i^{\mu\nu} = k_i^\mu J_i^\nu - k_i^\nu J_i^\mu$. At the quadratic order we find

$$\sum_{ij} k_{ij}^2 J_{ij}^\mu T^{a_i} T^{a_j} e^{k_{ij}\cdot x} = \sum_{ij} \Big(2(k_j \cdot J_i) J_j^\mu + k_i^\mu (J_i \cdot J_j)\Big) [T^{a_i}, T^{a_j}]\, e^{k_{ij}\cdot x}. \tag{4.2.10}$$

Because of the algebraic structure of the right-hand side, we find $J^\mu_{ij} = -J^\mu_{ji}$, so that the generators on the left-hand side organize into commutators. In this way, the sum can be efficiently rearranged as

$$\sum_{i<j} k^2_{ij}\, J^\mu_{ij}\, [T^{a_i}, T^{a_j}]\, e^{k_{ij}\cdot x} = \sum_{i<j} \Big(2(k_j \cdot J_i) J^\mu_j + k^\mu_i (J_i \cdot J_j) - (i \leftrightarrow j) \Big)\, [T^{a_i}, T^{a_j}]\, e^{k_{ij}\cdot x}. \quad (4.2.11)$$

Comparing the coefficients of each $[T^{a_i}, T^{a_j}]$, we find that J^μ_{ij} satisfies the recursion relation

$$s_{ij}\, J^\mu_{ij} = \left((k_j \cdot J_i) J^\mu_j + \frac{1}{2} k^\mu_i (J_i \cdot J_j) - (i \leftrightarrow j) \right), \quad (4.2.12)$$

that exactly matches the rank-two Berends-Giele current defined in (4.1.7). Note that in the equation above we made use of the Mandelstam invariant s_{ij} defined for a generic word in (4.1.5). Using the same symmetry argument for the rank-two field strength F^μ_{ij} and comparing the coefficient of each commutator, we obtain:

$$F^{\mu\nu}_{ij} = k^\mu_{ij} J^\nu_{ij} - k^\nu_{ij} J^\mu_{ij} - \left(J^\mu_i J^\nu_j - (i \leftrightarrow j) \right). \quad (4.2.13)$$

One can now use the same procedure at higher orders and find recursion relations for higher-rank perturbiners. In particular, from the symmetry properties of the equation of motion in (4.2.4), we have the shuffle symmetries

$$J^\mu_{P \sqcup\!\sqcup Q} = F^{\mu\nu}_{P \sqcup\!\sqcup Q} = 0, \qquad P, Q \neq 0. \quad (4.2.14)$$

With these symmetries, one finds compact expressions for the recursion relations for perturbiner coefficients. At this stage, it's not a surprise that the recursion relation for the coefficient J^μ_P is identical to (4.1.2), i.e. it reads as

$$s_P J^\mu_P = \sum_{XY=P} [J_X, J_Y]^\mu + \sum_{XYZ=P} \{J_X, J_Y, J_Z\}^\mu, \quad (4.2.15)$$

where the brackets are defined exactly as (4.1.3)-(4.1.4). In other words, we have recovered the recursion that computes the Berends-Giele currents using the different perspective of the perturbiner method. Here, we able to write a recursion also for the coefficients $F^{\mu\nu}_P$

$$F^{\mu\nu}_P = k^\mu_P J^\nu_P - k^\nu_P J^\mu_P - \sum_{XY=P} \left(J^\mu_X J^\nu_Y - J^\mu_Y J^\nu_X \right). \quad (4.2.16)$$

Again, we use the same notation of the previous section, the sum goes over all deconcatenations of the word P into non-empty ordered words X and Y.

As pointed out in the previous section, the Berends-Giele currents are linked to tree-level amplitude with one leg off-shell. Here the only ingredient that we miss in order to compute amplitudes via (4.1.6) are boundary conditions for the rank-one currents J_i^μ. We can obtain these from imposing that the linear order in (4.2.5) are solutions of the free-field equations, i.e. the one-particle states $J_i^\mu = \varepsilon_i^\mu$ are the polarization vectors, which satisfy the transversality condition $k_i \cdot \varepsilon_i = 0$. In general, this property is wider and one can check that $k_P^\mu J_P^\mu = 0$ holds for any P. Now we are able to use the compact formula (4.1.6) for on-shell color-ordered amplitudes in Yang-Mills, and the equivalence of the perturbiner method for Yang-Mills and the Berends-Giele technique is complete [100].

Before conc luding this section, it is worth doing some extra remarks about the Berends-Giele recursion rederived in (4.2.15) in the perturbiner context. Using the equation of motion in the second line of (4.2.4), we have been able to obtain a recursion in terms of the J_P^μ coefficients only, and, in this way, we have recovered the Berends-Giele formula where the contributions of cubic and quartic Feynman vertices in Yang-Mills are manifest. However, this choice was arbitrary and in principle we could have constructed a recursion by combining the perturbiners for A^μ and $F^{\mu\nu}$. This can be achieved by using the first line of (4.2.4) as the generating function for the recursion relation, and insert there the perturbiners of for A^μ and $F^{\mu\nu}$. In this way, we obtain an alternative formulation [100, 107] of the recursion for perturbiner coefficient J_P^μ as

$$s_P J_P^\mu = \frac{1}{2} \sum_{P=XY} \Big((J_X \cdot k_Y) J_Y^\mu + J_X^\lambda F_Y^{\lambda\mu} - (X \leftrightarrow Y) \Big) , \tag{4.2.17}$$

while the recursion for the coefficient $F_P^{\mu\nu}$ keeps the same of (4.2.16). Formally, this representation doesn't affect the value of the coefficient J_P^μ. However, here the triple deconcatenation, encoding quartic vertices, present in (4.2.15) disappears [107]. This leaves us with simpler symmetry properties for the currents J_P^μ, that will be accurately exploited later in the manuscript.

4.2.2 *Color-Dressed Perturbiners*

In the previous section we have made use the potentiality of color-stripped perturbiners to obtain recursions for the coefficients of the perturbiner expansion in Yang-Mills theory that correctly reproduce the color-ordered Berends-Giele formula in (4.1.2). Now our focus is on introducing color degrees of freedom within the coefficients of the perturbiner expansion, obtaining the so-called *color-dressed perturbiner*. In this way, we aim to obtain color-dressed currents that in turn can be used to construct full Yang-Mills amplitudes. The idea of color-dressed perturbiners takes origin from the study of theories without color ordering, e.g. special Galileon or Born–Infeld theory, where we need to introduce a different notion of perturbiner expansion. Unlike the previous section, where the matrix products T^P were used to organize the perturbiner expansion, here we lack Lie algebra generators, so the plane waves $e^{k_P \cdot x}$ are the only tools to separate the terms in the expansion.

SCALAR THEORY. As a pedagogical example, we consider the case of a cubic scalar theory. Here the equation of motion reads as

$$\Box\varphi = \varphi^2. \tag{4.2.18}$$

We use the following perturbiner expansion:

$$\varphi(x) = \sum_P \varphi_P e^{k_P \cdot x} = \sum_i \varphi_i e^{k_i \cdot x} + \sum_{i<j} \varphi_{ij} e^{k_{ij} \cdot x} + \sum_{i<j<k} \varphi_{ijk} e^{k_{ijk} \cdot x} + \dots \,. \tag{4.2.19}$$

Here the sum goes over non-empty ordered words $P = p_1 p_2 \dots p_m$ with $p_1 < p_2 < \dots < p_m$ to avoid double counting. Now, plugging this inside the equation of motion in (4.2.18), we obtain:

$$\begin{aligned} \Box\varphi &= \left(\sum_i \varphi_i e^{k_i \cdot x} + \sum_{i<j} \varphi_{ij} e^{k_{ij} \cdot x} + \dots \right) \left(\sum_p \varphi_p e^{k_p \cdot x} + \sum_{p<q} \varphi_{pq} e^{k_{pq} \cdot x} + \dots \right) \\ &= \sum_i \sum_p \varphi_i \varphi_p e^{k_{ip} \cdot x} + \sum_{i<j} \sum_p \varphi_{ij} \varphi_p e^{k_{ijp} \cdot x} + \sum_i \sum_{p<q} \varphi_i \varphi_{pq} e^{k_{ipq} \cdot x} + \dots \,. \end{aligned} \tag{4.2.20}$$

In order to have a proper match with the left-hand side, we have to reorganize the sums in the expression above as

$$\sum_i \sum_p = \sum_{i<p} + \sum_{p<i} , \qquad \sum_{i<j} \sum_p = \sum_{i<j<p} + \sum_{i<p<j} + \sum_{p<i<j} . \tag{4.2.21}$$

Also, diagonal terms are not allowed in the left-hand side of (4.2.20), so we need to take perturbiners φ_i to be nilpotent, i.e., $\varphi_i^2 = 0$, so that these terms do not contribute. With these ingredients, we can use the plane waves $e^{k_p \cdot x}$, $e^{k_{pq} \cdot x}$ and so on, as a bookkeeping device to construct the following recursions from (4.2.20):

$$k_r^2 \varphi_r = 0, \qquad k_{rs}^2 \varphi_{rs} = \varphi_r \varphi_s + \varphi_s \varphi_r, \tag{4.2.22}$$

$$k_{rst}^2 \varphi_{rst} = \varphi_{rs} \varphi_t + \varphi_{rt} \varphi_s + \varphi_{st} \varphi_r + \varphi_r \varphi_{st} + \varphi_s \varphi_{rt} + + \varphi_t \varphi_{rs}. \tag{4.2.23}$$

This straightforwardly generalizes to the following recursion for perturbiners in cubic scalar theory:

$$\varphi_P = \sum \frac{1}{2 s_P} \sum_{P = Q \cup R} \varphi_Q \varphi_R. \tag{4.2.24}$$

As it is clear from the organization of the sums in (4.2.19)-(4.2.21), the sum over $P = Q \cup R$ is performed over distributions of the letters of the ordered non-empty word P into ordered words Q and R.

$$\begin{aligned} P &= 12 && \rightarrow \quad (Q, R) = (1, 2), (2, 1) \\ P &= 123 && \rightarrow \quad (Q, R) = (12, 3), (13, 2), (23, 1), (1, 23), (2, 13), (3, 12). \end{aligned} \tag{4.2.25}$$

Fixing the initial conditions $\varphi_i = 1$, we can give examples of the first orders of the recursion in (4.2.24):

$$\varphi_{12} = \frac{1}{s_{12}}, \qquad \varphi_{123} = \frac{1}{s_{123}}\left(\frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}}\right). \tag{4.2.26}$$

The amplitudes are computed with standard formula seen in the previous section, that is

$$A_n^{\varphi^3} = \lim_{k_n^2 \to 0} s_{12\ldots n-1}\varphi_{12\ldots n-1}\varphi_n. \tag{4.2.27}$$

YANG-MILLS THEORY. We now focus our attention again on Yang-Mills theory and we consider an alternative perturbiner formulation using the prescription just seen for the scalar theory. In this case the perturbiner ansatz reads

$$A^{\mu,a}(x) = \sum_P J_P^{\mu,a} e^{k_P \cdot x} \qquad F^{\mu\nu,a}(x) = \sum_P F_P^{\mu\nu,a} e^{k_P \cdot x}, \tag{4.2.28}$$

where, as previously anticipated, we don't explicit the dependence on the generators T^a of the gauge group, but we absorb the color degrees of freedom inside the perturbiner coefficients $J_P^{\mu,a}$ and $F_P^{\mu\nu,a}$. Exploiting the same construction as in the scalar theory case, we use the perturbiner ansatz (4.2.28) inside the equations (4.2.3)-(4.2.4) and derive the recursions

$$J_P^{\mu,a} = \frac{1}{2s_P} f_{bc}{}^a \sum_{P=Q\cup R} \left(J_Q^b \cdot k_R J_Q^{\mu,c} + J_Q^{\nu,b} F_R^{\nu\mu,c}\right) \tag{4.2.29}$$

$$F_P^{\mu\nu,a} = k_P^\mu J_P^{\nu,a} - k_P^\nu J_P^{\mu,a} - f_{bc}{}^a \sum_{P=Q\cup R} J_Q^{\mu,b} J_R^{\nu,c}, \tag{4.2.30}$$

where for convenience we have modified the notation of the structure constants $f_{ab}{}^c$ with respect to (3.1.3). The initial conditions now read $A_i^{\mu,a} = \varepsilon_i^\mu \delta^{a,a_i}$ where we make explicit the dependence on the adjoint index a_i of the $i-$th particle. For example, the color-dressed rank-two current reads

$$\begin{aligned} J_{12}^{\mu,a} &= \frac{f_{bc}{}^a}{2s_{12}}\left(\varepsilon_1 \cdot k_2 \delta^{ba_1} \varepsilon_2^\mu \delta^{ca_2} + \varepsilon_1^\nu \delta^{ba_1}(k_2^\nu \varepsilon_2^\mu - k_2^\mu \varepsilon_2^\nu)\delta^{ca_2} + (1 \leftrightarrow 2)\right) \\ &= \frac{f_{a_1a_2}{}^a}{s_{12}}\left(\varepsilon_1 \cdot k_2\, \varepsilon_2^\mu + \frac{1}{2}\varepsilon_1 \cdot \varepsilon_2\, k_1^\mu - (1 \leftrightarrow 2)\right). \end{aligned} \tag{4.2.31}$$

We can recognize the coefficient of $f_{a_1a_2}{}^a$ to be rank-two color-stripped current J_{12}^μ met in (4.1.7) and later in (4.2.12). This is actually a broader statement, as higher-rank currents are systematically related to those met in section 4.2.1. Indeed, it can be shown that the following relation holds:

$$J_{12\ldots m}^{\mu,a} = \mathcal{F}_{12\ldots m}^a J_{12\ldots m}^\mu + \text{perm.}(2,3,\ldots,m), \tag{4.2.32}$$

where we have introduced the contraction

$$\mathcal{F}_{12\ldots m}^a = f_{a_1a_2}{}^b f_{ba_3}{}^c \cdots f_{za_m}{}^a. \tag{4.2.33}$$

Note that the permutations in (4.2.32) are restricted to the set perm. $(2, 3, \ldots, m)$, while the leg labeled 1 is fixed. This can be achieved as a result of an extensive use of Jacobi identities

$$f_{a_1 a_2}{}^{b} f_{b a_3}{}^{c} + \mathrm{cyc}(a_1, a_2, a_3) = 0. \tag{4.2.34}$$

Using this type of relations, we can always modify the contractions among structure constant in a way consistent with (4.2.33). Full amplitudes are computed through

$$\begin{aligned} \mathcal{A}_n^{\mathrm{tree}} &= \lim_{k_n^2 \to 0} s_{12\ldots n-1} J^{\mu,a}_{12\ldots n-1} J^{a}_{n,\mu} \\ &= \sum_{\sigma \in S_{n-2}} \mathcal{F}^{a_n}_{1\sigma(23\ldots n-1)} \mathcal{A}_n \left(1, \sigma(2, \ldots, n-1), n\right). \end{aligned} \tag{4.2.35}$$

In this construction the amplitudes come out naturally organized into the multi-peripheral basis of partial amplitudes already encountered in (3.2.7).

As a final remark in this chapter, we consider again the recursion in (4.2.29). Exploiting the antisymmetry $f_{a_1 a_2}{}^{b} = -f_{a_2 a_1}{}^{b}$, we can rewrite the recursion as

$$J_P^{\mu,a} = \frac{1}{2 s_P} f_{bc}{}^{a} \sum_{\substack{P = Q \cup R \\ |Q| > |R|}} \left(J_Q^b \cdot k_R J_Q^{\mu,c} + J_Q^{\nu,b} F_R^{\nu\mu,c} - (Q, b \leftrightarrow R, c) \right). \tag{4.2.36}$$

This representation is formally equivalent to the one in (4.2.29). However, here we have made manifest the symmetry $(Q, b \leftrightarrow R, c)$, that will play an important role in the following, when a novel method for the construction of perturbiner coefficients will be presented.

Part II

GRAVITATIONAL CORRECTIONS TO SCALAR QED AMPLITUDES

5 COMPTON-LIKE SCATTERING WITH N PHOTONS AND ONE GRAVITON

In this chapter we compute tree-level scattering amplitudes for a scalar particle coupled to an arbitrary number n of photons and a single graviton. We employ the worldline formalism as the main tool to compute the irreducible part of the amplitude, where all the photons and the graviton are directly attached to the scalar line. Next, we derive a *tree replacement rule* to construct the reducible parts of the amplitude. We test our construction by verifying on-shell properties of the amplitude, e.g. transversality at $n = 2$ among the others. Results discussed in this chapter are published in [108].

5.1 THE WORLDLINE PATH MEETS ONE GRAVITON

In section 2.4 we have reviewed the worldline description of scalar propagators dressed with different background fields. In particular, we focused our attention on a Maxwell background field, and, expressing it as a sum of plane waves, we have obtained a worldline master formula (2.4.13) for the two-scalar n-photon scattering. In our work, we have attempted to push forward the current state of the art of the worldline description of dressed propagators through the introduction of *gravity*. Gravity can be included inside the worldline path integral by extending it to a curved spacetime, i.e. we introduce a background metric $g_{\mu\nu}(x(\tau))$ and we replace the Minkowski flat metric $\eta_{\mu\nu}$ with

$$\eta_{\mu\nu} \quad \longrightarrow \quad g_{\mu\nu}(x(\tau)). \tag{5.1.1}$$

The action that enters in the Feynman representation of the propagator (2.4.5) is now modified to

$$S = \int_0^T d\tau \left(\frac{1}{4} g_{\mu\nu}(x(\tau)) \dot{x}^\mu \dot{x}^\nu + ie\, \dot{x} \cdot A(x(\tau)) + \xi R \right), \tag{5.1.2}$$

where we have replaced the free kinetic part by the geodesic one and ξ takes into account the non-minimal coupling of the scalar particle to gravity. This worldline action for curved spacetime leads to considerable mathematical subtleties, as discussed in [109]. In particular, the need for general covariance requires a proper redefinition of the standard path integral measure:

$$\mathcal{D}x(\tau) \quad \longrightarrow \quad \mathcal{D}x(\tau) \prod_{0 \le \tau \le T} \sqrt{\det g_{\mu\nu}(x(\tau))}. \tag{5.1.3}$$

We can rewrite these metric factors in a more efficient way by exponentiating them. A convenient way of doing this is by introducing auxiliary fields (Lee-Yang ghosts)[110] which are respectively commuting a^μ and anticommuting b^μ, c^μ with vanishing boundary conditions. This is obtained in practice via

$$\mathcal{D}x(\tau)\prod_{0\leq\tau\leq T}\sqrt{\det g_{\mu\nu}(x(\tau))}=\mathcal{D}x(\tau)\int DaDbDc\, e^{-S_{gh}(x;a,b,c)}, \quad (5.1.4)$$

where the ghost action is given by

$$S_{gh}(x;a,b,c)=\int_0^T d\tau\frac{1}{4}g_{\mu\nu}(x)\left(a^\mu a^\nu+b^\mu c^\nu\right). \quad (5.1.5)$$

Thus the final version of the path integral in (2.4.5) in curved spacetime is

$$\left\langle\phi(x')\bar{\phi}(x)\right\rangle_{A,g}=\int_0^\infty dTe^{-m^2T}\int_{x(0)=x}^{x(T)=x'}DxDaDbDc \times e^{-\int_0^T d\tau\left(\frac{1}{4}g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)+ie\dot{x}\cdot A(x)+\bar{\xi}R\right)}, \quad (5.1.6)$$

where $\bar{\xi}=\xi-1/4$, and $-R/4$ is the counterterm which arises employing worldline dimensional regularization [111]. For simplicity, here we consider the worldline minimal coupling $\bar{\xi}=0$, which renders the graviton vertex operator linear in $\epsilon_{\mu\nu}$. As usual in perturbative quantum gravity, we can rewrite the metric as the combination of flat space and a small perturbation:

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+\kappa\, h_{\mu\nu}(x), \quad (5.1.7)$$

where κ is the gravitational coupling constant. The graviton can be introduced by specifying

$$h_{\mu\nu}(x)=\epsilon_{\mu\nu}\, e^{ik_0\cdot x}. \quad (5.1.8)$$

Using the redefinition of the particle paths introduced in (2.4.8), we can read off the graviton vertex operator

$$V_g[\epsilon,k_0]=e^{ik_0\cdot x+\frac{1}{T^2}(x'-x)\cdot\epsilon\cdot(x'-x)} \times\int_0^T d\tau\, e^{ik_0\cdot\left((x'-x)\frac{\tau}{T}+y\right)+\epsilon_{\mu\nu}\left(\frac{2}{T}(x'-x)^\mu\dot{y}^\nu+\dot{y}^\mu\dot{y}^\nu+a^\mu a^\nu+b^\mu c^\nu\right)}\Bigg|_{\text{lin}}, \quad (5.1.9)$$

where we have re-exponentiated the polarization tensor using the trick in (2.3.24) —compare with the photon vertex operator in (2.4.10) and remember that only the part linear in all the polarizations (ε's and ϵ) has to be retained. According to the ghost action in (5.1.5), the path integrals over the auxiliary fields is Gaussian. The auxiliary fields propagators are

$$\left\langle a^\mu(\tau)a^\nu(\tau')\right\rangle=2\delta^{\mu\nu}\delta(\tau,\tau'), \quad (5.1.10)$$
$$\left\langle b^\mu(\tau)c^\nu(\tau')\right\rangle=-4\delta^{\mu\nu}\delta(\tau,\tau'), \quad (5.1.11)$$

and we can use them in the computation of the path integrals when a graviton vertex operator is involved. Now we have all the ingredients to construct the master formula for the irreducible part of the tree-level scalar propagator with the insertion of n photons and one graviton:

$$D^{(n,1)}(p,p';\varepsilon_1,k_1,\ldots,\varepsilon_n,k_n;\epsilon,k_0)=(-ie)^n\left(-\frac{\kappa}{4}\right)\int_0^\infty dTe^{-m^2T}$$
$$\times\int d^4x\int d^4x' e^{i(p\cdot x+p'\cdot x')-\frac{1}{4T}(x-x')^2}\frac{1}{(4\pi T)^{\frac{D}{2}}}\left\langle\prod_{l=1}^{n}V_A[\varepsilon_l,k_l]\,V_g[\epsilon,k_0]\right\rangle, \tag{5.1.12}$$

where we have Fourier-transformed the two external scalar legs to bring the dressed propagator in momentum space[1].

As pointed out above, the formula (5.1.12) picks up only the *irreducible* contribution to the full amplitude. In Feynman diagrams language, this means that the associated diagrams cannot be divided into two self-standing contributions by cutting an internal photon or graviton line. However, it has been already realized by *Gertsenshtein* [112] in the 1960s that the Einstein-Maxwell theory must include a vertex for the interaction of two photons and one graviton at the tree level, in both vacuum and in the presence of an external constant magnetic field, which is also imposed by general covariance. Therefore one needs to compute some *reducible* contributions to the full n-photon one-graviton amplitude, in which a graviton is siting on an external photon. In other words the photons, on their way to the scalar propagator, are picking up a full-energy graviton, and these contributions are not included in our above master formula. This occurs because, in our worldline description, the gravitational and photon fields are treated as background fields, and their dynamics is not captured by (5.1.12). Specifically, our formula fails to adequately describe the interaction between photons and gravitons mentioned earlier, and its contribution has to be included somehow. In the next sections, we will provide a specific recipe to handle this task and obtain a useful master formula for the full Feynman amplitude. As a final remark, we mention that this difficulty is similar to what happens in flat space scalar QCD, for which a worldline approach to the computation of the n-gluon scalar propagator was studied in [113]: it yields the irreducible part of the n-gluon two-scalar amplitude. However, the non-abelian nature of the theory implies that, in order to compute the full amplitude and guarantee transversality on the gluon lines, the latter must be completed with reducible parts [71].

1 Note that, in our Fourier transform convention, both the external momenta are taken incoming.

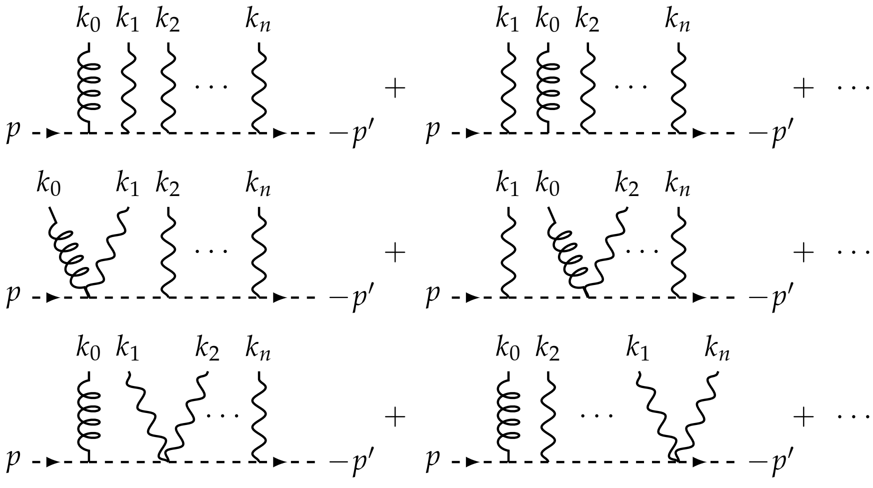


Fig. 5.1: The Feynman diagram representation for irreducible contributions to n-photon one-graviton amplitude. The diagrams in the second and third lines involve quartic vertices that, in the worldline approach, come from delta functions $\delta(\tau_i - \tau_j)$.

5.1.1 *Irreducible Part of the Amplitude*

In order to explicitly compute the irreducible part of the n-photon one-graviton amplitude (see figure 5.1) we find it convenient to parametrize the graviton polarization as

$$\epsilon_{\mu\nu} := \lambda_\mu \rho_\nu \,, \tag{5.1.13}$$

$$\varepsilon_{0\mu} := \lambda_\mu + \rho_\mu \,, \tag{5.1.14}$$

where, in (5.1.13), symmetrization between indices is implied. Such parametrization has to be understood as a simple bookkeeping device to combine photon and graviton insertions together[2]; at the end the graviton polarization is reconstructed from the term simultaneously linear in λ and ρ.

It is crucial now to point out that, with a single graviton insertion, the ghost contribution cancels against the singular part of the $\langle \dot{y}^\mu(\tau_0)\dot{y}^\nu(\tau_0)\rangle$ propagator that appears in the graviton vertex operator. Indeed, from the definitions (2.4.11)-(2.4.12), the double derivative ${}^{\bullet}\Delta^{\bullet}_{00}$ include a delta function with vanishing argument, that is precisely compensated by the ghost contribution. We can thus neglect the latter, provided we take $\langle \dot{y}^\mu(\tau_0)\dot{y}^\nu(\tau_0)\rangle \cong -\frac{2}{T}\delta^{\mu\nu}$ in the graviton sector. The graviton vertex operator can thus be written as

$$V_g[\epsilon, k_0] = e^{ik_0\cdot x + \frac{\varepsilon_0}{T}\cdot(x'-x)} \int_0^T d\tau_0\, e^{ik_0\cdot\left((x'-x)\frac{\tau_0}{T} + y(\tau_0)\right) + \varepsilon_0\cdot\dot{y}(\tau_0)}\Big|_{\text{lin.}\,\lambda,\rho}\,, \tag{5.1.15}$$

2 This trick looks like a reminiscence of the on-shell factorization of the graviton polarization tensor $\epsilon_{\mu\nu} = \varepsilon_\mu \bar{\varepsilon}_\nu$. However, in our convention the variables λ and ρ are simply bookkeeping devices and no on-shell condition is imposed.

which has the same form as the photon counterpart, with the only subtlety that the linear part in λ and ρ comes from the quadratic part in ε_0. After some straightforward algebra, we get the n-photon one-graviton scalar propagator

$$D^{(n,1)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n;\epsilon,k_0) = (-ie)^n\left(-\frac{\kappa}{4}\right)\int_0^\infty dTe^{-T(m^2+p'^2)}$$
$$\times\prod_{l=0}^{n}\int_0^T d\tau_l e^{(p'-p)\cdot\sum_{l=0}^n(-k_l\tau_l+i\varepsilon_l)+\sum_{l,l'=0}^n\left(k_l\cdot k_{l'}\Delta_{l-l'}-2i\varepsilon_l\cdot k_{l'}\dot{\Delta}_{l-l'}+\varepsilon_l\cdot\varepsilon_{l'}\ddot{\Delta}_{l-l'}\right)}\Big|_{\text{m.l.}} \tag{5.1.16}$$

where again we have to extract the linear contribution in all ε_l with $l=1,\dots,n$ and in λ and ρ. Here we recall that $\Delta_{l-l'}$, defined in (2.4.14), represents the translation-invariant part of the Green's function, and $\ddot{\Delta}_{0-0'}=0$ is implied. On the mass shell of the scalar particle, upon truncation of the external scalar lines, the formula (5.1.16) provides the irreducible contribution to the tree-level amplitude with n photons, one graviton and two scalars:

$$\begin{aligned}&\mathcal{D}^{(n,1)}_{irred}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n;\epsilon,k_0)\\&\quad=(p^2+m^2)(p'^2+m^2)D^{(n,1)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n;\epsilon,k_0).\end{aligned} \tag{5.1.17}$$

Let us single out some special cases of the previous formula which will be helpful in the following. We start by considering the case $n=0$, i.e. the *graviton-scalar vertex*:

$$\begin{aligned}&D^{(0,1)}(p,p';\epsilon,k_0)\\&\quad=\left(-\frac{\kappa}{4}\right)\int_0^\infty dT\,e^{-T(m^2+p'^2)}\int_0^T d\tau_0\,e^{(p'-p)\cdot(-k_0\tau_0+i\varepsilon_0)}\Big|_{\text{m.l.}}.\end{aligned} \tag{5.1.18}$$

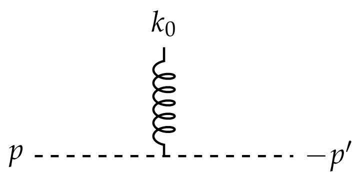
Using momentum conservation, that in the three-point scattering reads as $p+p'+k_0=0$, the latter can be reduced to

$$D^{(0,1)}(p,p';\epsilon,k_0)=\frac{\kappa}{4}(p'-p)^\mu\epsilon_{\mu\nu}(p'-p)^\nu\frac{1}{(p'^2+m^2)(p^2+m^2)}, \tag{5.1.19}$$

and, upon truncation, leads to the amplitude

$$\mathcal{D}^{(0,1)}(p,p';\epsilon,k_0)=\frac{\kappa}{4}(p'-p)^\mu\epsilon_{\mu\nu}(p'-p)^\nu. \tag{5.1.20}$$

This correctly reproduces the scalar-graviton vertex in field theory, depicted as the Feynman diagram



.

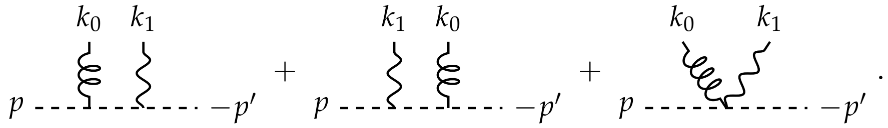
For $n = 1$, the irreducible part of the gravitational photoproduction amplitude can be easily obtained from

$$\begin{aligned} D^{(1,1)}(p,p';\varepsilon_1,k_1;\epsilon,k_0) &= (-ie)\left(-\frac{\kappa}{4}\right) \\ &\times \int_0^\infty dT e^{-T(m^2+p'^2)} \int_0^T d\tau_0 \int_0^T d\tau_1\, e^{(p'-p)\cdot(-k_0\tau_0 - k_1\tau_1 + i\varepsilon_0 + i\varepsilon_1)} \\ &\times e^{k_0\cdot k_1|\tau_0-\tau_1| + i(\varepsilon_1\cdot k_0 - \varepsilon_0\cdot k_1)\mathrm{sgn}(\tau_0-\tau_1) + 2\varepsilon_0\cdot\varepsilon_1\delta(\tau_0-\tau_1)}\Big|_{\mathrm{m.l.}}, \end{aligned} \tag{5.1.21}$$

where the $\delta(\tau_0 - \tau_1)$ part yields the seagull diagram, whereas the time ordered parts ($\tau_0 > \tau_1$ and $\tau_0 < \tau_1$) yield the diagrams where photon and graviton are singly emitted by the scalar line with the respective orderings. Imposing the on-shell conditions on the external legs, we get the following irreducible contribution to the scattering amplitude

$$\begin{aligned} \mathcal{D}^{(1,1)}_{irred}(p,p';\varepsilon_1,k_1;\epsilon,k_0) &= (p'^2+m^2)(p^2+m^2)\widetilde{D}^{(1,1)}(p,p';\varepsilon_1,k_1;\epsilon,k_0) \\ &= e\kappa\Big[(p-p')\cdot\epsilon\cdot\varepsilon_1 + \frac{\varepsilon_1\cdot p' p\cdot\epsilon\cdot p}{p\cdot k_0} - \frac{\varepsilon_1\cdot p\, p'\cdot\epsilon\cdot p'}{p\cdot k_1}\Big]. \end{aligned} \tag{5.1.22}$$

In Feynman diagram language, this contribution to the full amplitude corresponds to



Finally, let us consider the irreducible part of the two-photon one-graviton amplitude. Using the worldline approach, we are able to obtain a quite compact representation of this contribution. We report here the final result, which reads

$$\begin{aligned} \mathcal{D}^{(2,1)}_{irred}(p,p';\varepsilon_1,k_1,\varepsilon_2,k_2;\epsilon,k_0) &= \kappa e^2\Bigg\{2(\varepsilon_1\epsilon\varepsilon_2) + \Bigg(-2\frac{\varepsilon_1\cdot\varepsilon_2\,(p'\epsilon p')}{m^2+(p'+k_0)^2} \\ &+ 2\frac{\varepsilon_1\cdot p\,(\varepsilon_2\epsilon(p'-p-k_1))}{m^2+(p+k_1)^2} + 4\frac{(p'\epsilon p')\,\varepsilon_1\cdot(p+k_2)\,\varepsilon_2\cdot p}{(m^2+(p+k_2)^2)(m^2+(p'+k_0)^2)} \\ &+ 2\frac{\varepsilon_2\cdot p\,(\varepsilon_1\epsilon(p'-p-k_2))}{m^2+(p+k_2)^2} + 4\frac{(p'\epsilon p')\,\varepsilon_2\cdot(p+k_1)\,\varepsilon_1\cdot p}{(m^2+(p+k_1)^2)(m^2+(p'+k_0)^2)} \\ &+ 4\frac{((p+k_1)\epsilon(p'+k_2))\,\varepsilon_1\cdot p\,\varepsilon_2\cdot p'}{(m^2+(p+k_1)^2)(m^2+(p'+k_2)^2)} + \Big(p\leftrightarrow p'\Big)\Bigg)\Bigg\}, \end{aligned} \tag{5.1.23}$$

where the notation $(a\epsilon b) := a_\mu\epsilon^{\mu\nu}b_\nu$ has been introduced. The interested reader will find details of the computation in the appendix A.

5.1.2 *Reducible Part of the Amplitude*

In this section we tackle the problem of the reducible part of the amplitude. As formerly pointed out, the external graviton can couple

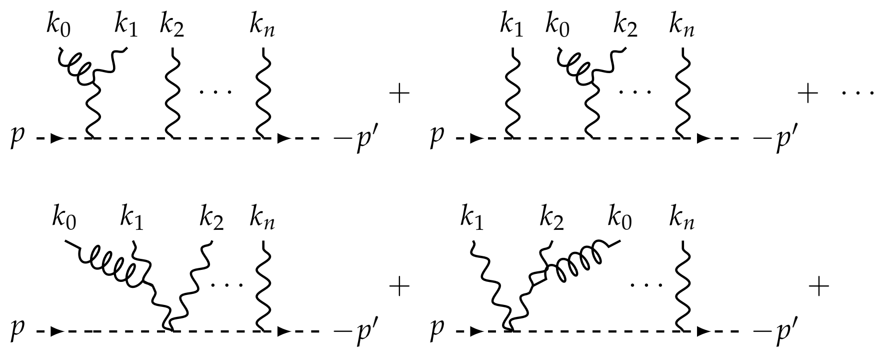


FIG. 5.2: The Feynman diagram representation of the reducible contribution to n-photon one-graviton amplitude.

directly to the scalar line, as reproduced by the formula described in (5.1.16) for the irreducible part of the scattering amplitude, but it can also couple to the photon lines, giving rise to reducible contributions —see figure 5.2 for the diagrammatic representation of these contributions. From a field theory view point the photon-graviton interaction is encoded in the vertex

$$\mathcal{V}[A,h] = \frac{\kappa}{2}\int d^4x h_{\mu\nu}T^{\mu\nu} = \frac{\kappa}{2}\int d^4x\, h_{\mu\nu}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\delta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{5.1.24}$$

which, using the tracelessness of the on-shell graviton, leads to the following tree-level amplitude between two photons and one graviton (to be called $\Gamma_{g\gamma\gamma}$)

$$\Gamma_{g\gamma\gamma}[\varepsilon,k,\varepsilon',k';\epsilon,k_0] = \kappa\Big[(k\epsilon k)\varepsilon\cdot\varepsilon' + (\varepsilon\epsilon\varepsilon')k\cdot k_0 - (\varepsilon\epsilon k)k\cdot\varepsilon' - (k\epsilon\varepsilon')\varepsilon\cdot k_0\Big]. \tag{5.1.25}$$

Here we have used the transversality conditions $k_{0\mu}\epsilon^{\mu\nu} = k_\mu\varepsilon^\mu = 0$ and conservation law $k' = -(k+k_0)$. The vertex in (5.1.25) can be used to construct the reducible part of the amplitude with the following recipe. Let us start from the one-photon two-scalar amplitude

$$\mathcal{D}^{(1)}(p,p';\varepsilon',k') = e\varepsilon'\cdot(p'-p), \tag{5.1.26}$$

which can be easily read off from (2.4.15). It yields the reducible part of the one-photon one-graviton two-scalar amplitude by simply multiplying expressions (5.1.25) and (5.1.26), and using the replacement rule

$$\varepsilon'^\alpha\varepsilon'^\beta \longrightarrow \frac{\delta^{\alpha\beta}}{k'^2}, \tag{5.1.27}$$

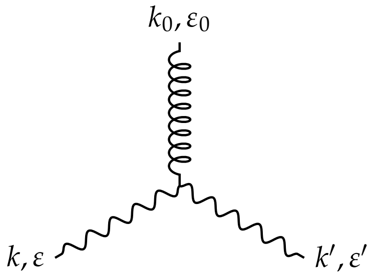
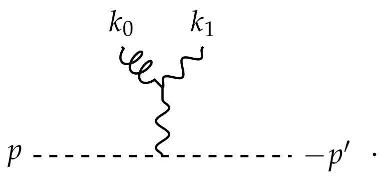


FIG. 5.3: $g\gamma\gamma$ vertex given by (5.1.25).

which is the photon propagator in the Feynman gauge. By renaming photon polarization and momentum as ε_1 and k_1, we thus get

$$\begin{aligned}&\mathcal{D}^{(1,1)}_{red}(p,p';\varepsilon_1,k_1;\epsilon,k_0)\\&= e\kappa(p'-p)_\mu \frac{\varepsilon_1^\mu(k_1\epsilon k_1)+(\varepsilon_1\epsilon)^\mu k_1\cdot k_0 - k_1^\mu(\varepsilon_1\epsilon k_1)-(k_1\epsilon)^\mu\varepsilon_1\cdot k_0}{2k_1\cdot k_0}.\end{aligned} \tag{5.1.28}$$

In the Feynman diagrams language, this corresponds to the contribution depicted as



For efficiency purposes, we can rewrite the amplitude (5.1.28) in a more compact form as

$$\mathcal{D}^{(1,1)}_{red}(p,p';\varepsilon_1,k_1;\epsilon,k_0) = \mathcal{D}^{(1)}(p,p';v_1,k_1+k_0), \tag{5.1.29}$$

where, starting from (5.1.26), we have defined the replacements

$$\begin{aligned}\varepsilon_1^\mu &\to v_1^\mu := \kappa\frac{\varepsilon_1^\mu(k_1\epsilon k_1)+(\varepsilon_1\epsilon)^\mu k_1\cdot k_0 - k_1^\mu(\varepsilon_1\epsilon k_1)-(k_1\epsilon)^\mu\varepsilon_1\cdot k_0}{2k_1\cdot k_0},\\ k_1^\mu &\to k_1^\mu + k_0^\mu.\end{aligned} \tag{5.1.30}$$

It is worth noticing here that the top line of (5.1.30) is transversal upon the replacement $\varepsilon_1 \to k_1$. This property is directly linked to the gauge transformation properties of the amplitude (5.1.25) and will be accurately exploited later in the manuscript. The rules introduced in (5.1.30) can be obviously extended to the n-photon two-scalar amplitude constructed above in (2.4.15), which thus yields the following reducible contribution

$$\begin{aligned}&\mathcal{D}^{(n,1)}_{red}(p,p';\varepsilon_1,k_1,\ldots,\varepsilon_n,k_n;\epsilon,k_0)\\&\qquad= \sum_{i=1}^n \mathcal{D}^{(n)}(p,p';\varepsilon_1,k_1,\ldots,v_i,k_i+k_0,\ldots\varepsilon_n,k_n)\,.\end{aligned} \tag{5.1.31}$$

Thus, the full tree-level amplitude with n photons, one graviton and two scalars reads

$$\mathcal{D}^{(n,1)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n;\epsilon,k_0) = \mathcal{D}^{(n,1)}_{irred}(p,p';\varepsilon_1,k_1,\dots;\epsilon,k_0) + \sum_{l=1}^{N}\mathcal{D}^{(n)}(p,p';\varepsilon_1,k_1,\dots,v_l,k_l+k_0,\dots,\varepsilon_n,k_n)\,, \quad (5.1.32)$$

where $\mathcal{D}^{(n,1)}_{irred}$ is given by eq. (5.1.16) truncated on the external scalar lines. For completeness, let us give the explicit expression for the reducible part of the amplitude with two photons. Let us start from the scalar Compton scattering amplitude, which can be easily obtained from (2.4.15) and reads

$$\mathcal{D}^{(2)}(p,p';\varepsilon_1,k_1,\varepsilon_2,k_2) = (-ie)^2\Bigg\{2\varepsilon_1\cdot\varepsilon_2 - \left(\frac{\varepsilon_1\cdot(p'-p-k_2)\varepsilon_2\cdot(p'-p+k_1)}{(p'+k_1)^2+m^2} + (1\leftrightarrow 2)\right)\Bigg\}\,. \quad (5.1.33)$$

By applying the replacement rule given above, we get

$$\mathcal{D}^{(2,1)}_{red}(p,p';\varepsilon_1,k_1,\varepsilon_2,k_2;\epsilon,k_0) = \mathcal{D}^{(2)}(p,p';v_1,k_1+k_0,\varepsilon_2,k_2) + \mathcal{D}^{(2)}(p,p';\varepsilon_1,k_1,v_2,k_2+k_0)\,. \quad (5.1.34)$$

Here, as an example, we report the value of the first contribution on the right hand side of the formula above:

$$\mathcal{D}^{(2)}(p,p';v_1,k_1+k_0,\varepsilon_2,k_2) = (-ie)^2\Bigg\{2\,v_1\cdot\varepsilon_2 - \frac{\varepsilon_2\cdot(p'-p+k_1+k_0)}{m^2+(p'+k_1+k_0)^2}\,v_1\cdot(p'-p-k_2) - \frac{\varepsilon_2\cdot(p'-p-k_1-k_0)}{m^2+(p'+k_2)^2}\,v_1\cdot(p'-p+k_2)\Bigg\}. \quad (5.1.35)$$

Below, in section 5.2, we will test the master formula (5.1.32) by checking the on-shell transversality conditions in the photon lines and graviton line. However, to conclude the present section, let us briefly review a factorization property that links graviton-photon amplitudes to photon amplitudes.

5.1.3 *On-Shell Factorization Property for the Graviton Photoproduction Amplitude*

For a mixed scattering with one graviton and one photon, i.e. for the graviton photoproduction process, the full amplitude involving both the irreducible contributions (5.1.22) and the reducible contribution (5.1.28), displays a very interesting factorization property in terms of the corresponding QED Compton amplitude. It can be easily seen by adopting the on-shell decomposition

$$\epsilon^{\mu\nu} \to \epsilon^{\mu}\epsilon^{\nu}\,, \tag{5.1.36}$$

which yields,

$$\begin{aligned}&\mathcal{D}^{(1,1)}(p,p';\varepsilon_1,k_1;\epsilon,k_0) = \\ &\frac{\kappa e}{k_0\cdot k_1}\Big[\epsilon\cdot p'\,k_0\cdot p - \epsilon\cdot p k_0\cdot p'\Big]\Big[\frac{\varepsilon_1\cdot p'\,\epsilon\cdot p}{p'\cdot k_1} + \frac{\varepsilon_1\cdot p\,\epsilon\cdot p'}{p'\cdot k_0} + \epsilon\cdot\varepsilon_1\Big].\end{aligned} \tag{5.1.37}$$

Using a more compact notation, we can write

$$\mathcal{D}^{(1,1)}(p,p';\varepsilon_1,k_1;\epsilon,k_0) = H\,\mathcal{D}^{(2)}(p,p';\epsilon,k_0,\varepsilon_1,k_1), \tag{5.1.38}$$

where

$$H = -\frac{\kappa}{2e}\frac{\epsilon\cdot p'\,k_0\cdot p - \epsilon\cdot p k_0\cdot p'}{k_0\cdot k_1}\,, \tag{5.1.39}$$

and $\mathcal{D}^{(1,1)}(p,p';\varepsilon_1,k_1;\epsilon,k_0)$ and $\mathcal{D}^{(2)}(p,p';\epsilon,k_0,\varepsilon_1,k_1)$ are respectively the on-shell versions of the graviton photoproduction amplitude and of the scalar QED Compton scattering given in equation (5.1.33). This factorization property was already studied in [114–118], and seems to be universal for four-body amplitudes with massless gauge bosons. However, beyond the four-particle level, such factorization property is not expected to hold due to the lack of enough conservation laws [114].

5.2 WARD IDENTITIES AND ON-SHELL TRANSVERSALITY

In this section we investigate the relevant Ward identities for the n-photon one-graviton amplitudes in scalar QED, making use of the formalism introduced in the previous section. The Ward identities are relations among scattering amplitudes of a given theory, and their existence ensures that the physical quantities preserve the symmetries of the theory. In general, Ward identities are derived by considering the infinitesimal transformation of a quantum field under a symmetry transformation and then using the fact that the vacuum state is invariant under this symmetry. This leads to relationships between vacuum expectation values of operators in the theory, which constrains and the form of the scattering amplitudes.

In the present discussion, the dressed propagator in (5.1.6) is covariant upon $U(1)$ gauge transformations and invariant under diffeomorphisms. The former is described by

$$\Big\langle\phi(x')\bar{\phi}(x)\Big\rangle_{A,g} \to \Big\langle\tilde{\phi}(x')\tilde{\bar{\phi}}(x)\Big\rangle_{\tilde{A},\tilde{g}} = e^{ie(\alpha(x)-\alpha(x'))}\Big\langle\phi(x')\bar{\phi}(x)\Big\rangle_{\tilde{A},\tilde{g}}. \tag{5.2.1}$$

Using that $\delta A_\mu = \partial_\mu \alpha$, the infinitesimal part of (5.2.1) becomes the electromagnetic Ward identity generator

$$\Big[\partial^y_\mu \frac{\delta}{\delta A_\mu(y)} + ie(\delta(y-x) - \delta(y-x'))\Big]\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g} = 0, \quad (5.2.2)$$

which holds off-shell. In momentum space, it yields an infinite set of Ward identities

$$\begin{aligned}&\mathcal{D}^{(n,1)}(p,p';-ik,k,\varepsilon_1,k_1,\ldots;\epsilon,k_0) = \\ &-ie\Big[\mathcal{D}^{(n-1,1)}(p+k,p';\varepsilon_1,k_1,\ldots;\epsilon,k_0) - \mathcal{D}^{(n-1,1)}(p,p'+k;\varepsilon_1,k_1,\ldots)\Big]\end{aligned} \quad (5.2.3)$$

which can be easily tested with the special cases singled out in section 5.1.1. On the other hand, on the scalar mass-shell the contact terms present in (5.2.2) do not have the correct pole structure and drop out upon truncation, whereas the first term leads to the on-shell transversality condition

$$\mathcal{D}^{(N,1)}_{irred}(p,p';\varepsilon_1,k_1,\ldots,-ik_l,k_l,\ldots;\epsilon,k_0) = 0\,, \quad (5.2.4)$$

which holds for any photon line. Moreover, the gauge invariance of scalar QED, in flat as well as in curved space, ensures that the full amplitude is transversal, i.e. the reducible part of the amplitude must result separately transversal. Indeed, given that (5.1.30) vanishes upon the replacement $\varepsilon_1 \to k_1$, this is enough to prove the transversality of the reducible part of the amplitude (5.1.2), as it can easily be checked for the expression (5.1.35).

Under infinitesimal diffeomorphisms, $x^\mu \to x^\mu - \xi^\mu(x)$, the dressed propagator transforms as

$$\begin{aligned}\Big\langle \tilde{\phi}(x')\tilde{\bar{\phi}}(x)\Big\rangle_{\tilde{A},\tilde{g}} &= \Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g} \\ &+ \int d^4y\, \xi^\mu(y)\big(\delta^{(4)}(y-x)\partial_\mu + \delta^{(4)}(y-x')\partial'_\mu\big)\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g}.\end{aligned} \quad (5.2.5)$$

However, using the worldline representation (5.1.6), one can as well get

$$\begin{aligned}\Big\langle \tilde{\phi}(x')\tilde{\bar{\phi}}(x)\Big\rangle_{\tilde{A},\tilde{g}} &= \Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g} + \int d^4y\Big[2\nabla_\mu\xi_\nu(y)\frac{\delta}{\delta g_{\mu\nu}(y)} \\ &+ (\xi^\alpha\partial_\alpha A_\mu(y) + \partial_\mu\xi^\alpha A_\alpha(y))\frac{\delta}{\delta A_\mu(y)}\Big]\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g},\end{aligned} \quad (5.2.6)$$

which, after some straightforward algebra and combining with expression (5.2.5), can be reduced to

$$\Big[-\nabla^y_\mu \frac{2g_{\nu\alpha}}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}(y)} + \frac{1}{\sqrt{g}}\Big(F_{\alpha\mu}\frac{\delta}{\delta A_\mu(y)} \\ - \delta^{(4)}(y-x)\partial_\alpha - \delta^{(4)}(y-x')\partial'_\alpha\Big)\Big]\Big\langle \phi(x')\bar{\phi}(x)\Big\rangle_{A,g} = 0, \quad (5.2.7)$$

which is the diffeomorphism Ward identity generator. Once again there are contact terms which drop out on the scalar particle mass-shell. The two left-over terms both contribute on-shell and thus the irreducible part of the n-photon one-graviton amplitude is not, by itself, transversal on the graviton line; rather it fulfills, even on-shell, an inhomogeneous Ward identity. Recalling the definition for the field strength tensor $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$ for each photon leg, and an *effective* photon polarization vector

$$\tilde{\varepsilon}_i^\mu = \kappa f_i^{\mu\nu}\xi_\nu, \quad (5.2.8)$$

this identity can be written concisely as follows (the same identity holds for the closed-loop case [119])

$$\mathcal{D}^{(n,1)}_{irred}(p,p';\varepsilon_1,k_1,\ldots;k_0\xi,k_0) \\ -\sum_{i=1}^{n}\mathcal{D}^{(n,0)}_{irred}(p,p';\varepsilon_1,k_1,\ldots,\tilde{\varepsilon}_i,k_i+k_0,\ldots,\varepsilon_n,k_n) = 0. \quad (5.2.9)$$

Here we have written the transformation of the (transverse traceless) polarization tensor as

$$\epsilon_{\mu\nu} \;\to\; \epsilon_{\mu\nu} + k_{0\mu}\xi_\nu + k_{0\nu}\xi_\mu, \quad k_0\cdot\xi = k_0^2 = 0, \quad (5.2.10)$$

and used $k_0\xi$ just a shortcut notation for the symmetrized product of the two vectors. However, the full amplitude is expected to be transversal on-shell, i.e.,

$$\mathcal{D}^{(n,1)}(p,p';\varepsilon_1,k_1,\ldots,\varepsilon_n,k_n;k_0\xi,k_0) = 0\ . \quad (5.2.11)$$

Using the *tree replacement* rule (5.1.30), it can be seen quite easily how this comes about: applying the transformation (5.2.10) to v_i^μ, the result can be written as

$$v_i^\mu \to -\tilde{\varepsilon}_i^\mu + \kappa\frac{k_0\cdot f_i\cdot\xi}{2k_i\cdot k_0}(k_0+k_i)^\mu\,. \quad (5.2.12)$$

The second term in brackets will drop out when inserted into the photon amplitude because of the transversality in the photon polarizations. The first one can be interpreted by combining it with the definition of the reducible contributions given in (5.1.31). In particular, we obtain

$$\begin{aligned}\mathcal{D}^{(n,1)}_{red}&(p,p';\varepsilon_1,k_1,\ldots,\varepsilon_n,k_n;k_0\xi,k_0)\\&=\sum_{i=1}^{n}\mathcal{D}^{(n,0)}_{irred}(p,p';\varepsilon_1,k_1,\ldots,-\tilde{\varepsilon}_i,k_i+k_0,\ldots\varepsilon_n,k_n).\end{aligned}\quad(5.2.13)$$

Exploiting the linearity of the amplitude with respect to the photon polarizations, it is easy now to convince ourselves that this contribution corresponds exactly to the second term in (5.2.9), and (5.2.11) is verified on the spot.

5.3 FINAL REMARKS

In this chapter we have described a novel worldline approach to the computation of the tree level scattering amplitudes associated to a scalar line coupled to electromagnetism and gravity. In particular, we have introduced a convenient parametrization for the graviton polarization and a replacement rule, which made it simple to calculate full amplitudes involving any number of photons and a single graviton. With this method, we have confirmed the on-shell factorization property for the one-graviton one-photon amplitude and the on-shell transversality of amplitudes with up to two photons and one graviton.

In the following chapter, we will make use of our novel technique to compute the one-loop correction to the graviton-scalar vertex in QED. Our worldline approach is off-shell, so the correction to the vertex can be calculated by examining the two-photon, one-graviton amplitude and sewing the photons together. To ensure a completely off-shell result for the vertex, we will need to slightly revisit the substitution rule in (5.1.30).

6 ONE-LOOP RADIATIVE CORRECTION TO THE GRAVITON VERTEX IN SCALAR QED

In this chapter we compute the one-loop QED radiative correction to the graviton coupled to a scalar particle in any covariant gauge. We use the worldline formalism to analyze the scattering amplitude of any number of off-shell photons and one graviton that are connected to a scalar propagator, focusing on the master formula derived in the previous chapter. In particular, to compute the one-loop correction to the graviton-scalar vertex (referred to as the *gss* vertex) in QED, we first re-derive the off-shell amplitude for two-photon and one-graviton interactions, then use the sewing procedure to calculate the radiative correction. There are three irreducible diagrams and two reducible ones. The first set of diagrams can be computed directly from the master formula following the sewing procedure, while the remaining diagrams can be derived using the previously obtained tree replacement rules.

6.1 THE GRAVITON GOES IN THE OFF-SHELL REALM

In chapter 5 we have studied the inclusion of a single graviton within the n-photon scalar propagator in the context of the worldline formalism. We have shown that in this approach the irreducible contributions are given directly by the master formula (5.1.16) and a nontrivial new replacement rule was introduced in (5.1.30) for the inclusion of the reducible contributions from lower irreducible pure photonic amplitudes. Here we want to make use of some properties of the one-graviton n-photon master formula for the scalar propagator. In particular we focus our analysis on the one-graviton two-photon amplitude, whose computation has been sketched in appendix A. Since our formalism holds off-shell, the idea is to sew the two external photons together (in any covariant gauge) and compute the one-loop scalar QED correction to the graviton-matter (spinless) vertex. Based on our knowledge, this has been studied only for the fermion propagator in [120] and repeated in [121]. As a starting point, we reconsider the the master formula (5.1.16), that we report again here for convenience

$$D^{(n,1)}(p,p';\varepsilon_1,k_1,\ldots,\varepsilon_n,k_n;\epsilon,k_0) = (-ie)^n\left(-\frac{\kappa}{4}\right)\int_0^\infty dT e^{-T(m^2+p'^2)}\times$$
$$\prod_{l=0}^{n}\int_0^T d\tau_l e^{(p'-p)\cdot\sum_{l=0}^n(-k_l\tau_l+i\varepsilon_l)+\sum_{l,l'=0}^n\left(k_l\cdot k_{l'}\Delta_{l-l'}-2i\varepsilon_l\cdot k_{l'}\dot{\Delta}_{l-l'}+\varepsilon_l\cdot\varepsilon_{l'}\ddot{\Delta}_{l-l'}\right)}\Big|_{\text{m.l.}}. \tag{6.1.1}$$

We remark the fact that this formula holds off-shell and therefore can be used to construct higher order corrections, as it will be discussed in what follows. The elegance of this key formula lies in its ability to merge all $(n+1)!$ arrangements (including the graviton) which may not appear very relevant at the tree-level, but it becomes evident in higher loops where it results in an integral representation of nontrivial sums of diagrams, as seen in the two- and three-loop corrections to the Euler-Heisenberg Lagrangian [122–124]. As it has been already pointed out in the previous chapter and in detail in [108], the master formula above describes only the irreducible part of the n-photon one-graviton scalar propagator. Previous chapter efficiently accounted for reducible contributions to the amplitude through the use of replacement rules, as defined in equation (5.1.30). These rules have to be applied directly to the n-photon scalar propagator, for which the formula (2.4.13) may be used —see (5.1.31) for the formal expression of the reducible contributions to the full amplitude.

Before we demonstrate how to obtain the radiative corrections to the gss vertex using our method, we need to make a few slight but crucial adjustments to the rules outlined in (5.1.30). Indeed, the rules in question were developed based on the tree-level amplitude of two photons and one on-shell graviton, as outlined in equation (5.1.25). The objective now is to calculate the radiative corrections to the gss vertex, but this time with the graviton being off-shell. Specifically, we begin again with the connection between the graviton and the electromagnetic energy-momentum tensor

$$\mathcal{V}[A,h] = \frac{\kappa}{2}\int d^4x h_{\mu\nu}T^{\mu\nu} = \frac{\kappa}{2}\int d^4x\, h_{\mu\nu}\Big(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\delta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\Big). \tag{6.1.2}$$

Now we require the two-photon one-graviton amplitude in (5.1.25) to be modified to[1]

$$\begin{aligned}\Gamma_{g\gamma\gamma}[\varepsilon,k;\varepsilon',k';\epsilon,k_0] \\ &= \kappa\Big[(k\epsilon(k+k_0))\varepsilon\cdot\varepsilon' - (\varepsilon\epsilon(k+k_0))k\cdot\varepsilon' + (\varepsilon\epsilon\varepsilon')k\cdot(k+k_0) \\ &\quad - (k\epsilon\varepsilon')\varepsilon\cdot(k+k_0)\Big] - \frac{\kappa}{2}\,\mathrm{tr}(\epsilon)\Big[k\cdot(k+k_0)\,\varepsilon\cdot\varepsilon' - \varepsilon\cdot(k+k_0)\,\varepsilon'\cdot k\Big],\end{aligned} \tag{6.1.3}$$

where only the energy-momentum conservation $k' = -(k+k_0)$ has been used, while all the external photon and graviton lines have been kept off-shell. As we have seen in section 5.1.2, the reducible part of the the n-photon one-graviton amplitude is obtained by sewing one of the photons in the above $g\gamma\gamma$ vertex in all possible way to the photons in the master formula (2.4.13). This fact has lead us to introduce the aforementioned replacement rule, which effectively handles all reducible

1 We are using a shorthand notation for the contraction of the graviton polarization tensor with vectors, specifically $a\cdot\epsilon\cdot b$ is denoted as $(a\epsilon b)$.

contributions for the n-photon one-graviton amplitude. Now our goal is to create a new replacement rule which will allow us to describe off-shell external photons and gravitons. However, the generalization in (6.1.3) is just the first step in our construction process. Indeed, it should also be noted that the replacement rule in (5.1.30) was obtained by sewing the photons through the propagator in the Feynman gauge, as seen in (5.1.27). As we are now working on calculating the one-loop correction to the gss vertex in a general covariant gauge, we will make use of the photon propagator in the form

$$\varepsilon'^{\mu}\varepsilon'^{\nu} \to \frac{\delta^{\mu\nu}}{k'^2} - (1-\xi)\frac{k'^{\mu}k'^{\nu}}{k'^4}, \tag{6.1.4}$$

where ξ is the gauge parameter, and $\xi = 0$ and $\xi = 1$ correspond to Landau and Feynman gauges respectively. Following the same procedure outlined in the previous chapter, we redefine the replacement rule to be applied to the pure QED amplitude. Specifically, the effective replacements for the momentum and polarization of the i-th photon, where the graviton is supposed to sit on, are now given by:

$$\begin{aligned}
\varepsilon_i^{\mu} \to \bar{v}_i^{\mu} &:= \kappa\Big[\varepsilon_i^{\alpha}(k_i\epsilon(k_i+k_0)) - k_i^{\alpha}(\varepsilon_i\epsilon(k_i+k_0)) + (\varepsilon_i\epsilon)^{\alpha}k_i\cdot(k_i+k_0)\\
&\quad - (k_i\epsilon)^{\alpha}\varepsilon_i\cdot(k_i+k_0) - \frac{1}{2}\mathrm{tr}\,(\epsilon)\Big(k_i\cdot(k_i+k_0)\varepsilon_i^{\alpha} - \varepsilon_i\cdot(k+k_0)k_i^{\alpha}\Big)\Big]\\
&\quad \times\Big(\frac{\delta_{\alpha}^{\mu}}{(k_i+k_0)^2} - (1-\xi)\frac{(k_i+k_0)^{\mu}(k_i+k_0)_{\alpha}}{(k_i+k_0)^4}\Big),\\
k_i^{\mu} &\to k_i^{\mu} + k_0^{\mu}.
\end{aligned} \tag{6.1.5}$$

where the effective polarization $\bar{v}_i^{\mu}$ is different from the one defined in (5.1.30), in order to take into account the off-shellness of the external graviton. However, this is not the end of the story, as it is easy to figure out that the above $\bar{v}_i^{\mu}$ has some interesting features when it is contracted with the gauge dependent part of the photon propagator. In particular, we can easily check that

$$\begin{aligned}
&(k_i+k_0)_{\alpha}\Big[\varepsilon_i^{\alpha}(k_i\epsilon(k_i+k_0)) - k_i^{\alpha}(\varepsilon_i\epsilon(k_i+k_0)) + (\varepsilon_i\epsilon)^{\alpha}k_i\cdot(k_i+k_0)\\
&- (k_i\epsilon)^{\alpha}\varepsilon_i\cdot(k_i+k_0) - \frac{1}{2}\mathrm{tr}\,(\epsilon)\Big(k_i\cdot(k_i+k_0)\varepsilon_i^{\alpha} - \varepsilon_i\cdot(k+k_0)k_i^{\alpha}\Big)\Big] = 0.
\end{aligned} \tag{6.1.6}$$

This means that the whole gauge dependent part in the replacement (6.1.5) is identically vanishing. This fact should not be a surprise at all, as can be easily understood looking at (6.1.2). Indeed, the $g\gamma\gamma$ vertex is formulated in terms of products of field strength tensors. These are manifestly gauge invariant quantities, thus gauge invariance on the photon lines is guaranteed at the level of the $g\gamma\gamma$ vertex. We have briefly mentioned this property in section 5.2, where transversality of the scalar propagator dressed with n photons and one graviton has been investigated. In summary, it is possible to set the variable $\xi = 1$ within the effective polarizations for the reducible contributions without affecting the overall results, and redefine

$$\bar{v}_i^\mu = \frac{\kappa}{(k_i+k_0)^2}\Big[\varepsilon_i^\mu(k_i\epsilon(k_i+k_0)) - k_i^\mu(\varepsilon_i\epsilon(k_i+k_0)) + (\varepsilon_i\epsilon)^\mu k_i\cdot(k_i+k_0) - (k_i\epsilon)^\mu\varepsilon_i\cdot(k_i+k_0) - \frac{1}{2}\mathrm{tr}\,(\epsilon)\Big(k_i\cdot(k_i+k_0)\varepsilon_i^\mu - \varepsilon_i\cdot(k+k_0)k_i^\mu\Big)\Big]. \tag{6.1.7}$$

Similarly, the isomorphism invariance also applies to the other photon leg in the expression above, as it can easily be seen that $\bar{v}_i^\mu$ is vanishing when ε_i is changed to k_i.

Now we have all the ingredients to write down the final formula for the n-photon one-graviton amplitude:

$$\mathcal{D}^{(n,1)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_n,k_n;\epsilon,k_0) = \mathcal{D}^{(n,1)}_{irred}(p,p';\varepsilon_1,k_1,\dots;\epsilon,k_0) + \sum_{l=1}^{N}\mathcal{D}^{(n)}(p,p';\varepsilon_1,k_1,\dots,\bar{v}_l,k_l+k_0,\dots,\varepsilon_n,k_n), \tag{6.1.8}$$

which, in contrast to (5.1.32), holds **fully** off-shell for all external photons and the graviton. Now we have all the ingredients to focus our attention on the two-photon one-graviton dressed propagator and use it properly for the one-loop QED correction to the graviton-matter coupling.

6.2 ONE-LOOP CORRECTION TO THE GRAVITON-MATTER VERTEX

As mentioned in the previous, our objective is to compute the one-loop QED correction to the gss vertex by utilizing the fully off-shell two-photon one-graviton amplitude $\mathcal{D}^{(2,1)}$, derived from (6.1.8). The details of such computation have been already discussed in appendix A. For convenience, we report here the master formula for the construction of the irreducible part of the amplitude:

$$\begin{aligned}\mathcal{D}^{(2,1)}_{irred}(p,p';\dots;\epsilon,k_0) &= (-ie)^2\left(-\frac{\kappa}{4}\right)\int_0^\infty dT\, e^{-T(m^2+p'^2)}\prod_{i=0}^{2}\int_0^T d\tau_i\\ &\times e^{(p'-p)\cdot(-k_0\tau_0-k_1\tau_1-k_2\tau_2+i\varepsilon_0+i\varepsilon_1+i\varepsilon_2)}\, e^{k_0\cdot k_1|\tau_0-\tau_1|+k_0\cdot k_2|\tau_0-\tau_2|+k_1\cdot k_2|\tau_1-\tau_2|}\\ &\times e^{i(\varepsilon_1\cdot k_0-\varepsilon_0\cdot k_1)\mathrm{sgn}(\tau_0-\tau_1)+i(\varepsilon_2\cdot k_0-\varepsilon_0\cdot k_2)\mathrm{sgn}(\tau_0-\tau_2)+i(\varepsilon_2\cdot k_1-\varepsilon_1\cdot k_2)\mathrm{sgn}(\tau_1-\tau_2)}\\ &\times e^{2\left[\varepsilon_0\cdot\varepsilon_1\delta(\tau_0-\tau_1)+\varepsilon_0\cdot\varepsilon_2\delta(\tau_0-\tau_2)+\varepsilon_1\cdot\varepsilon_2\delta(\tau_1-\tau_2)\right]}\Big|_{\text{m.l.}}.\end{aligned} \tag{6.2.1}$$

To keep track of the various contributions to the amplitude, we find it useful to consider the number of delta functions present. As a useful organizational tool, we adopt the notation of $\mathcal{Q}_i$ to denote[2] the portion

2 Here we don't follow the notation in appendix A: $\mathcal{Q}_i$ are the contributions to the integrand, while $\mathcal{E}_i$ are the contribution to the amplitude after integration.

of the integrand that contains i delta functions. Accordingly, we define the multilinear expansion of (6.2.1) as

$$\mathcal{Q}^{(2,1)} = \mathcal{Q}_0^{(2,1)} + \mathcal{Q}_1^{(2,1)} + \mathcal{Q}_2^{(2,1)}. \tag{6.2.2}$$

The irreducible contribution to the two-photon one-graviton amplitude can now be expressed as

$$\begin{aligned}\mathcal{D}^{(2,1)}_{irred}(p,p';k_1,\varepsilon_1,k_2,\varepsilon_2;k_0,\epsilon) &= (-ie)^2\left(-\frac{\kappa}{4}\right)\int_0^\infty dT\, e^{-T(m^2+p'^2)}\\ &\times \int_0^T d\tau_0 \int_0^T d\tau_1 \int_0^T d\tau_2\, e^{(p'-p)\cdot(-k_0\tau_0-k_1\tau_1-k_2\tau_2)}\\ &\times e^{k_0\cdot k_1|\tau_0-\tau_1|+k_0\cdot k_2|\tau_0-\tau_2|+k_1\cdot k_2|\tau_1-\tau_2|}\left(\mathcal{Q}_0^{(2,1)}+\mathcal{Q}_1^{(2,1)}+\mathcal{Q}_2^{(2,1)}\right).\end{aligned} \tag{6.2.3}$$

By making use of the parametrization of the graviton polarization tensor $\epsilon_{\mu\nu}$ and its connection to $\varepsilon_{0\mu}$ as outlined in (5.1.13)-(5.1.14), from (6.2.1) we can readily deduce

$$\begin{aligned}\mathcal{Q}_0^{(2,1)} =& \varepsilon_1\cdot(p'-p+k_0\sigma_{01}-k_2\sigma_{12})\,\varepsilon_2\cdot(p'-p+k_0\sigma_{02}+k_1\sigma_{12})\\ &\times(p'-p-k_1\sigma_{01}-k_2\sigma_{02})\cdot\epsilon\cdot(p'-p-k_1\sigma_{01}-k_2\sigma_{02}),\\ \mathcal{Q}_1^{(2,1)} =&\\ &-4\,\varepsilon_2\cdot(p'-p+k_0\sigma_{02}+k_1\sigma_{12})\,\varepsilon_1\cdot\epsilon\cdot(p'-p-k_1\sigma_{01}-k_2\sigma_{02})\,\delta_{01}\\ &-4\,\varepsilon_1\cdot(p'-p+k_0\sigma_{01}-k_2\sigma_{12})\,\varepsilon_2\cdot\epsilon\cdot(p'-p-k_1\sigma_{01}-k_2\sigma_{02})\,\delta_{02}\\ &-2\,(p'-p-k_1\sigma_{01}-k_2\sigma_{02})\cdot\epsilon\cdot(p'-p-k_1\sigma_{01}-k_2\sigma_{02})\,\varepsilon_1\cdot\varepsilon_2\delta_{12},\\ \mathcal{Q}_2^{(2,1)} =& 8\,\varepsilon_1\cdot\epsilon\cdot\varepsilon_2\,\delta_{01}\delta_{02}.\end{aligned} \tag{6.2.4}$$

where we have introduced the compact notation $\sigma_{ij} = \mathrm{sgn}(\tau_i - \tau_j)$ and $\delta_{ij} = \delta(\tau_i - \tau_j)$. By evaluating the parameter integrals for different orderings of τ_0, τ_1 and τ_2 and truncating the external scalar lines, we easily get the expression for the irreducible amplitude presented in (5.1.23) and computed in detail in (A10). However, for the future computations, we find it more beneficial to make use of the unintegrated form of the amplitude, and we will proceed from (6.2.4). As pointed out earlier in this chapter, the reducible contribution to the two-photon one-graviton amplitude is obtained from the pure off-shell two-photon amplitude where we apply suitable replacement rules (6.1.7). It is convenient to report here the expression of the pure two-photon amplitude as obtained from (2.4.13) by fixing $n = 2$, i.e.

$$\begin{aligned}&\mathcal{D}^{(2)}(p,p';k_1,\varepsilon_1,k_2,\varepsilon_2) =\\ &(-ie)^2\int_0^\infty dT\, e^{-T(m^2+p'^2)}\int_0^T d\tau_1\int_0^T d\tau_2\, e^{(p'-p)\cdot(-k_1\tau_1-k_2\tau_2)}\\ &\times e^{k_1\cdot k_2|\tau_1-\tau_2|+i\varepsilon_1\cdot(p'-p-k_2\mathrm{sgn}(\tau_1-\tau_2))+i\varepsilon_2\cdot(p'-p+k_1\mathrm{sgn}(\tau_1-\tau_2))}e^{2\varepsilon_1\cdot\varepsilon_2\delta(\tau_1-\tau_2)}\Big|_{\mathrm{m.l.}}.\end{aligned} \tag{6.2.5}$$

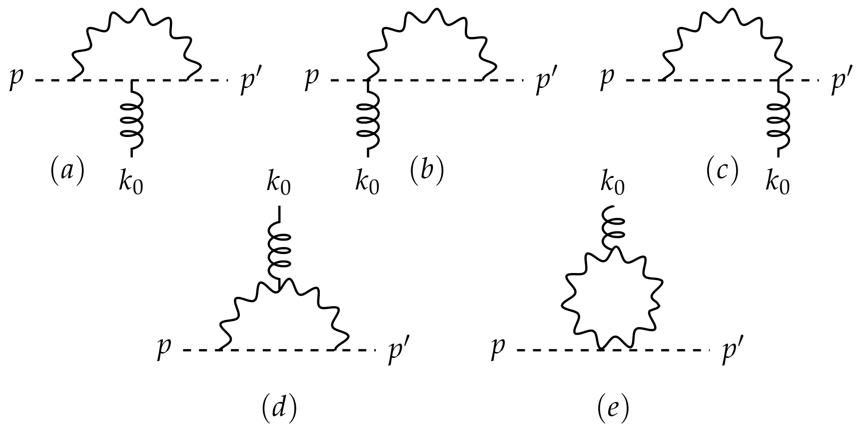


FIG. 6.1: Feynman diagrams contributing to the one-loop correction to the gss vertex after sewing two external photons from $\mathcal{D}^{(2,1)}$. The diagrams (a), (b) and (c) come from the irreducible part of $\mathcal{D}^{(2,1)}$, the last two (d) and (e) from the reducible one. All momenta are incoming by convention.

We establish the multilinear expansion $\mathcal{Q}^{(2)}$ by separating terms with either zero or one delta function, $\mathcal{Q}_0^{(2)}$ and $\mathcal{Q}_1^{(2)}$ respectively. The corresponding values can easily be computed as

$$\begin{aligned}\mathcal{Q}_0^{(2)} &= -\,\varepsilon_1 \cdot (p' - p - k_2 s_{12})\, \varepsilon_2 \cdot (p' - p + k_1 s_{12}) \\ \mathcal{Q}_1^{(2)} &= 2\,\varepsilon_1 \cdot \varepsilon_2\, \delta_{12},\end{aligned} \tag{6.2.6}$$

The overall result is

$$\begin{aligned}\mathcal{D}^{(2)}(p, p'; k_1, \varepsilon_1, k_2, \varepsilon_2) = (-ie)^2 \int_0^\infty dT\, e^{-T(m^2+p'^2)} \int_0^T d\tau_1 \int_0^T d\tau_2 \\ \times\, e^{(p'-p)\cdot(-k_1\tau_1 - k_2\tau_2) + k_1\cdot k_2|\tau_1-\tau_2|} \left(\mathcal{Q}_0^{(2)} + \mathcal{Q}_1^{(2)}\right) .\end{aligned} \tag{6.2.7}$$

The reducible part of the two-photon one-graviton amplitude is obtained from the above expression by applying the replacement rules in (6.1.7). In the computation of a scattering amplitude, such replacement rules can be further simplified by imposing on-shell conditions for the graviton ($\epsilon^\mu_\mu = k_0^\mu \epsilon_{\mu\nu} = 0$). Performing the parameter integrals in the different orderings for τ_1 and τ_2 and truncating the external scalar lines, one gets the reducible part of the two-photon one-graviton amplitude as expressed in (5.1.35). At the present time, we prefer to keep the amplitude in its unintegrated form, and use the replacement rules (6.1.7) to maintain the off-shell nature of the result.

We have now obtained all the tools required to calculate the radiative correction to the gss vertex with efficiency. To move forward, it is important to identify all the possible diagrams that can be constructed by combining external photons within $\mathcal{D}^{(2,1)}$. In the computation of

FIG. 6.2: Diagram (a) is obtained after sewing two external photons from the part of the integrand of $\mathcal{D}^{(2,1)}$ with zero delta functions involved.

the parameter integrals, we consider the special ordering $\tau_1 \geq \tau_0 \geq \tau_2$: as the photons have to be combined together, this ordering ensures to avoid redundancies in the computation of the loop-correction. By applying Feynman rules and straightforward combinatorial arguments, we identify five unique diagrams that play a role in the loop-correction. These diagrams are depicted in figure 6.1. The first three diagrams (a), (b) and (c) come from the irreducible part of $\mathcal{D}^{(2,1)}$, while and the last two (d) and (e) from the reducible one, after the sewing procedure takes place.

In the following portion of this section, we will outline the corrections associated to the different diagrams in a general covariant gauge of the internal photon, by identifying the corresponding contributions inside (6.2.3)-(6.2.7).

6.2.1 *Diagram (a)*

Diagram (a) represents the first contribution that we can obtain by sewing two photons in the irreducible part of the amplitude $\mathcal{D}^{(2,1)}$, once the ordering of the legs is fixed. This diagram comes from the $\mathcal{Q}_0^{(2,1)}$ part in (6.2.4), i.e. the part of the integrand that involves zero delta functions. Pictorially, we can represent the sewing mechanism that brings to the construction of diagram (a) as in figure 6.2. It is easy to convince ourselves that the diagram on the left side comes from the integrand that involves zero delta functions in $\mathcal{D}^{(2,1)}$, and where the ordering $\tau_1 \geq \tau_0 \geq \tau_2$ of the external legs has been fixed. More in detail, we can read off this contribution from (6.2.3):

$$\mathcal{D}_a(p, p'; \varepsilon_1, k_1, \varepsilon_2, k_2; \epsilon, k_0) = (-ie)^2 \left(-\frac{\kappa}{4}\right) \int_0^\infty dT e^{-T(m^2+p'^2)} \int_0^T d\tau_1$$
$$\times \int_0^{\tau_1} d\tau_0 \int_0^{\tau_0} d\tau_2 \, e^{-\tau_0 k_0 \cdot l_0 - \tau_1 k_1 \cdot l_1 - \tau_2 k_2 \cdot l_2} \, (l_0 \epsilon l_0)(\varepsilon_1 \cdot l_1)\,(\varepsilon_2 \cdot l_2), \quad (6.2.8)$$

where for convenience we have introduced the new quantities

$$l_0 = p' - p + k_1 - k_2\,,$$
$$l_1 = p' - p - k_0 - k_2\,,$$

$$l_2 = p' - p + k_0 + k_1. \tag{6.2.9}$$

It is now time to apply the sewing procedure on the external photons in (6.2.8). This process is inspired by the method used in [81] and can be summarized as:

1. Sew the two external photons together using the photon propagator in an arbitrary covariant gauge. Making use of (6.1.4), we replace the polarizations ε_1 and ε_2 with
$$\varepsilon_1^\mu \varepsilon_2^\nu \longrightarrow \frac{\delta^{\mu\nu}}{\ell^2} - (1-\xi)\frac{\ell^\mu \ell^\nu}{\ell^4}, \tag{6.2.10}$$
where ℓ^μ has to be understood as the momentum running inside the loop.

2. Replace the momenta k_1^μ and k_2^μ of the external photons with
$$k_1^\mu \longrightarrow \ell^\mu, \qquad k_2^\mu \longrightarrow -\ell^\mu. \tag{6.2.11}$$

3. Include the D-dimensional integral over the loop momentum ℓ, namely
$$\int \frac{d^D\ell}{(2\pi)^D}\,. \tag{6.2.12}$$

Following the sewing procedure outlined above, we obtain the contribution of the diagram (a) from (6.2.8) as

$$\begin{aligned}\widetilde{\Gamma}_a(p,p';\epsilon) = &-\frac{\kappa}{4}(-ie)^2 \int_0^\infty dT e^{-T(m^2+p'^2)} \int \frac{d^D\ell}{(2\pi)^D} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_0 \\ &\times \int_0^{\tau_0} d\tau_2\, (l_0\,\epsilon\, l_0) l_1^\mu l_2^\nu \left(\frac{\delta_{\mu\nu}}{\ell^2} - \frac{(1-\xi)\ell_\mu \ell_\nu}{\ell^4}\right) e^{-\tau_0 k_0\cdot l_0 - \tau_1 \ell\cdot l_1 + \tau_2 \ell\cdot l_2},\end{aligned} \tag{6.2.13}$$

where the notation $\widetilde{\Gamma}_a(p,p';\epsilon)$ indicates the untruncated[3] result for diagram (a). Note that the sewing procedure is responsible for the modification of the coefficients (6.2.9) to

$$\begin{aligned} l_0 &= p' - p + 2\ell\,, \\ l_1 &= p' - p - k_0 + \ell\,, \\ l_2 &= p' - p + k_0 + \ell\,. \end{aligned} \tag{6.2.14}$$

We focus now our attention on the gauge-independent part of (6.2.13), or equivalently the Feynman gauge result where $\xi = 1$ is fixed. The calculation of the vertex correction for diagram (a) is done as follows: first, we perform the integrals over the parameters τ_i, then we examine the product of the coefficients l_0, l_1, and l_2 and express it in terms

3 The integral (6.2.13) contains the propagators of the external scalars which can be truncated by multiplying by their respective inverse. These propagators can be easily regained if the vertex is part of a larger diagram.

of different powers of the loop momentum ℓ —different powers of ℓ in the numerator correspond to different master integrals. After truncating the external scalars, diagram (a) in the Feynman gauge corresponds to

$$\Gamma^{\mu\nu}_{a,\mathrm{Feyn}}(p,p';\epsilon) = (m^2+p'^2)(m^2+p^2)\widetilde{\Gamma}^{\mu\nu}_{a,\mathrm{Feyn}}(p,p';\epsilon), \tag{6.2.15}$$

where

$$\begin{aligned}
&\Gamma^{\mu\nu}_{a,\mathrm{Feyn}}(p,p';\epsilon) = \\
&\quad \frac{e^2\kappa}{4}\Big\{-4p'\cdot p\, q^\mu q^\nu J^{(0)}[p,p'] + \left(2\,q_\rho q^\mu\, q^\nu - 16p'\cdot p\, q^\mu\, \delta^\nu_\rho\right) J^{(1)\rho}[p,p'] \\
&\quad + \left(8q_\rho q^\mu - 16p'\cdot p\delta^\mu_\rho\right) J^{(2)\rho\nu}[p,p'] + 8q_\rho J^{(3)\rho\mu\nu}[p,p'] + q^\mu q^\nu K^{(0)}[p,p'] \\
&\quad + 4\,q^\mu K^{(1)\nu}[p,p'] + 4K^{(2)\mu\nu}[p,p']\Big\}.
\end{aligned} \tag{6.2.16}$$

In the above expression, the graviton polarization tensor $\epsilon^{\mu\nu}$ has been removed and the quantity $q^\mu = p'^\mu - p^\mu$ has been introduced. The shorthand notations $J^{(i)}$ and $K^{(i)}$ are used for Feynman integrals, the full expressions for which are given in appendix C. We want to emphasize that the vertex $\Gamma^{\mu\nu}_{a,\mathrm{Feyn}}$ has to be symmetric in the indices μ and ν, as expected after the contraction with the external polarization tensor $\epsilon^{\mu\nu}$. Thus, where not manifest, symmetrization is implicit in (6.2.16). This property is assumed to hold also for the diagrams that will be computed in the following of the section.

In a complete analogous way, we can consider the gauge dependent part of (6.2.13). Taking inspiration from a technique used in [81], we point out that the gauge-dependent part of (6.2.8) can be simply rewritten as a second derivative over the exponential, i.e.

$$\begin{aligned}
&(\xi-1)(l_0\,\epsilon\,l_0)\frac{l_1\cdot\ell\, l_2\cdot\ell}{\ell^4} e^{-\tau_0 k_0\cdot l_0 - \tau_1\ell\cdot l_1 + \tau_2\ell\cdot l_2} = \\
&\qquad\qquad -(\xi-1)(l_0\,\epsilon\,l_0)\frac{1}{\ell^4}\frac{\partial^2}{\partial\tau_1\partial\tau_2} e^{-\tau_0 k_0\cdot l_0 - \tau_1\ell\cdot l_1 + \tau_2\ell\cdot l_2}.
\end{aligned} \tag{6.2.17}$$

This suggests that the gauge-dependent part can be easily obtained by performing the calculation in the Feynman gauge, which in turn reduces the number of independent integrals required for the computation of the one-loop correction to the scalar QED vertex (see [81] for more details about the calculation). Using this prescription to solve the parameter integrals in terms of total derivatives, we compute the truncated gauge dependent contribution to be

$$\begin{aligned}
&\Gamma^{\mu\nu}_{a,\xi}(p,p') = \\
&\quad \frac{e^2\kappa}{4}(1-\xi)\Big\{(m^2+p'^2)\left(q^\mu q^\nu H^{(0)}[p'] + 4\,q^\nu H^{(1)\mu}[p'] + 4H^{(2)\mu\nu}[p']\right) \\
&\quad + (m^2+p^2)\Big[q^\mu q^\nu H^{(0)}[-p] + 4\,q^\nu H^{(1)\mu}[-p] + 4H^{(2)\mu\nu}[-p]\Big]\Big\} \\
&\quad - (m^2+p'^2)(m^2+p^2)\left(q^\mu q^\nu L^{(0)} + 4\,q^\mu L^{(1)\nu} + 4\,L^{(2)\mu\nu}\right),
\end{aligned} \tag{6.2.18}$$

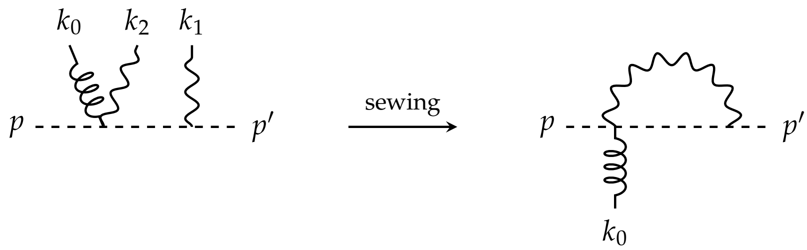


FIG. 6.3: Diagram (b) is obtained after sewing two external photons from the part of the integrand of $\mathcal{D}^{(2,1)}$ proportional to $\delta(\tau_0 - \tau_2)$.

where again we remark that the integrals of type $H^{(i)}$ and $L^{(i)}$ are listed in appendix C for convenience. Combining the expressions (6.2.16)-(6.2.18), we obtain the diagram (a) in any covariant gauge to be

$$\Gamma_a^{\mu\nu}(p,p') = \Gamma_{a,\text{Feyn}}^{\mu\nu}(p,p') + \Gamma_{a,\xi}^{\mu\nu}(p,p'). \tag{6.2.19}$$

6.2.2 *Diagrams $(b)-(c)$*

The next contribution we want to examine is the diagram (b) in figure 6.1. It is obtained from the part of $\mathcal{D}^{(2,1)}$ that includes a factor of $\delta(\tau_0 - \tau_2)$, as pictorially represented in figure 6.3. This diagram can be thought of as representing the merging of one photon and the graviton into the scalar line at the same point. At the tree level, this contribution is represented by the following expression

$$\begin{aligned}\mathcal{D}_b[p,p';k_1,\varepsilon_1;k_2,\varepsilon_2;k_0,\epsilon] &= -(\kappa e^2)^2 \int_0^\infty dT\, e^{-T(m^2+p'^2)} \\ \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_0\, & e^{-\tau_0(k_0+k_2)\cdot(l_0+k_2)-\tau_1 k_1\cdot l_1} \left(\varepsilon_2\cdot\epsilon\cdot(l_0+k_2)\right)\varepsilon_1\cdot l_1,\end{aligned} \tag{6.2.20}$$

where l_0 and l_1 can be read off from (6.2.9) and

$$l_0 + k_2 = p' - p + k_1. \tag{6.2.21}$$

In (6.2.20) again the delta function $\delta(\tau_0 - \tau_2)$ has been used to eliminate the integral over the parameter τ_2. Now we apply the sewing procedure on the external photons, as outlined in the previous computation. After sewing, diagram (b) reads as

$$\begin{aligned}\widetilde{\Gamma}_b(p,p';\epsilon) &= -e^2\kappa \int_0^\infty dT e^{-T(m^2+p^2)} \int \frac{d^D\ell}{(2\pi)^D} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_0 \\ \times\, e^{-\tau_0(k_0-\ell)\cdot(p'-p+\ell)-\tau_1\ell\cdot l_1} & \left(\epsilon\cdot(p'-p+\ell)\right)^\mu l_1^\nu \left(\frac{\delta_{\mu\nu}}{\ell^2} - \frac{(1-\xi)\ell_\mu\ell_\nu}{\ell^4}\right),\end{aligned} \tag{6.2.22}$$

where the coefficients take now the values (6.2.14). Upon evaluating the integral of the parameters, and separating the integrand into different powers of the loop momentum ℓ, the Feynman gauge diagram (b) can be expressed as

$$\begin{aligned}\Gamma^{\mu\nu}_{b,\,\mathrm{Feyn}}(p,p') =& \\ -(e^2\kappa)&\left(2p'^{\mu}q^{\nu}I^{(0)}[p'] + (2p'^{\mu}+q^{\mu})I^{(1)\nu}[p'] + I^{(2)\mu\nu}[p']\right), \end{aligned} \tag{6.2.23}$$

The above expression has had the graviton polarization tensor ϵ removed and the truncation of the scalar lines has been carried out. We have defined a new Feynman integral, $I^{(i)}$, which can be found in appendix C.

The calculation of the gauge-dependent portion in (6.2.22) is done in a similar manner to that of diagram (a). Again, we notice that the gauge dependent part can be (partially) rewritten in terms of derivatives as

$$\begin{aligned}(\xi-1)\left(\ell\epsilon(p'-p+\ell)\right)\frac{\ell\cdot l_1}{\ell^4}e^{-\tau_0(k_0-\ell)\cdot(p'-p+\ell)-\tau_1\ell\cdot l_1} =& \\ -(\xi-1)\left(\ell\epsilon(p'-p+\ell)\right)\frac{1}{\ell^4}\frac{\partial}{\partial\tau_1}&e^{-\tau_0(k_0-\ell)\cdot(p'-p+\ell)-\tau_1\ell\cdot l_1}. \end{aligned} \tag{6.2.24}$$

By taking advantage of this property, the gauge-dependent part (after truncation) can be computed as

$$\Gamma^{\mu\nu}_{b,\xi}(p,p') = -(e^2\kappa)(1-\xi)(m^2+p'^2)\left(q^{\nu}H^{(1)\mu}[p'] + H^{(2)\mu\nu}[p']\right). \tag{6.2.25}$$

Combining (6.2.23) and (6.2.25), we obtain the full expression of diagram (b) as

$$\Gamma^{\mu\nu}_{b}(p,p') = \Gamma^{\mu\nu}_{b,\,\mathrm{Feyn}}(p,p') + \Gamma^{\mu\nu}_{b,\,\xi}(p,p'). \tag{6.2.26}$$

It is now time to focus on diagram (c). However, It is easy to convince ourselves that we do not need to calculate the diagram from scratch. Indeed, it is clear from the diagrammatic representation in figure 6.1 that diagrams (b) and (c) are linked by Bose symmetry. In particular, we can obtain diagram (c) from diagram (b) simply by exchanging

$$p \;\longleftrightarrow\; p'. \tag{6.2.27}$$

In addition, we can also reverse the direction of the momentum ℓ that is flowing inside the loop through the replacement

$$\ell \;\longrightarrow\; -\ell. \tag{6.2.28}$$

This is useful in order to be consistent with the direction of the loop momentum. Note that the transformation (6.2.28) leaves the measure of the loop integral invariant, so the overall sign of the integrals only depends on the degree of ℓ in the numerator [4]. In detail, the vertex correction associated to diagram (c) reads in the Feynman gauge as

$$\Gamma^{\mu\nu}_{c,\,\mathrm{Feyn}}(p,p') = \\ -(e^2\kappa)\left(-2p^\mu q^\nu I^{(0)}[-p] + (-2p^\mu + q^\mu)I^{(1)\nu}[-p] + I^{(2)\mu\nu}[-p]\right), \tag{6.2.29}$$

while the gauge dependent part is

$$\Gamma^{\mu\nu}_{c,\xi}(p,p') = -(e^2\kappa)(1-\xi)(m^2+p^2)\left(q^\nu H^{(1)\mu}[-p] + H^{(2)\mu\nu}[-p]\right). \tag{6.2.30}$$

The combination of the two above contributions gives the full expression of the correction associated to diagram (c), that is

$$\Gamma^{\mu\nu}_{c}(p,p') = \Gamma^{\mu\nu}_{c,\,\mathrm{Feyn}}(p,p') + \Gamma^{\mu\nu}_{c,\xi}(p,p'). \tag{6.2.31}$$

6.2.3 *Diagram (d)*

In addition to the diagrams (a), (b), and (c) previously mentioned, there are two more diagrams that contribute to the one-loop QED correction to the gss vertex. These are diagrams (d) and (e) shown in figure 6.1, which come from reducible tree-level amplitudes after the sewing process. In the rest of this section, we will face the computation of these diagrams in the Feynman gauge. As previously mentioned, the effective polarizations constructed with the $g\gamma\gamma$ vertex in (6.1.2) are not affected by gauge transformations. This implies that diagrams (d) and (e) are gauge-invariant, thus they can be calculated in the Feynman gauge without any loss of generality for our purposes.

We start our analysis by considering diagram (d), which originates from the reducible part of the amplitude for two photons and one graviton as illustrated in Figure 6.4. In particular, by fixing the time ordering $\tau_1 > \tau_2$ in the pure two-photon amplitude in (6.2.1) and using the replacement rules described in (6.1.7), the tree-level contribution is given by

$$\mathcal{D}_d[p,p';k_1,\varepsilon_1;k_2,\varepsilon_2;k_0,\epsilon] = (\kappa e^2)\int_0^\infty dT e^{-T(m^2+p'^2)} \\ \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, e^{-\tau_1(k_1+k_0)\cdot(l_0-k_1)-\tau_2 k_2\cdot l_2}\,(\varepsilon_2\cdot l_2)\,\bar{v}_1\cdot(l_0-k_1), \tag{6.2.32}$$

where the off-shell effective polarization $\bar{v}^\mu_i$ is defined in (6.1.7), and the coefficients l_i are the same of (6.2.9). It is convenient now to modify the expression of the effective polarization $\bar{v}^\mu_i$ to

$$\bar{v}^\mu_i \equiv \frac{\kappa}{(k_i+k_0)^2}\,\varepsilon_{i\nu}\,\bar{v}^{\mu\nu}_i, \tag{6.2.33}$$

4 It is easy to check that the net effect of the transformations (6.2.27)-(6.2.28) corresponds to the replacement $p \leftrightarrow -p'$.

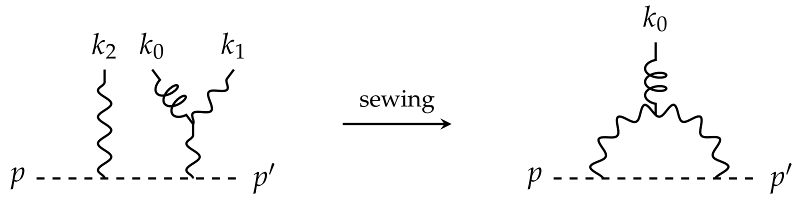


FIG. 6.4: Diagram (d) is obtained by sewing the external photons from the reducible diagram on the left, which is included in $\mathcal{D}^{(2,1)}$. This diagram is derived from the pure photon amplitude $\mathcal{D}^{(2)}$ using the standard replacement rules.

where the polarization ε_i and the propagator associated with the internal edge have been removed from $\bar{v}_i^{\mu}$ in order to make the sewing procedure more efficient. In particular, the rank-two vertex $\bar{v}_i^{\mu\nu}$ reads

$$\begin{aligned}\bar{v}_i^{\mu\nu} &= k_i\epsilon(k_i+k_0)\delta^{\mu\nu} + \epsilon^{\mu\nu}k_i\cdot(k_i+k_0) - ((k_i+k_0)\epsilon)^{\nu}k_i^{\mu}\\ &- (k_i\epsilon)^{\mu}(k_i+k_0)^{\nu} - \frac{\operatorname{tr}(\epsilon)}{2}\left(k_i\cdot(k_i+k_0)\delta^{\mu\nu} - k_i^{\mu}(k_i+k_0)^{\nu}\right). \qquad (6.2.34)\end{aligned}$$

Following the sewing procedure and performing the parameter integrals contained in (6.2.32), diagram (d) is given by the following expression

$$\Gamma_{d,\mathrm{Feyn}} = (\kappa e^2)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{(p'-p+\ell)_\mu\bar{v}_1^{\mu\nu}(p'-p+\ell+k_0)_\nu}{\ell^2(\ell+k_0)^2(m^2+(p-\ell)^2)}, \qquad (6.2.35)$$

where the photon propagator in the Feynman gauge $\varepsilon_1^{\mu}\varepsilon_2^{\nu}\to\delta^{\mu\nu}/\ell^2$ has been used to connect the two external photons. After the sewing procedure, it is worth mentioning that the effective vertex $\bar{v}_1^{\mu\nu}$ now has been changed to

$$\begin{aligned}\bar{v}_1^{\mu\nu} &= \ell\,\epsilon\,(\ell+k_0)\delta^{\mu\nu} + \epsilon^{\mu\nu}\ell\cdot(\ell+k_0) - \ell^{\mu}((\ell+k_0)\,\epsilon)^{\nu}\\ &- (\ell\,\epsilon)^{\mu}(\ell+k_0)^{\nu} - \frac{\operatorname{tr}(\epsilon)}{2}\left(\ell\cdot(\ell+k_0)\delta^{\mu\nu} - \ell^{\mu}(\ell+k_0)^{\nu}\right). \qquad (6.2.36)\end{aligned}$$

Combining the gauge invariance of the $g\gamma\gamma$ vertex with the expression above, it is easy to notice that $\ell_\nu\bar{v}_1^{\mu\nu} = (\ell+k_0)_\mu\bar{v}_1^{\mu\nu} = 0$. This allows us to simplify the expression of diagram (d) to

$$\Gamma_{d,\mathrm{Feyn}} = -4\,(\kappa e^2)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{p'_\mu\bar{v}_1^{\mu\nu}p_\nu}{\ell^2(\ell+k_0)^2(m^2+(p-\ell)^2)}. \qquad (6.2.37)$$

Finally, we can strip off the graviton polarization tensor from the expression of the vertex, and, after organizing the terms inside the loop integral in terms of powers of the loop momentum, we obtain diagram (d) to be

$$\Gamma^{\mu\nu}_{d,\mathrm{Feyn}}[p,p';k_0] = \Gamma^{\mu\nu}_{d,1} + \Gamma^{\mu\nu}_{d,2} + \Gamma^{\mu\nu}_{d,3}, \qquad (6.2.38)$$

where for convenience in the notation we have introduced the three coefficients

$$\Gamma^{\mu\nu}_{d,1} = -4\,(\kappa e^2)\Big(p'\cdot p\Big[M^{(2)\mu\nu}[k_0,p] + M^{(1)\mu}[k_0,p]\,k_0^\nu\Big] \\ + p'^\mu p^\nu\Big[N^{(0)}[k_0,p] + M^{(1)\rho}[k_0,p]\,k_{0\rho}\Big]\Big) \quad (6.2.39)$$

$$\Gamma^{\mu\nu}_{d,2} = 4\,(\kappa e^2)\Big(p'_\rho\,p^\mu\Big[M^{(2)\rho\nu}[k_0,p] + M^{(1)\rho}[k_0,p]\,k_0^\nu\Big] \\ + p'^\mu\,p_\rho\Big[M^{(2)\rho\nu}[k_0,p] + M^{(1)\mu}[k_0,p]\,k_0^\rho\Big]\Big) \quad (6.2.40)$$

$$\Gamma^{\mu\nu}_{d,3} = 4\,(\kappa e^2)\frac{\delta^{\mu\nu}}{2}\Big(p'\cdot p\Big[N^{(0)}[k_0,p] + M^{(1)\rho}[k_0,p]k_{0\rho}\Big] \\ - p'_\rho p_\sigma\Big[M^{(2)\rho\sigma}[k_0,p] + M^{(1)\rho}[k_0,p]k_0^\sigma\Big]\Big). \quad (6.2.41)$$

The complete expressions of the different loop integrals has been relegated as usual to appendix C.

6.2.4 *Diagram (e)*

The last contribution we have to compute to finish our analysis about the one-loop correction to the gss vertex in scalar QED corresponds to the diagram (e) contained in 6.1. This diagram can be obtained from the part of the pure photon amplitude $\mathcal{D}^{(2)}$ in (6.2.1) proportional to $\delta(\tau_1-\tau_2)$. In other words this corresponds to the diagram where the two photons merge into the scalar line at same point (seagull vertex). The graviton is inserted using the well known replacement rule (6.1.7) for the construction of reducible contributions. At tree-level this diagram is given by

$$\mathcal{D}_e[p,p';k_1,\varepsilon_1;k_2,\varepsilon_2;k_0,\epsilon] = \\ -2(\kappa e^2)\int_0^\infty dTe^{-T(m^2+p^2)}\int_0^T d\tau_1\,e^{-(p-p')\cdot(k_1+k_2+k_0)\tau_1}\,\frac{\varepsilon_{2\mu}\bar{v}_1^{\mu\nu}\varepsilon_{1\nu}}{(k_1+k_0)^2}, \quad (6.2.42)$$

where the prescription in (6.2.33) has been used for the effective polarization. Finally, sewing the two photons using the Feynman gauge through $\varepsilon_1^\mu\varepsilon_2^\nu \to \delta^{\mu\nu}/\ell^2$, we can strip off the graviton polarization tensor and organize the integrand in terms of powers of the loop

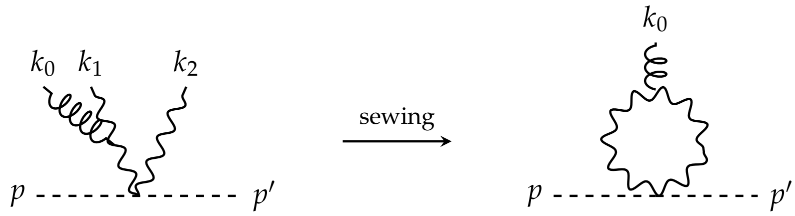


FIG. 6.5: Diagram (e) is obtained after sewing the two external photons in the part of the pure photon $\mathcal{D}^{(2)}$ where the photons are attached in same point, i.e. the diagram associated to a seagull vertex. The graviton is inserted in one photon line using the known replacement rules.

momentum. After the parameter integral has been performed, this leads to the diagram in figure 6.5:

$$\Gamma^{\mu\nu}_{e,\text{Feyn}}[k_0] = -2\,(\kappa e^2) \times \Big[(D-2)\left(O^{(2)\mu\nu}[k_0] + k_0^\mu O^{(1)\nu}[k_0]\right) + \frac{3-D}{2}\eta^{\mu\nu}k_{0\rho}O^{(1)\rho}[k_0]\Big], \tag{6.2.43}$$

where D is referred to the arbitrary dimension over which the loop integral is performed. Finally, we remark that, as previously discussed, the gauge dependent part of the diagram (e) vanishes because of the gauge invariance of the $g\gamma\gamma$ vertex. The integrals introduced in the latter expression can be found in appendix (C).

6.2.5 *Full One-Loop Correction to the Graviton Vertex in Scalar QED*

To sum up, we can collect all the results of the previous sections to construct the full one-loop correction to the graviton vertex in scalar QED. In the Feynman gauge, the one-loop vertex is given by the following contributions

$$\Gamma^{\mu\nu}_{\text{Feyn}}[p,p';k_0] = \Gamma^{\mu\nu}_{a,\text{Feyn}} + \Gamma^{\mu\nu}_{b,\text{Feyn}} + \Gamma^{\mu\nu}_{c,\text{Feyn}} + \Gamma^{\mu\nu}_{d,\text{Feyn}} + \Gamma^{\mu\nu}_{e,\text{Feyn}}, \tag{6.2.44}$$

where the single terms can be found in (6.2.16), (6.2.23), (6.2.29), (6.2.38) and (6.2.43) respectively. This vertex can be extended to any covariant gauge with the addition of an extra term of type

$$\Gamma^{\mu\nu}_{\xi}[p,p';k_0] = \Gamma^{\mu\nu}_{a,\xi} + \Gamma^{\mu\nu}_{b,\xi} + \Gamma^{\mu\nu}_{c,\xi}, \tag{6.2.45}$$

where the gauge dependent contributions can be found in (6.2.18), (6.2.25) and (6.2.30). Finally, combining the two expressions above, we obtain the full one-loop vertex in any covariant gauge:

$$\Gamma^{\mu\nu}_{\text{loop}}[p,p';k_0] = \Gamma^{\mu\nu}_{\text{Feyn}} + \Gamma^{\mu\nu}_{\xi}. \tag{6.2.46}$$

6.3 TRANSVERSALITY OF THE ONE-LOOP gss VERTEX

In this section we perform a preliminary check of our result by testing the **transversality** of the one-loop graviton-scalar amplitude constructed using the vertex in (6.2.46). Transversality is a property that originates directly from the equation of motion of the graviton. It requires that the amplitude remains unchanged if the polarization tensor is substituted with

$$\epsilon_{\mu\nu} \;\to\; \epsilon_{\mu\nu} + k_{0\mu}\xi_\nu + k_{0\nu}\xi_\mu, \quad k_0 \cdot \xi = k_0^2 = 0, \tag{6.3.1}$$

where k_0^μ is the momentum of the graviton, and ξ^μ an arbitrary vector that satisfies $k_0 \cdot \xi = 0$. In particular, we want to verify that the amplitude

$$\mathcal{D}^{(1)}_{\text{loop}}[p, p'; \epsilon, k_0] = \epsilon_{\mu\nu}\Gamma^{\mu\nu}_{\text{loop}}[p, p'; k_0] \tag{6.3.2}$$

is invariant under the replacement (6.3.1), namely

$$\mathcal{D}^{(1)}_{\text{loop}}[p, p'; \xi k_0, k_0] = 0. \tag{6.3.3}$$

The expression $k_0\xi$ in (6.3.3) is a shorthand for the symmetric combination of the two vectors. Transversality is a property of on-shell amplitudes, and the equation of motion, $p^2 = p'^2 = -m^2$, should be imposed also on the scalar lines. In the following, we will use the expressions of the individual diagrams in the Feynman gauge. Indeed transversality is a gauge-invariant statement, so in principle this should be sufficient. However, it is also easy to see that the gauge dependent corrections in (6.2.18), (6.2.25) and (6.2.30) vanish on the mass-shell $p^2 = p'^2 = -m^2$, which adds to the argument.

DIAGRAM (a) The contribution from diagram (a) to the transversality of the full one-loop gss amplitude is computed by contracting the vertex correction in (6.2.16) with the symmetrized product of the vectors $k_0\,\xi$. We obtain

$$\begin{aligned} k_0 \cdot \Gamma_{a,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{a,\text{Feyn}} \cdot k_0 = (\kappa e^2)\Big[&-4p' \cdot p\, \xi \cdot (p' - p)\, J^{(1)} \cdot k_0 \\ &- 8p' \cdot p\, k_0 \cdot J^{(2)} \cdot \xi + 2(p' - p) \cdot \xi\, (p' - p) \cdot J^{(2)} \cdot k_0 + 2k_0 \cdot K^{(2)} \cdot \xi \\ &+ 4(p'_\rho - p_\rho)\, k_{0\mu}\xi_\nu J^{(3)\rho\mu\nu} + (p' - p) \cdot \xi\, K^{(1)} \cdot k_0\Big], \end{aligned} \tag{6.3.4}$$

where the missing terms have been simplified through the relation

$$(p' - p) \cdot k_0 = -(p' - p) \cdot (p' + p) = p^2 - p'^2 = 0, \tag{6.3.5}$$

that holds on-shell. Also, the total four-momentum conservation $p + p' + k_0 = 0$ is used if needed. The result above can be further manipulated by exploiting the vector products with external momenta to complete the square inside the momentum integrals in the on-shell frame. Namely, we express

$$\begin{aligned} p' \cdot \ell &= \frac{1}{2}(\ell^2 + 2\,p' \cdot \ell) - \frac{1}{2}\ell^2 \\ p \cdot \ell &= \frac{1}{2}\ell^2 - \frac{1}{2}(\ell^2 - 2\,p \cdot \ell) \\ k_0 \cdot \ell &= \frac{1}{2}(\ell^2 - 2\,p \cdot \ell) - \frac{1}{2}(\ell^2 + 2\,p' \cdot \ell), \end{aligned} \tag{6.3.6}$$

where ℓ is the momentum flowing inside the loop, and momentum conservation has been used to obtain the last equality. Using systematically the relations above, one can easily show that the integrals of type $J^{(i)}$ and $K^{(i)}$ are all reduced to combinations of integrals of type $I^{(i)}$ and $G^{(i)}$. The latter are defined in appendix C. Finally, applying the reduction to master integrals described in (C3) for specific forms, we rewrite (6.3.4) as

$$\begin{aligned} k_0 \cdot \Gamma_{a,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{a,\text{Feyn}} \cdot k_0 &= (\kappa e^2)\, \xi \cdot k_0 \, \frac{2\,(1+D)\,m^2 + D\,k_0^2}{2m^2\,(-1+D)}\, G^{(0)} \\ &= 0, \end{aligned} \tag{6.3.7}$$

where the condition $\xi \cdot k_0 = 0$ has been used to obtain a vanishing result.

DIAGRAMS (b)-(c) The calculation of the transversality of the full one-loop gss amplitude involves the contribution from diagram (b), as shown in equation (6.2.23), after it is combined with the symmetrized product of the vectors k_0, ξ. In detail, this contraction produces

$$\begin{aligned} k_0 \cdot \Gamma_{b,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{b,\text{Feyn}} \cdot k_0 &= \\ -\,(\kappa e^2) \Big[2p' \cdot k_0\,(p'-p) \cdot \xi\, I^{(0)}[p'] &+ (3p'-p) \cdot k_0\, I^{(1)}[p'] \cdot \xi \\ + (3p'-p) \cdot \xi\, I^{(1)}[p'] \cdot k_0 &+ 2\,\xi \cdot I^{(2)}[p'] \cdot k_0 \Big]. \end{aligned} \tag{6.3.8}$$

This expression can be further simplified using the reductions for the integrals presented in (C3). We obtain

$$\begin{aligned} \Gamma_{b,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{b,\text{Feyn}} \cdot k_0 &= -(\kappa e^2) \\ \times \frac{\xi \cdot k_0\,\left(4m^2\,(-3+D) + k_0^2\,(-1+D)^2\right) + 2\,\xi \cdot p'\,k_0^2\,(1+D)}{4m^2\,(-3+D)\,(-1+D)} &\, G^{(0)}, \end{aligned} \tag{6.3.9}$$

and finally

$$\Gamma_{b,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{b,\text{Feyn}} = 0 \tag{6.3.10}$$

by using the relations $k_0^2 = \xi \cdot k_0 = 0$.

We can proceed in an analogous way in order to derive the contribution originating from diagram (c). This is obtained from the previous expression (6.3.9) under the replacement $p' \leftrightarrow p$ (the inversion of the loop momentum $\ell \to -\ell$ has no effects on the master integrals). We easily get

$$\begin{aligned}&\Gamma_{c,\mathrm{Feyn}} \cdot \xi + \xi \cdot \Gamma_{c,\mathrm{Feyn}} \cdot k_0 = -(\kappa e^2) \\ &\times \frac{\xi \cdot k_0 \left(4m^2 (-3+D) + k_0^2 (-1+D)^2\right) + 2\, \xi \cdot p\, k_0^2 (1+D)}{4m^2 (-3+D)(-1+D)} G^{(0)},\end{aligned} \tag{6.3.11}$$

and

$$\Gamma_{c,\mathrm{Feyn}} \cdot \xi + \xi \cdot \Gamma_{c,\mathrm{Feyn}} = 0 \tag{6.3.12}$$

from the on-shell conditions.

DIAGRAM (d) Now we compute the contribution of diagram (d) to the transversality. This can be determined by analyzing the result of the contraction of the vertex in equation (6.2.38) with the symmetrized product of k_0, ξ, namely

$$\begin{aligned}&k_0 \cdot \Gamma_{d,\mathrm{Feyn}} \cdot \xi + \xi \cdot \Gamma_{d,\mathrm{Feyn}} \cdot k_0 = \\ &4\,(\kappa e^2) \Big[k_0 \cdot \xi \left(p' \cdot p I^{(0)}[p'] - p' \cdot M^{(2)} \cdot p \right) + p \cdot k_0\, p' \cdot M^{(2)} \cdot \xi \\ &+ k_0 \cdot \xi \, \left(I^{(0)}[p'] + M^{(1)} \cdot k_0 \right) + p' \cdot \xi \left(p \cdot M^{(2)} \cdot k_0 + p \cdot k_0\, M^{(1)} \cdot k_0 \right) \\ &+ p \cdot \xi \left(p' \cdot M^{(2)} \cdot k_0 + k_0^2 M^{(1)} \cdot p' \right) - 2\, p' \cdot p\, k_0 \cdot M^{(2)} \cdot \xi \\ &- p' \cdot p\, k_0^2\, M^{(1)} \cdot \xi + p' \cdot k_0 \left(p \cdot M^{(2)} \cdot \xi + p \cdot k_0\, M^{(1)} \cdot \xi \right) \Big].\end{aligned} \tag{6.3.13}$$

Using an analogous technique to (6.3.6), we can further simplify the loop integrals that enter in the expression above. In particular, we can complete the square inside specific integrals using now

$$\begin{aligned}\ell \cdot k_0 &= \frac{1}{2}(2\ell \cdot k_0 + \ell^2) - \frac{1}{2}\ell^2 - \frac{1}{2}k_0^2 \\ \ell \cdot p &= \frac{1}{2}\ell^2 - \frac{1}{2}(\ell^2 - 2\,p \cdot \ell) \\ \ell \cdot p' &= \frac{1}{2}(\ell^2 - 2\,p \cdot \ell) - \frac{1}{2}(2\ell \cdot k_0 + \ell^2) + \frac{1}{2}k_0^2,\end{aligned} \tag{6.3.14}$$

Combining those relations when necessary together with the conservation of the total four-momentum and the tensor-reduction formulae for integrals of type $I^{(1)}$ described in (C3), we rewrite (6.3.13) as

$$k_0 \cdot \Gamma_{d,\mathrm{Feyn}} \cdot \xi + \xi \cdot \Gamma_{d,\mathrm{Feyn}} \cdot k_0 = -\frac{(\kappa e^2)}{2}\, \xi \cdot k_0 (2\, G^{(0)} + k_0^2\, O^{(0)}[k_0]). \tag{6.3.15}$$

Finally, using on-shell conditions, we obtain

$$k_0 \cdot \Gamma_{d,\mathrm{Feyn}} \cdot \xi + \xi \cdot \Gamma_{d,\mathrm{Feyn}} \cdot k_0 = 0. \tag{6.3.16}$$

DIAGRAM (e) The final contribution that must be taken into account in the transversality calculation of the one-loop gss vertex comes from diagram (e), shown in equation (6.2.43). By contracting it with the symmetrized k_0, ξ, we obtain

$$
\begin{aligned}
&k_0 \cdot \Gamma_{e,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{e,\text{Feyn}} \cdot k_0 = \\
&-(\kappa e^2)\Big\{(D-2)\Big(2k_0 \cdot O^{(2)}[k_0] \cdot \xi + k_0^2\, O^{(1)}[k_0] \cdot \xi + k_0 \cdot \xi\, O^{(1)}[k_0] \cdot k_0\Big) \\
&+ (3-D)\, k_0 \cdot \xi\, O^{(1)}[k_0] \cdot k_0\Big\}
\end{aligned}
\tag{6.3.17}
$$

Through the help of the reductions in (C3), one immediately verifies that the expression above can be simplified to

$$
k_0 \cdot \Gamma_{e,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{e,\text{Feyn}} \cdot k_0 = \frac{(\kappa e^2)}{2}\, \xi \cdot k_0\, k_0^2\, O^{(0)}[k_0], \tag{6.3.18}
$$

and, using on-shell conditions, we obtain as expected

$$
k_0 \cdot \Gamma_{e,\text{Feyn}} \cdot \xi + \xi \cdot \Gamma_{e,\text{Feyn}} \cdot k_0 = 0. \tag{6.3.19}
$$

TRANSVERSALITY In this section we have verified the on-shell transversality of the one-loop graviton-scalar amplitude constructed using the vertex in (6.2.46). In particular, we have tested the transversality by examining the individual contributions from the diagrams in figure (6.1), all of which vanish separately. In other words, we don't have any interplay among diagrams in the transversality test. This may arise from specific properties three-particle systems, that, once on-shell conditions are imposed on all external legs, become over-constrained and transversality often turns out to be simple. In the previous chapter, transversality of the full amplitude on the graviton line was derived as a direct result of the Ward identities generated by (5.2.7) once on-shell conditions are imposed. In the future, a possible task is to make us of generating functions of the form (5.2.7) to provide off-shell relations that could serve as a rigorous test of our results.

6.4 FINAL REMARKS

In this chapter, we have applied (and expanded upon) the method presented in chapter 5 to calculate the radiative one-loop correction to the scalar-scalar-graviton vertex in any dimension. This was accomplished by sewing together the two external photons in the scalar amplitude with one graviton and two photons, using any covariant gauge. Specifically, we have computed the radiative correction at the one-loop level for the vertex by examining the relevant diagrams and computing their contributions using worldline techniques. A revised replacement rule in (6.1.7) was required to ensure a completely off-shell outcome. Our construction has been tested by verifying the on-shell transversality.

In the following part of the thesis, we will set aside the study of dressed propagators while still utilizing the worldline formalism as the core principle. In particular, we will examine recent developments in color-kinematics duality and double copy for scattering amplitudes by exploring how these nicely combine with well-established worldline methods.

Part III

COLOR-KINEMATICS DUALITY AND DOUBLE COPY FROM THE STRING-INSPIRED FORMALISM

7 COLOR-KINEMATICS FROM THE STRING-INSPIRED FORMALISM

In this chapter we present a novel procedure to construct Berends-Giele currents using the Bern-Kosower formalism for one-loop gluon amplitudes. Applying the pinch procedure of that formalism to a suitable special case, the currents are naturally obtained in terms of multiparticle fields and obeying color-kinematics duality. Constructed as composite fields in the labels of the external states, the multiparticle fields, firstly introduced in [125, 126] and stabilized in [127], have proven to be a useful and efficient method for building scattering amplitudes, specifically at high multiplicities. In this chapter, we review how color-kinematics duality can be conveniently restated in the multiparticle language by means of the so-called Generalized Jacobi Identities (GJI). Furthermore, combining the multiparticle approach with techniques inspired by the Bern-Kosower formalism, we obtain building blocks for the construction of tree-level gluon amplitudes, that will naturally appear in a color-kinematic-dual representation. Results discussed in this chapter are published in [76].

7.1 THE STRUCTURE OF WORLDLINE INTEGRANDS AND PINCH OPERATORS

In the preceding parts of this manuscript we have introduced the Bern-Kosower formalism as simple and direct method to construct one-loop on-shell n-gluon amplitudes. In particular, in section 2.3.3 we have briefly reviewed the set of rules provided by this formalism for the construction of parameter integrals that build the amplitude. For the purposes of this chapter, the most relevant aspect of these Bern-Kosower rules is that they allow one to reconstruct the integrands of the reducible contributions to the amplitude from the one of the irreducible one by a pinching procedure. The latter is encoded in the Bern-Kosower master formula for one-particle irreducible amplitudes

$$\Gamma(k_1,\varepsilon_1;\dots;k_n,\varepsilon_n) = (-ig)^n \mathrm{Tr}(T^{a_1}\cdots T^{a_n}) \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}} \mathrm{e}^{-m^2T} \int_0^T d\tau_1 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \exp\Big\{ \sum_{i,j=1}^n \Big(\frac{1}{2} G_{ij} k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j \Big) \Big\}\Big|_{\varepsilon_1\dots\varepsilon_n}, \tag{7.1.1}$$

which we have already encountered in (2.3.30). The n-gluon contribution to the amplitude is obtained expanding the exponential above keeping only the terms linear in each polarization, namely

$$\exp\left\{\cdot\right\}|_{\varepsilon_1\ldots\varepsilon_n} \equiv (-i)^n P_n\left(\dot{G}_{ij}, \ddot{G}_{ij}\right) e^{\frac{1}{2}\sum_{i,j=1}^{n}\frac{1}{2}G_{ij}k_i\cdot k_j}. \tag{7.1.2}$$

According to the procedure presented in section 2.3.3, the application of the Bern-Kosower pinch rules requires one to first perform certain partial integrations to the integrand that effectively remove quartic vertices. These partial integrations have to be performed using the *symmetric partial integration* algorithm introduced in section 2.3.4, that allows for the replacement

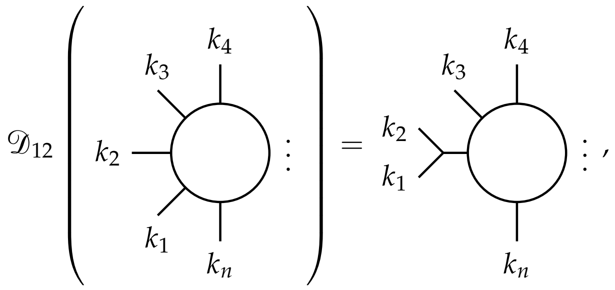
$$P_n\left(\dot{G}_{ij}, \ddot{G}_{ij}\right) \to Q_n\left(\dot{G}_{ij}\right). \tag{7.1.3}$$

The general structure of the resulting integrand $Q_n\left(\dot{G}_{ij}\right)$ is remarkable: as we have seen in section 2.3.4, it is symmetric under permutations of the external legs and homogeneous in the polarizations ε_i and momenta k_i. This allows to completely redefine the integrand in terms of cycles $Z_k(i_1, i_2, \ldots, i_k)$, built as traces of products of gluon field strength tensors $f_i^{\mu\nu} = k_i^\mu\varepsilon_i^\nu - k_i^\nu\varepsilon_i^\mu$, and leftover terms, called *tails* and denoted as $T(i_1, i_2.\ldots, i_l)$. The tails are not manifestly transversal objects, however they turn into total derivatives whenever any of the polarization ε_{i_m} contained in them is replaced by k_{i_m}. These properties have important consequences, as it will be clear in the following. For the time being, we just point out that clearly the pinching procedure in the Bern-Kosower formalism must include the full information on the Berends-Giele currents attached to the loop. The main goal of this chapter is the analysis of these currents, that, within this construction, show generalized structures that manifestly exhibit color-kinematics duality —see section 3.3.

The main ingredients of our approach are the symmetric partial integration algorithm and the Bern-Kosower replacement rules mentioned above. Once the algorithm has been applied to produce the permutation invariant integrand $Q_n\left(\dot{G}\right)$, we can construct reducible contributions through the rules outlined in 2.3.3. In this process, it is useful to introduce a *pinch operator* acting on Q_n, that synthesizes the effect of the pinching rules: for two adjacent legs i and j with $i < j$, it is defined as

$$\mathscr{D}_{ij}Q_n = \frac{\partial}{\partial\dot{G}_{ij}}Q_n\bigg|_{\substack{\dot{G}_{ij}=0\\ \dot{G}_{jk}\to\dot{G}_{ik}}}, \tag{7.1.4}$$

i.e. it acts only on terms linear in $\dot{G}_{ij}$. It is worth mentioning that, inside the pinch operator, we haven't included the factors specified by the Bern-Kosower rules that guarantee the correct pole structure of the amplitude —see (2.3.32). For the time being, we are mostly interested in the algebraic structure of the integrand when one (or more) pinch operators act on it. The correct pole structure will be recovered later in the chapter, when we will compute full scattering amplitudes. Diagrammatically, we can depict the action of the pinch operator on the integrand Q_n as



where two adjacent legs 1 and 2 have been pinched together.

In order to understand the basic link between the pinch operator and color-kinematics duality, let us first define the so-called **generalized Jacobi identities** (GJI) [16, 128]. Let P be a word, i.e. a multiparticle label of type $P = 12\ldots n$ —see 4.1 for the notation. Consider the free Lie algebra Lie$[1,\ldots,n]$ generated by all the words in the letters in P, and introduce ℓ as the left-to-right bracketing on Lie$[1,\ldots,n]$. The latter is recursively defined as

$$\begin{aligned}\ell(i_1 i_2 \cdots i_k) &= \ell(i_1 i_2 \cdots i_{k-1})\, i_k - i_k\, \ell(i_1 i_2 \cdots i_{k-1})\\ \ell(i) &= i\\ \ell(\varnothing) &= 0.\end{aligned} \tag{7.1.5}$$

The generalized Jacobi identities correspond to the elements in the kernel of ℓ. For example

$$\ell(12+21) = 0, \qquad \ell(123+231+312) = 0, \tag{7.1.6}$$

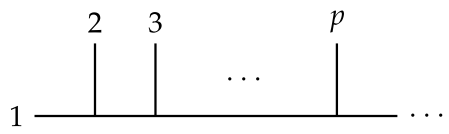
which correspond with the antisymmetry and Jacobi identity of the Lie bracket.

Using the identity $\ell(P\ell(Q)) = [\ell(P), \ell(Q)]$, it is easy to see that $\ell(A\ell(B) + B\ell(A)) = 0$ for any words A and B. In addition, due to the recursive definition of ℓ if $\ell(P) = 0$ it also follows that $\ell(PQ) = 0$ for any word Q. Therefore, for objects labeled by words, the generalized Jacobi identities can be characterized by an abstract operator $\mathcal{L}_k$

$$\mathcal{L}_k \circ K_{ABC} \equiv K_{A\ell(B)C} + K_{B\ell(A)C}, \tag{7.1.7}$$

where this definition holds $\forall A, B \neq \varnothing$ and $\forall C$ such that $|A| + |B| = k$. The partition of non-empty words A and B in the above definition is arbitrary, while C can be empty. This leads a non-unique operator $\mathcal{L}_k$, e.g.

$$\begin{aligned}\mathcal{L}_3 \circ K_{123} =& K_{123} - K_{132} + K_{231}, &&\text{for } A = 1,\ B = 23,\ C = \varnothing\\ \mathcal{L}_3 \circ K_{123} =& K_{123} + K_{312} - K_{321}, &&\text{for } A = 12,\ B = 3,\ C = \varnothing.\end{aligned} \tag{7.1.8}$$



$\leftrightarrow \quad K_{12\ldots p} = K_P$

FIG. 7.1: The correspondence between local multiparticle superfields $K_{12\ldots p} = K_P$ and tree-level subdiagrams.

Note that the two expression agree if $\mathcal{L}_2 \circ K_{123} = 0$ is guaranteed. In general, we say that the objects K_P satisfy generalized Jacobi identities iff

$$\mathcal{L}_k \circ K_P = 0, \qquad \forall k \leq |P|. \tag{7.1.9}$$

The generalized Jacobi identities are also called *BCJ symmetries*. We can give

$$\begin{aligned} K_{12C} + K_{21C} &= 0, \qquad \forall C, \\ K_{123C} + K_{231C} + K_{312C} &= 0, \qquad \forall C, \\ K_{1234C} + K_{2143C} + K_{3412C} + K_{4321C} &= 0, \qquad \forall C, \end{aligned} \tag{7.1.10}$$

where we have already used the fact that K_P satisfies the BCJ symmetries $\mathcal{L}_k \circ K_P = 0$ for all $k \leq |P|$ to simplify the appearance of the above.

It is easy to convince ourselves that the BCJ symmetries in (7.1.10) are equivalent to the symmetries obeyed by the following string of structure constants —see figure 7.1. We have a precise correspondence

$$K_{12\ldots p} \ \leftrightarrow \ f^{a_1 a_2 b} f^{b a_3 c} \cdots f^{z a_p a}, \tag{7.1.11}$$

that we can exploit if we look at the identities in (7.1.10). The first two lines of of the latter are the counterparts of the antisymmetry $f^{a_1 a_2 a} = -f^{a_2 a_1 a}$ and the Jacobi identities $f^{a_1 a_2 a} f^{a a_3 b} + \mathrm{cyc}(1,2,3) = 0$. More generally, if the objects K_P represent multiparticle fields that contain the kinematic information of the external particles, the correspondence (7.1.11) lines up with the BCJ duality between color and kinematics introduced in section 3.3. Accordingly, multiparticle fields that satisfy the symmetries (7.1.9) are said to be in the **BCJ gauge** [100].

Since the fields K_P in the BCJ gauge satisfy the same generalized Jacobi symmetries as nested brackets $\ell(P) = [[\ldots[[p_1, p_2], p_3], \ldots], p_n]$, it is convenient to use a notation where this is manifest. To this effect, a word P is understood as having a nested bracket structure $P \to \ell(P)$ and we define

$$K_P \equiv K_{\ell(P)}. \tag{7.1.12}$$

For instance, $K_{12} = K_{[1,2]}$ and $K_{123} = K_{[[1,2],3]}$. The Jacobi symmetry allows the definition of local superfields with a even more general bracketing structure. Using the identity

$$[\ell(A), \ell(B)] = \ell(A\ell(B)), \tag{7.1.13}$$

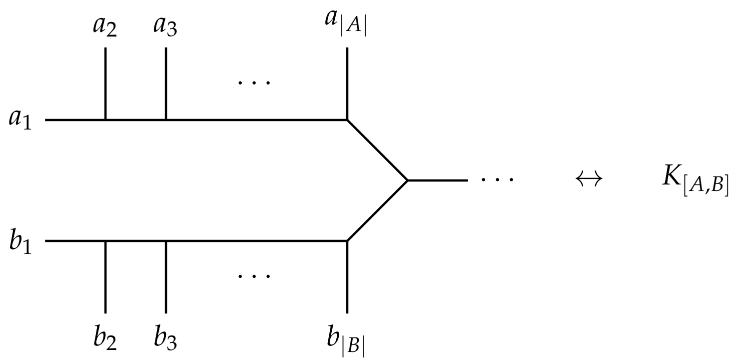


FIG. 7.2: The planar binary tree associated with the multiparticle field $K_{[A,B]}$.

it is always possible to flatten brackets within multiparticle fields,

$$K_{[A,B]} \equiv K_{[\ell(A),\ell(B)]} = K_{\ell(A\ell(B))} \equiv K_{A\ell(B)}. \tag{7.1.14}$$

For example,

$$\begin{aligned} K_{[[1,2],[3,4]]} &= K_{\ell(12\ell(34))} = K_{[[[1,2],3],4]} - K_{[[[1,2],4],3]} \\ K_{[1,[[2,3],4]]} &= K_{\ell(1\ell(234))} \\ &= K_{[[[1,2],3],4]} - K_{[[[1,3],2],4]} - K_{[[[1,4],2],3]} + K_{[[[1,4],3],2]}. \end{aligned} \tag{7.1.15}$$

Identities of this type will be extremely helpful in the following, where manipulations of multiparticle fields in the BCJ gauge will be necessary in the computation of amplitudes. They can be visualized as the systematic use of Jacobi identities to flatten out of the planar binary tree associated with the two branches.

Now that we have obtained the GJI and have exploited some of their basic properties, we can explore the connection between these and the pinch operators introduced in (7.1.4) in more detail. For this, we need to understand a little better the structure of the polynomial Q_n obtained after IBP in (7.1.3). Given a map $\alpha\colon \{1,\dots,n\} \to \{1,\dots,n\}$, consider the following polynomial of degree n on the $\dot{G}_{ij}$'s

$$Q_n^{(\alpha)} = \sum_{\text{perm.}} C_{12\dots n} \dot{G}_{1\alpha(1)} \dot{G}_{2\alpha(2)} \cdots \dot{G}_{n\alpha(n)}, \tag{7.1.16}$$

where the coefficients $C_{12\dots n}$ depend on the polarizations and momenta. Then, making use of the permutation symmetry of Q_n, we can write it as a sum of polynomials of the form (7.1.16). Therefore, to understand how the pinch operators act on Q_n, it will be enough to consider their action on such polynomials. For this, it will be convenient first to examine a specific example. Take $n = 4$ and α such that $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 2$ and $\alpha(4) = 3$. By a straightforward calculation, one finds that the action of the pinch operator $\mathscr{D}_{12}$ on the resulting polynomial $Q_4^{(\alpha)}$ yields

$$(C_{3214} - C_{3124})\dot{G}_{13}^2\dot{G}_{14} + (C_{4312} - C_{4321})\dot{G}_{13}\dot{G}_{34}^2 + (3 \leftrightarrow 4). \qquad (7.1.17)$$

We can check directly that each of the coefficients is antisymmetric in 1 and 2, i.e. they satisfy the GJI of order 1 —see first line of (7.1.10). Let us next apply the pinch operator $\mathscr{D}_{13}$ to (7.1.17). The result is

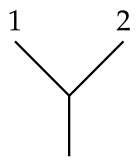
$$(C_{4213} - C_{4123} + C_{4312} - C_{4321})\dot{G}_{14}^2. \qquad (7.1.18)$$

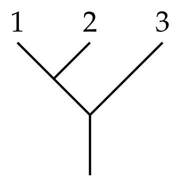
Now the coefficient satisfies the Jacobi identity in 1, 2 and 3, i.e. the GJI of order 2 —see second line of (7.1.10). Returning to the general case, we may infer that the iterated action of the pinch operators $\mathscr{D}_{12}, \mathscr{D}_{13}, \ldots, \mathscr{D}_{1(n-1)}$ on a polynomial of the form (7.1.16) will produce a monomial in $\dot{G}_{1n}$. Explicitly,

$$\mathscr{D}_{1(n-1)} \cdots \mathscr{D}_{13}\mathscr{D}_{12}Q_n^{(\alpha)} = \tilde{C}_{12\ldots n}\dot{G}_{1n}^2, \qquad (7.1.19)$$

where the coefficient $\tilde{C}_{12\cdots n}$ satisfies the GJI of order $n-1$ in $1, 2, \ldots, n-1$. We have checked that this property holds up to degree $n = 9$.

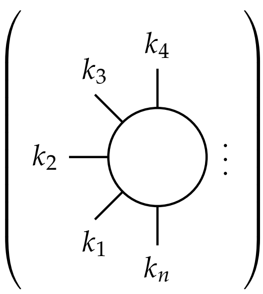
We make a final remark on how the above can be represented diagrammatically. Using the correspondence (7.1.11) and the identification (7.1.12) with nested brackets, we interpret the left-to-right bracketing in Lie$[1, \ldots, n]$ as a planar binary tree and vice versa using the correspondence (7.1.11). For example,

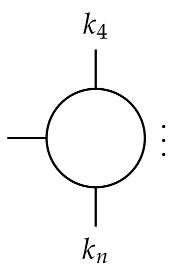
$\ell(12) =$ 

, $\ell(12) =$ 

, $\ldots$. (7.1.20)

Using this notation, we find, for instance, that the iterated action of $\mathscr{D}_{12}$ and $\mathscr{D}_{13}$ on Q_n can be graphically represented as

$\mathscr{D}_{13}\mathscr{D}_{12}$ 

$= \ell(123)$ 

. (7.1.21)

7.2 MULTIPARTICLE FIELDS FROM PINCHING

Making use of the properties of the Bern-Kosower integrands discussed in the previous section, we are ready now to present our novel technique for the construction of the coefficients of Berends-Giele currents for Yang-Mills directly in the BCJ gauge. As we have pointed out in (7.1.4), we omit the tree propagators in the implementation of the conventional Bern-Kosower pinch rules: at this point, we prefer to exploit the symmetries in the integrands, and use these to obtain all the coefficients of the Berends-Giele currents. We will include propagators later in this chapter, where computation of full scattering amplitudes will be carried out.

The two main quantities that we want to compute are the coefficients of the multiparticle currents J^{μ}_{P} and the field strength currents $F^{\mu\nu}_{P}$, introduced in chapter 4 as tools for the calculation of scattering amplitudes. The idea is to compute these quantities not relying on standard recursion relations, e.g. (4.2.15)-(4.2.16), but using a novel technique based on Bern-Kosower pinch rules. In this way, we will be able to obtain multiparticle numerators in the BCJ gauge on the spot.

The appropriate quantity for finding the field strength multiparticle coefficients is the sum of the terms in polynomial Q_n with a single one-cycle component. Using the notation (2.3.40), the one-cycle components are only those with one single label as superscript, e.g.

$$\tilde{Q}_n = Q^2_n + Q^3_n + \ldots + Q^n_n. \tag{7.2.1}$$

Using now (7.1.19), we obtain

$$\begin{aligned} \mathcal{D}_{1(n-1)} \cdots \mathcal{D}_{13}\mathcal{D}_{12}\tilde{Q}_n &= Z_2(12\ldots n-1, n)\dot{G}^2_{1n} \\ &= \tfrac{1}{2} f^{\mu\nu}_{12\cdots(n-1)} f_{n\nu\mu}\dot{G}^2_{1n}, \end{aligned} \tag{7.2.2}$$

where $f^{\mu\nu}_{12\cdots(n-1)}$ satisfy the GJI, i.e. we identify

$$f^{\mu\nu}_{12\cdots(n-1)} \equiv f^{\mu\nu}_{[[\cdots[1,2],\cdots],(n-1)]}, \tag{7.2.3}$$

in agreement with the property derived in (7.1.19). The identification of $f^{\mu\nu}_{12\cdots(n-1)}$ with the multiparticle generalization of the field strength tensor comes naturally if we look at the integrand Q_2, representing the bubble diagram with two external legs merging into the loop —see the first diagram in figure 7.3. This is simply given by

$$Q_2 = Z_2(1,2)\dot{G}^2_{12} = \tfrac{1}{2} f^{\mu\nu}_{1} f_{2\nu\mu}\dot{G}^2_{12}, \tag{7.2.4}$$

where we have used the global definition of cycles given in (2.3.37). Here $f^{\mu\nu}_{1}$ is nothing the usual abelian one-particle field strength tensor

$$f^{\mu\nu}_{i} = k^{\mu}_{i}\varepsilon^{\nu}_{i} - k^{\nu}_{i}\varepsilon^{\mu}_{i}, \tag{7.2.5}$$

already encountered previously in this manuscript. Using now the maximal pinch prescription on a n-point integrand Q_n, we recover a bubble integrand with the same structure of (7.2.4). Here all the kinematic information of the $n-1$ particles are all contained in one single

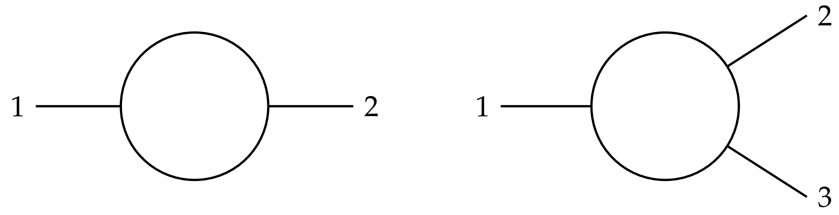


FIG. 7.3: Bubble and triangle diagrams describing the integrands Q_2 and Q_3 respectively.

field, that is naturally interpreted as the multiparticle generalization of the field strength tensor —see figure 7.4.

From the field strength tensors one can also extract the multiparticle polarizations, using the expression (4.2.16) that relates the field strength coefficients to the Berends-Giele ones. This procedure will be clarified later in the chapter, where the relation among multiparticle fields and Berends-Giele currents will be exploited. However it turns out that those can alternatively be obtained applying pinch operators just to the tails[1]

$$\mathscr{D}_{1(n-1)}\mathscr{D}_{1(n-2)}\cdots\mathscr{D}_{13}\mathscr{D}_{12}T(1,2,\ldots,n-2)=\varepsilon_{12\cdots(n-2)}\cdot k_{n-1}. \quad (7.2.6)$$

The multiparticle polarizations obtained in either way will satisfy the corresponding GJI, i.e.

$$\varepsilon^{\mu}_{12\cdots(n-2)}\equiv\varepsilon^{\mu}_{[[\cdots[1,2],\cdots],(n-2)]}, \quad (7.2.7)$$

where again the property in (7.1.19) is exploited. Note, however, that the second tail-pinching method requires one to know the tails to one order higher than is necessary for the first cycle-pinching approach. Again we naturally obtain the interpretation (7.2.6) looking at the simplest case where a one-tail appears, i.e. the triangle integrand Q_3 —see the second diagram in figure 7.3. Its expression contains

$$\dot{G}(2,3)T(1)=Z_2(2,3)\dot{G}^2_{23}T(1), \quad (7.2.8)$$

where

$$T(i)=\sum_r \varepsilon_i\cdot k_r\dot{G}_{1r}. \quad (7.2.9)$$

If now we apply the maximal pinch prescription on a n-point integrand Q_n, and we look at the term $\dot{G}(n-1,n)T(1,2,\ldots,n-2)$, the pinch acts only on the $(n-2)$-tail. The final result must have the same structure of (7.2.9), where all the kinematic information of the $n-2$ particles are contained in one single field, that is interpreted as the multiparticle generalization of the single particle polarization —see figure 7.5.

1 Here the last pinch operator $\mathscr{D}_{1(n-1)}$ is applied to get a more compact defining expression for the polarization $\varepsilon_{12\cdots(n-2)}$, but it doesn't affect the algebraic structure of the polarization itself. See [76] for an alternative definition.

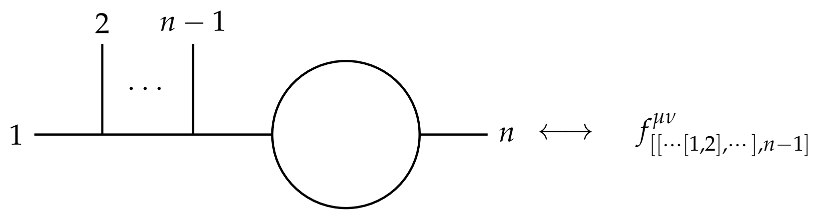
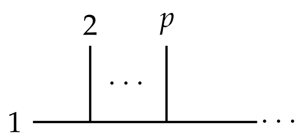


FIG. 7.4: Multiparticle generalization of the bubble integrand, that we can exploit to compute the multiparticle field strength tensor $f^{\mu\nu}_{[[\cdots[1,2],\cdots],n-1]}$.

Up to now, we have obtained an intuitive interpretation of the fields $f^{\mu\nu}_P$ and ε^μ_P, and we have understood why these should be considered as multiparticle generalizations of field strength tensor and to the gluon polarization respectively. Now it is important to clarify how these fields are related to the standard Berends-Giele currents $F^{\mu\nu}_P$ and J^μ_P introduced in chapter 4: when the link with the Berends-Giele formulation is understood, we are finally able to compute full scattering amplitudes. Clearly, the fields $f^{\mu\nu}_P$ and ε^μ_P cannot contain the full information of the Berends-Giele currents for two simple reasons:

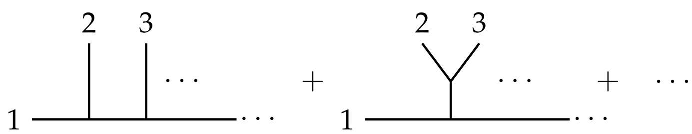
- In the pinching procedure we have deliberately ignored the locality structure of the multiparticle fields. We want to reintroduce somehow the propagators if we want to have any chance to reproduce the Berends-Giele formula.

- According to the convention in Bern-Kosower formalism detailed in section 2.3.3, the pinching procedure starts with the outermost vertices and recursively removes the trees attached to the loop in an ordered manner. Thus, the coefficients $f^{\mu\nu}_P$ and ε^μ_P in (7.2.2) and (7.2.6) should encode information only about the diagram



(7.2.10)

On the other hand, Berends-Giele currents describe the full set of color-ordered diagrams, each characterized by a specific locality structure, like



(7.2.11)

This information is clearly not included in the single diagram (7.2.10), and must be recovered somehow. Note that in the diagrams above the dotted line represents the leg that merges into

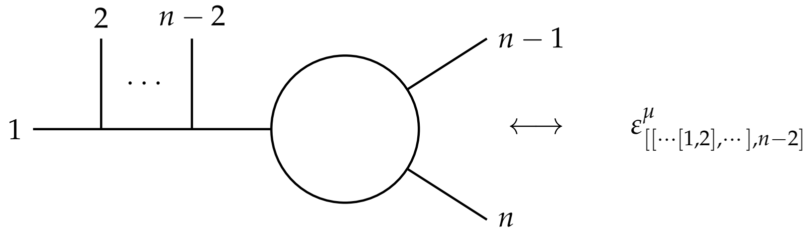


FIG. 7.5: Multiparticle generalization of the triangle integrand, that is used to compute the multiparticle polarization $\varepsilon^{\mu}_{[[\cdots[1,2],\cdots],n-2]}$.

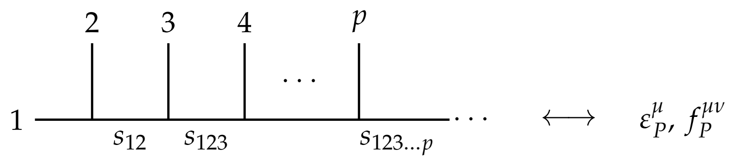


FIG. 7.6: Cubic diagram whose kinematic representatives are the multiparticle fields ε^{μ}_{P}, $f^{\mu\nu}_{P}$. Note that the $n = p+1$ leg is maintained off-shell, in a consistent way with the Berends-Giele currents formulation.

the loop. If we cut this line from the loop and we identify it with an external on-shell leg, we expect to recover the corresponding partial amplitude.

In order to do some progress, we have first of all to do a precise mathematical characterization of the quantities $f^{\mu\nu}_{P}$ and ε^{μ}_{P}, i.e. we have to understand which quantities they exactly represent. The intuition given previously in this section is an hint. However, we can characterize them with no margin of error by direct comparison of quantities contained in [98, 127]. Explicit examples of the coefficients $f^{\mu\nu}_{P}$ and ε^{μ}_{P} will be given in the next section. For the time being, we limit ourselves to say that these multiparticle fields should be identified as the kinematic representatives of the diagram of type (7.2.10), where p legs are on-shell and the $n = p+1$ leg is maintained off-shell. In other words, they represent exactly the numerators of the coefficients inside $F^{\mu\nu}_{P}$ and J^{μ}_{P} associated to the aforementioned diagram, i.e. the coefficients in the Berends-Giele formula with the corresponding locality structure —see figure 7.6. In the next section, we will give explicit examples of the numerators $f^{\mu\nu}_{P}$ and ε^{μ}_{P}, while at the end of the chapter we will see how these objects can be used to construct amplitudes through full Berends-Giele currents.

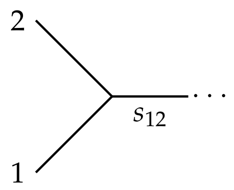


FIG. 7.7: Diagrammatic interpretation of the rank-two fields $f^{\mu\nu}_{[1,2]}$ and $\varepsilon^{\mu}_{[1,2]}$.

7.3 EXAMPLES

In this section we will now work out the technique presented in the previous section to compute the multiparticle fields $f_P^{\mu\nu}$ and ε_P^{μ} up to $n = 5$.

7.3.1 *Two-Particle Case*

At rank two, the numerator $f^{\mu\nu}_{[1,2]}$ is extracted from the Q_3 integrand (2.3.38), where only one-cycles are involved. To obtain the bubble integrand numerator we only have to pinch the two legs 1 and 2, and use (7.2.2)

$$\mathcal{D}_{12} Q_3 = Z_2(12,3)\dot{G}^2_{13}. \tag{7.3.1}$$

The explicit expression for the Lorentz two-cycle for this case is

$$Z_2(12,3) = \varepsilon_2 \cdot k_1 Z_2(1,3) - \tfrac{1}{2} Z_3(1,2,3) - (1 \leftrightarrow 2). \tag{7.3.2}$$

We can immediately see that $f_3^{\nu\mu}$ can be factorized out to give the two-current field strength numerator

$$f^{\mu\nu}_{[1,2]} = \varepsilon_2 \cdot k_1 f_1^{\mu\nu} - (f_1 f_2)^{\mu\nu} - (1 \leftrightarrow 2)\,. \tag{7.3.3}$$

From the definition of the two-tail in (2.3.43) and using (7.2.6), we compute

$$\mathcal{D}_{13}\mathcal{D}_{12} T(1,2) = \varepsilon_{[1,2]} \cdot k_3, \tag{7.3.4}$$

and extract the two-particle polarization

$$\varepsilon^{\mu}_{[1,2]} = \tfrac{1}{2}\left[\varepsilon_2 \cdot k_1 \varepsilon_1^{\mu} - \varepsilon_{1\rho} f_2^{\rho\mu} - (1 \leftrightarrow 2)\right]. \tag{7.3.5}$$

It is not hard to check that, at rank two, the multiparticle fields are proportional to their Berends–Giele counterparts. This is plausible, since the latter only describe a single cubic diagram —see figure 7.7. Evidently the fields $f^{\mu\nu}_{[1,2]}$ and $\varepsilon^{\mu}_{[1,2]}$ are antisymmetric in 1 and 2, and obey to the order-1 GJI:

$$\begin{aligned} \varepsilon^{\mu}_{[1,2]} + \varepsilon^{\mu}_{[2,1]} &= 0 \\ f^{\mu\nu}_{[1,2]} + f^{\mu\nu}_{[2,1]} &= 0. \end{aligned} \tag{7.3.6}$$

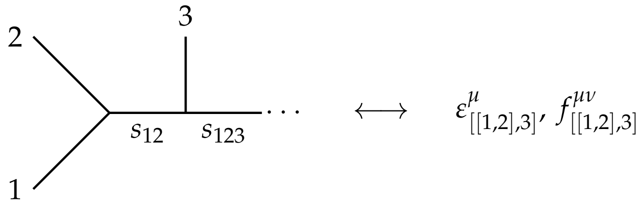


FIG. 7.8: Diagrammatic interpretation of the rank-three multiparticle fields $\varepsilon^{\mu}_{[[1,2],3]}, f^{\mu\nu}_{[[1,2],3]}$ subject to Jacobi relations such as $\varepsilon^{\mu}_{[[1,2],3]} + \mathrm{cyc}(1,2,3) = 0$.

7.3.2 *Three-Particle Case*

Considering now rank three, the numerator $f^{\mu\nu}_{[[1,2],3]}$ is obtained starting from the one-cycle components of Q_4, i.e.

$$\tilde{Q}_4 = Q_4^4 + Q_4^3 + Q_4^2. \tag{7.3.7}$$

The latter will immediately deliver the field strength three-current. Using the defining relation

$$\mathscr{D}_{13}\mathscr{D}_{12}\tilde{Q}_4 = Z_2(123,4)\dot{G}_{14}^2, \tag{7.3.8}$$

we easily extract

$$f^{\mu\nu}_{[[1,2],3]} = k^{\mu}_{123}\varepsilon^{\nu}_{[[1,2],3]} - k_{12}\cdot k_3\varepsilon^{\mu}_{[1,2]}\varepsilon^{\nu}_3 \\ - k_1\cdot k_2(\varepsilon^{\mu}_1\varepsilon^{\nu}_{[2,3]} + \varepsilon^{\mu}_{[1,3]}\varepsilon^{\nu}_2) - (\mu \leftrightarrow \nu). \tag{7.3.9}$$

Here for compactness we have included the multiparticle polarization $\varepsilon^{\nu}_{[[1,2],3]}$. We compute the latter using the expression for $T(1,2,3)$ that can be found in appendix C of [37]. In particular, making use of (7.2.6)

$$\mathscr{D}_{14}\mathscr{D}_{13}\mathscr{D}_{12}\, T(1,2,3) = \varepsilon_{[[1,2],3]}\cdot k_4, \tag{7.3.10}$$

we obtain

$$\varepsilon^{\mu}_{[[1,2],3]} = \tfrac{1}{2}\Big[\left(k_3\cdot\varepsilon_{[1,2]}\right)\varepsilon^{\mu}_3 - (k_{12}\cdot\varepsilon_3)\,\varepsilon^{\mu}_{[1,2]} \\ + \varepsilon_{[1,2]\nu}f_3^{\nu\mu} - \varepsilon_{3\nu}f^{\nu\mu}_{[1,2]}\Big] - k^{\mu}_{123}h_{123}, \tag{7.3.11}$$

where we have included the extra term

$$h_{123} = \tfrac{1}{4}\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot(k_2 - k_1)\,. \tag{7.3.12}$$

We employ the h_P scalars here for notational compactness and to compare with the results of [98, 127], where those were introduced to take the numerators from the Lorenz gauge to the BCJ gauge. The expressions for the h_P's are different from the ones in [98, 127], but the difference can be understood as a residual generalized gauge transformation that doesn't affect the symmetry properties of the coefficients. However, the non-trivial fact that we want to point out here is that in [98, 127] the scalars h_P's were introduced as a result of non-linear transformations in order to achieve numerators in the BCJ gauge. This is not our case: the scalars h_P's appear naturally as a result of the pinch algorithm, and further transformations are *not* needed to obtain coefficients in the desired representation —see figure 7.8. Indeed it easy to check that both (7.3.9) and (7.3.11) satisfy the GJI

$$\begin{aligned} \varepsilon^{\mu}_{[[1,2],3]} + \varepsilon^{\mu}_{[[2,1],3]} &= 0 \\ \varepsilon^{\mu}_{[[1,2],3]} + \varepsilon^{\mu}_{[[2,3],1]} + \varepsilon^{\mu}_{[[3,1],2]} &= 0. \end{aligned} \tag{7.3.13}$$

7.3.3 *Four-Particle Case*

Following the same procedure at rank four (see appendix C of [37] for the expression of Q_5), we arrive at the field strength numerator in the four-particle case

$$\begin{aligned} f^{\mu\nu}_{[[[1,2],3],4]} =& k^{\mu}_{1234}\varepsilon^{\nu}_{[[[1,2],3],4]} + k_{123}\cdot k_4 \varepsilon^{\nu}_{[[1,2],3]}\varepsilon^{\mu}_4 + k_{12}\cdot k_3\left[\varepsilon^{\nu}_{[1,2]}\varepsilon^{\mu}_{[3,4]}\right. \\ &\left.+\varepsilon^{\nu}_{[[1,2],4]}\varepsilon^{\mu}_3\right] + k_1\cdot k_2\left[\varepsilon^{\nu}_1\varepsilon^{\mu}_{[[2,3],4]} + \varepsilon^{\nu}_{[[1,3],4]}\varepsilon^{\mu}_2 + \varepsilon^{\nu}_{[1,3]}\varepsilon^{\mu}_{[2,4]}\right. \\ &\left.+\varepsilon^{\nu}_{[1,4]}\varepsilon^{\mu}_{[2,3]}\right] - (\mu \leftrightarrow \nu). \end{aligned} \tag{7.3.14}$$

We also find the numerator of the four-particle polarization using tail $T(1,2,3,4)$ that again can be found in the appendix C of [37]. Computing

$$\mathcal{D}_{15}\mathcal{D}_{14}\mathcal{D}_{13}\mathcal{D}_{12}\, T(1,2,3,4) = \varepsilon_{[[[1,2],3],4]}\cdot k_5, \tag{7.3.15}$$

we extract

$$\begin{aligned} \varepsilon^{\mu}_{[[[1,2],3],4]} =& \tfrac{1}{2}\left[\varepsilon^{\mu}_4\left(\varepsilon_{[[1,2],3]}\cdot k_4\right) - \varepsilon^{\mu}_{[[1,2],3]}\left(\varepsilon_4\cdot k_{123}\right) + \varepsilon_{[[1,2],3]\nu} f^{\nu\mu}_4\right. \\ &\left. - \varepsilon_{4\nu} f^{\nu\mu}_{[[1,2],3]}\right] + \varepsilon^{\mu}_3 (k_{12}\cdot k_3) h_{124} + k_1\cdot k_2\big(\varepsilon^{\mu}_2 h_{134} \\ & - \varepsilon^{\mu}_1 h_{234}\big) - k^{\mu}_{1234} h_{1234}, \end{aligned} \tag{7.3.16}$$

where the new scalar h_{1234} is defined as

$$\begin{aligned} h_{1234} =& \tfrac{1}{4}\big[\varepsilon_1\cdot\varepsilon_2\, \varepsilon_3\cdot k_2 \varepsilon_4\cdot\, (k_1 - k_{23}) \\ & + \tfrac{1}{2}\left(\varepsilon_1\cdot\varepsilon_2\, \varepsilon_3\cdot\varepsilon_4\, k_2\cdot k_3\right) - (123 \to 312)\big] - (1\leftrightarrow 2). \end{aligned} \tag{7.3.17}$$

A corresponding multiparticle field with the BCJ property was also obtained in [98, 127]: here a two-step procedure involving a BRST-inspired transformation was necessary to compute the numerators in the desired gauge. It is worth noting that this transformation is more complicated than the one used at rank three to compute h_{123}, i.e. it requires an additional step to obtain the final result. Moreover, as one can expect, the complexity of the transformations increases at increasing rank. Thus, the benefit of our procedure is really far from trivial:

the numerators are obtained immediately in BCJ gauge regardless of the rank we are considering, and complex transformations are avoided at all. We briefly mention the GJI satisfied by the numerators:

$$\begin{aligned} \varepsilon^{\mu}_{[[[1,2],3],4]} + \varepsilon^{\mu}_{[[[2,1],3],4]} &= 0 \\ \varepsilon^{\mu}_{[[[1,2],3],4]} + \varepsilon^{\mu}_{[[[3,1],2],4]} + \varepsilon^{\mu}_{[[[2,3],1],4]} &= 0 \\ \varepsilon^{\mu}_{[[[1,2],3],4]} - \varepsilon^{\mu}_{[[[1,2],4],3]} + \varepsilon^{\mu}_{[[[3,4],1],2]} - \varepsilon^{\mu}_{[[[3,4],2],1]} &= 0. \end{aligned} \tag{7.3.18}$$

One noteworthy property is related to some symmetry properties exhibited by the scalars h_P that are computed using our pinch technique. Indeed, looking at the first examples (7.3.12)-(7.3.17), it is clear that the scalars $h_{12\ldots n}$ satisfy the GJI of order $n-1$, e.g.

$$h_{123} + h_{213} = 0 \tag{7.3.19}$$

and

$$\begin{aligned} h_{1234} + h_{2134} &= 0 \\ h_{1234} + h_{2314} + h_{3124} &= 0. \end{aligned} \tag{7.3.20}$$

This property will continue to hold for the $n = 5$ case, as it will be clear in the following. This fact should not be surprising. This result should not come as a surprise, since in the original derivation of the h_P fields (e.g. see [127]), it was shown that, for any $|P| = p$, h_P satisfies all Lie symmetries $\mathcal{L}_k$ with $k \leq p-1$. Our pinch technique confirms this property.

7.3.4 *Five-Particle Case*

As a final example, we present the rank-five numerators computed through the pinch technique. Again, looking at the one-cycle components of Q_6, always contained in [37], we extract the multiparticle field strength

$$\begin{aligned} f^{\mu\nu}_{[[[[1,2],3],4],5]} =& k^{\mu}_{12345}\varepsilon^{\nu}_{[[[[1,2],3],4],5]} + k_{1234}\cdot k_5 \varepsilon^{\mu}_5 \varepsilon^{\nu}_{[[[1,2],3],4]} \\ &+ k_{123}\cdot k_4 \big(\varepsilon^{\mu}_4 \varepsilon^{\nu}_{[[[1,2],3],5]} + \varepsilon^{\mu}_{45}\varepsilon^{\nu}_{[[1,2],3]}\big) \\ &+ k_{12}\cdot k_3 \big(\varepsilon^{\mu}_3 \varepsilon^{\nu}_{[[[1,2],4],5]} + \varepsilon^{\mu}_{35}\varepsilon^{\nu}_{[[1,2],4]} + \varepsilon^{\mu}_{34}\varepsilon^{\nu}_{[[1,2],5]} + \varepsilon^{\mu}_{345}\varepsilon^{\nu}_{[1,2]}\big) \\ &+ k_1\cdot k_2 \big(\varepsilon^{\mu}_2 \varepsilon^{\nu}_{[[[1,3],4],5]} + \varepsilon^{\mu}_{25}\varepsilon^{\nu}_{[[1,3],4]} + \varepsilon^{\mu}_{24}\varepsilon^{\nu}_{[[1,3],5]} + \varepsilon^{\mu}_{23}\varepsilon^{\nu}_{[[1,4],5]} \\ &+ \varepsilon^{\mu}_{245}\varepsilon^{\nu}_{[1,3]} + \varepsilon^{\mu}_{235}\varepsilon^{\nu}_{[1,4]} + \varepsilon^{\mu}_{234}\varepsilon^{\nu}_{[1,5]} + \varepsilon^{\mu}_{2345}\varepsilon^{\nu}_{1}\big) - (\mu \leftrightarrow \nu), \end{aligned} \tag{7.3.21}$$

where

$$\begin{aligned}\varepsilon^{\mu}_{[[[[1,2],3],4],5]} = \frac{1}{2}\Big[&\varepsilon^{\mu}_{5}\left(\varepsilon_{[[[1,2],3],4]}\cdot k_5\right) - \varepsilon^{\mu}_{[[[1,2],3],4]}\left(\varepsilon_5\cdot k_{1234}\right)\\ &+ \varepsilon_{[[[1,2],3],4]\nu} f_5^{\nu\mu} - \varepsilon_{5\nu} f^{\nu\mu}_{[[[1,2],3],4]}\Big] + (k_{123}\cdot k_4)\varepsilon^{\mu}_{4} h_{1235}\\ &+ (k_{12}\cdot k_3)\big(\varepsilon^{\mu}_{3} h_{1245} + \varepsilon^{\mu}_{[3,4]} h_{125} - \varepsilon^{\mu}_{[1,2]} h_{345}\big)\\ &+ (k_1\cdot k_2)\big(\varepsilon^{\mu}_{2} h_{1345} + \varepsilon^{\mu}_{[2,3]} h_{145} + \varepsilon^{\mu}_{[2,4]} h_{135}\\ &- \varepsilon^{\mu}_{1} h_{2345} - \varepsilon^{\mu}_{[1,3]} h_{245} - \varepsilon^{\mu}_{[1,4]} h_{235}\big) - k^{\mu}_{12345} h_{12345}. \end{aligned} \tag{7.3.22}$$

For the sake of brevity, we omit here the expression of the scalar h_{12345}, that can be found in [76, 129]. Above the expression of the multiparticle polarization $\varepsilon_{[[[[1,2],3],4],5]}$ has been computed using the rule (7.2.6) through the five-point tail $T(1,2,3,4,5)$, whose expression is contained in [76]. It should be noted that a closed formula for the construction of currents of the form (7.3.21) with any number of particles already exists in the literature, e.g. as described in [130].

Finally, we explicit the GJI obeyed by the numerators (7.3.22) and (7.3.21):

$$\begin{aligned} \varepsilon^{\mu}_{[[[[1,2],3],4],5]} + \varepsilon^{\mu}_{[[[[2,1],3],4],5]} &= 0\\ \varepsilon^{\mu}_{[[[[1,2],3],4],5]} + \varepsilon^{\mu}_{[[[[3,1],2],4],5]} + \varepsilon^{\mu}_{[[[[2,3],1],4],5]} &= 0,\\ \varepsilon^{\mu}_{[[[[1,2],3],4],5]} - \varepsilon^{\mu}_{[[[[1,2],4],5],3]} + \varepsilon^{\mu}_{[[[[3,4],1],2],5]} - \varepsilon^{\mu}_{[[[[3,4],2],1],5]} &= 0,\\ \varepsilon^{\mu}_{[[[[1,2],3],4],5]} - \varepsilon^{\mu}_{[[[[1,2],3],5],4]} +&\\ \varepsilon^{\mu}_{[[[[4,5],1],2],3]} - \varepsilon^{\mu}_{[[[[4,5],2],1],53]} - \varepsilon^{\mu}_{[[[[4,5],3],1],2]} + \varepsilon^{\mu}_{[[[[4,5],3],2],1]} &= 0. \end{aligned} \tag{7.3.23}$$

In this section we have showed a simple and algorithmic method to compute numerators ε^{μ}_{P} and $f^{\mu\nu}_{P}$ for cubic off-shell diagrams. We have computed these objects using techniques borrowed from the Bern-Kosower formalism, that allow us to obtain fields directly in the BCJ gauge. The connection of the multiparticle fields to Berends-Giele currents will be further explained in the following section. Now we just want to mention that we have explicitly checked the combinatorial properties of the numerators ε^{μ}_{P} and $f^{\mu\nu}_{P}$ up to length $|P| = 7$, and the procedure for the construction of such quantities is not expected to break at higher orders. In particular, the construction of tails, that in turn provide the expression for ε^{μ}_{P}, requires increasing computational time at increasing rank, but still the symmetric IBP algorithm applies and, in principle, we are able to obtain numerators at any number of points.

7.4 MULTIPARTICLE FIELDS AND BERENDS-GIELE CURRENTS

Armed with the results of the previous section, we want now to relate the Jacobi-satisfying numerators for cubic off-shell diagrams to

Berends–Giele currents in the BCJ gauge. The idea is relatively simple: we want to construct alternative currents combining cubic diagrams and dressing them with multiparticle polarizations and propagators. In particular, we take inspiration from the recursion relations in (4.2.15) and (4.2.29), where color-stripped and color-dressed currents respectively are computed. Now we want to compute similar currents using the multiparticle fields seen in the previous section. The final goal is to obtain expression for color-ordered and full Yang-Mills amplitudes in such a way that color-kinematics duality is satisfied. The relevant quantity for the computation of these objects is the multiparticle polarization ε_P^μ, so in the following we can restrict our attention to it.

7.4.1 *Color-Stripped Berends-Giele Currents*

In order to compute color-stripped (or color-ordered) Berends-Giele currents, it is very convenient to introduce a new tool termed *color-stripped Berends-Giele map*, or *binary tree map* as in [16, 131]. This represents a combinatorial artifact that allows us to identify all the planar binary trees in a given color-ordered amplitude and that helps to keep track of the correspondence between these trees and nested Lie brackets. It is defined as the map b_{cs} acting on all words and determined recursively by

$$\begin{aligned} b_{\text{cs}}(i) &= i, \\ b_{\text{cs}}(P) &= \frac{1}{s_P} \sum_{P=QR} [b_{\text{cs}}(Q), b_{\text{cs}}(R)], \end{aligned} \tag{7.4.1}$$

where s_P is the Mandelstam invariant, and where $\sum_{P=QR}$ denotes the sum over all possible deconcatenations of the word P into Q and R. We denote $p = |P|$ as the length of the word P. The sum in (7.4.1) has to be understood as reproducing all the possible color-ordered cubic diagrams that can be constructed with $n = p + 1$ external legs, or equivalently as all the possible nested brackets built with the ordered word P. Also as a matter of notation, for an arbitrary labeled object U_P, such as the multiparticle polarization fields ε_P^μ, we bring the definition from [16] for the replacement of words by such object as

$$[\![U]\!] \circ P = U_P. \tag{7.4.2}$$

With this background in mind, the color-stripped Berends-Giele polarization current can be defined from a purely combinatorial point of view based on the map $b_{\text{cs}}(P)$ acting on the multiparticle polarization fields ε_P^μ, that in our notation represent the numerators of specific coefficients within the Berends-Giele current. In particular, the Berends-Giele current is defined as

$$J_P^\mu = [\![\varepsilon^\mu]\!] \circ b_{\text{cs}}(P). \tag{7.4.3}$$

As an example, the color-stripped Berends-Giele polarization currents up to multiplicity four would read

$$\begin{aligned}
J^{\mu}_{1} &= \varepsilon^{\mu}_{1}, \\
J^{\mu}_{12} &= \frac{\varepsilon^{\mu}_{[1,2]}}{s_{12}}, \\
J^{\mu}_{123} &= \frac{\varepsilon^{\mu}_{[[1,2],3]}}{s_{12}s_{123}} + \frac{\varepsilon^{\mu}_{[1,[2,3]]}}{s_{23}s_{123}}, \\
J^{\mu}_{1234} &= \frac{\varepsilon^{\mu}_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{\varepsilon^{\mu}_{[[1,[2,3]],4]}}{s_{123}s_{1234}s_{23}} + \frac{\varepsilon^{\mu}_{[[1,2],[3,4]]}}{s_{12}s_{1234}s_{34}} + \frac{\varepsilon^{\mu}_{[1,[[2,3],4]]}}{s_{1234}s_{23}s_{234}} \\
&\quad + \frac{\varepsilon^{\mu}_{[1,[2,[3,4]]]}}{s_{1234}s_{234}s_{34}}.
\end{aligned} \tag{7.4.4}$$

In these expressions, the multiparticle polarization fields ε^{μ}_{P} in first position in each line represent respectively the numerators computed earlier in (7.3.5), (7.3.11) and (7.3.16) respectively. Clearly, this cannot be the end of the story, as other numerators with different nested bracket structures, i.e. corresponding to different diagrams with respect to figure 7.6, are present. However, this is not a problem at all, as the multiparticle polarizations ε^{μ}_{P} satisfy the GJI! In other words, we can use the very same symmetries of the commutators within each nested bracket that labels the numerators. In particular, it is easy to obtain

$$\varepsilon^{\mu}_{[1,[2,3]]} = -\varepsilon^{\mu}_{[[2,3],1]}, \tag{7.4.5}$$

and

$$\begin{aligned}
\varepsilon^{\mu}_{[[1,[2,3]],4]} &= -\varepsilon^{\mu}_{[[[2,3],1],4]} \\
\varepsilon^{\mu}_{[1,[[2,3],4]]} &= -\varepsilon^{\mu}_{[[[2,3],4],1]} \\
\varepsilon^{\mu}_{[1,[2,[3,4]]]} &= \varepsilon^{\mu}_{[[[3,4],2],1]} \\
\varepsilon^{\mu}_{[[1,2],[3,4]]} &= \varepsilon^{\mu}_{[[[1,2],3],4]} - \varepsilon^{\mu}_{[[[1,2],4],3]},
\end{aligned} \tag{7.4.6}$$

so that these multiparticle polarization fields in (7.4.4) are obtained from the formulae in (7.3.5), (7.3.11) and (7.3.16) by a simple relabeling. At this point, however, we should perhaps emphasize that the way of representing the color-stripped Berends-Giele polarization currents in (7.4.3) is always possible regardless of whether or not the multiparticle polarization fields ε^{μ}_{P} satisfy the GJI. When they do, as it is the case in the present discussion, we can exploit these identities to reduce the computational effort and obtain the full current from a single numerator ε^{μ}_{P}. We note, moreover, that the GJI satisfied by the multiparticle polarization fields ε^{μ}_{P} translate directly into the shuffle symmetry $J^{\mu}_{Q\sqcup R} = 0$ of the currents in (7.4.3), as expected from the relation obtained earlier in (4.1.10). In section 7.3 we have carried out the computation of the rank-five multiparticle polarization $\varepsilon^{\mu}_{[[[[1,2],3],4],5]}$.

For brevity, we won't report here the full Berends-Giele current associated to such numerator, as it can be easily obtained combining (7.3.22), (7.4.3) and GJI's when needed.

Now that we have an explicit form of the color-stripped Berends-Giele polarization currents, the next task is to write down the color-stripped perturbiner expansion. This is a simple matter: we just set it to be generating series

$$A^\mu(x) = \sum_P J_P^\mu e^{k_P \cdot x} T^{a_P}, \tag{7.4.7}$$

where the sum is performed over the set of non-empty words $P = 12\ldots n$ with different length. It is important to note that the shuffle symmetry satisfied by the constituent currents J_P^μ guarantees that the generating series (7.4.7) is a Lie algebra-valued field. This expansion does not come directly from the Yang-Mills action, since in our case we have only trivalent vertices with no use of auxiliary fields.

To complete our discussion we must also remember how the color-stripped Berends-Giele polarization currents J_P^μ are related to the scattering amplitudes in Yang-Mills theory. At tree level, the color-ordered partial amplitude of n gluons is determined through the Berends-Giele formula

$$A_n(1,2,\ldots,n) = s_{12\cdots(n-1)} J^\mu_{12\cdots(n-1)} J_{n\mu}, \tag{7.4.8}$$

as already seen in (4.1.6). The factor $s_{12\cdots(n-1)}$ is inserted to cancel the off-shell propagator inside $J_{12\cdots(n-1)}$. Now that we are assuming momentum conservation and have on-shell external legs, there are off-shell terms that cancel out, e.g. the ones of the form $k_P^\mu h_P$ at the end of each polarization. Finally it may be remarked that, by virtue of the shuffle symmetry, the partial amplitudes in the form of (7.4.8) satisfy the Kleiss-Kuijf relations [85] —see (3.2.8).

7.4.2 *Color-Dressed Berends-Giele Currents*

Now we turn our attention to obtaining the color-dressed Berends-Giele polarization currents from the multiparticle polarization fields ε_P^μ. These type of currents, suitable for the computation of full Yang-Mills amplitudes, have been presented earlier in section 4.2.2, where the perturbiner methods have been used. The procedure to compute the color-dressed Berends-Giele currents will be similar to the one showed in the previous section, with some small, but important, changes. In the first place, we need to modify the color-stripped Berends-Giele map (7.4.1) by a color-dressed version of it, which we write as b_{cd}. Here we borrow the prescription already depicted in [132]. Namely, we define b_{cd} as the map acting on all ordered words and determined recursively by

$$\begin{aligned} b_{\mathsf{cd}}(i) &= i, \\ b_{\mathsf{cd}}(P) &= \frac{1}{2s_P} \sum_{P=Q\cup R} [b_{\mathsf{cd}}(Q), b_{\mathsf{cd}}(R)], \end{aligned} \tag{7.4.9}$$

where $\sum_{P=Q\cup R}$ denotes the sum over all possible ways of distributing the letters of the ordered word P into non-empty ordered words Q and R. We remark that the factor of 2 in the denominator can be dropped if we impose the condition that $|Q| \geq |R|$. The decomposition of type $\sum_{P=Q\cup R}$ has already been encountered in the recursion (4.2.29) from color-dressed perturbiner, where we have checked that such decomposition correctly reproduces the locality structure of the full Yang-Mills amplitude. In the second place, for each ordered word $P = i_1 i_2 \cdots i_n$ of length n, we employ the notation c^a_P to indicate the product of color factors determined by

$$c^a_P = \tilde{f}_{a_{i_1} a_{i_2}}{}^{b} \tilde{f}_{b a_{i_3}}{}^{c} \cdots \tilde{f}_{d a_{i_{n-1}}}{}^{e} \tilde{f}_{e a_{i_n}}{}^{a}, \tag{7.4.10}$$

with the understanding that $c^a_i = \delta^a{}_{a_i}$. We further put

$$c^a_{[P,Q]} = \tilde{f}_{bc}{}^{a} c^b_P c^c_Q \tag{7.4.11}$$

for any pair of ordered words P and Q. In the third place, given two arbitrary labeled objects U_P and V_P, we define the replacement of ordered words by the product of such objects as

$$[\![U \otimes V]\!] \circ P = U_P V_P. \tag{7.4.12}$$

By making use of the foregoing, one can show that we can write the color-dressed Berends-Giele polarization currents in the form

$$J^{a\mu}_P = [\![c^a \otimes \varepsilon^\mu]\!] \circ b_{\mathsf{cd}}(P). \tag{7.4.13}$$

As we pointed out in the previous section, the multiparticle polarization fields ε^μ_P entering in this representation of the color-dressed Berends-Giele currents are not restricted to satisfy the GJI. When they do, as it is the case in the present discussion, we see that such identities mirror the GJI satisfied by the color factor c^a_P. Hence, we are led to the conclusion that the *factorization* of the color-dressed Berends-Giele polarization currents given in (7.4.13) is a realization of the color-kinematics duality. In the next chapter, we will see that in terms of this factorization, the double-copy prescription is straightforward to phrase.

We shall now proceed to write down explicitly the color-dressed Berends-Giele polarization currents up to multiplicity four, in order to familiarize ourselves with formula (7.4.13). We first consider the single-particle case in which $P = 1$. Then we at once obtain

$$J^{a\mu}_1 = \delta^a{}_{a_1} \varepsilon^\mu_1. \tag{7.4.14}$$

Next we consider the two-particle case in which $P = 12$. In this case, the only possible way of distributing the letters is $(Q, R) = (1, 2)$, and thus we find that color-dressed Berends-Giele polarization current $J_{12}^{a\mu}$ acquires the form

$$J_{12}^{a\mu} = \frac{c^a_{[1,2]}\varepsilon^\mu_{[1,2]}}{s_{12}}, \tag{7.4.15}$$

with color factor $c^a_{[1,2]} = \tilde{f}_{a_1a_2}{}^a$ and two-particle polarization field $\varepsilon^\mu_{[1,2]}$ given by (7.3.5). Let us next take up the three-particle case in which $P = 123$. In this case, the possible ways of distributing the letters that contribute to the sum are $(Q, R) = (12, 3), (13, 2), (23, 1)$. Therefore, after a straightforward calculation making use of the recursion (7.4.9) we obtain for the color-dressed Berends-Giele polarization current $J_{123}^{a\mu}$ the formula

$$J_{123}^{a\mu} = \frac{c^a_{[[1,2],3]}\varepsilon^\mu_{[[1,2],3]}}{s_{12}s_{123}} + \frac{c^a_{[[1,3],2]}\varepsilon^\mu_{[[1,3],2]}}{s_{13}s_{123}} + \frac{c^a_{[[2,3],1]}\varepsilon^\mu_{[[2,3],1]}}{s_{23}s_{123}}, \tag{7.4.16}$$

with color factors $c^a_{[[1,2],3]} = \tilde{f}_{a_1a_2}{}^b\tilde{f}_{ba_3}{}^a$, $c^a_{[[1,3],2]} = \tilde{f}_{a_1a_3}{}^b\tilde{f}_{ba_2}{}^a$, $c^a_{[[2,3],1]} = \tilde{f}_{a_2a_3}{}^b\tilde{f}_{ba_1}{}^a$ and three-particle polarization fields obtained by (7.3.11) after relabeling when needed. Finally, we consider the four-particle case in which $P = 1234$. In this case, the possible ways of distributing the letters that contribute to the sum are $(Q, R) = (123, 4)$, $(124, 3)$, $(134, 2)$, $(234, 1)$, $(12, 34)$, $(13, 24)$, $(23, 14)$. By analogy with the calculation leading to (7.4.16), we find that the color-dressed Berends-Giele polarization current $J_{1234}^{a\mu}$ may be represented in the form

$$\begin{aligned} J_{1234}^{a\mu} = &\frac{c^a_{[[[1,2],3],4]}\varepsilon^\mu_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{c^a_{[[[1,2],4],3]}\varepsilon^\mu_{[[[1,2],4],3]}}{s_{12}s_{124}s_{1234}} + \frac{c^a_{[[[1,3],4],2]}\varepsilon^\mu_{[[[1,3],4],2]}}{s_{13}s_{134}s_{1234}} \\ &+ \frac{c^a_{[[[2,3],4],1]}\varepsilon^\mu_{[[[2,3],4],1]}}{s_{23}s_{234}s_{1234}} + \frac{c^a_{[[[1,3],2],4]}\varepsilon^\mu_{[[[1,3],2],4]}}{s_{13}s_{123}s_{1234}} + \frac{c^a_{[[[1,4],2],3]}\varepsilon^\mu_{[[[1,4],2],3]}}{s_{14}s_{124}s_{1234}} \\ &+ \frac{c^a_{[[[1,4],3],2]}\varepsilon^\mu_{[[[1,4],3],2]}}{s_{14}s_{134}s_{1234}} + \frac{c^a_{[[[2,3],1],4]}\varepsilon^\mu_{[[[2,3],1],4]}}{s_{23}s_{123}s_{1234}} + \frac{c^a_{[[[2,4],1],3]}\varepsilon^\mu_{[[[2,4],1],3]}}{s_{24}s_{124}s_{1234}} \\ &+ \frac{c^a_{[[[2,4],3],1]}\varepsilon^\mu_{[[[2,4],3],1]}}{s_{24}s_{234}s_{1234}} + \frac{c^a_{[[[3,4],1],2]}\varepsilon^\mu_{[[[3,4],1],2]}}{s_{34}s_{134}s_{1234}} + \frac{c^a_{[[[3,4],2],1]}\varepsilon^\mu_{[[[3,4],2],1]}}{s_{34}s_{234}s_{1234}} \\ &+ \frac{c^a_{[[1,2],[3,4]]}\varepsilon^\mu_{[[1,2],[3,4]]}}{s_{12}s_{34}s_{1234}} + \frac{c^a_{[[1,3],[2,4]]}\varepsilon^\mu_{[[1,3],[2,4]]}}{s_{13}s_{24}s_{1234}} + \frac{c^a_{[[1,4],[2,3]]}\varepsilon^\mu_{[[1,4],[2,3]]}}{s_{14}s_{23}s_{1234}}. \end{aligned} \tag{7.4.17}$$

Here the color factors are easily determined from (7.4.10) and (7.4.11) as

$$\begin{aligned} c^a_{[[[1,2],3],4]} &= \tilde{f}_{a_1a_2}{}^b\tilde{f}_{ba_3}{}^c\tilde{f}_{ca_4}{}^a, & c^a_{[[[1,2],4],3]} &= \tilde{f}_{a_1a_2}{}^b\tilde{f}_{ba_4}{}^c\tilde{f}_{ca_3}{}^a, \\ c^a_{[[[1,3],4],2]} &= \tilde{f}_{a_1a_3}{}^b\tilde{f}_{ba_4}{}^c\tilde{f}_{ca_2}{}^a, & c^a_{[[[2,3],4],1]} &= \tilde{f}_{a_2a_3}{}^b\tilde{f}_{ba_4}{}^c\tilde{f}_{ca_1}{}^a, \\ c^a_{[[[1,3],2],4]} &= \tilde{f}_{a_1a_3}{}^b\tilde{f}_{ba_2}{}^c\tilde{f}_{ca_4}{}^a, & c^a_{[[[1,4],2],3]} &= \tilde{f}_{a_1a_4}{}^b\tilde{f}_{ba_2}{}^c\tilde{f}_{ca_3}{}^a, \\ c^a_{[[[1,4],3],2]} &= \tilde{f}_{a_1a_4}{}^b\tilde{f}_{ba_3}{}^c\tilde{f}_{ca_2}{}^a, & c^a_{[[[2,3],1],4]} &= \tilde{f}_{a_2a_3}{}^b\tilde{f}_{ba_1}{}^c\tilde{f}_{ca_4}{}^a, \end{aligned}$$

$$
\begin{aligned}
c^a_{[[[2,4],1],3]} &= \tilde{f}_{a_2a_4}{}^{b}\tilde{f}_{ba_1}{}^{c}\tilde{f}_{ca_3}{}^{a}, & c^a_{[[[2,4],3],1]} &= \tilde{f}_{a_2a_4}{}^{b}\tilde{f}_{ba_3}{}^{c}\tilde{f}_{ca_1}{}^{a},\\
c^a_{[[[3,4],1],2]} &= \tilde{f}_{a_3a_4}{}^{b}\tilde{f}_{ba_1}{}^{c}\tilde{f}_{ca_2}{}^{a}, & c^a_{[[[3,4],2],1]} &= \tilde{f}_{a_3a_4}{}^{b}\tilde{f}_{ba_2}{}^{c}\tilde{f}_{ca_1}{}^{a},\\
c^a_{[[1,2],[3,4]]} &= \tilde{f}_{a_1a_2}{}^{b}\tilde{f}_{a_3a_4}{}^{c}\tilde{f}_{bc}{}^{a}, & c^a_{[[1,3],[2,4]]} &= \tilde{f}_{a_1a_3}{}^{b}\tilde{f}_{a_2a_4}{}^{c}\tilde{f}_{bc}{}^{a},\\
c^a_{[[1,4],[2,3]]} &= \tilde{f}_{a_1a_4}{}^{b}\tilde{f}_{a_2a_3}{}^{c}\tilde{f}_{bc}{}^{a}. && \qquad (7.4.18)
\end{aligned}
$$

As for the four-particle polarization fields, keeping in mind the identities

$$
\begin{aligned}
\varepsilon^{\mu}_{[[1,2],[3,4]]} &= \varepsilon^{\mu}_{[[[1,2],3],4]} - \varepsilon^{\mu}_{[[[1,2],4],3]}\\
\varepsilon^{\mu}_{[[1,3],[2,4]]} &= \varepsilon^{\mu}_{[[[1,3],2],4]} - \varepsilon^{\mu}_{[[[1,3],4],2]}\\
\varepsilon^{\mu}_{[[1,4],[2,3]]} &= \varepsilon^{\mu}_{[[[1,4],2],3]} - \varepsilon^{\mu}_{[[[1,4],3],2]},
\end{aligned}
\qquad (7.4.19)
$$

they are all determined by (7.3.16) with the necessary relabelings. For brevity, the example of the color-dressed Berends-Giele polarization current at a rank of five is included in appendix D.

Having obtained the expression (7.4.13) for the color-dressed Berends-Giele polarization currents, we can of course then obtain the color-dressed perturbiner expansion. This is simply given as the generating series

$$
A^{a\mu}(x) = \sum_P J^{a\mu}_P e^{k_P\cdot x}, \qquad (7.4.20)
$$

where the sum is performed over the set of non-empty words $P = 12\ldots n$. Finally we remark that, despite we focused out attention on the multiparticle polarization field ε^{μ}_P, a color-dressed perturbiner expansion analogous to (7.4.20) can be obtained for the field strength: using the numerators $f^{\mu\nu}_P$ in the BCJ gauge obtained in section 7.3, we can compute color-dressed Berends-Giele field strength current associated with the multiparticle field strength.

Before leaving this section, let us comment on the role the color-dressed Berends-Giele polarization currents $J^{a\mu}_P$ play in the determination of the scattering amplitudes for Yang-Mills theory. As seen in (4.2.35), the Berends-Giele formula can be employed to obtain the full, or color-dressed, n-point amplitude

$$
\mathcal{A}^{\text{tree}}_n = s_{12\cdots(n-1)} J^{a\mu}_{12\cdots(n-1)} J_{na\mu} \qquad (7.4.21)
$$

where again we assume momentum conservation. It is also interesting to note that we may rewrite the amplitude (7.4.21) as

$$
\mathcal{A}^{\text{tree}}_n = \sum_{j\in\,\text{trivalent}} \frac{c_j n_j}{\prod_{i_j} s_{i_j}}. \qquad (7.4.22)
$$

where the sum goes over all $(2n-5)!!$ trivalent trees with propagators s_{i_j} associated to each internal edge i of the diagram. This corresponds exactly to the representation introduced in (3.1.6).

7.5 FINAL REMARKS

In this chapter we introduced a new approach for building Berends-Giele currents by making use of the Bern-Kosower formalism and a special pinch contribution to n-gluon amplitudes. By employing the multiparticle fields technology and combining them with the string-based rules, we have computed Berends-Giele numerators up to the five-point case. Interestingly, we have shown that the latter obey the GJI required by BCJ gauge, i.e. they satisfy color-kinematics duality. The most attractive feature of our formalism is that it never becomes necessary to determine gauge transformation terms to modify the numerators, as they appear naturally in the desired gauge. This is a significant advantage of our construction over others in literature, as seen in [16, 98, 127]. See [133] for comparison with a different technique recently appeared in literature. In turn, the multiparticle polarization vectors can be used as numerators of Berends-Giele currents, and, exploiting suitable symmetry properties, full tree-level Yang-Mills amplitudes are obtained from a single basic calculation.

In the following chapter, we will use an analogous approach to show that the multiparticle generalization of the gravity polarization tensor arise naturally as product of multiparticle fields, if we consider the Bern-Dunbar-Shimada formalism for one-loop gravity amplitudes. This will allow us to establish a new double-copy prescription for Berends-Giele currents in gravity.

8

DOUBLE COPY FROM THE STRING-INSPIRED FORMALISM

The relation for the gravity polarization tensor as the tensor product of two gluon polarization vectors has been well-known for a long time, but a version of this relation for multiparticle fields is presently still not known. Using the results presented in the previous chapter, we show that the multiparticle generalization of the gravity polarization tensor as product of multiparticle fields arise naturally in the Bern-Dunbar-Shimada formalism for one-loop gravity amplitudes, which is the gravitational counterpart of the Bern Kosower formalism described earlier. This allows us to formulate a revisited prescription for double-copy at the level of gravity Berends-Giele currents, and to obtain the gravitational Berends-Giele currents explicitly in the BCJ gauge. Results discussed in this chapter are published in [129].

8.1 MULTIPARTICLE POLARIZATION TENSORS FROM THE BERN-DUNBAR-SHIMADA FORMALISM

In the previous chapter we have presented an efficient way to construct Berends-Giele currents packed in the BCJ gauge, that is, they naturally display color-kinematics duality. This was made explicit by identifying Berends-Giele numerators, called multiparticle polarizations, which satisfy the generalized Jacobi identities (GJI). In order to extend such constructions to gravity, a possible strategy is to rely upon the well-known perturbative gauge-gravity duality. In chapter 3, we have seen some examples of how this duality is realized in field theory and how this can be related to established results in string theory. In particular, in section 3.6 we have seen that the gauge-gravity duality can be understood as a consequence of open-closed duality of string theory. At the perturbative string level, the latter gives rise to the KLT relations between open string amplitudes and closed string amplitudes. In the particle limit of string theory $\alpha' \to 0$, it leads to relations between tree-level graviton amplitudes and tree-level gluon amplitudes in Yang-Mills theories, which are often summarized as

$$\text{Gravity} = (\text{Gauge Theory})^2. \tag{8.1.1}$$

As seen in section 3.5, Bern, Carrasco and Johansson discovered a direct way of constructing gravity amplitudes from gauge theory amplitudes after organizing the latter in terms of cubic diagrams only and in such a way that the amplitude numerators respect color-kinematics duality —see (3.3.1)-(3.3.2). The great advantage of this representation is that the calculation of the associated gravity amplitudes is automatic: through the so-called double copy prescription, these are obtained in terms of the gauge theory information simply by replacing the color factors by another copy of the kinematic numerators and summing over the same cubic diagrams —see (3.5.2).

Earlier in this manuscript, we have deeply examined the Bern-Kosower formalism for the one-loop gluon scattering amplitudes. In particular, in the previous chapter we have investigated the *pinching procedure* of the Bern-Kosower formalism, that allows us to construct the reducible parts of the one-loop amplitudes from the irreducible ones at the level of the Feynman.-Schwinger integrands. We have implemented the pinching procedure with the introduction of a differential operator, the pinch operator defined in (7.1.4), and we have used it in a suitable way on the Bern-Kosower integrand in order to extract the multiparticle field strengths and polarizations that in this way appear naturally in the BCJ gauge. Using these numerators, we have been able to write down specific Berends-Giele currents, that in turn provide Yang-Mills amplitudes that naturally display color-kinematics duality.

As mentioned few lines above, in such a representation gravity amplitudes can be obtained on the spot using double copy, but we want to do more. Specifically, we consider now the Bern-Dunbar-Shimada formalism, introduced in section 2.3.5 as a double copy extension to gravity of the Bern-Kosower string-based rules, that takes origin from the decomposition of the closed string modes into left-movers and right-movers. Here we want to use the same algebraic tools with respect to the gluonic case in order to extract the multiparticle polarization tensor for gravity and establish a revised *double-copy prescription* at the level of the Berends-Giele currents within the perturbiner approach.

As point of departure, we first briefly discuss a systematic procedure, exactly analogous to the one described in section 7.2, to obtain the multiparticle polarization fields on the gravity side from the Bern-Dunbar-Shimada formalism for one-loop graviton amplitudes. In particular, we reconsider the symmetric partial integration algorithm and the Bern-Dunbar-Shimada rules explained in sections 2.3.4 and 2.3.5 respectively. In the latter, we have mentioned few extra details that have to be considered when symmetric IBP is applied to the Bern-Dunbar-Shimada formalism. In particular, it is generally not possible to remove all the double derivatives $\ddot{G}_{ij}$ and $\ddot{\bar{G}}_{ij}$ through integration by parts without generating extra terms involving the function $H_{ij} = H(\tau_i - \tau_j)$, that represent the coupling of the left- and right-movers through the zero mode of the string. However, this is not a problem at all, as we can order the integrand according to the powers of H_{ij}: from this, the part of the integrand containing no H_{ij}'s can still be factorized into the permutation invariant polynomial $\bar{Q}_n(\dot{\bar{G}})Q_n(\dot{G})$. The latter can be understood as the double copy generalization of the polynomial $Q_n(\dot{G})$, and comes into play when left- and right-movers conspire for gravity calculations. In addition, just as in (7.1.4) in the previous chapter, where we have associated for two adjacent legs i and j with $i<j$ a pinch operator, we may likewise define a *double pinch operator* acting on $\bar{Q}_n(\dot{\bar{G}})Q_n(\dot{G})$ as

$$\bar{\mathscr{D}}_{ij}\mathscr{D}_{ij}\bar{Q}_n(\dot{\bar{G}})Q_n(\dot{G}) = \left(\frac{\partial}{\partial\dot{\bar{G}}_{ij}}\bar{Q}_n(\dot{\bar{G}})\Big|_{\substack{\dot{\bar{G}}_{ij}=0\\ \dot{\bar{G}}_{jk}\to\dot{\bar{G}}_{ik}}}\right)\left(\frac{\partial}{\partial\dot{G}_{ij}}Q_n(\dot{G})\Big|_{\substack{\dot{G}_{ij}=0\\ \dot{G}_{jk}\to\dot{G}_{ik}}}\right). \tag{8.1.2}$$

Note that this new double pinch operator agrees with the replacement rule (2.3.50) provided by the BDS formalism. Moreover, as in (7.1.4), we are omitting the propagators in the pinching operator —these will be recovered later in the chapter. It is easy to see that the above double pinch operator is identical to the one for Yang-Mills applied independently to both the left- and right-mover parts of the integrand expression.

Now our goal is to find the multiparticle polarization tensors by iterated action of double pinch operators. Here we may borrow from the analysis carried out in the Yang-Mills case in the previous chapter, where we have learned that the part of the polynomial $Q_n(\dot{G})$ relevant to the multiparticle polarizations is the $(n-2)$-tail. This makes it feasible in the present situation to also consider the $(n-2)$-tail $\bar{T}(1,2,\ldots,n-2)T(1,2,\ldots,n-2)$. Applying the double pinch operator consecutively $n-2$ times to the latter in an analogous way to (7.2.6), one finds

$$\begin{aligned}&\bar{\mathscr{D}}_{1(n-1)}\mathscr{D}_{1(n-1)}\bar{\mathscr{D}}_{1(n-2)}\mathscr{D}_{1(n-2)}\cdots\bar{\mathscr{D}}_{13}\mathscr{D}_{13}\bar{\mathscr{D}}_{12}\mathscr{D}_{12}\\ &\quad\bar{T}(1,\ldots,n-2)T(1,\ldots,n-2) = \bar{\varepsilon}^{\mu}_{12\cdots(n-2)}\varepsilon^{\nu}_{12\cdots(n-2)}k_{(n-1)\mu}.\end{aligned} \tag{8.1.3}$$

This relation ensures that the quantity $\bar{\varepsilon}^{\mu}_{12\cdots(n-2)}\varepsilon^{\nu}_{12\cdots(n-2)}$ exactly describes the multiparticle polarization tensor, as we have intuitively understood in the Yang-Mills case in (7.2.8). We may also remark that, by construction, each of the individual factors $\bar{\varepsilon}^{\mu}_{12\cdots(n-2)}$ and $\varepsilon^{\nu}_{12\cdots(n-2)}$ satisfies the generalized Jacobi identity of order $n-2$ in $1,2,\ldots,n-2$. In other words, we can identify

$$\bar{\varepsilon}^{\mu}_{12\cdots(n-2)}\varepsilon^{\nu}_{12\cdots(n-2)} \equiv \bar{\varepsilon}^{\mu}_{[[\cdots[1,2],\cdots],(n-2)]}\varepsilon^{\nu}_{[[\cdots[1,2],\cdots],(n-2)]} \tag{8.1.4}$$

It is trivial to specify that this relation can exactly be identified with the *square* of the Yang-Mills multiparticle polarization fields derived upon using (7.2.6). Specific examples of multiparticle polarization tensors can be carried out simply by considering double copy versions of the formulae in (7.3.5), (7.3.11) and (7.3.16) at rank two, three and four respectively.

One further thing to be noted is this. In our preliminary discussion of the Bern-Dunbar-Shimada formalism, we indicated that when bringing into play the pinching rules we no longer have an ordering of the tree legs. This means that the tree attached to the loop is obtained by taking all possible pinches with all the possible orderings of the legs, which is an exceedingly tedious and onerous task. The main point to be stressed in connection with (8.1.3) is that we may infer directly the existence of a *double-copy* version of the Berends-Giele polarization currents: using the multiparticle polarization tensors in (8.1.3) as the numerators of a new gravitational Berends-Giele current, we can circumvent the need to determine the contribution of the various trees directly using the pinching procedure.

8.2 DOUBLE-COPY PERTURBINER EXPANSION

In the present section, we will discuss all the underlying principles that are necessary for treating the double-copy polarization currents and the corresponding perturbiner expansion. In the previous section, we have used the double pinch operator to provide a recipe for the computation of multiparticle polarization tensors in (8.1.3). Following the procedure in section 7.4, we use these as numerators within a revised prescription for gravitational Berends-Giele currents. Borrowing the color-dressed Berends-Giele map from (7.4.9), we can use it to correctly reproduce the locality structure of the gravity currents. Note that the color-stripped Berends-Giele map from (7.4.1) is not suitable for this task, as it only reproduces the kinematic poles that appear in a color-ordered amplitude. This is not the case for gravity amplitudes, where all the possible orderings of the external legs have to be considered: thus, the map in (7.4.9) is the correct way to proceed. Now, we have to point out the main difference in our discussion with respect to Yang-Mills calculations: in (7.4.13) we have constructed currents by dressing the kinematic poles with the corresponding numerators and color factors. For gravity Berends-Giele currents, the notion of color factor disappears and we build them by dressing numerators with a suitable locality structure. To be more precise, the multiparticle polarization tensors of type (8.1.4) are exactly the numerators that enter in the gravity Berends-Giele currents. Taking advantage of the map (7.4.9), we express a double-copy polarization current, which we denote by $\mathcal{G}_P^{\mu\nu}$, in the form

$$\mathcal{G}_P^{\mu\nu} = [\![\bar{\varepsilon}^{\mu} \otimes \varepsilon^{\nu}]\!] \circ b_{\mathsf{cd}}(P). \tag{8.2.1}$$

Examining the expression for the color-dressed Berends-Giele polarization current (7.4.13), it is readily verified that the double-copy polarization currents may be obtained by replacing the color factor c^a with another copy of the multiparticle polarization field $\bar{\varepsilon}^{\mu}$. This provides a realization of the off-shell double-copy that arises naturally in the string-based formalism, as an alternative to previous approaches [106, 134] that mimic the KLT relations adapting them to Berends-Giele currents.

As some examples, bringing to mind (7.4.14), (7.4.15), (7.4.16) and (7.4.17), the first instances of the double-copy polarization current up to multiplicity four are given by

$$\mathcal{G}_1^{\mu\nu} = \bar{\varepsilon}_1^{\mu}\varepsilon_1^{\nu}, \tag{8.2.2}$$

$$\mathcal{G}_{12}^{\mu\nu} = \frac{\bar{\varepsilon}^{\mu}_{[1,2]}\varepsilon^{\nu}_{[1,2]}}{s_{12}}, \tag{8.2.3}$$

$$\mathcal{G}_{123}^{\mu\nu} = \frac{\bar{\varepsilon}^{\mu}_{[[1,2],3]}\varepsilon^{\nu}_{[[1,2],3]}}{s_{12}s_{123}} + \frac{\bar{\varepsilon}^{\mu}_{[[1,3],2]}\varepsilon^{\nu}_{[[1,3],2]}}{s_{13}s_{123}} + \frac{\bar{\varepsilon}^{\mu}_{[[2,3],1]}\varepsilon^{\nu}_{[[2,3],1]}}{s_{23}s_{123}}, \tag{8.2.4}$$

$$\begin{aligned}\mathcal{G}_{1234}^{\mu\nu} = {} & \frac{\bar{\varepsilon}^{\mu}_{[[[1,2],3],4]}\varepsilon^{\nu}_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[1,2],4],3]}\varepsilon^{\nu}_{[[[1,2],4],3]}}{s_{12}s_{124}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[1,3],4],2]}\varepsilon^{\nu}_{[[[1,3],4],2]}}{s_{13}s_{134}s_{1234}} \\ & + \frac{\bar{\varepsilon}^{\mu}_{[[[2,3],4],1]}\varepsilon^{\nu}_{[[[2,3],4],1]}}{s_{23}s_{234}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[1,3],2],4]}\varepsilon^{\nu}_{[[[1,3],2],4]}}{s_{13}s_{123}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[1,4],2],3]}\varepsilon^{\nu}_{[[[1,4],2],3]}}{s_{14}s_{124}s_{1234}} \\ & + \frac{\bar{\varepsilon}^{\mu}_{[[[1,4],3],2]}\varepsilon^{\nu}_{[[[1,4],3],2]}}{s_{14}s_{134}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[2,3],1],4]}\varepsilon^{\nu}_{[[[2,3],1],4]}}{s_{23}s_{123}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[2,4],1],3]}\varepsilon^{\nu}_{[[[2,4],1],3]}}{s_{24}s_{124}s_{1234}} \\ & + \frac{\bar{\varepsilon}^{\mu}_{[[[2,4],3],1]}\varepsilon^{\nu}_{[[[2,4],3],1]}}{s_{24}s_{234}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[3,4],1],2]}\varepsilon^{\nu}_{[[[3,4],1],2]}}{s_{34}s_{134}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[[3,4],2],1]}\varepsilon^{\nu}_{[[[3,4],2],1]}}{s_{34}s_{234}s_{1234}} \\ & + \frac{\bar{\varepsilon}^{\mu}_{[[1,2],[3,4]]}\varepsilon^{\nu}_{[[1,2],[3,4]]}}{s_{12}s_{34}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[1,3],[2,4]]}\varepsilon^{\nu}_{[[1,3],[2,4]]}}{s_{13}s_{24}s_{1234}} + \frac{\bar{\varepsilon}^{\mu}_{[[1,4],[2,3]]}\varepsilon^{\nu}_{[[1,4],[2,3]]}}{s_{14}s_{23}s_{1234}}.\end{aligned} \tag{8.2.5}$$

For brevity, the double-copy polarization current at rank five is included in appendix D. We reiterate that the crucial step in the double-copy procedure we have just argued is the construction of the multiparticle polarization tensor $\bar{\varepsilon}^{\mu}_{P}\varepsilon^{\nu}_{P}$, where the single fields separately satisfy the GJI as pointed out by color-kinematics duality. In particular, all the fields in the currents above can be obtained from the canonical numerators (7.3.5), (7.3.11) and (7.3.16) using symmetry properties.

Since we have already obtained the double-copy polarization currents we can now readily obtain the double-copy perturbiner expansion, which is nothing but the generating series

$$G^{\mu\nu}(x) = \sum_{P} \mathcal{G}_P^{\mu\nu} e^{k_P\cdot x}, \tag{8.2.6}$$

where the sum is performed over the set of non-empty words $P = 12\ldots n$. Like in the Yang-Mills case, (8.2.6) is not a solution of the Einstein field equations, for it has been "strictified" to include exclusively cubic interactions.

Going on-shell now, it remains to say a word about the scattering amplitudes in the double-copy theory. Recalling the color-dressed amplitude (7.4.21), the Berends-Giele formula for the n-point gravity amplitude reads

$$\mathcal{M}_n^{\text{tree}} = s_{12\cdots(n-1)}\mathcal{G}^{\mu\nu}_{12\cdots(n-1)}\mathcal{G}_{n\mu\nu}. \tag{8.2.7}$$

Not surprisingly, the previous expression takes the well-known form for gravity amplitudes in its double-copy version

$$\mathcal{M}_n^{\text{tree}} = \sum_{j\in\,\text{trivalent}} \frac{\bar{n}_j n_j}{\prod_{i_j} s_{i_j}}, \tag{8.2.8}$$

exactly as introduced in (3.5.3). We also checked our result up to degree $n = 5$ for particular polarizations. At any rate, the outcome of this approach is that we can calculate the amplitudes for the double-copy theory in a relatively straightforward manner, without the need for applying the pinching procedure multiple times to obtain the local BCJ numerators. This attribute was not apparent in previous approaches using the perturbiner method, since the generating series of Berends-Giele currents is usually presented in its color-stripped version for the BCJ gauge.

8.3 ADDITIONAL EXAMPLES

Now that we found a prescription for the double-copy perturbiners, let us apply it to other theories beyond Yang-Mills and gravity. In principle it can be applied to any theory as soon as we guarantee multiparticle fields in the BCJ gauge. One first example should be the case where the BCJ gauge originally appeared, ten-dimensional $\mathcal{N} = 1$ super Yang-Mills in [100] (more recently from a new approach in [135]), but for now we will restrict our presentation only to cases without supersymmetry.

8.3.1 α'-Deformations

For the first example we will calculate the currents and amplitudes for the deformations of general relativity that come from the α' corrections of the closed bosonic string, also referred to as GR+R^2+R^3. The amplitudes for this theory were calculated using the KLT relations for string theory [12, 136]. The action was found in [137] and it reads

$$\begin{aligned} S_{\substack{\text{closed}\\ \text{bosonic}}} \sim \int \mathrm{d}^D x \sqrt{g}\Big\{ & R - 2(\partial_\mu\varphi)^2 - \frac{1}{12}H^2 + \frac{\alpha'}{4}e^{-2\varphi}\big(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} \\ & - 4R_{\mu\nu}R^{\mu\nu} + R^2\big) + \alpha'^2 e^{-4\varphi}\Big(\frac{1}{16}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu} \\ & - \frac{1}{12}R^{\mu\nu}{}_{\alpha\beta}R^{\nu\lambda}{}_{\beta\rho}R^{\lambda\mu}{}_{\rho\alpha}\Big) + \mathcal{O}(\alpha'^3)\Big\}, \end{aligned} \tag{8.3.1}$$

where here φ represents the dilaton and $H = dB$ represents the field strength of the B-field. The gauge field theory for the double-copy is the deformed Yang-Mills theory that comes from the low energy limit of the open string. The action, compatible with color-kinematics duality [138], is the following

$$S_{\mathrm{YM}+F^3+F^4} = \int \mathrm{d}^D x \, \mathrm{tr}\Big\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{2\alpha'}{3}F_{\mu}{}^{\nu}F_{\nu}{}^{\lambda}F_{\lambda}{}^{\mu} \\ + \frac{\alpha'^2}{4}[F_{\mu\nu},F_{\lambda\rho}][F^{\mu\nu},F^{\lambda\rho}]\Big\}, \quad (8.3.2)$$

which has the following equations of motion in the Lorenz gauge, $\partial_\mu A^\mu = 0$,

$$\Box A^{\lambda} = [A^{\mu},\partial_{\mu}A^{\lambda}] + [A_{\mu},F^{\mu\lambda}] + 2\alpha'\{[\nabla_{\mu}F^{\mu\nu},F_{\nu}{}^{\lambda}] + [F^{\mu\nu},\nabla_{\mu}F_{\nu}{}^{\lambda}]\} \\ + 2\alpha'^2\Big\{\big[[\nabla_{\mu}F^{\mu\lambda},F_{\rho\sigma}],F^{\rho\sigma}\big] + \big[[F^{\mu\lambda},\nabla_{\mu}F_{\rho\sigma}],F^{\rho\sigma}\big] \\ + \big[[F^{\mu\lambda},F_{\rho\sigma}],\nabla_{\mu}F^{\rho\sigma}\big]\Big\}. \quad (8.3.3)$$

In [98], the authors conducted a detailed analysis for the calculation of the currents in this gauge using the perturbiner approach [106, 107, 139]. Then, they applied the non-linear gauge transformation studied in [100] in order to obtain currents in the BCJ gauge. In general the expressions for the α'-deformed multiparticle polarizations have the following structure

$$a^{\mu}_{P} = \varepsilon^{\mu}_{P} + \alpha'\varepsilon^{(1)\mu}_{P} + \alpha'^2\varepsilon^{(2)\mu}_{P}. \quad (8.3.4)$$

We invite the reader to have a look at the explicit expressions in [98].

Our double-copy perturbiner for the α'-deformation of general relativity comes out to be

$$G^{(\alpha')\mu\nu}(x) = \sum_{P}\mathcal{G}^{(\alpha')\mu\nu}_{P}\,\mathrm{e}^{\mathrm{i}k_P\cdot x}, \quad (8.3.5)$$

where the Berends-Giele currents is given by

$$\mathcal{G}^{(\alpha')\mu\nu}_{P} = [\![a^{\mu}\otimes\bar{a}^{\nu}]\!]\circ b_{cd}(P). \quad (8.3.6)$$

Naturally, in complete analogy with (8.2.7), the corresponding amplitude reads

$$\mathcal{M}^{(\alpha')\mathrm{tree}}_{n} = s_{1\ldots n-1}\mathcal{G}^{(\alpha')\mu\nu}_{1\ldots n-1}\mathcal{G}^{(\alpha')}_{n\mu\nu}. \quad (8.3.7)$$

This has been checked using the explicit expressions for a^{μ}_{P} from [98].

8.3.2 *Zeroth-Copy*

Another example whose perturbiner can be obtained in a very straightforward manner with our approach is the one for the bi-adjoint scalar model. For this model, originally found in [140], we have a scalar field that takes values in the tensor product $SU(N) \otimes SU(N')$, and is expressible in terms of the generators as $\Phi = \Phi_{aa'} T^a \otimes T'^{a'}$. The corresponding action takes the form

$$S_{\text{bi-adjoint}} = \int \mathrm{d}^D x \left\{ -\tfrac{1}{2} \Phi^{aa'} \Box \Phi_{aa'} + \frac{1}{3!} \tilde{f}^{abc} \tilde{f}'^{a'b'c'} \Phi_{aa'} \Phi_{bb'} \Phi_{cc'} \right\}. \tag{8.3.8}$$

Its Berends-Giele currents were found for the first time in [107] in the color-stripped version and the color-dressed version in [106], both cases using the perturbiner approach. Here we can obtain it simply by applying the zeroth-copy [141], now in its analogue perturbiner version. Therefore, for the bi-adjoint perturbiner we have

$$\Phi^{aa'}(x) = \sum_P \phi_P^{aa'} e^{\mathrm{i} k_P \cdot x}, \tag{8.3.9}$$

where for the Berends-Giele currents read

$$\phi_P^{aa'} = [\![c^a \otimes c'^{a'}]\!] \circ b_{cd}(P) \tag{8.3.10}$$

The expressions for the currents are exactly like the ones in (8.2.2) but replacing the polarizations by the color factors presented in section 7.3. The color-dressed amplitudes can also be calculated directly using the Berends-Giele formula in (8.2.7). In [131], a different method of constructing the Berends-Giele currents in (8.3.10) is presented, which involves the use of scalar products of words.

8.4 FINAL REMARKS

In this chapter, we have extended the construction presented in chapter (7) to build multiparticle polarization tensors from the full pinching of the tails in Bern-Dunbar-Shimada formalism. As in the Yang-Mills case, the polarization tensors can be used as numerators of gravitational Berends-Giele currents. This allowed us to present a revised prescription for the off-shell double-copy, that has applications to theories beyond the ones that we can represent by the infinite string tension limit.

This chapter ends the part of the thesis dedicated to the results of the thesis. Throughout, we have made use of the worldline formalism as a primary tool for various calculations, ranging from computing dressed propagators in curved spacetime scalar QED to exploring the recent color-kinematics duality and double copy. In this journey, We hope to have demonstrated to the reader the potential of the worldline approach in enhancing calculations in conventional quantum field theory, and also the important role that it can have for the advancements of more recent techniques in the study of scattering amplitudes.

Part IV

EPILOGUE

9 EPILOGUE

In this manuscript we have presented various results in scattering amplitudes in quantum field theory. Thanks to techniques acquired from the worldline, or string-inspired, formalism, we have been able to carry out different computations with interesting insights in QED, gauge theories and gravity.

In the first part we have described a novel worldline approach to the computation of the tree level scattering amplitudes associated to the scalar line coupled to electromagnetism and gravity. In particular, we have provided a convenient parametrization for the graviton polarization and a replacement rule, which allowed us to easily compute full amplitudes with an arbitrary number of photons and one graviton. The on-shell transversality of the amplitudes was explicitly checked.

A priori, our technique can be as well implemented to compute amplitudes with an arbitrary number of gravitons. However, in that case more care is needed in the treatment of chains of contractions between the Lee-Yang ghost fields that represent the non trivial measure [110, 142]: the implementation of this generalization is a task for future work. On the other hand amplitudes with gravitons have always been the subject of extensive studies. In particular, theorems which involve gravitons with low momentum have long been analyzed [143] and, in the recent past, various soft-graviton theorems have been studied [144], due to their connections to the infrared structure of gauge theory and gravity —see [145]. The present results intend to provide a novel approach towards the computation of amplitudes with gravitons, which may shed new light on the structure of such quantities.

Later in the manuscript, we have used the aforementioned approach for the computation of the radiative one-loop correction to the scalar-scalar-graviton vertex in arbitrary dimension. This has been obtained by sewing two external photons in an arbitrary covariant gauge. This procedure has been possible since the worldline formalism is an off-shell approach, and the external legs in a given amplitude can be sewed to provide loop contributions. In particular, we have computed the different diagrams that build the radiative one-loop correction to the scalar-scalar-graviton vertex, and our construction has been checked by verifying the on-shell transversality. In the near future, we aim to provide stronger tests for the validity of the result by constructing appropriate Ward identities for the off-shell *gss* vertex. Furthermore, the study of renormalization and the calculation of the associated form factor are also on the agenda.

In the second part of the manuscript we have presented a novel method of constructing Berends-Giele currents using the Bern-Kosower formalism and a specific pinch contribution to the n-gluon amplitudes. Using the technology of multiparticle fields and combining them with the string-based rules, we have computed Berends-Giele numerators up to the five-point case, and we have shown that the latter obey the GJI required by BCJ gauge, indicative of color-kinematics duality. The most attractive feature of our formalism is that it never becomes necessary to determine gauge transformation terms to modify the numerators, as these appear naturally in the desired gauge. In turn, the multiparticle polarization vectors can be used as numerators of Berends-Giele currents, and, exploiting suitable symmetry properties, full tree-level Yang-Mills amplitudes are obtained from a single basic calculation. It can be argued that using these Berends-Giele currents as words in generalized Lorentz cycles, and the associated multiparticle polarization vectors in generalized tails, provides an extremely attractive approach towards absorbing the effect of the Bern-Kosower pinching procedure into multiparticle tensor structures. A possible task in the future is to obtain along these lines a representation of the one loop n-gluon amplitudes that would be ultracompact as well as exhibit manifest color-kinematics duality.

We have also seen that double-copy arise quite naturally in the form of multiparticle fields within the string-inspired formalism. Using a similar construction to the Yang-Mills case, we have been able to build multiparticle polarization tensors from the full pinching of the tails in Bern-Dunbar-Shimada formalism. As in the Yang-Mills scenario, the polarization tensors can be used as numerators of gravitational Berends-Giele currents. This allowed us to presented a revisited prescription for the off-shell double-copy, that has applications to theories beyond the ones that we can represent by the infinite string tension limit, as we have demonstrated with the examples of α'-deformed gravity and the bi-adjoint scalar model. Previously, we have argued that feeding the obtained multiparticle vectors back into the Bern-Kosower formalism, can make the whole pinching procedure unnecessary. It is not obvious whether this aspect of our approach can be generalized to the gravity case, since here the existence of the cross terms seems to start making a real difference. This task is left for future studies. A possible application is the calculation of Berends-Giele currents for gravity coupled to matter fields along the lines of [146, 147], that could be compared with [148]. Another application for the near future is to some cases of supergravity, where Berends-Giele currents have been found for $\mathcal{N} = 1$ Super-Yang-Mills in the BCJ gauge in [16, 127].

Part V

APPENDIX

10 APPENDIX

A TWO-PHOTON ONE-GRAVITON SCALAR PROPAGATOR

We use the master formula (5.1.16) to compute the two-photon one-graviton scalar propagator, and the related *irreducible* part of the two-photon one-graviton two-scalar amplitude, whose Feynman diagrams are depicted in figure 10.1. It is described through

$$
\begin{aligned}
D^{(2,1)}(p,p';\dots;\epsilon,k_0) &= (-ie)^2\left(-\frac{\kappa}{4}\right)\int_0^\infty dT\, e^{-T(m^2+p'^2)}\prod_{i=0}^{2}\int_0^T d\tau_i \\
&\times e^{(p'-p)\cdot(-k_0\tau_0-k_1\tau_1-k_2\tau_2+i\varepsilon_0+i\varepsilon_1+i\varepsilon_2)}\, e^{k_0\cdot k_1|\tau_0-\tau_1|+k_0\cdot k_2|\tau_0-\tau_2|+k_1\cdot k_2|\tau_1-\tau_2|} \\
&\times e^{i(\varepsilon_1\cdot k_0-\varepsilon_0\cdot k_1)\mathrm{sgn}(\tau_0-\tau_1)+i(\varepsilon_2\cdot k_0-\varepsilon_0\cdot k_2)\mathrm{sgn}(\tau_0-\tau_2)+i(\varepsilon_2\cdot k_1-\varepsilon_1\cdot k_2)\mathrm{sgn}(\tau_1-\tau_2)} \\
&\times e^{2\left[\varepsilon_0\cdot\varepsilon_1\delta(\tau_0-\tau_1)+\varepsilon_0\cdot\varepsilon_2\delta(\tau_0-\tau_2)+\varepsilon_1\cdot\varepsilon_2\delta(\tau_1-\tau_2)\right]}\Big|_{\text{m.l.}}\,. \qquad \text{(A1)}
\end{aligned}
$$

Firstly, let us consider contributions involving delta functions, which are linked to seagull diagrams. We find it convenient to grade the different contributions in terms of how many delta functions occur. There is only one double-delta term —see the last diagram in figure 10.1, i.e.

$$
\mathcal{E}_2^{(2,1)} = e^2\kappa\int_0^\infty dT\, e^{-T(m^2+p'^2)}\int_0^T d\tau_0\, e^{\tau_0(p'^2-p^2)}\,\varepsilon_0\cdot\varepsilon_1\varepsilon_0\cdot\varepsilon_2\big|_{\text{m.l.}}, \quad \text{(A2)}
$$

where the notation $\mathcal{E}_i$ specifies contributions to the worldline integrand with a number i of δ-functions involved. Using (5.1.13) and (5.1.14) and truncating over the external lines, the expression (A1) reduces to

$$
\mathcal{E}_2^{(2,1)} = e^2\kappa\, 2\,(\varepsilon_1\epsilon\varepsilon_2), \qquad \text{(A3)}
$$

that represents the Feynman amplitude related to the diagram where two photons and one graviton are emitted at the same point of the scalar line. Note that, also for an arbitrary number n of photons, this is the largest number of particles that can be emitted at the same point of the scalar line together with a single graviton.

There are three terms with a single delta function —see the second and third diagrams and their permutations in figure 10.1. These correspond to the six Feynman diagrams where the emission of a pair of particles (either two photons or one photon and the graviton) takes place from the same point of the scalar line, and the remaining particle emitted from another point on the line. Let us, for example consider the term that involves $\delta(\tau_1-\tau_2)$, that corresponds to the third diagram

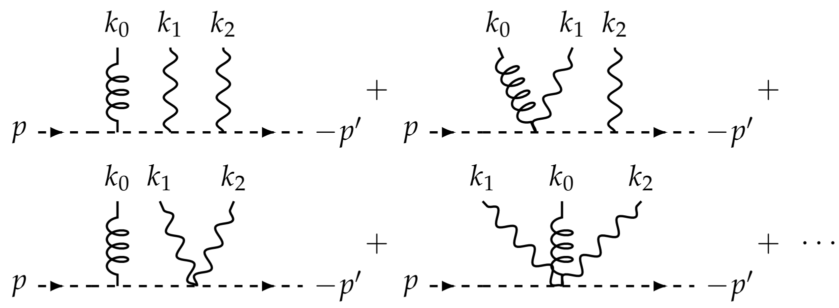


FIG. 10.1: Irreducible contributions to the two-photon one-graviton amplitude. The remaining terms in the sum of diagrams refer to permutations between the photon lines and among the emission points.

in figure 10.1, and which yields the diagrams where two photons are emitted at the same point. The integrand reads

$$\mathcal{E}^{(2,1)}_{1,1} = (-ie)^2\left(-\frac{\kappa}{4}\right)\varepsilon_1\cdot\varepsilon_2\,\left[i\varepsilon_0\cdot(p'-p-(k_1+k_2)\mathrm{sgn}(\tau_0-\tau_1))\right]^2 \times e^{(p-p')\cdot(k_0\tau_0+(k_1+k_2)\tau_1)+k_0\cdot(k_1+k_2)|\tau_0-\tau_1|}, \quad \text{(A4)}$$

which provides two diagrams, according to whether $\tau_1 < \tau_0$ or $\tau_0 < \tau_1$. After some straightforward algebra that corresponds to the Schwinger integral parametrization of the diagrams, we obtain

$$\mathcal{E}^{(2,1)}_{1,1} = -2e^2\kappa\varepsilon_1\cdot\varepsilon_2\Big[\frac{(p'\epsilon p')}{m^2+(p'+k_0)^2}+\frac{(p\epsilon p)}{m^2+(p+k_0)^2}\Big], \quad \text{(A5)}$$

where we have truncated the external scalar lines. Similarly, the other terms with single delta functions $\delta(\tau_0-\tau_1)$ and $\delta(\tau_0-\tau_2)$ give

$$\begin{aligned}\mathcal{E}^{(2,1)}_{1,2} = 2e^2\kappa\Big[&\frac{\varepsilon_1\cdot p\,(\varepsilon_2\epsilon(p'-p-k_1))}{m^2+(p+k_1)^2}+\frac{\varepsilon_1\cdot p'\,(\varepsilon_2\epsilon(p-p'-k_1))}{m^2+(p'+k_1)^2}\\ &+\frac{\varepsilon_2\cdot p\,(\varepsilon_1\epsilon(p'-p-k_2))}{m^2+(p+k_2)^2}+\frac{\varepsilon_2\cdot p'\,(\varepsilon_1\epsilon(p-p'-k_2))}{m^2+(p'+k_2)^2}\Big].\end{aligned} \quad \text{(A6)}$$

The full contribution to the worldline integrand from terms with one single δ-funciton is given by the combination of (A5)-(A6), i.e.

$$\mathcal{E}^{(2,1)}_{1} = \mathcal{E}^{(2,1)}_{1,1}+\mathcal{E}^{(2,1)}_{1,2}. \quad \text{(A7)}$$

The term without delta functions corresponds to the leftover six Feynman diagrams where the two photons and the graviton and emitted singly by the scalar line (the first diagram in figure 10.1 and its permutations), six being the number of permutations of the three particles, which in the present worldline representation correspond to the different orderings of the three times τ_i. The integrand in this case reads

$$\begin{aligned}\mathcal{E}_0^{(2,1)} &= (-ie)^2\left(-\frac{\kappa}{4}\right)\int_0^\infty dT\, e^{-T(m^2+p'^2)}\int_0^T d\tau_0\int_0^T d\tau_1\int_0^T d\tau_2 \\ &\times e^{(p'-p)\cdot(-k_0\tau_0-k_1\tau_1-k_2\tau_2)}\, e^{k_0\cdot k_1|\tau_0-\tau_1|+k_0\cdot k_2|\tau_0-\tau_2|+k_1\cdot k_2|\tau_1-\tau_2|} \\ &\times \varepsilon_1\cdot(p'-p+k_0\mathrm{sgn}(\tau_0-\tau_1)-k_2\mathrm{sgn}(\tau_1-\tau_2)) \\ &\times \varepsilon_2\cdot(p'-p+k_0\mathrm{sgn}(\tau_0-\tau_2)+k_1\mathrm{sgn}(\tau_1-\tau_2)) \\ &\times \frac{1}{2}\left[\varepsilon_0\cdot(p'-p-k_1\mathrm{sgn}(\tau_0-\tau_1)-k_2\mathrm{sgn}(\tau_0-\tau_2))\right]^2, \end{aligned} \tag{A8}$$

and yields

$$\begin{aligned}\mathcal{E}_0^{(2,1)} = 4e^2\kappa\Big[&\frac{(p'\epsilon p')\,\varepsilon_1\cdot(p+k_2)\,\varepsilon_2\cdot p}{((p+k_2)^2+m^2)((p'+k_0)^2+m^2)} + (1\leftrightarrow 2) \\ &+\frac{(p\epsilon p)\,\varepsilon_1\cdot(p'+k_2)\,\varepsilon_2\cdot p'}{((p+k_0)^2+m^2)((p'+k_2)^2+m^2)} + (1\leftrightarrow 2) \\ &+\frac{((p+k_1)\epsilon(p'+k_2))\,\varepsilon_1\cdot p\,\varepsilon_2\cdot p'}{((p+k_1)^2+m^2)((p'+k_2)^2+m^2)} + (1\leftrightarrow 2)\Big]\,. \end{aligned} \tag{A9}$$

Thus,

$$\mathcal{D}_{irred}^{(2,1)}(p,p';\varepsilon_1,k_1,\varepsilon_2,k_2;\epsilon,k_0) = \mathcal{E}_0^{(2,1)} + \mathcal{E}_1^{(2,1)} + \mathcal{E}_2^{(2,1)} \tag{A10}$$

is the irreducible part of the two-scalar two-photon one-graviton amplitude, that corresponds exactly to the expression presented in (5.1.23).

B TRANSVERSALITY OF THE AMPLITUDES WITH ONE GRAVITON AND $n \leq 2$ PHOTONS

Let us here check how the transversality of the graviton line explicitly works for $n \leq 2$. For the $n = 0$ amplitude of equation (5.1.20) we have

$$\mathcal{D}^{(0,1)}(p,p';k_0\xi,k_0) = \frac{\kappa}{2}(p'-p)\cdot k_0\,(p'-p)\cdot\xi\,, \tag{B1}$$

which vanishes on-sell because $k_0 = -(p+p')$. For $n = 1$, using on-shellness, the momentum conservation and the transversality conditions $k_{0\mu}\epsilon^{\mu\nu} = k_\mu\varepsilon^\mu = 0$, we have

$$\begin{aligned}\mathcal{D}_{red}^{(1,1)}(p,p';\varepsilon_1,k_1;k_0\xi,k_0) &= -\mathcal{D}_{irred}^{(1,1)}(p,p';\varepsilon_1,k_1;k_0\xi,k_0) \\ &= e\kappa(p'-p)_\mu\left(\varepsilon_1^\mu k_1\cdot\xi + k_0^\mu\varepsilon_1\cdot\xi\right), \end{aligned} \tag{B2}$$

so that

$$\mathcal{D}^{(1,1)}(p,p';\varepsilon_1,k_1;k_0\xi,k_0) = 0, \tag{B3}$$

as expected.

The computation for the $n = 2$ case is of course more complicated. However, let us sketch some details. An useful way to proceed is to identify different *kind* of terms in both the reducible and irreducible parts of the amplitude, that must sum up to zero separately.

Let us first consider the part of the amplitude proportional to the product $\varepsilon_1 \cdot \varepsilon_2$. After performing the substitution described in equation (5.2.10), and denoting the corresponding reducible and irreducible contributions as $\mathcal{D}^{\varepsilon_1\varepsilon_2}_{red}$ and $\mathcal{D}^{\varepsilon_1\varepsilon_2}_{irred}$, we obtain

$$\begin{aligned}
\mathcal{D}^{\varepsilon_1\varepsilon_2}_{irred} &= -\frac{2\varepsilon_1 \cdot \varepsilon_2}{p \cdot k_0}\left(p \cdot k_0 p \cdot \xi\right) - \frac{2\varepsilon_1 \cdot \varepsilon_2}{p' \cdot k_0}\left(p' \cdot k_0 p' \cdot \xi\right) \\
&= -2\varepsilon_1 \cdot \varepsilon_2 \xi \cdot (p + p'), \qquad \text{(B4)} \\
\mathcal{D}^{\varepsilon_1\varepsilon_2}_{red} &= -\frac{2\varepsilon_1 \cdot \varepsilon_2}{k_1 \cdot k_0}\left(k_1 \cdot k_0 k_1 \cdot \xi\right) - \frac{2\varepsilon_1 \cdot \varepsilon_2}{k_2 \cdot k_0}\left(k_2 \cdot k_0 k_2 \cdot \xi\right) = \\
&= -2\varepsilon_1 \cdot \varepsilon_2 \xi \cdot (k_1 + k_2) = 2\varepsilon_1 \cdot \varepsilon_2 \xi \cdot (p + p') = -\mathcal{D}^{\varepsilon_1\varepsilon_2}_{irred}, \qquad \text{(B5)}
\end{aligned}$$

where in the last line we have used the conservation of total energy-momentum together with the transversality condition given in equation (5.2.10). Thus, we get

$$\mathcal{D}^{\varepsilon_1\varepsilon_2}_{irred} + \mathcal{D}^{\varepsilon_1\varepsilon_2}_{red} = 0, \qquad \text{(B6)}$$

as expected.

Similarly we could consider the part in the total amplitude proportional to $\varepsilon_1 \cdot \xi$, and we indicate with $\mathcal{D}^{\varepsilon_1\xi}_{red}$ and $\mathcal{D}^{\varepsilon_1\xi}_{irred}$ respectively the reducible and irreducible contributions. After some manipulations, we obtain

$$\begin{aligned}
\mathcal{D}^{\varepsilon_1\xi}_{irred} =& \frac{\varepsilon_2 \cdot p'}{k_2 \cdot p'} p \cdot k_0 \varepsilon_1 \cdot \xi + 2\varepsilon_1 \cdot \xi \varepsilon_2 \cdot k_0 + \frac{\varepsilon_2 \cdot p'}{k_2 \cdot p'}(p + k_1) \cdot k_0 \varepsilon_1 \cdot \xi \\
&+ \frac{\varepsilon_2 \cdot p}{k_2 \cdot p} p' \cdot k_0 \varepsilon_1 \cdot \xi + \frac{\varepsilon_2 \cdot p}{k_2 \cdot p}(p' + k_1) \cdot k_0 \varepsilon_1 \cdot \xi \\
=& \frac{\varepsilon_2 \cdot p'}{k_2 \cdot p'} p \cdot k_0 \varepsilon_1 \cdot \xi + 2\varepsilon_1 \cdot \xi \varepsilon_2 \cdot k_0 - \frac{\varepsilon_2 \cdot p'}{k_2 \cdot p'} p \cdot k_1 \varepsilon_1 \cdot \xi + \varepsilon_1 \cdot \xi \varepsilon_2 \cdot p' \\
&+ \frac{\varepsilon_2 \cdot p}{k_2 \cdot p} p' \cdot k_0 \varepsilon_1 \cdot \xi - \frac{\varepsilon_2 \cdot p}{k_2 \cdot p} p' \cdot k_1 \varepsilon_1 \cdot \xi + \varepsilon_1 \cdot \xi \varepsilon_2 \cdot p \\
=& \frac{\varepsilon_2 \cdot p'}{k_2 \cdot p'} \varepsilon_1 \cdot \xi p \cdot (k_0 - k_1) + \frac{\varepsilon_2 \cdot p}{k_2 \cdot p} \varepsilon_1 \cdot \xi p' \cdot (k_0 - k_1) \\
&+ \varepsilon_1 \cdot \xi \varepsilon_2 \cdot (k_0 - k_1). \qquad \text{(B7)}
\end{aligned}$$

Notice that in the last equality we have exploited the conservation of total energy-momentum, while in the second equality we have used the relations

$$\begin{aligned}
k_0 \cdot (p + k_1) &= -p \cdot k_1 + p' \cdot k_2, \\
k_0 \cdot (p' + k_1) &= -p' \cdot k_1 + p \cdot k_2. \qquad \text{(B8)}
\end{aligned}$$

The contribution coming from the reducible part of the amplitude is obtained as

$$\begin{aligned}\mathcal{D}_{red}^{\varepsilon_1\xi} =& \frac{\varepsilon_2\cdot p'}{p'\cdot k_2 k_0\cdot k_1}\varepsilon_1\cdot\xi(p\cdot k_1 k_0\cdot k_1 - p\cdot k_0 k_0\cdot k_1)\\ &+2\left(\frac{\varepsilon_1\cdot\xi}{2k_0\cdot k_1}(k_0\cdot k_1\varepsilon_2\cdot k_1 - k_0\cdot k_1\varepsilon_2\cdot k_0)\right)\\ &+\frac{\varepsilon_2\cdot p}{p\cdot k_2 k_0\cdot k_1}\varepsilon_1\cdot\xi(p'\cdot k_1 k_0\cdot k_1 - p'\cdot k_0 k_0\cdot k_1)\\ =& -\frac{\varepsilon_2\cdot p'}{k_2\cdot p'}\varepsilon_1\cdot\xi p\cdot(k_0-k_1) - \frac{\varepsilon_2\cdot p}{k_2\cdot p}\varepsilon_1\cdot\xi p'\cdot(k_0-k_1)\\ &-\varepsilon_1\cdot\xi\varepsilon_2\cdot(k_0-k_1),\end{aligned} \tag{B9}$$

and the sum of the reducible and irreducible contribution vanishes, that is

$$\mathcal{D}_{irred}^{\varepsilon_1\xi} + \mathcal{D}_{red}^{\varepsilon_1\xi} = 0. \tag{B10}$$

By Bose symmetry the contributions proportional to $\varepsilon_2\cdot\xi$ can be obtained from the latter with the replacements $\varepsilon_1 \leftrightarrow \varepsilon_2$ and $k_1 \leftrightarrow k_2$. Now we are ready to write down all the remaining terms that enter in the transversality expression for the total amplitude. We find it convenient to organize them in terms of their different denominators, which are scalar product of momenta. We thus use the notation $\mathcal{D}_{rem}^{pk}$ to indicate those terms that have the common denominator $p\cdot k$ and similarly with others. We have,

$$\begin{aligned}\mathcal{D}_{rem}^{p'k_2} =& -\frac{\varepsilon_2\cdot p'}{p'\cdot k_2}2p\cdot\xi\varepsilon_1\cdot(p+k_0) + \frac{\varepsilon_2\cdot p'}{p'\cdot k_2}\varepsilon_1\cdot k_0 p\cdot\xi\\ &+\frac{\varepsilon_2\cdot p'}{p'\cdot k_2}\varepsilon_1\cdot k_0\xi\cdot(p+k_1) + \frac{\varepsilon_2\cdot p'}{p'\cdot k_2}2\varepsilon_1\cdot p\xi\cdot(p+k_1)\\ &-\frac{\varepsilon_2\cdot p'}{p'\cdot k_2}2p\cdot\varepsilon_1\xi\cdot k_1 - \frac{\varepsilon_2\cdot p'}{p'\cdot k_2}\varepsilon_1\cdot k_0\xi\cdot k_1 = 0,\end{aligned} \tag{B11}$$

$$\begin{aligned}\mathcal{D}_{rem}^{pk_1} =& -\frac{\varepsilon_1\cdot p}{p\cdot k_1}\varepsilon_2\cdot k_0\xi\cdot(p+k_1) - \frac{\varepsilon_1\cdot p}{p\cdot k_1}2p'\cdot\xi\varepsilon_2\cdot(p'+k_0)\\ &+\frac{\varepsilon_1\cdot p}{p\cdot k_1}\varepsilon_2\cdot k_0\xi\cdot p' - \frac{\varepsilon_1\cdot p}{p\cdot k_1}2\varepsilon_2\cdot p'\xi\cdot(p+k_1)\\ &+\frac{\varepsilon_1\cdot p}{p\cdot k_1}2(p+k_1)\cdot\varepsilon_2\xi\cdot k_2 + \frac{\varepsilon_1\cdot p}{p\cdot k_1}\varepsilon_2\cdot k_0\xi\cdot k_2 = 0,\end{aligned} \tag{B12}$$

$$\begin{aligned}\mathcal{D}_{rem}^{p'k_1} =& -\frac{\varepsilon_1\cdot p'}{p'\cdot k_1}2p\cdot\xi\varepsilon_2\cdot(p+k_0) + \frac{\varepsilon_1\cdot p'}{p'\cdot k_1}p\cdot\xi\varepsilon_2\cdot k_0\\ &-\frac{\varepsilon_1\cdot p'}{p'\cdot k_1}\varepsilon_2\cdot k_0\xi\cdot(p'+k_1) - \frac{\varepsilon_1\cdot p'}{p'\cdot k_1}2\varepsilon_2\cdot p\xi\cdot(p'+k_1)\\ &+\frac{\varepsilon_1\cdot p'}{p'\cdot k_1}2(p'+k_1)\cdot\varepsilon_2\xi\cdot k_2 + \frac{\varepsilon_1\cdot p'}{p'\cdot k_1}\varepsilon_2\cdot k_0\xi\cdot k_2 = 0,\end{aligned} \tag{B13}$$

$$\begin{aligned}
\mathcal{D}^{pk_2}_{rem} = &- \frac{\varepsilon_2 \cdot p}{p \cdot k_2} 2p' \cdot \xi \varepsilon_1 \cdot (p' + k_0) + \frac{\varepsilon_2 \cdot p}{p \cdot k_2} \varepsilon_1 \cdot k_0 \xi \cdot p' \\
&+ \frac{\varepsilon_2 \cdot p}{p \cdot k_2} \varepsilon_1 \cdot k_0 \xi \cdot (p' + k_1) + \frac{\varepsilon_2 \cdot p}{p \cdot k_2} 2\varepsilon_1 \cdot p' \xi \cdot (p' + k_1) \\
&- \frac{\varepsilon_2 \cdot p}{p \cdot k_2} 2p' \cdot \varepsilon_1 \xi \cdot k_1 - \frac{\varepsilon_2 \cdot p}{p \cdot k_2} \varepsilon_1 \cdot k_0 \xi \cdot k_1 = 0\,,
\end{aligned} \tag{B14}$$

$$\begin{aligned}
\mathcal{D}^{k_0 k_1}_{rem} = &\frac{\varepsilon_1 \cdot k_0 \xi \cdot k_1}{k_0 \cdot k_1} \varepsilon_2 \cdot (k_0 + k_1) + \frac{\varepsilon_1 \cdot k_0}{k_0 \cdot k_1} \varepsilon_2 \cdot p' \xi \cdot k_1 + \frac{\varepsilon_1 \cdot k_0}{k_0 \cdot k_1} \varepsilon_2 \cdot p \xi \cdot k_1 \\
= &\frac{\varepsilon_1 \cdot k_0}{k_0 \cdot k_1} \xi \cdot k_1 \varepsilon_2 \cdot (p + p' + k_0 + k_1) \propto \varepsilon_2 \cdot k_2 = 0\,,
\end{aligned} \tag{B15}$$

$$\begin{aligned}
\mathcal{D}^{k_0 k_2}_{rem} = &\frac{\varepsilon_2 \cdot k_0 \xi \cdot k_2}{k_0 \cdot k_2} \varepsilon_1 \cdot (k_0 + k_2) + \frac{\varepsilon_2 \cdot k_0}{k_0 \cdot k_2} \varepsilon_1 \cdot p \xi \cdot k_2 + \frac{\varepsilon_2 \cdot k_0}{k_0 \cdot k_2} \varepsilon_1 \cdot p' \xi \cdot k_2 \\
= &\frac{\varepsilon_2 \cdot k_0}{k_0 \cdot k_2} \xi \cdot k_2 \varepsilon_1 \cdot (p + p' + k_0 + k_2) \propto \varepsilon_1 \cdot k_1 = 0\,,
\end{aligned} \tag{B16}$$

where $\varepsilon_i \cdot k_i = 0$ and momentum conservation have used when needed. Thus, all the different contributions sum up to zero and the *transversality* of the total amplitude is proven, i.e.,

$$\mathcal{D}^{(2,1)}\left(p, p'; \varepsilon_1, k_1; \varepsilon_2, k_2; k_0 \xi, k_0\right) = 0\,. \tag{B17}$$

C MOMENTUM INTEGRALS

In this appendix we list the integrals that we have introduced in chapter 6 for the computation of the one-loop radiative correction to the *gss* vertex from the worldline approach.

$$\begin{aligned}
J^{(0)}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
J^{(1)\mu}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^2 (m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
J^{(2)\mu\nu}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^2 (m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
J^{(3)\mu\nu\rho}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu \ell^\rho}{\ell^2 (m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
K^{(0)}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
K^{(1)\mu}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu}{(m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
K^{(2)\mu\nu}[p, p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{(m^2 + (p' + \ell)^2)(m^2 + (p - \ell)^2)} \\
I^{(0)}[p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (m^2 + (p' + \ell)^2)} \\
I^{(1)\mu}[p'] &= \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^2 (m^2 + (p' + \ell)^2)}
\end{aligned}$$

$$
\begin{aligned}
I^{(2)\mu\nu}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^2(m^2+(p'+\ell)^2)} \\
H^{(0)}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^4(m^2+(p'+\ell)^2)} \\
H^{(1)\mu}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^4(m^2+(p'+\ell)^2)} \\
H^{(2)\mu\nu}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^4(m^2+(p'+\ell)^2)} \\
L^{(0)}[p,p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^4(m^2+(p'+\ell)^2)(m^2+(p-\ell)^2)} \\
L^{(1)\mu}[p,p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^4(m^2+(p'+\ell)^2)(m^2+(p-\ell)^2)} \\
L^{(2)\mu\nu}[p,p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^4(m^2+(p'+\ell)^2)(m^2+(p-\ell)^2)} \\
M^{(1)\mu}[k_0,p] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^2(k_0+\ell)^2(m^2+(p-\ell)^2)} \\
M^{(2)\mu\nu}[k_0,p] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^2(k_0+\ell)^2(m^2+(p-\ell)^2)} \\
N^{(0)}[k_0,p] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(k_0+\ell)^2(m^2+(p-\ell)^2)} \\
O^{(1)\mu}[k_0] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^2(\ell+k_0)^2} \\
O^{(2)\mu\nu}[k_0] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+k_0)^2}
\end{aligned} \tag{C1}
$$

The list of Feynman integrals presented is overcomplete. One can easily observe this by examining the integral $N^{(0)}[k_0, p]$, which can be transformed into $I^{(0)}[p']$ by making a simple change of variable, $\ell \rightarrow \ell - k_0$. This example is simple, however, the complexity can escalate when working with Feynman integrals. Feynman integrals are essential components that arise in contemporary elementary particle physics, and the calculations involving them currently result in the need to evaluate millions of such integrals. A classical approach is to apply *integration by parts* (IBP) relations [149] and reduce all integrals to a smaller set, the so-called *master integrals*. Currently there is a number of computer codes that can solve IBP relations and perform Feynman integral reduction. In our work, we are making use of the program `FIRE6` [150], developed in `Wolfram Mathematica`. Currently, the program has primarily been applied for the reduction of simple integral forms, to verify the gauge-invariant nature of the one-loop graviton amplitude in scalar QED, utilizing the expression of the vertex found in equation (6.2.44). In future studies, we plan to consistently employ the program to compute the form factor for the one-loop *gss* vertex.

Before applying the reduction to master integrals to specific contribution, it is useful to introduce additional integrals

$$
\begin{aligned}
G^{(0)}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(m^2 + (p' + \ell)^2)} \\
G^{(1)\mu}[p'] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{l^\mu}{(m^2 + (p' + \ell)^2)} \\
O^{(0)}[k_0] &= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell + k_0)^2},
\end{aligned}
\tag{C2}
$$

where, $G^{(0)}[p']$ and $O^{(0)}[k_0]$ represent the master integrals required for the transversality calculation, where the mass-shell conditions $p^2 = p'^2 = -m^2$ are imposed. It is worth noting that $G^{(0)}[p'] = G^{(0)}[-p]$ after a shift in the loop momentum, and for the sake of simplicity, in the following we will refer to $G^{(0)}[p']$ as $G^{(0)}$ without loss of generality. The systematic use of relations of type (6.3.6)-(6.3.14) allows to simplify all the integrals of type $J^{(i)}$, $K^{(i)}$ and $M^{(i)}$ in the transversality calculation. Thus, only a limited set of integrals requires the reduction to simpler forms:

$$
\begin{aligned}
I^{(0)\mu}[p'] &= -\frac{1}{2m^2} \frac{(-2+D)}{(-3+D)} G^{(0)}[p'] \\
I^{(1)\mu}[p'] &= \frac{1}{2m^2} p'^\mu G^{(0)}[p'] \\
I^{(2)\mu\nu}[p'] &= \frac{1}{2m^2} \frac{m^2 \delta^{\mu\nu} - (-2+D)\, p'^\mu p'^\nu}{(-1+D)} G^{(0)}[p'] \\
G^{(1)\mu}[p'] &= -\, p'^\mu G^{(0)} \\
O^{(1)\mu}[k_0] &= -\frac{1}{2} k_0^\mu O^{(0)}[k_0] \\
O^{(2)\mu\nu}[k_0] &= \frac{-k_0^2 \delta^{\mu\nu} + D\, k_0^\mu k_0^\nu}{4(-1+D)} O^{(0)}[k_0].
\end{aligned}
\tag{C3}
$$

The calculation of form factors will involve a massive reduction of the integrals listed in equation (C1). Here, the absence of on-shell conditions on the scalar lines will increase the number of master integrals, and the capabilities of the program FIRE6 will be exploited.

D BERENDS-GIELE CURRENTS OF MULTIPLICITY FIVE

In this appendix, we use the method outlined in section 7.4.2 to calculate color-dressed Berends-Giele polarization currents in the BCJ gauge and provide the full expression for the current at a multiplicity five. The computation follows from (7.4.13), where in the five-particle case $P = 12345$. In this case, the word decomposition reads

$$
\begin{aligned}
(Q,R) =& (1234)(5), (1235)(4), (1245)(3), (1345)(2), (2345)(1), \\
& (123)(45), (124)(35), (125)(34), (134)(25), (145)(23), \\
& (135)(24), (234)(15), (235)(14), (245)(13), (345)(12).
\end{aligned}
\tag{D1}
$$

Therefore we obtain for the color-dressed Berends-Giele polarization current $J^{a\mu}_{12345}$ the formula

$$
\begin{aligned}
J^{a\mu}_{12345} =& \frac{c^a_{[[[[1,2],3],4],5]}\varepsilon^\mu_{[[[[1,2],3],4],5]}}{s_{12}s_{123}s_{1234}s_{12345}} + \frac{c^a_{[[[[1,3],2],4],5]}\varepsilon^\mu_{[[[[1,3],2],4],5]}}{s_{13}s_{123}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[[2,3],1],4],5]}\varepsilon^\mu_{[[[[2,3],1],4],5]}}{s_{23}s_{123}s_{1234}s_{12345}} + \frac{c^a_{[[[[1,2],4],3],5]}\varepsilon^\mu_{[[[[1,2],4],3],5]}}{s_{12}s_{124}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[[1,4],2],3],5]}\varepsilon^\mu_{[[[[1,4],2],3],5]}}{s_{14}s_{124}s_{1234}s_{12345}} + \frac{c^a_{[[[[2,4],1],3],5]}\varepsilon^\mu_{[[[[2,4],1],3],5]}}{s_{24}s_{124}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[[1,3],4],2],5]}\varepsilon^\mu_{[[[[1,3],4],2],5]}}{s_{13}s_{134}s_{1234}s_{12345}} + \frac{c^a_{[[[[1,4],3],2],5]}\varepsilon^\mu_{[[[[1,4],3],2],5]}}{s_{14}s_{134}s_{1234}s_{12345}} + \\
&+ \frac{c^a_{[[[[3,4],1],2],5]}\varepsilon^\mu_{[[[[3,4],1],2],5]}}{s_{34}s_{134}s_{1234}s_{12345}} + \frac{c^a_{[[[[2,3],4],1],5]}\varepsilon^\mu_{[[[[2,3],4],1],5]}}{s_{23}s_{234}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[[2,4],3],1],5]}\varepsilon^\mu_{[[[[2,4],3],1],5]}}{s_{24}s_{234}s_{1234}s_{12345}} + \frac{c^a_{[[[[3,4],2],1],5]}\varepsilon^\mu_{[[[[3,4],2],1],5]}}{s_{34}s_{234}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[1,2],[3,4]],5]}\varepsilon^\mu_{[[[1,2],[3,4]],5]}}{s_{12}s_{34}s_{1234}s_{12345}} + \frac{c^a_{[[[1,3],[2,4]],5]}\varepsilon^\mu_{[[[1,3],[2,4]],5]}}{s_{13}s_{24}s_{1234}s_{12345}} \\
&+ \frac{c^a_{[[[1,4],[2,3]],5]}\varepsilon^\mu_{[[[1,4],[2,3]],5]}}{s_{14}s_{23}s_{1234}s_{12345}} \\
&+ \Big((1234)(5) \leftrightarrow (1235)(4)\Big) + \Big((1234)(5) \leftrightarrow (1245)(3)\Big) \\
&+ \Big((1234)(5) \leftrightarrow (1345)(2)\Big) + \Big((1234)(5) \leftrightarrow (2345)(1)\Big) \\
&+ \frac{c^a_{[[[1,2],3],[4,5]]}\varepsilon^\mu_{[[[1,2],3],[4,5]]}}{s_{12}s_{123}s_{45}s_{12345}} + \frac{c^a_{[[[2,3],1],[4,5]]}\varepsilon^\mu_{[[[2,3],1],[4,5]]}}{s_{23}s_{123}s_{45}s_{12345}} \\
&+ \frac{c^a_{[[[1,3],2],[4,5]]}\varepsilon^\mu_{[[[1,3],2],[4,5]]}}{s_{13}s_{123}s_{45}s_{12345}} + \Big((123)(45) \leftrightarrow (124)(35)\Big) \\
&+ \Big((123)(45) \leftrightarrow (125)(34)\Big) + \Big((123)(45) \leftrightarrow (134)(25)\Big) \\
&+ \Big((123)(45) \leftrightarrow (145)(23)\Big) + \Big((123)(45) \leftrightarrow (135)(24)\Big) \\
&+ \Big((123)(45) \leftrightarrow (234)(15)\Big) + \Big((123)(45) \leftrightarrow (235)(14)\Big) \\
&+ \Big((123)(45) \leftrightarrow (245)(13)\Big) + \Big((123)(45) \leftrightarrow (345)(12)\Big), \qquad \text{(D2)}
\end{aligned}
$$

where the color factors have structures of type

$$
\begin{aligned}
c^a_{[[[[1,2],3],4],5]} &= \tilde{f}_{a_1a_2}{}^b \tilde{f}_{ba_3}{}^c \tilde{f}_{ca_4}{}^d \tilde{f}_{da_5}{}^a \\
c^a_{[[[1,2],[3,4]],5]} &= \tilde{f}_{a_1a_2}{}^b \tilde{f}_{a_3a_4}{}^c \tilde{f}_{bc}{}^d \tilde{f}_{da_5}{}^a \\
c^a_{[[[1,2],3],[4,5]]} &= \tilde{f}_{a_1a_2}{}^b \tilde{f}_{ba_3}{}^c \tilde{f}_{a_4a_5}{}^d \tilde{f}_{cd}{}^a. \qquad \text{(D3)}
\end{aligned}
$$

Note that, as for the lower-point polarization fields, $\varepsilon^\mu_{[[[[1,2],3],4],5]}$ is the only five-particle polarization field needed in (D2). Indeed, using the identities

$$
\varepsilon^\mu_{[[[1,2],[3,4]],5]} = \varepsilon^\mu_{[[[[1,2],3],4],5]} - \varepsilon^\mu_{[[[[1,2],4],3],5]},
$$

$$\varepsilon^{\mu}_{[[[1,2],3],[4,5]]} = \varepsilon^{\mu}_{[[[[1,2],3],4],5]} - \varepsilon^{\mu}_{[[[[1,2],3],5],4]}, \tag{D4}$$

all the five-particle polarization fields appearing in (D2) are obtained from the polarization computed in (7.3.22) by a simple relabelling.

Finally, we report here the formula for the double-copy polarization current at five point $\mathcal{G}^{\mu\nu}_{12345}$. As pointed out in section 8.2, this is simply obtained by substituting the color factors c^a in (D2) with another copy of the multiparticle polarization field $\bar{\varepsilon}^{\mu}$. The resulting expression for $\mathcal{G}^{\mu\nu}_{12345}$ is thus given by

$$\begin{aligned}
\mathcal{G}^{\mu\nu}_{12345} =& \frac{\varepsilon^{\mu}_{[[[[1,2],3],4],5]}\bar{\varepsilon}^{\nu}_{[[[[1,2],3],4],5]}}{s_{12}s_{123}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[1,3],2],4],5]}\bar{\varepsilon}^{\nu}_{[[[[1,3],2],4],5]}}{s_{13}s_{123}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[[2,3],1],4],5]}\bar{\varepsilon}^{\nu}_{[[[[2,3],1],4],5]}}{s_{23}s_{123}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[1,2],4],3],5]}\bar{\varepsilon}^{\nu}_{[[[[1,2],4],3],5]}}{s_{12}s_{124}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[[1,4],2],3],5]}\bar{\varepsilon}^{\nu}_{[[[[1,4],2],3],5]}}{s_{14}s_{124}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[2,4],1],3],5]}\bar{\varepsilon}^{\nu}_{[[[[2,4],1],3],5]}}{s_{24}s_{124}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[[1,3],4],2],5]}\bar{\varepsilon}^{\nu}_{[[[[1,3],4],2],5]}}{s_{13}s_{134}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[1,4],3],2],5]}\bar{\varepsilon}^{\nu}_{[[[[1,4],3],2],5]}}{s_{14}s_{134}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[[3,4],1],2],5]}\bar{\varepsilon}^{\nu}_{[[[[3,4],1],2],5]}}{s_{34}s_{134}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[2,3],4],1],5]}\bar{\varepsilon}^{\nu}_{[[[[2,3],4],1],5]}}{s_{23}s_{234}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[[2,4],3],1],5]}\bar{\varepsilon}^{\nu}_{[[[[2,4],3],1],5]}}{s_{24}s_{234}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[[3,4],2],1],5]}\bar{\varepsilon}^{\nu}_{[[[[3,4],2],1],5]}}{s_{34}s_{234}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[1,2],[3,4]],5]}\bar{\varepsilon}^{\nu}_{[[[1,2],[3,4]],5]}}{s_{12}s_{34}s_{1234}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[1,3],[2,4]],5]}\bar{\varepsilon}^{\nu}_{[[[1,3],[2,4]],5]}}{s_{13}s_{24}s_{1234}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[1,4],[2,3]],5]}\bar{\varepsilon}^{\nu}_{[[[1,4],[2,3]],5]}}{s_{14}s_{23}s_{1234}s_{12345}} \\
&+ \Big((1234)(5) \leftrightarrow (1235)(4)\Big) + \Big((1234)(5) \leftrightarrow (1245)(3)\Big) \\
&+ \Big((1234)(5) \leftrightarrow (1345)(2)\Big) + \Big((1234)(5) \leftrightarrow (2345)(1)\Big) \\
&+ \frac{\varepsilon^{\mu}_{[[[1,2],3],[4,5]]}\bar{\varepsilon}^{\nu}_{[[[1,2],3],[4,5]]}}{s_{12}s_{123}s_{45}s_{12345}} + \frac{\varepsilon^{\mu}_{[[[2,3],1],[4,5]]}\bar{\varepsilon}^{\nu}_{[[[2,3],1],[4,5]]}}{s_{23}s_{123}s_{45}s_{12345}} \\
&+ \frac{\varepsilon^{\mu}_{[[[1,3],2],[4,5]]}\bar{\varepsilon}^{\nu}_{[[[1,3],2],[4,5]]}}{s_{13}s_{123}s_{45}s_{12345}} + \Big((123)(45) \leftrightarrow (124)(35)\Big) \\
&+ \Big((123)(45) \leftrightarrow (125)(34)\Big) + \Big((123)(45) \leftrightarrow (134)(25)\Big) \\
&+ \Big((123)(45) \leftrightarrow (145)(23)\Big) + \Big((123)(45) \leftrightarrow (135)(24)\Big) \\
&+ \Big((123)(45) \leftrightarrow (234)(15)\Big) + \Big((123)(45) \leftrightarrow (235)(14)\Big) \\
&+ \Big((123)(45) \leftrightarrow (245)(13)\Big) + \Big((123)(45) \leftrightarrow (345)(12)\Big).
\end{aligned} \tag{D5}$$

[1] Georges Aad et al. "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC." In: *Phys. Lett. B* 716 (2012), pp. 1–29. DOI: 10.1016/j.physletb.2012.08.020. arXiv: 1207.7214 [hep-ex].

[2] B. P. Abbott et al. "Observation of Gravitational Waves from a Binary Black Hole Merger." In: *Phys. Rev. Lett.* 116.6 (2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. arXiv: 1602.03837 [gr-qc].

[3] Stephen J. Parke and T. R. Taylor. "An Amplitude for n Gluon Scattering." In: *Phys. Rev. Lett.* 56 (1986), p. 2459. DOI: 10.1103/PhysRevLett.56.2459.

[4] Ruth Britto, Freddy Cachazo, and Bo Feng. "New recursion relations for tree amplitudes of gluons." In: *Nucl. Phys. B* 715 (2005), pp. 499–522. DOI: 10.1016/j.nuclphysb.2005.02.030. arXiv: hep-th/0412308.

[5] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. "Direct proof of tree-level recursion relation in Yang-Mills theory." In: *Phys. Rev. Lett.* 94 (2005), p. 181602. DOI: 10.1103/PhysRevLett.94.181602. arXiv: hep-th/0501052.

[6] Zvi Bern, Lance J. Dixon, David C. Dunbar, and David A. Kosower. "One loop n point gauge theory amplitudes, unitarity and collinear limits." In: *Nucl. Phys. B* 425 (1994), pp. 217–260. DOI: 10.1016/0550-3213(94)90179-1. arXiv: hep-ph/9403226.

[7] Ruth Britto, Freddy Cachazo, and Bo Feng. "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills." In: *Nucl. Phys. B* 725 (2005), pp. 275–305. DOI: 10.1016/j.nuclphysb.2005.07.014. arXiv: hep-th/0412103.

[8] A. A. Rosly and K. G. Selivanov. "On amplitudes in selfdual sector of Yang-Mills theory." In: *Phys. Lett. B* 399 (1997), pp. 135–140. DOI: 10.1016/S0370-2693(97)00268-2. arXiv: hep-th/9611101.

[9] K. G. Selivanov. "SD perturbiner in Yang-Mills + gravity." In: *Phys. Lett.* B420 (1998), pp. 274–278. DOI: 10.1016/S0370-2693(97)01514-1. arXiv: hep-th/9710197 [hep-th].

[10] Frits A. Berends and W. T. Giele. "Recursive Calculations for Processes with n Gluons." In: *Nucl. Phys. B* 306 (1988), pp. 759–808. DOI: 10.1016/0550-3213(88)90442-7.

[11] Z. Bern, J. J. M. Carrasco, and Henrik Johansson. "New Relations for Gauge-Theory Amplitudes." In: *Phys. Rev. D* 78 (2008), p. 085011. DOI: 10.1103/PhysRevD.78.085011. arXiv: 0805.3993 [hep-ph].

[12] H. Kawai, D. C. Lewellen, and S. H. H. Tye. "A Relation Between Tree Amplitudes of Closed and Open Strings." In: *Nucl. Phys.* B269 (1986), pp. 1–23. DOI: 10.1016/0550-3213(86)90362-7.

[13] Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson. "Perturbative Quantum Gravity as a Double Copy of Gauge Theory." In: *Phys. Rev. Lett.* 105 (2010), p. 061602. DOI: 10.1103/PhysRevLett.105.061602. arXiv: 1004.0476 [hep-th].

[14] Carlos R. Mafra, Oliver Schlotterer, and Stephan Stieberger. "Explicit BCJ Numerators from Pure Spinors." In: *JHEP* 07 (2011), p. 092. DOI: 10.1007/JHEP07(2011)092. arXiv: 1104.5224 [hep-th].

[15] Yi-Jian Du and Fei Teng. "BCJ numerators from reduced Pfaffian." In: *JHEP* 04 (2017), p. 033. DOI: 10.1007/JHEP04(2017)033. arXiv: 1703.05717 [hep-th].

[16] Elliot Bridges and Carlos R. Mafra. "Algorithmic construction of SYM multiparticle superfields in the BCJ gauge." In: *JHEP* 10 (2019), p. 022. DOI: 10.1007/JHEP10(2019)022. arXiv: 1906.12252 [hep-th].

[17] Z. Bern, J. J. M. Carrasco, Lance J. Dixon, H. Johansson, and R. Roiban. "The Complete Four-Loop Four-Point Amplitude in N=4 Super-Yang-Mills Theory." In: *Phys. Rev. D* 82 (2010), p. 125040. DOI: 10.1103/PhysRevD.82.125040. arXiv: 1008.3327 [hep-th].

[18] Carlos R. Mafra and Oliver Schlotterer. "Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace." In: *JHEP* 10 (2015), p. 124. DOI: 10.1007/JHEP10(2015)124. arXiv: 1505.02746 [hep-th].

[19] Song He, Ricardo Monteiro, and Oliver Schlotterer. "String-inspired BCJ numerators for one-loop MHV amplitudes." In: *JHEP* 01 (2016), p. 171. DOI: 10.1007/JHEP01(2016)171. arXiv: 1507.06288 [hep-th].

[20] Song He, Oliver Schlotterer, and Yong Zhang. "New BCJ representations for one-loop amplitudes in gauge theories and gravity." In: *Nucl. Phys. B* 930 (2018), pp. 328–383. DOI: 10.1016/j.nuclphysb.2018.03.003. arXiv: 1706.00640 [hep-th].

[21] Zvi Bern, John Joseph Carrasco, Wei-Ming Chen, Alex Edison, Henrik Johansson, Julio Parra-Martinez, Radu Roiban, and Mao Zeng. "Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops." In: *Phys. Rev. D* 98.8 (2018), p. 086021. DOI: 10.1103/PhysRevD.98.086021. arXiv: 1804.09311 [hep-th].

[22] Ricardo Monteiro, Donal O'Connell, and Chris D. White. "Black holes and the double copy." In: *JHEP* 12 (2014), p. 056. DOI: 10.1007/JHEP12(2014)056. arXiv: 1410.0239 [hep-th].

[23] Andrés Luna, Ricardo Monteiro, Isobel Nicholson, Alexander Ochirov, Donal O'Connell, Niclas Westerberg, and Chris D. White. "Perturbative spacetimes from Yang-Mills theory." In: *JHEP* 04 (2017), p. 069. DOI: 10.1007/JHEP04(2017)069. arXiv: 1611.07508 [hep-th].

[24] Tim Adamo, Eduardo Casali, Lionel Mason, and Stefan Nekovar. "Scattering on plane waves and the double copy." In: *Class. Quant. Grav.* 35.1 (2018), p. 015004. DOI: 10.1088/1361-6382/aa9961. arXiv: 1706.08925 [hep-th].

[25] Jan Plefka, Jan Steinhoff, and Wadim Wormsbecher. "Effective action of dilaton gravity as the classical double copy of Yang-Mills theory." In: *Phys. Rev. D* 99.2 (2019), p. 024021. DOI: 10.1103/PhysRevD.99.024021. arXiv: 1807.09859 [hep-th].

[26] Zvi Bern, Clifford Cheung, Radu Roiban, Chia-Hsien Shen, Mikhail P. Solon, and Mao Zeng. "Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order." In: *Phys. Rev. Lett.* 122.20 (2019), p. 201603. DOI: 10.1103/PhysRevLett.122.201603. arXiv: 1901.04424 [hep-th].

[27] Zvi Bern, Clifford Cheung, Radu Roiban, Chia-Hsien Shen, Mikhail P. Solon, and Mao Zeng. "Black Hole Binary Dynamics from the Double Copy and Effective Theory." In: *JHEP* 10 (2019), p. 206. DOI: 10.1007/JHEP10(2019)206. arXiv: 1908.01493 [hep-th].

[28] Gabriele Travaglini et al. "The SAGEX review on scattering amplitudes*." In: *J. Phys. A* 55.44 (2022), p. 443001. DOI: 10.1088/1751-8121/ac8380. arXiv: 2203.13011 [hep-th].

[29] R. P. Feynman. "Mathematical formulation of the quantum theory of electromagnetic interaction." In: *Phys. Rev.* 80 (1950). Ed. by L. M. Brown, pp. 440–457. DOI: 10.1103/PhysRev.80.440.

[30] Michael B. Green, John H. Schwarz, and Lars Brink. "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories." In: *Nucl. Phys. B* 198 (1982), pp. 474–492. DOI: 10.1016/0550-3213(82)90336-4.

[31] Joseph A. Minahan. "One Loop Amplitudes on Orbifolds and the Renormalization of Coupling Constants." In: *Nucl. Phys. B* 298 (1988), pp. 36–74. DOI: 10.1016/0550-3213(88)90303-3.

[32] Vadim S. Kaplunovsky. "One Loop Threshold Effects in String Unification." In: *Nucl. Phys. B* 307 (1988). [Erratum: Nucl.Phys.B 382, 436–438 (1992)], pp. 145–156. DOI: 10.1016/0550-3213(88)90526-3. arXiv: hep-th/9205068.

[33] Zvi Bern and David A. Kosower. "A New Approach to One Loop Calculations in Gauge Theories." In: *Phys. Rev. D* 38 (1988), p. 1888. DOI: 10.1103/PhysRevD.38.1888.

[34] Zvi Bern and David A. Kosower. "Efficient calculation of one loop QCD amplitudes." In: *Phys. Rev. Lett.* 66 (1991), pp. 1669–1672. DOI: 10.1103/PhysRevLett.66.1669.

[35] Zvi Bern and David A. Kosower. "The Computation of loop amplitudes in gauge theories." In: *Nucl. Phys. B* 379 (1992), pp. 451–561. DOI: 10.1016/0550-3213(92)90134-W.

[36] Matthew J. Strassler. "Field theory without Feynman diagrams: One loop effective actions." In: *Nucl. Phys. B* 385 (1992), pp. 145–184. DOI: 10.1016/0550-3213(92)90098-V. arXiv: hep-ph/9205205.

[37] Christian Schubert. "Perturbative quantum field theory in the string inspired formalism." In: *Phys. Rept.* 355 (2001), pp. 73–234. DOI: 10.1016/S0370-1573(01)00013-8. arXiv: hep-th/0101036.

[38] Michael G. Schmidt and Christian Schubert. "Multiloop calculations in the string inspired formalism: The Single spinor loop in QED." In: *Phys. Rev. D* 53 (1996), pp. 2150–2159. DOI: 10.1103/PhysRevD.53.2150. arXiv: hep-th/9410100.

[39] Holger Gies, Kurt Langfeld, and Laurent Moyaerts. "Casimir effect on the worldline." In: *JHEP* 06 (2003), p. 018. DOI: 10.1088/1126-6708/2003/06/018. arXiv: hep-th/0303264.

[40] Fiorenzo Bastianelli and Andrea Zirotti. "Worldline formalism in a gravitational background." In: *Nucl. Phys. B* 642 (2002), pp. 372–388. DOI: 10.1016/S0550-3213(02)00683-1. arXiv: hep-th/0205182.

[41] Fiorenzo Bastianelli, Olindo Corradini, and Andrea Zirotti. "dimensional regularization for N=1 supersymmetric sigma models and the worldline formalism." In: *Phys. Rev. D* 67 (2003), p. 104009. DOI: 10.1103/PhysRevD.67.104009. arXiv: hep-th/0211134.

[42] Fiorenzo Bastianelli, Paolo Benincasa, and Simone Giombi. "Worldline approach to vector and antisymmetric tensor fields." In: *JHEP* 04 (2005), p. 010. DOI: 10.1088/1126-6708/2005/04/010. arXiv: hep-th/0503155.

[43] Fiorenzo Bastianelli and Christian Schubert. "One loop photon-graviton mixing in an electromagnetic field: Part 1." In: *JHEP* 02 (2005), p. 069. DOI: 10.1088/1126-6708/2005/02/069. arXiv: gr-qc/0412095.

[44] Youngjai Kiem, Soo-Jong Rey, Haru-Tada Sato, and Jung-Tay Yee. "Anatomy of one loop effective action in noncommutative scalar field theories." In: *Eur. Phys. J. C* 22 (2002), pp. 757–770. DOI: 10.1007/s100520100829. arXiv: hep-th/0107106.

[45] R. Bonezzi, O. Corradini, S. A. Franchino Vinas, and P. A. G. Pisani. "Worldline approach to noncommutative field theory." In: *J. Phys. A* 45 (2012), p. 405401. DOI: 10.1088/1751-8113/45/40/405401. arXiv: 1204.1013 [hep-th].

[46] Paul Mansfield. "The fermion content of the Standard Model from a simple world-line theory." In: *Phys. Lett. B* 743 (2015), pp. 353–356. DOI: 10.1016/j.physletb.2015.02.061. arXiv: 1410.7298 [hep-ph].

[47] James P. Edwards. "Unified theory in the worldline approach." In: *Phys. Lett. B* 750 (2015), pp. 312–318. DOI: 10.1016/j.physletb.2015.09.038. arXiv: 1411.6540 [hep-th].

[48] Canxin Shi and Jan Plefka. "Classical double copy of worldline quantum field theory." In: *Phys. Rev. D* 105.2 (2022), p. 026007. DOI: 10.1103/PhysRevD.105.026007. arXiv: 2109.10345 [hep-th].

[49] Gustav Mogull, Jan Plefka, and Jan Steinhoff. "Classical black hole scattering from a worldline quantum field theory." In: *JHEP* 02 (2021), p. 048. DOI: 10.1007/JHEP02(2021)048. arXiv: 2010.02865 [hep-th].

[50] David Tong. "String Theory." In: (Jan. 2009). arXiv: 0908.0333 [hep-th].

[51] E. S. Fradkin and Arkady A. Tseytlin. "Nonlinear Electrodynamics from Quantized Strings." In: *Phys. Lett. B* 163 (1985), pp. 123–130. DOI: 10.1016/0370-2693(85)90205-9.

[52] Joel Scherk. "Zero-slope limit of the dual resonance model." In: *Nucl. Phys. B* 31 (1971), pp. 222–234. DOI: 10.1016/0550-3213(71)90227-6.

[53] A. Neveu and Joel Scherk. "Connection between Yang-Mills fields and dual models." In: *Nucl. Phys. B* 36 (1972), pp. 155–161. DOI: 10.1016/0550-3213(72)90301-X.

[54] T. Yoneya. "Quantum gravity and the zero slope limit of the generalized Virasoro model." In: *Lett. Nuovo Cim.* 8 (1973), pp. 951–955. DOI: 10.1007/BF02727806.

[55] Joel Scherk and John H. Schwarz. "Dual Models for Non-hadrons." In: *Nucl. Phys. B* 81 (1974), pp. 118–144. DOI: 10.1016/0550-3213(74)90010-8.

[56] Zvi Bern, Lance J. Dixon, and David A. Kosower. "One loop corrections to five gluon amplitudes." In: *Phys. Rev. Lett.* 70 (1993), pp. 2677–2680. DOI: 10.1103/PhysRevLett.70.2677. arXiv: hep-ph/9302280.

[57] Zvi Bern, David C. Dunbar, and Tokuzo Shimada. "String based methods in perturbative gravity." In: *Phys. Lett. B* 312 (1993), pp. 277–284. DOI: 10.1016/0370-2693(93)91081-W. arXiv: hep-th/9307001.

[58] David C. Dunbar and Paul S. Norridge. "Calculation of graviton scattering amplitudes using string based methods." In: *Nucl. Phys. B* 433 (1995), pp. 181–208. DOI: 10.1016/0550-3213(94)00385-R. arXiv: hep-th/9408014.

[59] E. Fradkin. "Application of functional methods in quantum field theory and quantum statistics (II)." In: *Nucl. Phys.* 76.3 (1966), pp. 588–624. DOI: 10.1016/0029-5582(66)90200-8.

[60] L. Brink, P. Di Vecchia, and Paul S. Howe. "A Lagrangian Formulation of the Classical and Quantum Dynamics of Spinning Particles." In: *Nucl. Phys. B* 118 (1977), pp. 76–94. DOI: 10.1016/0550-3213(77)90364-9.

[61] Y. Ohnuki and T. Kashiwa. "Coherent States of Fermi Operators and the Path Integral." In: *Prog. Theor. Phys.* 60 (1978), p. 548. DOI: 10.1143/PTP.60.548.

[62] Eric D'Hoker and Darius G. Gagne. "Worldline path integrals for fermions with scalar, pseudoscalar and vector couplings." In: *Nucl. Phys. B* 467 (1996), pp. 272–296. DOI: 10.1016/0550-3213(96)00125-3. arXiv: hep-th/9508131.

[63] Eric D'Hoker and Darius G. Gagne. "Worldline path integrals for fermions with general couplings." In: *Nucl. Phys. B* 467 (1996), pp. 297–312. DOI: 10.1016/0550-3213(96)00126-5. arXiv: hep-th/9512080.

[64] Martin Reuter, Michael G. Schmidt, and Christian Schubert. "Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals." In: *Annals Phys.* 259 (1997), pp. 313–365. DOI: 10.1006/aphy.1997.5716. arXiv: hep-th/9610191.

[65] Taco Nieuwenhuis and J. A. Tjon. "Nonperturbative study of generalized ladder graphs in a phi**2 chi theory." In: *Phys. Rev. Lett.* 77 (1996), pp. 814–817. DOI: 10.1103/PhysRevLett.77.814. arXiv: hep-ph/9606403.

[66] A. I. Karanikas, C. N. Ktorides, and N. G. Stefanis. "Effective field theory description of Green and vertex functions in the infrared domain." In: *Phys. Rev. D* 52 (1995), pp. 5898–5916. DOI: 10.1103/PhysRevD.52.5898.

[67] A. I. Karanikas, C. N. Ktorides, N. G. Stefanis, and S. M. H. Wong. "Off mass shell Sudakov suppression factor for the fermionic four point function in QCD." In: *Phys. Lett. B* 455 (1999), pp. 291–299. DOI: 10.1016/S0370-2693(99)00397-4. arXiv: hep-ph/9812335.

[68] Dmitri Antonov. "Testing nonperturbative ansatze for the QCD field strength correlator." In: *Phys. Lett. B* 479 (2000), pp. 387–394. DOI: 10.1016/S0370-2693(00)00337-3. arXiv: hep-ph/0001193.

[69] Luis Alvarez-Gaume. "Supersymmetry and the Atiyah-Singer Index Theorem." In: *Commun. Math. Phys.* 90 (1983). Ed. by W. E. Brittin, K. E. Gustafson, and W. Wyss, p. 161. DOI: 10.1007/BF01205500.

[70] Luis Alvarez-Gaume and Edward Witten. "Gravitational Anomalies." In: *Nucl. Phys. B* 234 (1984). Ed. by A. Salam and E. Sezgin, p. 269. DOI: 10.1016/0550-3213(84)90066-X.

[71] Filippo Maria Balli. "Wordline Computation of Tree-level QED and QCD Amplitudes and Transversality." Unpublished M.Sc. Thesis (2018).

[72] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. International Series In Pure and Applied Physics. New York: McGraw-Hill, 1980. ISBN: 978-0-486-44568-7.

[73] Mark Srednicki. *Quantum field theory*. Cambridge University Press, 2007.

[74] L. s. Schulman. *Techniques and Applications OF Path Integration*. 1981.

[75] Zvi Bern and David A. Kosower. "Color decomposition of one loop amplitudes in gauge theories." In: *Nucl. Phys. B* 362 (1991), pp. 389–448. DOI: 10.1016/0550-3213(91)90567-H.

[76] Naser Ahmadiniaz, Filippo Maria Balli, Cristhiam Lopez-Arcos, Alexander Quintero Velez, and Christian Schubert. "Color-kinematics duality from the Bern-Kosower formalism." In: *Phys. Rev. D* 104.4 (2021), p. L041702. DOI: 10.1103/PhysRevD.104.L041702. arXiv: 2105.06745 [hep-th].

[77] A. P. Balachandran, Per Salomonson, Bo-Sture Skagerstam, and Jan-Olof Winnberg. "Classical Description of Particle Interacting with Nonabelian Gauge Field." In: *Phys. Rev. D* 15 (1977), pp. 2308–2317. DOI: 10.1103/PhysRevD.15.2308.

[78] Fiorenzo Bastianelli, Roberto Bonezzi, Olindo Corradini, and Emanuele Latini. "Particles with non abelian charges." In: *JHEP* 10 (2013), p. 098. DOI: 10.1007/JHEP10(2013)098. arXiv: 1309.1608 [hep-th].

[79] Fiorenzo Bastianelli, Roberto Bonezzi, Olindo Corradini, Emanuele Latini, and Khaled Hassan Ould-Lahoucine. "A worldline approach to colored particles." In: *J. Phys. Conf. Ser.* 1208.1 (2019). Ed. by Adnan Bashir and Christian Schubert, p. 012004. DOI: 10.1088/1742-6596/1208/1/012004. arXiv: 1504.03617 [hep-th].

[80] K. Daikouji, M. Shino, and Y. Sumino. "Bern-Kosower rule for scalar QED." In: *Phys. Rev. D* 53 (1996), pp. 4598–4615. DOI: 10.1103/PhysRevD.53.4598. arXiv: hep-ph/9508377.

[81] Naser Ahmadiniaz, Adnan Bashir, and Christian Schubert. "Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED." In: *Phys. Rev. D* 93.4 (2016), p. 045023. DOI: 10.1103/PhysRevD.93.045023. arXiv: 1511.05087 [hep-ph].

[82] Michelangelo L. Mangano and Stephen J. Parke. "Multiparton amplitudes in gauge theories." In: *Phys. Rept.* 200 (1991), pp. 301–367. DOI: 10.1016/0370-1573(91)90091-Y. arXiv: hep-th/0509223.

[83] Michelangelo L. Mangano, Stephen J. Parke, and Zhan Xu. "Duality and Multi - Gluon Scattering." In: *Nucl. Phys. B* 298 (1988), pp. 653–672. DOI: 10.1016/0550-3213(88)90001-6.

[84] Vittorio Del Duca, Lance J. Dixon, and Fabio Maltoni. "New color decompositions for gauge amplitudes at tree and loop level." In: *Nucl. Phys. B* 571 (2000), pp. 51–70. DOI: 10.1016/S0550-3213(99)00809-3. arXiv: hep-ph/9910563.

[85] Ronald Kleiss and Hans Kuijf. "Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders." In: *Nucl. Phys. B* 312 (1989), pp. 616–644. DOI: 10.1016/0550-3213(89)90574-9.

[86] Dieter Blessenohl and Hartmut Laue. "Generalized Jacobi Identities." In: *Note di Matematica* 8.1 (1988), pp. 111–121.

[87] N. E. J. Bjerrum-Bohr, Poul H. Damgaard, Thomas Søndergaard, and Pierre Vanhove. "Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes." In: *JHEP* 06 (2010), p. 003. DOI: 10.1007/JHEP06(2010)003. arXiv: 1003.2403 [hep-th].

[88] S. H. Henry Tye and Yang Zhang. "Dual Identities inside the Gluon and the Graviton Scattering Amplitudes." In: *JHEP* 06 (2010). [Erratum: JHEP 04, 114 (2011)], p. 071. DOI: 10.1007/JHEP06(2010)071. arXiv: 1003.1732 [hep-th].

[89] Zvi Bern, Tristan Dennen, Yu-tin Huang, and Michael Kiermaier. "Gravity as the Square of Gauge Theory." In: *Phys. Rev. D* 82 (2010), p. 065003. DOI: 10.1103/PhysRevD.82.065003. arXiv: 1004.0693 [hep-th].

[90] Daniel Z. Freedman and Antoine Van Proeyen. *Supergravity*. Cambridge, UK: Cambridge Univ. Press, May 2012. ISBN: 978-1-139-36806-3, 978-0-521-19401-3.

[91] S. J. Gates, Marcus T. Grisaru, M. Rocek, and W. Siegel. *Superspace Or One Thousand and One Lessons in Supersymmetry*. Vol. 58. Frontiers in Physics. 1983. ISBN: 978-0-8053-3161-5. arXiv: hep-th/0108200.

[92] Zvi Bern, John Joseph Carrasco, Marco Chiodaroli, Henrik Johansson, and Radu Roiban. "The Duality Between Color and Kinematics and its Applications." In: (Sept. 2019). arXiv: 1909.01358 [hep-th].

[93] Hadleigh Frost, Carlos R. Mafra, and Lionel Mason. "A Lie bracket for the momentum kernel." In: (Dec. 2020). arXiv: 2012.00519 [hep-th].

[94] S. Stieberger. "A Relation between One-Loop Amplitudes of Closed and Open Strings (One-Loop KLT Relation)." In: (Dec. 2022). arXiv: 2212.06816 [hep-th].

[95] Song He and Oliver Schlotterer. "New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level." In: *Phys. Rev. Lett.* 118.16 (2017), p. 161601. DOI: 10.1103/PhysRevLett.118.161601. arXiv: 1612.00417 [hep-th].

[96] Alex Edison, Max Guillen, Henrik Johansson, Oliver Schlotterer, and Fei Teng. "One-loop matrix elements of effective superstring interactions: α'-expanding loop integrands." In: *JHEP* 12 (2021), p. 007. DOI: 10.1007/JHEP12(2021)007. arXiv: 2107.08009 [hep-th].

[97] Carlos R. Mafra and Oliver Schlotterer. "Berends-Giele recursions and the BCJ duality in superspace and components." In: *JHEP* 03 (2016), p. 097. DOI: 10.1007/JHEP03(2016)097. arXiv: 1510.08846 [hep-th].

[98] Lucia M. Garozzo, Leonel Queimada, and Oliver Schlotterer. "Berends-Giele currents in Bern-Carrasco-Johansson gauge for F^3- and F^4-deformed Yang-Mills amplitudes." In: *JHEP* 02 (2019), p. 078. DOI: 10.1007/JHEP02(2019)078. arXiv: 1809.08103 [hep-th].

[99] Frits A. Berends and W. T. Giele. "Multiple Soft Gluon Radiation in Parton Processes." In: *Nucl. Phys. B* 313 (1989), pp. 595–633. DOI: 10.1016/0550-3213(89)90398-2.

[100] Seungjin Lee, Carlos R. Mafra, and Oliver Schlotterer. "Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality." In: *JHEP* 03 (2016), p. 090. DOI: 10.1007/JHEP03(2016)090. arXiv: 1510.08843 [hep-th].

[101] Manfred Schocker. "Lie elements and Knuth relations." In: *Canadian Journal of Mathematics* 56.4 (2004), pp. 871–882.

[102] A. A. Rosly and K. G. Selivanov. "Gravitational SD perturbiner." In: (Oct. 1997). arXiv: hep-th/9710196.

[103] K. G. Selivanov. "Gravitationally dressed Parke-Taylor amplitudes." In: *Mod. Phys. Lett. A* 12 (1997), pp. 3087–3090. DOI: 10.1142/S0217732397003204. arXiv: hep-th/9711111.

[104] A Rosly and K Selivanov. "On form-factors in sinh-Gordon theory." In: *Phys. Lett. B* 426 (1998), pp. 334–338. DOI: 10.1016/S0370-2693(98)00280-9. arXiv: hep-th/9801044.

[105] K. G. Selivanov. "On tree form-factors in (supersymmetric) Yang-Mills theory." In: *Commun. Math. Phys.* 208 (2000), pp. 671–687. DOI: 10.1007/s002200050006. arXiv: hep-th/9809046.

[106] Sebastian Mizera and Barbara Skrzypek. "Perturbiner Methods for Effective Field Theories and the Double Copy." In: *JHEP* 10 (2018), p. 018. DOI: 10.1007/JHEP10(2018)018. arXiv: 1809.02096 [hep-th].

[107] Carlos R. Mafra. "Berends-Giele recursion for double-color-ordered amplitudes." In: *JHEP* 07 (2016), p. 080. DOI: 10.1007/JHEP07(2016)080. arXiv: 1603.09731 [hep-th].

[108] Naser Ahmadiniaz, Filippo Maria Balli, Olindo Corradini, José Manuel Dávila, and Christian Schubert. "Compton-like scattering of a scalar particle with N photons and one graviton." In: *Nucl. Phys. B* 950 (2020), p. 114877. DOI: 10.1016/j.nuclphysb.2019.114877. arXiv: 1908.03425 [hep-th].

[109] F. Bastianelli, O. Corradini, and P. van Nieuwenhuizen. "Dimensional regularization of nonlinear sigma models on a finite time interval." In: *Phys. Lett. B* 494 (2000), pp. 161–167. DOI: 10.1016/S0370-2693(00)01180-1. arXiv: hep-th/0008045.

[110] Fiorenzo Bastianelli. "The Path integral for a particle in curved spaces and Weyl anomalies." In: *Nucl. Phys. B* 376 (1992), pp. 113–126. DOI: 10.1016/0550-3213(92)90070-R. arXiv: hep-th/9112035.

[111] F. Bastianelli and P. van Nieuwenhuizen. *Path integrals and anomalies in curved space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Sept. 2006. ISBN: 978-0-511-21772-2, 978-0-521-12050-0, 978-0-521-84761-2. DOI: 10.1017/CBO9780511535031.

[112] ME Gertsenshtein. "Wave resonance of light and gravitional waves." In: *Sov Phys JETP* 14 (1962), pp. 84–85.

[113] Naser Ahmadiniaz, Fiorenzo Bastianelli, and Olindo Corradini. "Dressed scalar propagator in a non-Abelian background from the worldline formalism." In: *Phys. Rev. D* 93.2 (2016). [Addendum: Phys.Rev.D 93, 049904 (2016)], p. 025035. DOI: 10.1103/PhysRevD.93.025035. arXiv: 1508.05144 [hep-th].

[114] C. J. Goebel, F. Halzen, and J. P. Leveille. "Angular zeros of Brown, Mikaelian, Sahdev, and Samuel and the factorization of tree amplitudes in gauge theories." In: *Phys. Rev. D* 23 (1981), pp. 2682–2685. DOI: 10.1103/PhysRevD.23.2682.

[115] S. Y. Choi, J. S. Shim, and H. S. Song. "Factorization and polarization in linearized gravity." In: *Phys. Rev. D* 51 (1995), pp. 2751–2769. DOI: 10.1103/PhysRevD.51.2751. arXiv: hep-th/9411092.

[116] Barry R. Holstein. "Factorization in graviton scattering and the 'natural' value of the g-factor." In: (July 2006). arXiv: gr-qc/0607058.

[117] N. E. J. Bjerrum-Bohr, Barry R. Holstein, Ludovic Planté, and Pierre Vanhove. "Graviton-Photon Scattering." In: *Phys. Rev. D* 91.6 (2015), p. 064008. DOI: 10.1103/PhysRevD.91.064008. arXiv: 1410.4148 [gr-qc].

[118] Naser Ahmadiniaz, Olindo Corradini, José Manuel Dávila, and Christian Schubert. "Gravitational Compton Scattering from the Worldline Formalism." In: *Int. J. Mod. Phys. Conf. Ser.* 43 (2016). Ed. by Sanghyeon Chang, Kiwoon Choi, and San Pyo Kim, p. 1660201. DOI: 10.1142/S2010194516602015.

[119] F. Bastianelli, O. Corradini, J. M. Dávila, and C. Schubert. "On the low-energy limit of one-loop photon–graviton amplitudes." In: *Phys. Lett. B* 716 (2012), pp. 345–349. DOI: 10.1016/j.physletb.2012.08.030. arXiv: 1202.4502 [hep-th].

[120] Robert Delbourgo and P. Phocas-Cosmetatos. "Radiative corrections to the electron graviton vertex." In: *Lett. Nuovo Cim.* 5 (1972), pp. 420–422. DOI: 10.1007/BF02905266.

[121] Frits A. Berends and R. Gastmans. "Quantum Electrodynamical Corrections to Graviton-Matter Vertices." In: *Annals Phys.* 98 (1976), p. 225. DOI: 10.1016/0003-4916(76)90245-1.

[122] I. Huet, D. G. C. McKeon, and C. Schubert. "Euler-Heisenberg lagrangians and asymptotic analysis in 1+1 QED, part 1: Two-loop." In: *JHEP* 12 (2010), p. 036. DOI: 10.1007/JHEP12(2010)036. arXiv: 1010.5315 [hep-th].

[123] Idrish Huet, Michel Rausch de Traubenberg, and Christian Schubert. "The Euler-Heisenberg Lagrangian Beyond One Loop." In: *Int. J. Mod. Phys. Conf. Ser.* 14 (2012). Ed. by M. Asorey, M. Bordag, and Emilio Elizalde, pp. 383–393. DOI: 10.1142/S2010194512007507. arXiv: 1112.1049 [hep-th].

[124] Idrish Huet, Michel Rausch De Traubenberg, and Christian Schubert. "Three-loop Euler-Heisenberg Lagrangian in 1+1 QED, part 1: single fermion-loop part." In: *JHEP* 03 (2019), p. 167. DOI: 10.1007/JHEP03(2019)167. arXiv: 1812.08380 [hep-th].

[125] Carlos R. Mafra, Oliver Schlotterer, Stephan Stieberger, and Dimitrios Tsimpis. "A recursive method for SYM n-point tree amplitudes." In: *Phys. Rev. D* 83 (2011), p. 126012. DOI: 10.1103/PhysRevD.83.126012. arXiv: 1012.3981 [hep-th].

[126] Carlos R. Mafra, Oliver Schlotterer, and Stephan Stieberger. "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation." In: *Nucl. Phys. B* 873 (2013), pp. 419–460. DOI: 10.1016/j.nuclphysb.2013.04.023. arXiv: 1106.2645 [hep-th].

[127] Carlos R. Mafra and Oliver Schlotterer. "Multiparticle SYM equations of motion and pure spinor BRST blocks." In: *JHEP* 07 (2014), p. 153. DOI: 10.1007/JHEP07(2014)153. arXiv: 1404.4986 [hep-th].

[128] Christophe Reutenauer. "Free lie algebras." In: *Handbook of algebra*. Vol. 3. Elsevier, 2003, pp. 887–903.

[129] Naser Ahmadiniaz, Filippo Maria Balli, Olindo Corradini, Cristhiam Lopez-Arcos, Alexander Quintero Velez, and Christian Schubert. "Manifest colour-kinematics duality and double-copy in the string-based formalism." In: *Nucl. Phys. B* 975 (2022), p. 115690. DOI: 10.1016/j.nuclphysb.2022.115690. arXiv: 2110.04853 [hep-th].

[130] Carlos R. Mafra and Oliver Schlotterer. "Tree-level amplitudes from the pure spinor superstring." In: (Oct. 2022). arXiv: 2210.14241 [hep-th].

[131] Carlos R. Mafra. "Planar binary trees in scattering amplitudes." In: Nov. 2020. DOI: 10.4171/205-1/6. arXiv: 2011.14413 [math.CO].

[132] Valentina Guarin Escudero, Cristhiam Lopez-Arcos, and Alexander Quintero Velez. "Homotopy double copy and the Kawai-Lewellen-Tye relations for the non-abelian and tensor Navier-Stokes equations." In: (Jan. 2022). arXiv: 2201.06047 [math-ph].

[133] Alex Edison and Fei Teng. "Efficient Calculation of Crossing Symmetric BCJ Tree Numerators." In: *JHEP* 12 (2020), p. 138. DOI: 10.1007/JHEP12(2020)138. arXiv: 2005.03638 [hep-th].

[134] Konglong Wu and Yi-Jian Du. "Off-shell extended graphic rule and the expansion of Berends-Giele currents in Yang-Mills theory." In: *JHEP* 01 (2022), p. 162. DOI: 10.1007/JHEP01(2022)162. arXiv: 2109.14462 [hep-th].

[135] Maor Ben-Shahar and Max Guillen. "10D super-Yang-Mills scattering amplitudes from its pure spinor action." In: *JHEP* 12 (2021), p. 014. DOI: 10.1007/JHEP12(2021)014. arXiv: 2108.11708 [hep-th].

[136] N. E. J. Bjerrum-Bohr. "String theory and the mapping of gravity into gauge theory." In: *Phys. Lett. B* 560 (2003), pp. 98–107. DOI: 10.1016/S0370-2693(03)00373-3. arXiv: hep-th/0302131.

[137] R. R. Metsaev and Arkady A. Tseytlin. "Curvature Cubed Terms in String Theory Effective Actions." In: *Phys. Lett. B* 185 (1987), pp. 52–58. DOI: 10.1016/0370-2693(87)91527-9.

[138] Johannes Broedel and Lance J. Dixon. "Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators." In: *JHEP* 10 (2012), p. 091. DOI: 10.1007/JHEP10(2012)091. arXiv: 1208.0876 [hep-th].

[139] Cristhiam Lopez-Arcos and Alexander Quintero Vélez. "L_∞-algebras and the perturbiner expansion." In: *JHEP* 11 (2019), p. 010. DOI: 10.1007/JHEP11(2019)010. arXiv: 1907.12154 [hep-th].

[140] Freddy Cachazo, Song He, and Ellis Ye Yuan. "*Scattering in Three Dimensions from Rational Maps*." In: *JHEP* 10 (2013), p. 141. DOI: 10.1007/JHEP10(2013)141. arXiv: 1306.2962 [hep-th].

[141] Chris D. White. "Twistorial Foundation for the Classical Double Copy." In: *Phys. Rev. Lett.* 126.6 (2021), p. 061602. DOI: 10.1103/PhysRevLett.126.061602. arXiv: 2012.02479 [hep-th].

[142] Fiorenzo Bastianelli and Peter van Nieuwenhuizen. "Trace anomalies from quantum mechanics." In: *Nucl. Phys. B* 389 (1993), pp. 53–80. DOI: 10.1016/0550-3213(93)90285-W. arXiv: hep-th/9208059.

[143] Steven Weinberg. "Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass." In: *Phys. Rev.* 135 (1964), B1049–B1056. DOI: 10.1103/PhysRev.135.B1049.

[144] Freddy Cachazo and Andrew Strominger. "Evidence for a New Soft Graviton Theorem." In: (Apr. 2014). arXiv: 1404.4091 [hep-th].

[145] Andrew Strominger. "Lectures on the Infrared Structure of Gravity and Gauge Theory." In: (Mar. 2017). arXiv: 1703.05448 [hep-th].

[146] Humberto Gomez, Renann Lipinski Jusinskas, Cristhiam Lopez-Arcos, and Alexander Quintero Velez. "The L_∞ structure of gauge theories with matter." In: *JHEP* 02 (2021), p. 093. DOI: 10.1007/JHEP02(2021)093. arXiv: 2011.09528 [hep-th].

[147] Henrik Johansson and Alexander Ochirov. "Double copy for massive quantum particles with spin." In: *JHEP* 09 (2019), p. 040. DOI: 10.1007/JHEP09(2019)040. arXiv: 1906.12292 [hep-th].

[148] Humberto Gomez and Renann Lipinski Jusinskas. "Multiparticle solutions to Einstein's equations." In: (June 2021). arXiv: 2106.12584 [hep-th].

[149] K. G. Chetyrkin and F. V. Tkachov. "Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops." In: *Nucl. Phys. B* 192 (1981), pp. 159–204. DOI: 10.1016/0550-3213(81)90199-1.

[150] A. V. Smirnov and F. S. Chuharev. "FIRE6: Feynman Integral REduction with Modular Arithmetic." In: *Comput. Phys. Commun.* 247 (2020), p. 106877. DOI: 10.1016/j.cpc.2019.106877. arXiv: 1901.07808 [hep-ph].