Journal of Nonlinear Science (2026) 36:90
https://doi.org/10.1007/s00332-026-10307-2

Journal of Nonlinear Science

# Variational Analysis of Sharp Folds and Cusps in Shells: A Generalized Functional Setting with Atomic Measures

**Paolo Maria Mariano**[1] · **Domenico Mucci**[2]

Received: 27 October 2025 / Accepted: 14 July 2026 / Published online: 3 August 2026


**Abstract**
We characterize the functional structure of deformations that describe the occurrence of possibly irrecoverable sharp folds and cusps in shell-like bodies, while they avoid self-penetration of matter. The combination of special bounded variation maps with one-dimensional jump set and atomic measures provides a functional environment in which it is possible to represent configurations not otherwise described by special bounded Hessian maps, often used for energy minimization of shells. The introduction of atomic measures provides necessary control over the singular component of the generalized gradient measure, ensuring the robust compactness required by the direct method. In the proposed framework, we prove the existence of minimizers under Dirichlet boundary conditions for a class of energies that are appropriate for an ambient setting foreseeing large strains. Although we treat a traditional topic, the analysis goes beyond the classical format of nonlinear elasticity.

**Keywords** Continuum mechanics · Calculus of variations · Shells · Energy minimization

**Mathematics Subject Classification** 74K25 · 49Q20 · 49J45 · 26B25

## 1 Introduction

To be determined along the deformation of thin shells, possibly irrecoverable sharp folds and cusps require large strains at least locally, namely at least in some of

Communicated by Peter Sternberg.

✉ Paolo Maria Mariano
paolomaria.mariano@unifi.it

Domenico Mucci
domenico.mucci@unimore.it

[1] DICEA, Università di Firenze, via Santa Marta 3, 50139 Firenze, Italy

[2] DSFIM, Università degli Studi di Modena e Reggio Emilia, via Giuseppe Campi 213/b, 41125 Modena, Italy

their ‘small’ neighborhoods. The adjective “irrecoverable" refers to permanence after unloading.

We look at shells in terms of Gauss graphs; that is, we consider the “middle surface" and the associated normal vector field. Thus, we do not account for through-the-thickness shear, which we neglect due to the smallness of the thickness considered. So, we refer to two-dimensional bodies with additional structure embedded in three-dimensional space.

We give a functional characterization of pertinent deformations that avoid self-penetration of matter and may describe the formation of cusps and sharp folds, granting compactness.

Two-dimensional models of shells with additional structure, as the one provided by the normal vector field, can be justified in different ways. One is ‘top-to-bottom’: It means that we collapse a body ‘around a surface’ evaluating the energy. Plates are a paradigmatic case: Appropriate pertinent descriptions emerge by shrinking a parallelepiped up to a plane. More precisely, consider a domain $\Omega \subset \mathbb{R}^2$ and the ‘thick’ region $\Omega_h := \Omega \times \left(-h/2,\ h/2\right)$, with $h > 0$ the thickness. Thus, deformations are maps $\tilde{u} : \Omega_h \to \mathbb{R}^3$ assumed to be one-to-one, differentiable, and such that they preserve the local spatial orientation. For the elastic energy, we then write

$$\mathscr{E}_h := \int_{\Omega_h} e\,(Du\,(x))\ \mathrm{d}x\ ,$$

with $e\,(\cdot)$ the energy density; it is objective in the sense that $e\,(Du) = e\,(QDu)$ for any $Q \in SO(3)$. (Under large strains, objectivity renders the energy density physically incompatible with a quadratic dependence on $Du$ alone.) A possible program is then to think of plate models as the result of an energy limit by letting $h$ to zero in the sense of De Giorgi’s $\Gamma$-convergence. A key factor is the scaling, that is, how the ratio $\mathscr{E}_h/h$ behaves. Another essential rôle is the scaling of the applied loads, that is, the ratio between force densities and shell thickness. In-plane and out-of-plane energy components must be evaluated differently. The result for homogeneous materials has been obtained in Friesecke et al. (2006), based on analyses in Friesecke et al. (2003) and Le Dret and Raoult (2000). Precisely: (1) For $(\mathscr{E}_h/h) \sim 1$, the membrane theory emerges. (2) $(\mathscr{E}_h/h) \sim h^2$ gives the energy pertaining to shells that bend without stretching $\Omega$. (3) Condition $(\mathscr{E}_h/h) \sim h^4$ leads to the nonlinear von Kármán’s theory. (4) The scaling $(\mathscr{E}_h/h) \sim h^\gamma$, with $\gamma > 4$, yields the linearized version of case (3); when the material is linearly elastic and homogeneous, a biharmonic equation describes equilibrium. (5) Finally, the scaling $(\mathscr{E}_h/h) \sim h^\gamma$, with $2 < \gamma < 4$, implies non-standard theories of plates. For analogous analysis in the heterogeneous case, see (Schmidt 2007); see also (Neff 2010; Saem et al. 2023, 2025).

Else, as before we can shrink to the plane a parallelepiped made of a second-grade elastic material: This means that the energy density is a function of the first and second deformation gradients, namely

$$\mathscr{E}_{h2} := \int_{\Omega_h} e\left(Dy\left(x\right), D^2 y\left(x\right)\right)\, \mathrm{d}x\ .$$

When $(\mathscr{E}_{h2}/h) \sim 1$, the shrinking process leads to a surface with out-of-tangent-plane directors that can rotate independently from each other (Bhattacharya and James 1999) (see also (Anzellotti et al. 1994; Le Dret and Raoult 1996)), a Cosserat surface indeed.

Beyond elasticity, $\Gamma$-convergence analyses leading to plate and shell models have also been carried out considering deformations as special bounded Hessian (SBH) maps (Demengel 1984, 1989), which may represent peculiar aspects of elastic–plastic behavior (Percivale 1990; Acerbi and Percivale 1992; Tomarelli et al. 1992; Carriero et al. 1992, 1994; Percivale and Tomarelli 2002). Clearly, such a variational view does not account for plastic flows unless the related minimization problems are inserted in De Giorgi's minimizing movements procedure. Also, resorting to SBH maps does not account for the possible formation of cusps, those we describe here in terms of atomic measures. For example, when looking at cuspidate prismatic shells, concentrated actions may appear, as pointed out from a completely different path in Chinchaladze et al. (2011).

Always in variational terms, we also have a bottom-to-top approach to shell models based on a superposition of atomic layers. The result is still a Cosserat surface but with multiple out-of-tangent-plane directors, one for each atomic layer (Friesecke and James 2000). (An analysis that foresees a growth to infinity of the atoms in each layer is in Schmidt (2008); see also (Schmidt 2006).) If we assume that the $n$-tuple of directors is defined pointwise to within a permutation, we obtain a scheme that is adequate for the representation of shells or thin films made of solid mixtures, because it accounts for the unknown component mixing (Mariano 2018).

Beyond $\Gamma$-convergence, pointwise asymptotic procedures can be adopted to derive plate and shell models (Ciarlet 2001; Ciarlet and Mardare 2019; Steigmann 1999). Another efficient way is to exploit the theory of internal constraint in the $3D$ setting (Podio-Guidugli 1989; Antman and Marlow 1991; Lembo and Podio-Guidugli 2007).

It is also possible to derive a model for the mechanics of shells by looking only at the geometry of surfaces embedded in a $3D$ Euclidean point space, which is Koiter's proposal (Koiter 1966, 1960). Such an approach, however, does not account for the through-the-thickness behavior that is at least roughly incorporated in the Cosserat-surface-based view, which can be adopted *directly*, as Ericksen and Truesdell made first in 1958 (Ericksen and Truesdell 1958), opening a fruitful research path, the one of so-called *direct models* (e.g., (Simo et al. 1988; Simo and Fox 1989; Di Carlo et al. 2001; Steigmann 2013, 2018; Mariano and Mucci 2023b); a wide view on shell models is in the treatises (Antman 1995; Ciarlet 1997; Villaggio 1997)).

The viewpoint adopted here falls within the context of direct models (Ericksen and Truesdell 1958). We consider deformations of the middle shell surface as continuous maps with sufficiently regular derivatives. However, detaching from a pure SBH setting, we consider conical points (cusps) that can emerge in the folding. As already mentioned, to characterize cusps, we consider atomic measures supported by a countable set of points. To our knowledge, existential analyses of minimum problems for shell models that account for the emergence of cusps under large strains do not appear to have been considered so far in the available literature. The normal vector field is

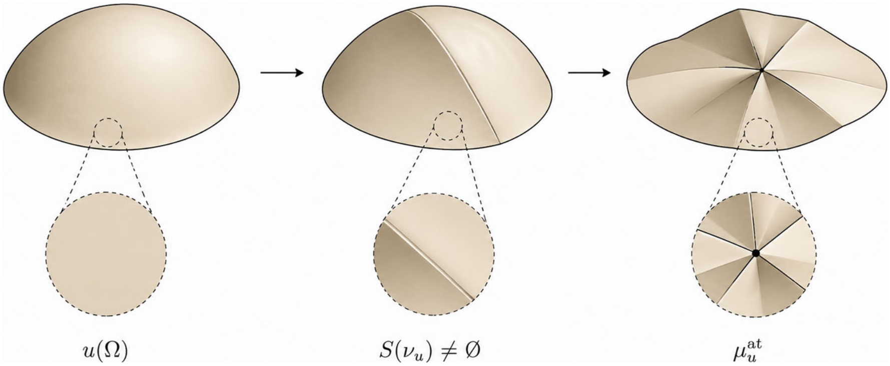


**Fig. 1** Typical geometries considered in the paper: from smooth shell configurations to sharp folds and cusp-like singularities associated with concentrated curvature measures

considered to be given by a special function of bounded variation (SBV). So, at every point along a sharp fold, we only consider the two normal vectors that limit the normal cone of a fold.

The energy class we look at is surface polyconvex; also, it involves the gradient of the normal, considered as an SBV map with jump set endowed with peculiar line energy; the possible presence of an external potential is considered to cover cases in which there is activation of shape morphing. For this energy class, we prove the existence of minimizers under Dirichlet-type boundary conditions, intended in the sense of traces, in the functional setting that we propose.

The conceptual path that we follow is summarized in Fig. 1: from smooth deformation, to those with nonzero singular one-dimensional set, to the introduction of atomic measures.

Shell singularities emerge in dimensional reduction and energy scaling in terms of $\Gamma$-convergence. Cusps are seen as results of metric incompatibility associated with embedding, scaling energy and relaxation (Lewicka 2020; Lewicka and Müller 2011). These scaling approaches to shells are linked with the hierarchy analysis proposed in Friesecke et al. (2003).

Always in the scaling perspective, a question tackled in Müller and Olbermann (2014); Olbermann (2016, 2017, 2018) concerns the amount of energy that we need to pay in order to get a cusp. We tackle another question: What's the functional structure that is necessary to represent cusps, maintaining compactness?

We adopt tools from geometric measure theory. On one side, it is a technical convenience: It allows us to avoid cancellations, control limits of the Gauss map and, at the same time, folds and cusps, obtaining compactness and eventually using the Federer–Fleming theorem. However, there is something more at the interplay of analytical and mechanical aspects. These tools allow us to select a subspace of $W^{1,p}$, with $p > 1$, of maps that avoid the non-required presence of holes, as the very concept of elasticity imposes. (Otherwise we were referring to elastic-brittle materials.) These maps have the minimal analytical regularity needed to formulate the mechanical setting. These aspects point out that the choice of a functional class has per se constitutive character.

Notions from geometric measure theory are necessary tools for analysis; we recall them in Sect. 2. Then, we first analyze smooth deformations in Sect. 3, clarifying how we impose a condition that avoids self-penetration of matter and how our approach prevents a possible cancelation in the Gauss graph when two distant shell parts come into contact with opposite orientation. Then, in Sect. 4 we prove summability properties of not necessarily smooth deformations that form cusps but are convex in a neighborhood of the cusps themselves; the analysis, however, is not reduced to such deformations, although they have a special status. In Sect. 5, we describe possibly irrecoverable folds in terms of vector-valued measures concentrated on 1-dimensional sets where jumps of the unit normal to the deformed shell may appear. The representation of cusps in terms of atomic measures supported by a countable point set is discussed in Sect. 6. An energy functional that accounts for the above-mentioned geometries and the energy weight of each of the corresponding features is introduced in Sect. 7; for it we prove the existence of minimizers.

## 2 Background Material

Basic tools of geometric measure theory and the theory of smooth surface Gauss graphs are recalled below for the sake of reader's convenience.

### 2.1 Special Functions of Bounded Variation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\mathscr{L}^n$ and $\mathscr{H}^k$ be Lebesgue and Hausdorff measure, respectively. We denote by $BV(\Omega, \mathbb{R}^N)$ the class of summable functions $\nu : \Omega \to \mathbb{R}^N$ whose distributional derivative $D\nu$ is an $\mathbb{R}^{N\times n}$-valued finite Borel measure. Denoting by $S(\nu)$ the complement of the set of Lebesgue's points of $\nu$, a mutually singular decomposition holds,

$$D\nu = \nabla\nu\, \mathscr{L}^n \llcorner \Omega + D^C\nu + D^J\nu\,, \tag{2.1}$$

where $\nabla\nu \in L^1(\Omega, \mathbb{R}^{N\times n})$ is the approximate gradient function, $D^C\nu$ is the Cantor component that satisfies $|D^C\nu|(B) = 0$ for every Borel set $B \subset \Omega$ such that $\mathscr{H}^{n-1}(B) < \infty$, and the jump component is given by

$$D^J\nu = [\nu] \otimes \mathbf{n}\, \mathscr{H}^{n-1} \llcorner S(\nu)\,, \tag{2.2}$$

where the jump set $S(\nu)$ is countably $(n-1)$-rectifiable, $\mathbf{n}$ is a unit normal to $S(\nu)$, and $[\nu](x) := \nu^+(x) - \nu^-(x) \in \mathbb{R}^N$ is the jump of $\nu$ at $\mathscr{H}^{n-1}$-almost every (shortly a.e.) point $x \in S(\nu)$, where $\nu^\pm(x)$ are the one-sided limits of $\nu$ at $x$ with respect to the normal $\mathbf{n}(x)$. Then, the total variation $|D\nu|$ of $D\nu$ is given by

$$|D\nu|(\Omega) = \int_\Omega |\nabla\nu|\, \mathrm{d}\mathscr{L}^n + |D^C\nu|(\Omega) + \int_{S(\nu)} |[\nu](x)|\, \mathrm{d}\mathscr{H}^{n-1}(x)\,. \tag{2.3}$$

If, in particular, $D^C\nu = 0$, the map $\nu$ is called a special function of bounded variation, and we write $\nu \in SBV(\Omega, \mathbb{R}^N)$, see (Ambrosio et al. 2000).

The case considered deals with $n = 2$, $N = 3$, while $\nu$ is identified with the unit normal to a sufficiently smooth parametric surface, so that $\nu : \Omega \to \mathbb{S}^2$, where $\mathbb{S}^2$ is the Gauss sphere $\mathbb{S}^2 := \{z \in \mathbb{R}^3 : |z| = 1\}$. For $X = C^2$, $W^{1,p}$, $BV$, or $SBV$, we thus denote

$$X(\Omega, \mathbb{S}^2) := \{\nu \in X(\Omega, \mathbb{R}^3) : |\nu(x)| = 1 \text{ for } \mathscr{L}^2\text{-a.e. } x \in \Omega\}.$$

### 2.2 Rectifiable Currents

For $U \subset \mathbb{R}^m$ an open set, and $k = 0, \ldots, m$, we write $\mathscr{D}_k(U)$ for the dual to the space $\mathscr{D}^k(U)$ of compactly supported smooth $k$-forms in $U$. Elements of $\mathscr{D}_k(U)$ are called *currents*. Notions of mass and boundary current are associated with them. Precisely, the *mass* $\mathbf{M}(T)$ of a current $T$ in $\mathscr{D}_k(U)$ is defined to be

$$\mathbf{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \mathscr{D}^k(U),\ \ \|\omega\| \le 1\},$$

while (when $k \ge 1$) the *boundary* of $T$ is the current $\partial T$ in $\mathscr{D}_{k-1}(U)$ given by

$$\langle \partial T, \eta\rangle := \langle T, \mathrm{d}\eta\rangle, \qquad \forall \eta \in \mathscr{D}^{k-1}(U),$$

where $\mathrm{d}\eta$ is the exterior differential of $\eta$. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathscr{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h, \omega\rangle = \langle T, \omega\rangle, \qquad \forall\, \omega \in \mathscr{D}^k(U).$$

If $T_h \rightharpoonup T$, by lower semicontinuity we also have

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h).$$

For $k \ge 1$, a $k$-current $T$ with finite mass is called *rectifiable* if

$$\langle T, \omega\rangle = \int_{\mathscr{N}} \theta\, \langle \omega, \xi\rangle\, \mathrm{d}\mathscr{H}^k, \qquad \forall\, \omega \in \mathscr{D}^k(U),$$

where $\mathscr{N}$ is a countably $k$-rectifiable set in $U$, $\xi : \mathscr{N} \to \bigwedge^k \mathbb{R}^m$ is a $\mathscr{H}^k \llcorner \mathscr{N}$-measurable function such that $\xi(w)$ is a *simple* unit $k$-vector orienting the approximate tangent space to $\mathscr{N}$ at $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$, and $\theta : \mathscr{N} \to [0, +\infty)$ is an $\mathscr{H}^k \llcorner \mathscr{N}$-summable and nonnegative function, called a *multiplicity*. Thus, for $T$ a rectifiable current, we have

$$\mathbf{M}(T) = \int_{\mathscr{M}} \theta\, \mathrm{d}\mathscr{H}^k < \infty,$$

and the short-hand notation $T = [\![ \mathcal{N}, \xi, \theta ]\!]$ is commonly adopted. The *size* of $T$ is defined by

$$\mathbf{S}(T) := \mathcal{H}^k\{w \in \mathcal{N} \mid \theta(w) > 0\}.$$

When $\theta$ is integer-valued and $T$ is rectifiable, we say that $T$ is an *integer rectifiable* (in short *i.m.*) current, a class indicated by $\mathcal{R}_k(U)$ (Giaquinta et al. 1998a).

More generally, $T$ is called a *size bounded* current if $T = [\![ \mathcal{N}, \xi, \theta ]\!]$ is rectifiable and with finite size, $\mathbf{S}(T) < \infty$. We indicate by $\mathcal{S}_k(U)$ the pertinent class. Any current $T$ in $\mathcal{R}_k(U)$ satisfies $\mathbf{S}(T) \le \mathbf{M}(T)$, and therefore, it belongs to $\mathcal{S}_k(U)$.

### 2.3 Basic Examples

If $\mathcal{N}$ is a smooth, oriented, and compact Riemannian $k$-manifold embedded in $U$, a current $[\![ \mathcal{N} ]\!]$ in $\mathcal{R}_k(U)$ is naturally associated with $\mathcal{N}$ by setting

$$\langle [\![ \mathcal{N} ]\!], \omega \rangle := \int_{\mathcal{N}} \omega, \quad \omega \in \mathcal{D}^k(U).$$

In that case, we have $\mathbf{M}([\![ \mathcal{N} ]\!]) = \mathcal{H}^k(\mathcal{N}) < \infty$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected set and $u \in C^2(\overline{\Omega}, \mathbb{R}^3)$. With $U = \Omega \times \mathbb{R}^3$, and $\mathcal{G}_u$ the naturally oriented graph of $u$, the *graph current* $G_u := [\![ \mathcal{G}_u ]\!]$ belongs to $\mathcal{R}_2(\Omega \times \mathbb{R}^3)$, and by the area formula we get

$$\mathbf{M}(G_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + |\partial_1 u \times \partial_2 u|^2}\, \mathrm{d}\mathcal{L}^2 < \infty \tag{2.4}$$

where $\partial_1 u$ and $\partial_2 u$ are the column vectors of the gradient matrix $\nabla u$. Notice that $|\partial_1 u \times \partial_2 u| = |\operatorname{adj}_2 \nabla u|$, where $\operatorname{adj}_2 \nabla u$ is the vector in $\mathbb{R}^3$ with entries given by the determinant of the $2 \times 2$ minors of $\nabla u$. Also, Stokes theorem implies the null-boundary condition

$$\langle \partial G_u, \eta \rangle = 0\,, \quad \forall\, \eta \in \mathcal{D}^1(\Omega \times \mathbb{R}^3). \tag{2.5}$$

More generally, if $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ for some $p \ge 2$, the graph current is defined by $G_u := (\mathrm{Id} \bowtie u)_\#[\![ \Omega ]\!]$, where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ is the graph map. More precisely, the pullback being defined $\mathcal{L}^2$-a.e. in $\Omega$ in terms of the approximate gradient of $u$, we set

$$\langle G_u, \omega \rangle := \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega, \quad \omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3).$$

Since $\partial_1 u \times \partial_2 u \in L^{p/2}(\Omega, \mathbb{R}^3)$, where $p/2 \ge 1$, it turns out that $G_u$ belongs to $\mathcal{R}_2(\Omega \times \mathbb{R}^3)$, and the mass formula (2.4) still holds true. Moreover, a density argument implies that the null-boundary condition (2.5) is still valid.

If $v \in BV(\Omega, \mathbb{S}^2)$ is such that $\partial_1 v \times \partial_2 v \in L^1(\Omega, \mathbb{R}^3)$, the graph current $G_v$ is defined as in the Sobolev case, in terms of the approximate gradient $\nabla v$, and actually, by Federer's flatness theorem, $G_\nu \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$. Its finite mass is given again by formula (2.4), with $u = \nu$. However, even in the Sobolev case, in general the null-boundary condition (2.5) is violated.

Thus, we have $\mathbf{M}(\partial G_\nu) = \infty$ if $|D^C \nu|(\Omega) > 0$. A necessary and sufficient condition ensuring that a $BV$ map $\nu : \Omega \to \mathbb{S}^2$ belongs to $SBV(\Omega, \mathbb{S}^2)$, with a jump set of finite size, namely $\mathscr{H}^1(S(\nu)) < \infty$, is given by the requirement that the currents $G_{\nu^j}$ in $\mathscr{R}_1(\Omega \times \mathbb{R})$ carried by the graph of the component $BV$ functions $\nu^j$ have finite boundary mass (Giaquinta et al. 1998a), that is,

$$\mathbf{M}(\partial G_{\nu^j}) < \infty\,, \quad \forall\, j \in \{1, 2, 3\}\,. \tag{2.6}$$

Furthermore, if (2.6) holds, a result by Ambrosio et al. (1998) implies that the trace functions $\nu^\pm : S(\nu) \to \mathbb{S}^2$ are approximately differentiable $\mathscr{H}^1$-almost everywhere on the jump set $S(\nu)$.

## 2.4 Size Bounded Vector-Valued Currents

We will exploit $\mathbb{R}^3$-valued size bounded rectifiable currents defined on a bounded domain $U \subset \bar{\mathbb{R}}^3 \simeq \mathbb{R}^3$, where $\mathbf{x} = (x_1, x_2, x_3)$ (compare (Mariano and Mucci 2023a)).

First recall that $k$-forms $\overline{\omega}$ in $[\mathscr{D}^k(U)]^3$, with $0 \le k \le 3$ integer, are triplets $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$, where $\omega_j \in \mathscr{D}^k(U)$, for $j = 1, 2, 3$. More precisely, vector fields $\phi$ in $C_c^\infty(U, \mathbb{R}^3)$ agree with 0-forms $\overline{\omega}$ in $[\mathscr{D}^0(U)]^3$, say $\overline{\omega} = \omega_\phi^{(0)}$. A 1-form $\overline{\omega} \in [\mathscr{D}^1(U)]^3$ is identified by a fully covariant tensor-valued field $\psi \in C_c^\infty(U, \mathbb{R}^{3\times 3})$ with components $\psi_{jA}$ by letting $\omega_j = \sum_{A=1}^3 \psi_{jA} \mathrm{d}x^A$ for $j = 1, 2, 3$. In this case, we write $\overline{\omega} = \omega_\psi^{(1)}$. In a similar way, a 2-form $\overline{\omega} \in [\mathscr{D}^2(U)]^3$ is identified by a tensor-valued field $\zeta \in C_c^\infty(U, \mathbb{R}^{3\times 3})$ with components $\zeta_{jA}$ by letting $\omega_j = \sum_{A=1}^3 (-1)^{A-1} \zeta_{jA} \widehat{\mathrm{d}x^A}$, where $\widehat{\mathrm{d}x^A}$ is the 2-vector that complements $\mathrm{d}x^A$ to complete $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, so that $\mathrm{d}x^A \wedge \widehat{\mathrm{d}x^A} = (-1)^{A-1} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In this case, we write $\overline{\omega} = \omega_\zeta^{(2)}$. Finally, a 3-form $\overline{\omega} \in [\mathscr{D}^3(U)]^3$ is identified by a covector field $\eta \in C_c^\infty(U, \mathbb{R}^3)$ with $\eta = (\eta_1, \eta_2, \eta_3)$, by letting $\omega_j = \eta_j \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, and we write $\overline{\omega} = \omega_\eta^{(3)}$. We thus have

$$\mathrm{d}\omega_\phi^{(0)} = \omega_{\nabla\phi}^{(1)}\,, \qquad \mathrm{d}\omega_\psi^{(1)} = \omega_{\operatorname{curl}\psi}^{(2)}\,, \qquad \mathrm{d}\omega_\zeta^{(2)} = \omega_{\operatorname{div}\zeta}^{(3)}$$

and hence, the identities $\operatorname{curl} \nabla\phi = 0$ and $\operatorname{div}(\operatorname{curl}\psi) = 0$ are equivalent to the closure relations $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$ for 0-forms and 1-forms, respectively.

For $k = 1, 2$, a *size bounded* current $\bar{T} \in [\mathscr{S}_k(U)]^3$ is identified by a triplet $(\mathscr{N}, \mathbf{b}, \xi)$, where $\mathscr{N} \subset U$ is a $k$-rectifiable set of $U$ satisfying $\mathscr{H}^k(\mathscr{N}) < \infty$, the $\mathbb{R}^3$-valued multiplicity function $\mathbf{b} : \mathscr{N} \to \mathbb{R}^3$ is $\mathscr{H}^k \llcorner \mathscr{N}$-summable, and $\xi : \mathscr{N} \to \bigwedge^k(\mathbb{R}^3)$ is an $\mathscr{H}^k \llcorner \mathscr{M}$-measurable function such that $\xi(w)$ is a simple unit $k$-vector yielding an orientation to the approximate tangent $k$-space to $\mathscr{N}$ at $w$, for $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$. Without loss of generality, we assume $\mathbf{b}(w) \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^k$-a.e. $w \in \mathscr{N}$, so that the *size* of the current $\bar{T}$ is equal to the $k$-measure of $\mathscr{N}$,

$$\mathbf{S}(\bar{T}) = \mathscr{H}^k(\mathscr{N}) < \infty.$$

The action of $\bar{T}$ is defined componentwise by the formula

$$\langle \bar{T}, \overline{\omega} \rangle := \sum_{j=1}^{3} \langle T^j, \omega_j \rangle, \quad \overline{\omega} \in [\mathscr{D}^k(U)]^3,$$

where $T^j = [\![ \mathscr{N}, |b^j|, \sigma^j \xi ]\!]$ if $\mathbf{b} = (b^1, b^2, b^3)$, and $\sigma^j := 1$ if $b^j \geq 0$, whereas $\sigma^j := -1$ if $b^j < 0$. Therefore, we have

$$\langle T^j, \omega_j \rangle = \int_{\mathscr{N}} |b^j| \langle \omega_j, \sigma^j \xi \rangle \, \mathrm{d}\mathscr{H}^k, \quad \forall \omega_j \in \mathscr{D}^k(U).$$

We thus assume that every component $T^j$ has finite *mass*, namely

$$\mathbf{M}(T^j) = \int_{\mathscr{N}} |b^j| \, \mathrm{d}\mathscr{H}^k < \infty, \quad j = 1, 2, 3,$$

and the mass of $\bar{T}$ is given by $\mathbf{M}(\bar{T}) := \sum_{j=1}^{3} \mathbf{M}(T^j)$. With the previous notations, we thus write $\bar{T} = (T^1, T^2, T^3) = [\![ \mathscr{N}, \mathbf{b}, \xi ]\!]$.

Finally, setting $\mathrm{d}\overline{\omega} := (\mathrm{d}\omega_1, \mathrm{d}\omega_2, \mathrm{d}\omega_3)$, the boundary current is defined by

$$\langle \partial \bar{T}, \overline{\omega} \rangle := \langle \bar{T}, \mathrm{d}\overline{\omega} \rangle \,, \quad \forall \overline{\omega} \in [\mathscr{D}^{k-1}(U)]^3 \,.$$

### 2.5 Gauss Graphs of Smooth Surfaces

Given a smooth (say $C^2$), bounded, and oriented surface $\mathscr{M} \subset \mathbb{R}^3$, with smooth boundary $\partial\mathscr{M}$, the *Gauss map* $\nu : \mathscr{M} \to \mathbb{S}^2$ associates with each point $y \in \mathscr{M}$ the unit normal $\nu(y) \in \mathbb{S}^2$. The graph of such a map, or *Gauss graph*, is the 2-dimensional surface in $\mathbb{R}^6$ given by

$$\mathscr{G}\mathscr{M} := \{(y, z) \mid y \in \mathscr{M} \,, \, z = \nu(y)\} \subset \mathscr{M} \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \hat{\mathbb{R}}^3 \simeq \mathbb{R}^6 \,,$$

where $\hat{\mathbb{R}}^3$ is the isomorphic copy of $\mathbb{R}^3$ in which we consider embedded the sphere $\mathbb{S}^2$. We are interested in the case of a parametric surface

$$\mathscr{M} = \mathscr{M}_u := u(\Omega) \quad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \,.$$

Therefore, we shall denote by $(\varepsilon_1, \varepsilon_2)$ and $(e_1, e_2, e_3)$ the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and by $(\varepsilon^1, \varepsilon^2)$ and $(e^1, e^2, e^3)$ the natural dual bases defined by $\varepsilon_i \cdot \varepsilon^j = \delta_i^j$ and $e_h \cdot e^k = \delta_h^k$, where the interposed dot means dual pairing. In particular, with $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$, we will identify the covector bases in the cotangent spaces to $\Omega$ and $\mathbb{R}^3$ at every $x \in \Omega$ and $y \in \mathbb{R}^3$ with $(\mathrm{d}x^1, \mathrm{d}x^2)$ and $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3)$; they are naturally dual to the bases $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$ and $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$ in the pertinent tangent spaces. We also correspondingly denote by $(\epsilon_1, \epsilon_2, \epsilon_3)$ the canonical basis of $\hat{\mathbb{R}}^3$, and by $(\mathrm{d}z^1, \mathrm{d}z^2, \mathrm{d}z^3)$ the associated covector basis, where $z = (z_1, z_2, z_3)$.

Referring to Anzellotti et al. (1990) or Anzellotti (1988) for further details, we only recall that if $\Phi : \mathscr{M} \to \mathbb{R}^3 \times \hat{\mathbb{R}}^3$ is the graph map $\Phi(y) := (y, \nu(y))$, the *tangential Jacobian* $J_\Phi^{\mathscr{M}}$ is given by

$$J_\Phi^{\mathscr{M}}(y) = \Big(1 + (\mathbf{k}_1{}^2 + \mathbf{k}_2{}^2) + (\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2}, \qquad y \in \mathscr{M}$$

where $\mathbf{k}_{1,2} = \mathbf{k}_{1,2}(y)$ are the *principal curvatures* at $y \in \mathscr{M}$. *Mean curvature* and *Gauss curvature* are given by

$$\mathbf{H} := \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2), \qquad \mathbf{K} := \mathbf{k}_1\mathbf{k}_2,$$

so that $\mathbf{k}_{1,2} = \mathbf{H} \pm \sqrt{\mathbf{H}^2 - \mathbf{K}}$. Therefore, we equivalently have

$$(J_\Phi^{\mathscr{M}})^2 = 1 + (2\mathbf{H})^2 - 2\mathbf{K} + \mathbf{K}^2 = 4\mathbf{H}^2 + (1 - \mathbf{K})^2$$

and hence, by the area formula, the area of the Gauss graph is

$$\begin{aligned} \mathscr{H}^2(\mathscr{G}\mathscr{M}) &= \int_{\mathscr{M}} \Big(1 + (\mathbf{k}_1{}^2 + \mathbf{k}_2{}^2) + (\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2} \mathrm{d}\mathscr{H}^2 \\ &= \int_{\mathscr{M}} \sqrt{1 + (4\mathbf{H}^2 - 2\mathbf{K}) + \mathbf{K}^2}\, \mathrm{d}\mathscr{H}^2 < \infty . \end{aligned} \tag{2.7}$$

It agrees with the mass of the current $[\![ \mathscr{G}\mathscr{M} ]\!] = \Phi_\#[\![ \mathscr{M} ]\!]$ in $\mathscr{R}_2(\mathbb{R}^3 \times \mathbb{S}^2)$ carried by the Gauss graph.

We finally recall that the *curvature functional* $\|\mathscr{M}\|$ of a smooth surface $\mathscr{M} \subset \mathbb{R}^3$ is defined in reference (Anzellotti et al. 1990) to be

$$\|\mathscr{M}\| := \mathscr{H}^2(\mathscr{M}) + \int_{\mathscr{M}} \sqrt{\mathbf{k}_1^2 + \mathbf{k}_2^2}\, \mathrm{d}\mathscr{H}^2 + \int_{\mathscr{M}} |\mathbf{k}_1\mathbf{k}_2|\, \mathrm{d}\mathscr{H}^2 . \tag{2.8}$$

It can be equivalently written as

$$\|\mathscr{M}\| := \int_{\mathscr{M}} \big(1 + \sqrt{4\mathbf{H}^2 - 2\mathbf{K}} + |\mathbf{K}|\big)\, \mathrm{d}\mathscr{H}^2,$$

so that by formula (2.7) we obtain bounds such as

$$\frac{1}{2}\|\mathscr{M}\| \le \mathscr{H}^2(\mathscr{G}\mathscr{M}) \le \|\mathscr{M}\|, \qquad \mathscr{H}^2(\mathscr{G}\mathscr{M}) = \mathbf{M}([\![ \mathscr{G}\mathscr{M} ]\!]).$$

## 3 Smooth Shell Deformations

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $\mathfrak{h} > 0$ a fixed small thickness parameter, and

$$U = U_{\mathfrak{h}} := \Omega \times I_{\mathfrak{h}} \,, \quad I_{\mathfrak{h}} = (-\mathfrak{h}/2, \mathfrak{h}/2)\,.$$

We adopt the notation $x = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{x} = (x_1, x_3, x_3) \in \bar{\mathbb{R}}^3 = \mathbb{R}^2 \times \mathbb{R}$, with $\bar{\mathbb{R}}^3$ an isomorphic copy of $\mathbb{R}^3$, the isomorphism being simply the identification. This choice is justified by the will of maintaining distinguished, although identical, the space in which we select the reference configuration with the one in which we detect the deformed configurations. We also identify $\mathrm{d}x$ with $\mathrm{d}\mathscr{L}^2$ or $\mathrm{d}\mathbf{x}$ with $\mathrm{d}\mathscr{L}^3$ inside the integrals defined over $\Omega$ or $U$, respectively.

Consider a deformation $u : \bar{\Omega} \to \mathbb{R}^3$ of class $C^2(\bar{\Omega}, \mathbb{R}^3)$ such that

$$\partial_1 u(x) \times \partial_2 u(x) \neq 0_{\mathbb{R}^3} \tag{3.1}$$

for every $x \in \Omega$. The unit normal to the oriented deformed surface $\mathscr{M}_u := u(\Omega)$ at $y = u(x)$ is given by

$$z = \nu_u(x) := \frac{\partial_1 u(x) \times \partial_2 u(x)}{|\partial_1 u(x) \times \partial_2 u(x)|} \in \mathbb{S}^2 \subset \hat{\mathbb{R}}^3. \tag{3.2}$$

### 3.1 Avoiding Self Penetration

When we consider the deformation of a $3D$ body into $3D$ space, two equivalent conditions express a constraint that avoids self-penetration of matter. One is due to Ciarlet and Nečas (1987), the other to Giaquinta et al. (1989) (see also (Giaquinta et al. 1998b, Sec. 2.3.2)).

For a sufficiently smooth map $v : U \to \mathbb{R}^3$ with $\det \nabla v > 0$ a.e. in $U$, the *global invertibility condition* above mentioned reads

$$\int_U f(\mathbf{x}, v(\mathbf{x})) \, \det \nabla v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} \sup_{\mathbf{x} \in U} f(\mathbf{x}, y) \, \mathrm{d}y \tag{3.3}$$

for every compactly supported smooth function $f : U \times \mathbb{R}^3 \to [0, +\infty)$. It is preserved by weak convergence in terms of currents and implies $\mathscr{L}^3$-a.e. injectivity (Giaquinta et al. 1989).

For a given parameter $0 < s < 1$, we thus denote by $v_{u,s} : U \to \mathbb{R}^3$ the deformation map

$$v_{u,s}(\mathbf{x}) := u(x_1, x_2) + s\, x_3\, \nu_u(x_1, x_2)\,. \tag{3.4}$$

Writing $u = (u^1, u^2, u^3)$ and $\nu_u = (\nu_u^1, \nu_u^2, \nu_u^3)$, we have

$$\nabla v_{u,s} = \begin{pmatrix} \partial_1 u^1 + s\, x_3\, \partial_1 \nu_u^1 & \partial_2 u^1 + s\, x_3\, \partial_2 \nu_u^1 & s\, \nu_u^1 \\ \partial_1 u^2 + s\, x_3\, \partial_1 \nu_u^2 & \partial_2 u^2 + s\, x_3\, \partial_2 \nu_u^1 & s\, \nu_u^2 \\ \partial_1 u^3 + s\, x_3\, \partial_1 \nu_u^3 & \partial_2 u^3 + s\, x_3\, \partial_2 \nu_3 & s\, \nu_u^3 \end{pmatrix}, \tag{3.5}$$

so that

$$\lim_{s \to 0^+} \frac{1}{s} \int_U \nabla v_{u,s}\, \mathrm{d}\mathbf{x} = \mathfrak{h} \int_\Omega \mathbf{G}\, \mathrm{d}x\,, \tag{3.6}$$

where $\mathbf{G} \in C^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3})$ has components

$$\mathbf{G}_A^j(x) := \begin{cases} \partial_A u^j(x) & \text{if} \quad A = 1, 2 \\ 0 & \text{if} \quad A = 3\,, \end{cases} \qquad j = 1, 2, 3\,. \tag{3.7}$$

We thus compute for each $s$

$$\det \nabla v_{u,s}(\mathbf{x}) = s\, |(\partial_1 u \times \partial_2 u)(x)| + s^2\, x_3\, \lambda_u^{(1)}(x) + s^3\, x_3^2\, \lambda_u^{(2)}(x)\,, \tag{3.8}$$

where

$$\begin{aligned} \lambda_u^{(1)}(x) := {} & \nu_u^1 \left(\partial_1 u^2\, \partial_2 \nu_u^3 - \partial_2 u^2\, \partial_1 \nu_u^3 + \partial_1 \nu_u^2\, \partial_2 u^3 - \partial_2 \nu_u^2\, \partial_1 u^3\right) \\ & + \nu_u^2 \left(\partial_1 u^3\, \partial_2 \nu_u^1 - \partial_2 u^3\, \partial_1 \nu_u^1 + \partial_1 \nu_u^3\, \partial_2 u^1 - \partial_2 \nu_u^3\, \partial_1 u^1\right) \\ & + \nu_u^3 \left(\partial_1 u^1\, \partial_2 \nu_u^2 - \partial_2 u^1\, \partial_1 \nu_u^2 + \partial_1 \nu_u^1\, \partial_2 u^2 - \partial_2 \nu_u^1\, \partial_1 u^2\right), \\ \lambda_u^{(2)}(x) := {} & \nu_u^1 \left(\partial_1 \nu_u^2\, \partial_2 \nu_u^3 - \partial_2 \nu_u^2\, \partial_1 \nu_u^3\right) \\ & + \nu_u^2 \left(\partial_1 \nu_u^3\, \partial_2 \nu_u^1 - \partial_2 \nu_u^3\, \partial_1 \nu_u^1\right) \\ & + \nu_u^3 \left(\partial_1 \nu_u^1\, \partial_2 \nu_u^2 - \partial_2 \nu_u^1\, \partial_1 \nu_u^2\right), \end{aligned} \tag{3.9}$$

so that we have

$$\lim_{s \to 0^+} \frac{1}{s} \int_U \det \nabla v_{u,s}\, \mathrm{d}\mathbf{x} = \mathfrak{h} \int_\Omega |\partial_1 u \times \partial_2 u|\, \mathrm{d}x\,. \tag{3.10}$$

Furthermore, since the functions $\lambda_u^{(k)}(x)$ are bounded, we get the uniform convergence $s^{-1} \det \nabla v_{u,s} \to |\partial_1 u \times \partial_2 u|$ as $s \to 0^+$ on $U$, and hence, we find

$$\det \nabla v_{u,s}(\mathbf{x}) > 0, \qquad \forall\, \mathbf{x} \in U,$$

if $s > 0$ is sufficiently small. Therefore, injectivity is recovered by requiring property (3.3), with $v = v_{u,s}$, for $s > 0$ sufficiently small (Mariano and Mucci 2023c)

### 3.2 Gauss Graphs of Smooth Parametric Surfaces

In the present setting, the Gauss graph of the surface $\mathscr{M}_u$ is described by

$$\mathscr{G}\mathscr{M}_u = \{\Phi_u(x) \mid x \in \Omega\}\,, \tag{3.11}$$

where $\Phi_u : \Omega \to \mathbb{R}^3 \times \hat{\mathbb{R}}^3$ is the (smooth) Gauss graph map

$$\Phi_u(x) := (u(x), \nu_u(x)) . \tag{3.12}$$

The mean curvature of $\mathscr{M}_u$ at $u(x)$ becomes

$$\mathbf{H}_u = \frac{1}{2g_u} \left( |\partial_1 u|^2 \partial^2_{2,2} u + |\partial_2 u|^2 \partial^2_{1,1} u - 2(\partial_1 u \bullet \partial_2 u) \partial^2_{1,2} u \right) \bullet \nu_u , \tag{3.13}$$

where

$$g_u := |\partial_1 u \times \partial_2 u|^2 = |\operatorname{adj}_2 \nabla u|^2 \tag{3.14}$$

and the interposed bullet means scalar product. The Gauss curvature $\mathbf{K}_u$ reads

$$\mathbf{K}_u = \frac{1}{{g_u}^{3/2}} \left( (\partial^2_{1,1} u \bullet \nu_u)(\partial^2_{2,2} u \bullet \nu_u) - (\partial^2_{1,2} u \bullet \nu_u)^2 \right) . \tag{3.15}$$

The tangent space at a point $(y, z) = (u(x), \nu_u(x))$ in the Gauss graph $\mathscr{G}\mathscr{M}_u$ is oriented for every $x \in \Omega$ by the wedge product

$$\xi_u(x) := \partial_1 \Phi_u(x) \wedge \partial_2 \Phi_u(x) \in \bigwedge\nolimits^2 (\mathbb{R}^3 \times \hat{\mathbb{R}}^3) . \tag{3.16}$$

According to the basis chosen, we can write the stratification

$$\xi_u = \xi_u^{(0)} + \xi_u^{(1)} + \xi_u^{(2)},$$

where on account of (3.12) we have:

$$\begin{aligned}
\xi_u^{(0)} &= \sum_{1 \le i < j \le 3} \det \begin{pmatrix} \partial_1 u^i & \partial_2 u^i \\ \partial_1 u^j & \partial_2 u^j \end{pmatrix} e_i \wedge e_j , \\
\xi_u^{(1)} &= \sum_{i,j=1}^{3} \det \begin{pmatrix} \partial_1 u^i & \partial_2 u^i \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{pmatrix} e_i \wedge \epsilon_j , \\
\xi_u^{(2)} &= \sum_{1 \le i < j \le 3} \det \begin{pmatrix} \partial_1 {\nu_u}^i & \partial_2 {\nu_u}^i \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{pmatrix} \epsilon_i \wedge \epsilon_j .
\end{aligned} \tag{3.17}$$

Moreover, by the expression (2.7), the area formula implies that the Gauss graph area $\mathscr{H}^2(\mathscr{G}\mathscr{M}_u)$ is equal to

$$\begin{aligned}
&\int_\Omega \sqrt{g_u} \sqrt{1 + (4\mathbf{H}_u^2 - 2\mathbf{K}_u) + \mathbf{K}_u^2} \, \mathrm{d}x = \\
&\qquad \int_{\mathscr{M}_u} \sqrt{1 + (4\mathbf{H}_u^2 - 2\mathbf{K}_u) + \mathbf{K}_u^2} \, \mathrm{d}\mathscr{H}^2 = \int_\Omega |\xi_u| \, \mathrm{d}x .
\end{aligned} \tag{3.18}$$

Furthermore, on account of (2.8), the curvature functional $\|\mathscr{G}\mathscr{M}_u\|$ coincides with

$$\mathscr{E}(u) := \int_\Omega \sqrt{g_u}\left(1 + \sqrt{4\mathbf{H}_u^2 - 2\mathbf{K}_u} + |\mathbf{K}_u|\right) \mathrm{d}x \,.$$

We thus get

$$\mathscr{E}(u) = \int_\Omega \left(|\xi_u^{(0)}| + |\xi_u^{(1)}| + |\xi_u^{(2)}|\right) \mathrm{d}x \;, \tag{3.19}$$

where by (3.17) we obtain

$$\begin{aligned}
|\xi_u^{(0)}|^2 &= g_u = |\partial_1 u \times \partial_2 u|^2 = |\operatorname{adj}_2 \nabla u|^2 \;, \\
|\xi_u^{(1)}|^2 &= g_u\,(4\mathbf{H}_u^2 - 2\mathbf{K}_u) = \sum_{i,j=1}^{3} \left(\partial_1 u^i\, \partial_2 {\nu_u}^j - \partial_2 u^i\, \partial_1 {\nu_u}^j\right)^2 \;, \\
|\xi_u^{(2)}|^2 &= g_u\, \mathbf{K}_u^2 = \sum_{1\le i<j\le 3} \left(\partial_1 {\nu_u}^i\, \partial_2 {\nu_u}^j - \partial_2 {\nu_u}^i\, \partial_1 {\nu_u}^j\right)^2 .
\end{aligned} \tag{3.20}$$

Let us now consider the 2-forms $\Theta_1$ and $\Theta_2$ in $\mathbb{R}^3 \times \hat{\mathbb{R}}^3$ given by

$$\begin{aligned}
\Theta_1 &:= \frac{1}{2}\Big(z^1(\mathrm{d}y^2 \wedge \mathrm{d}z^3 - \mathrm{d}y^3 \wedge \mathrm{d}z^2) + z^2(\mathrm{d}y^3 \wedge \mathrm{d}z^1 - \mathrm{d}y^1 \wedge \mathrm{d}z^3) \\
&\qquad + z^3(\mathrm{d}y^1 \wedge \mathrm{d}z^2 - \mathrm{d}y^2 \wedge \mathrm{d}z^1)\Big)\,, \\
\Theta_2 &:= z^1 \mathrm{d}z^2 \wedge \mathrm{d}z^3 + z^2 \mathrm{d}z^3 \wedge \mathrm{d}z^1 + z^3 \mathrm{d}z^1 \wedge \mathrm{d}z^2 \,,
\end{aligned} \tag{3.21}$$

the latter a volume form in $\mathbb{S}^2$. The pullback through the graph map $\Phi_u$ is given by

$$-\,{\Phi_u}^{\#}\Theta_1 = \sqrt{g_u}\, \mathbf{H}_u\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,, \qquad {\Phi_u}^{\#}\Theta_2 = \sqrt{g_u}\, \mathbf{K}_u\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,, \tag{3.22}$$

where $g_u$ is in formula (3.14).

Finally, on account of properties (3.5), (3.8), (3.9), definition (3.12), and relations (3.20), we readily obtain the estimate

$$|\nabla v_{u,s}| + |\operatorname{adj}_2 \nabla v_{u,s}| + |\det \nabla v_{u,s}| \le c\left(|D\Phi_u| + |\xi_u^{(0)}| + |\xi_u^{(1)}| + |\xi_u^{(2)}|\right) \tag{3.23}$$

where $c > 0$ is some absolute real constant not depending on $u$.

### 3.3 Absence of Cancellations

Assume now the shell thickness to be vanishing, namely infinitesimal. Consider also the circumstance in which two distant pieces of the deformed shell $\mathscr{M}_u = u(\Omega)$ are in contact, with the same or opposite orientation. In that case, we have two pairwise disjoint and connected open sets $A, B \subset \Omega$ such that both the restrictions $u_{|A}$ and $u_{|B}$ are injective, but $u(A) = u(B)$.

Therefore, one of the two following alternatives holds: either $u_\#[\![ A ]\!] = u_\#[\![ B ]\!]$, so that the image current $u_\#[\![ A \cup B ]\!]$ has multiplicity two, or $u_\#[\![ A ]\!] = -u_\#[\![ B ]\!]$, whence $u_\#[\![ A \cup B ]\!] = 0$. In the second case, a cancellation occurs and actually

$$\mathbf{M}(u_\#[\![ \Omega ]\!]) < \int_\Omega |\xi_u^{(0)}|\, \mathrm{d}x, \quad |\xi_u^{(0)}| = |\partial_1 u \times \partial_2 u|.$$

However, since the unit normal $\nu_u$ at corresponding points in $u(A)$ and $u(B)$ has opposite sign, by the expressions (3.17) it turns out that no cancellation occurs in the Gauss graph current $[\![ \mathcal{GM}_u ]\!]$ (Mariano and Mucci 2023c). In conclusion, even in the presence of cancellations for the current $u_\#[\![ \Omega ]\!]$, we always have

$$\mathbf{M}([\![ \mathcal{GM}_u ]\!]) = \mathcal{H}^2(\mathcal{GM}_u) = \int_\Omega |\xi_u|\, \mathrm{d}x < \infty \tag{3.24}$$

and hence, on account of the expression (3.19) and (3.24),

$$\frac{1}{2}\,\mathcal{E}(u) \le \mathbf{M}([\![ \mathcal{GM}_u ]\!]) \le \mathcal{E}(u).$$

## 4 New Summability Properties

Relevant to what we discuss here is the class of convex surfaces. For them, a summability result of the Gauss graph map holds (Theorem 4.1) together with some properties of conical points recently analyzed in Mucci and Saracco (2024). These results motivate the assumptions we shall adopt.

Since we use local arguments, we restrict to the nonparametric case, where the deformation is given by the graph map $x \mapsto u(x) := (x, g(x))$ of some concave, positive, and bounded function $g : B^2 \to \mathbb{R}$ defined on the unit disk $B^2$. The image $u(B^2)$, that is, the graph $\mathcal{G}_g = \{(x, g(x)) \mid x \in B^2\}$ of $g$, may be seen as a piece of the boundary of a convex set

$$K(g) := \{(x,t) \in \mathbb{R}^3 \mid x \in B^2,\ 0 \le t \le g(x)\}.$$

The function $g$ is continuous, it belongs to the Sobolev class $W^{1,1}(B^2)$, and its gradient $\nabla g := (\partial_1 g, \partial_2 g)$ is a vector-valued function of bounded variation, that is, $\nabla g \in BV(B^2, \mathbb{R}^2)$. By the chain rule formula, the outward unit normal in (3.2) is given for $\mathcal{L}^2$-a.e. $x \in B^2$ by

$$\nu_u(x) = \frac{(-\partial_1 g(x), -\partial_2 g(x), 1)}{\sqrt{1 + |\nabla g(x)|^2}}\,, \tag{4.1}$$

and actually $\nu_u \in BV(B^2, \mathbb{S}^2)$. However, we cannot conclude that $\nu_u$ is a special function of bounded variation, in general.

***Example 4.1*** If $V : (0,1) \to \mathbb{R}$ is the classical Cantor–Vitali function, and $v(\rho) := \int_0^\rho V(t)\, dt$, the positive function $g(x) = 1 - v(|x|)$ is concave on $B^2$, and $\nabla g(x) =$

$-V(|x|)\,x/|x|$ for $x \neq 0_{\mathbb{R}^2}$. We thus have

$$\nu_u(x) = \frac{\big(V(\rho)x_1, V(\rho)x_2, \rho\big)}{\rho\sqrt{1+V(\rho)^2}}, \quad \rho = |x|, \quad x \neq 0_{\mathbb{R}^2},$$

so that by the chain rule formula $D^C \nu_u \neq 0$, and hence, $\nu_u \notin SBV(B^2, \mathbb{S}^2)$.

Using known results (Anzellotti et al. 1990; Anzellotti and Ossanna 1995; Mucci 2021), we obtain new summability properties concerning the approximate derivatives of the unit normal $\nu_u$.

**Theorem 4.1** *Given $g : B^2 \to \mathbb{R}$ a bounded, positive, and concave function, let $\Phi_u$ denote the Gauss graph map in (3.12), where $u(x) = (x, g(x))$ and $\nu_u$ is given by (4.1). Moreover, let $\xi_u(x)$ be the 2-vector given for $\mathscr{L}^2$-a.e. $x \in B^2$ by (3.16). Then, $\xi_u \in L^1(B^2, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$. In particular, we have*

*i)* $\partial_1 g \partial_2 \nu_u^j - \partial_2 g \partial_1 \nu_u^j \in L^1(B^2)$, *for* $j = 1, 2, 3$;

*ii)* $\partial_1 \nu_u^{j_1} \partial_2 \nu_u^{j_2} - \partial_2 \nu_u^{j_1} \partial_1 \nu_u^{j_2} \in L^1(B^2)$, *for* $1 \leq j_1 < j_2 \leq 3$.

***Proof*** For $h \in \mathbb{N}^+$, denote by $K(g)_h$ the set of points in $\mathbb{R}^3$ whose distance from the convex set $K(g)$ is lower than $1/h$. Its boundary $\partial K(g)_h$ is a surface of class $C^{1,1}$. By setting $B_r^2 := \{x \in \mathbb{R}^2 : |x| < r\}$, we have

$$K(g)_h = \{(x,t) \in \mathbb{R}^3 \mid x \in B^2_{1+1/h}, \quad -\alpha_h(x) \leq t \leq \beta_h(x)\},$$

where $\alpha_h(x) = 1/h$ if $|x| \leq 1$, and $\alpha_h(x) = \sqrt{h^{-2} - 1 + 2|x| - |x|^2}$ if $1 \leq |x| \leq 1 + 1/h$, whereas $\beta_h : B^2_{1+1/h} \to \mathbb{R}$ is a smooth, positive, concave function.

Therefore, the currents $\Sigma_h$ carried by the Gauss graph of the (naturally oriented) smooth surfaces $\partial K(g)_h$ are i.m. rectifiable currents with equibounded masses and zero boundary. By Federer and Fleming's closure theorem, a subsequence of $\{\Sigma_h\}$ weakly converges to a current $\Sigma$ (compare (Anzellotti et al. 1990)). On the other hand, setting $g_h := \beta_{h|B^2}$, the sequence $\{g_h\}$ of Lipschitz-continuous functions converges to $g$ uniformly in $B^2$.

As a consequence, by the relationship between $\Sigma_h$ and $g_h$, it turns out that the concave function $g$ has finite relaxed energy in the sense of Mucci (2021), and hence, the assertion readily follows. Expressions $(i)$ and $(ii)$ in the theorem follow from (3.17). Further details are omitted. □

Something more can be said concerning conical points of the convex surface given by $u(B^2) = \mathscr{G}_g$. To this end, we recall that $g$ is differentiable at $\mathscr{L}^2$-a.e. point $x \in B^2$, and that the unit normal at regular points is given by the Lebesgue value $\nu_u(x) \in \mathbb{S}^2$, see (4.1).

If $g$ is not differentiable at $x$, a weak notion of unit normal at the point $P = u(x) = (x, g(x))$ is given in Mucci and Saracco (2024). Denote by $\mathbf{c}_P$ the geodesically convex spherical curve obtained by intersecting the tangent cone to $u(B^2) = \mathscr{G}_g$ at $P$ with the unit sphere centered at $P$. By exploiting (through approximating inscribed geodesically convex polygonals) a polarity argument concerning the curve $\mathbf{c}_P$, it is possible to define a function $\nu_P : [0, 2\pi[ \to \mathbb{S}^2$ satisfying the following properties:

- $\nu_P$ is a function of bounded variation, $\nu_P \in BV((0,2\pi),\mathbb{S}^2)$;
- the geodesically convex hull of the rectifiable image of $\nu_P$ in $\mathbb{S}^2$ gives a good notion of normal cone $\mathscr{N}_P$ to $u(B^2) = \mathscr{G}_g$ at $P$;
- with $\mathscr{AD}(P)$ the area of normal cone $\mathscr{N}_P$, also called an *angle defect*, if $P$ is a ridge or a regular point, $\mathscr{AD}(P) = 0$ and $\mathscr{AD}(P) > 0$ otherwise, that is, at any conical point $P$;
- for every $P \in u(B^2)$, we can find a sequence $\{\Sigma_h\}$ of smooth surfaces converging in Hausdorff distance to the tangent cone to $u(B^2)$ at $P$ and such that, denoting by $\mathbf{K}_h$ the Gauss curvature of $\Sigma_h$, we have

$$\lim_{h\to\infty}\int_{\Sigma_h} |\mathbf{K}_h|\, d\mathscr{H}^2 = \mathscr{AD}(P).$$

Referring to Mucci and Saracco (2024) for further details, it turns out that at every point $P = u(x)$, a Dirac measure with weight equal to the angle defect $\mathscr{AD}(P)$ is well defined, and $\mathscr{AD}(P) = 0$ if and only if $P$ is either a regular or a ridge point.

## 5 Possibly Irrecoverable Sharp Folds

We consider the occurrence of possibly irrecoverable sharp folds. For them, we propose a description in terms of vector-valued measures concentrated on 1-dimensional sets where jumps of the unit normal to the deformed shell may appear. Such measures are introduced in terms of size bounded currents $\bar{T}$ in $[\mathscr{S}_2(U)]^3$, where $U \subset \bar{\mathbb{R}}^3$ is a bounded open set.

To this purpose, we first point out that a tensor-valued measure $\bar{F} = \bar{F}(\bar{T})$ in $\mathscr{M}_b(U,\mathbb{R}^{3\times 3})$ is naturally associated with a current $\bar{T} \in [\mathscr{S}_2(U)]^3$, by letting

$$\langle \bar{F}, \zeta\rangle := \langle \bar{T}, \omega_\zeta^{(2)}\rangle, \quad \forall\, \zeta \in C_c^\infty(U,\mathbb{R}^{3\times 3})\,.$$

With the previous notation, we denote $\bar{T} = [\![\, \mathscr{N}, \mathbf{b}, \xi\,]\!]$, where $\xi = \xi_1\varepsilon_2\wedge\varepsilon_3 + \xi_2\varepsilon_3\wedge\varepsilon_1 + \xi_3\varepsilon_1\wedge\varepsilon_2$, with $(\varepsilon_A)_{A=1}^3$ the canonical basis in $\bar{\mathbb{R}}^3$. Then, writing $\bar{\omega} = \omega_\zeta^{(2)}$ we have

$$\langle T^j, \omega_j\rangle = \int_{\mathscr{N}} b^j \sum_{A=1}^{3} (-1)^{A-1}\zeta_{jA}\xi_A \,\mathrm{d}\mathscr{H}^2\,, \quad j = 1,2,3\,.$$

Therefore, recalling that $*\xi = \sum_{A=1}^3 \xi_A\varepsilon_A$, where $*$ is the Hodge operator in $\bar{\mathbb{R}}^3$, we get

$$\bar{F} = \mathbf{b}\otimes\mathbf{m}\,\mathscr{H}^2 \llcorner \mathscr{N} \quad \mathbf{m} := *\xi\,.$$

Finally, for every $\psi \in C_c^\infty(U,\mathbb{R}^{3\times 3})$ we have:

$$\langle \operatorname{curl}\bar{F}, \psi\rangle := \langle \bar{F}, \operatorname{curl}\psi\rangle = \langle \bar{T}, \omega^{(2)}_{\operatorname{curl}\psi}\rangle = \langle \partial\bar{T}, \omega_\psi^{(1)}\rangle\,. \tag{5.1}$$

### 5.1 A Relevant Example

**Definition 5.1** Let $\mathscr{P}\mathscr{F}$ be the class of triplets $(\mathscr{I}, \mathbf{n}, \Theta)$ endowed with the following properties:

(1) $\mathscr{I}$ is a 1-rectifiable subset of $\Omega$;
(2) $\mathbf{n} : \mathscr{I} \to \mathbb{R}^3$ is an $\mathscr{H}^1 \llcorner \mathscr{I}$-measurable unit vector field that is normal to the approximate tangent 1-space to $\mathscr{I}$ at $\mathscr{H}^1$-a.e. $(x_1, x_2) \in \mathscr{I}$;
(3) $\Theta : \mathscr{I} \to \mathbb{R}^3$ is $\mathscr{H}^1 \llcorner \mathscr{I}$-summable, with components $\Theta = (\theta^1, \theta^2, \theta^3)$, such that $|\Theta| \leq 2$.

For a given triplet $(\mathscr{I}, \mathbf{n}, \Theta)$ in $\mathscr{P}\mathscr{F}$, we denote by $\mathbf{t} = \mathbf{n}^\perp$ the unit vector

$$\mathbf{t} := (t_1, t_2) = (-n_2, n_1) \quad \text{if} \quad \mathbf{n} = (n_1, n_2)\,, \tag{5.2}$$

that is tangent to $\mathscr{I}$ at $\mathscr{H}^1 \llcorner \mathscr{I}$-a.e. point.

An $\mathbb{R}^{3\times 2}$-valued rectifiable measure $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ is naturally associated to the triplet $(\mathscr{I}, \mathbf{n}, \Theta)$, with $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ defined by

$$\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta} := \Theta \otimes \mathbf{n}^\perp\, \mathscr{H}^1 \llcorner \mathscr{I}\,.$$

More precisely, a test function $\bar{f} \in C^0_b(\Omega, \mathbb{R}^{3\times 2})$ is given by $\bar{f} = (\bar{f}_{jA})$, where $j = 1, 2, 3$ and $A = 1, 2$, and each $\bar{f}_{jA}$ is a bounded and continuous function on $\Omega$. With $\mathbf{t} = \mathbf{n}^\perp$, we have

$$\langle \mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}, \bar{f} \rangle = \sum_{j=1}^{3} \sum_{A=1}^{2} \int_{\mathscr{I}} \theta^j\, \bar{f}_{jA} t_A \,\mathrm{d}\mathscr{H}^1\,. \tag{5.3}$$

Moreover, for any open set $V \subset \Omega$, the total variation of $\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}$ is bounded, namely

$$|\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}|(V) = \int_{V\cap\mathscr{I}} |\Theta| \,\mathrm{d}\mathscr{H}^1 < \infty\,.$$

For $0 < s < 1$, we let $\bar{T}_s = (T^1_s, T^2_s, T^3_s) := [\![ \mathscr{N}, \mathbf{b}_s, \xi ]\!] \in [\mathscr{S}_2(U)]^3$, where $\mathscr{N} := \mathscr{I} \times I_{\mathfrak{h}}$, $\mathbf{b}_s(x_1, x_2, x_3) := s\,\Theta(x_1, x_2)$, and $\xi := *\mathbf{m}$, where $\mathbf{m} := (\mathbf{n}, 0)$, so that $\xi = n_1\, \varepsilon_2 \wedge \varepsilon_3 + n_2\, \varepsilon_3 \wedge \varepsilon_1$, and $*\xi = \mathbf{m}$. In particular, the tensor-valued measure in $\mathscr{M}_b(U, \mathbb{R}^{3\times 3})$ naturally associated with $\bar{T}_s$ is given by

$$\bar{F}_s = s\, x_3\, \Theta \otimes \mathbf{m}\, \mathscr{H}^2 \llcorner (\mathscr{I} \times I_{\mathfrak{h}})\,. \tag{5.4}$$

Finally, we choose a smooth test function $\varphi \in C^\infty_c(I_{\mathfrak{h}})$ that satisfies the constraint $\int_{I_{\mathfrak{h}}} \varphi(x_3)\,\mathrm{d}x_3 = 1$, so that

$$\int_{I_{\mathfrak{h}}} x_3\, \varphi'(x_3)\,\mathrm{d}x_3 = -\int_{I_{\mathfrak{h}}} \varphi(x_3)\,\mathrm{d}x_3 = -1\,, \tag{5.5}$$

for a given $\bar{f} \in C_b^0(\Omega, \mathbb{R}^{3\times 2})$ we introduce the notation $\psi = \bar{f}\varphi$ if $\psi_{jA}(\mathbf{x}) := \bar{f}_{jA}(x_1, x_2)\,\varphi(x_3)$, when $A = 1, 2$, and $\psi_{j3}(\mathbf{x}) \equiv 0$, for $j = 1, 2, 3$. Notice that $f\varphi \in C_c^\infty(U, \mathbb{R}^{3\times 3})$ if $f \in C_c^\infty(\Omega, \mathbb{R}^{3\times 2})$.

**Theorem 5.1** *With the previous notation, for any $(\mathscr{I}, \mathbf{n}, \Theta) \in \mathscr{P}\mathscr{F}$ we have*

$$\frac{1}{s}\,\langle \mathrm{curl}\,\bar{F}_s, \bar{f}\varphi\rangle = \langle \mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}, \bar{f}\rangle \quad \forall\, \bar{f} \in C_b^0(\Omega, \mathbb{R}^{3\times 2})\,.$$

***Proof*** Using that $T_s^j = [\![\, \mathscr{N}, s\,|\theta^j|, \sigma^j \xi\,]\!]$, if $\overline{\omega} = \omega_\zeta^{(2)}$, we get

$$\langle T_s^j, \omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\,\big(n_1\,\zeta_{j1} + n_2\zeta_{j2}\big)\,\mathrm{d}\mathscr{H}^2$$

for $j = 1, 2, 3$, and hence, taking $\overline{\omega} = \omega_\psi^{(1)} \in [\mathscr{D}^1(U)]^3$, we compute

$$\langle \partial T_s^j, \omega_j\rangle := \langle T_s^j, \mathrm{d}\omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\,\big(n_1\,(\mathrm{curl}\,\psi_j)^1 + n_2\,(\mathrm{curl}\,\psi_j)^2\big)\,\mathrm{d}\mathscr{H}^2$$

where $(\mathrm{curl}\,\psi_j)^1 = \partial_{x_2}\psi_{j3} - \partial_{x_3}\psi_{j2}$ and $(\mathrm{curl}\,\psi_j)^2 = \partial_{x_3}\psi_{j1} - \partial_{x_1}\psi_{j3}$. Therefore, with $\psi = f\varphi$, since $f_{j3} = 0$, we get

$$\langle \partial T_s^j, \omega_j\rangle = s\int_{\mathscr{N}} x_3\,\theta^j\,\big(-n_1\,f_{j2}\,\varphi' + n_2\,f_{j1}\,\varphi'\big)\,\mathrm{d}\mathscr{H}^2.$$

Recalling that $\mathscr{N} = \mathscr{I} \times I_{\mathfrak{h}}$, by Fubini's theorem and (5.5) we obtain

$$\langle \partial T_s^j, \omega_j\rangle = s\sum_{A=1}^{2}\int_{\mathscr{I}} \theta^j\,f_{jA}t_A\,\mathrm{d}\mathscr{H}^1 \quad \forall\, j = 1, 2, 3\,.$$

On account of equations (5.1) and (5.3), we have just shown the thesis when $f \in C_c^\infty(U, \mathbb{R}^{3\times 2})$. Since the involved measures have finite total variation, the assertion follows through a density argument. □

### 5.2 Possibly Irrecoverable Sharp Folds in the Deformed Surface

Let now $u : \bar{\Omega} \to \mathbb{R}^3$ be a continuous function such that

(1) $u \in W^{1,1}(\Omega, \mathbb{R}^3)$;
(2) property (3.1) holds true for $\mathscr{L}^2$-a.e. $x \in \Omega$;
(3) the unit normal $\nu_u$ given by (3.2) belongs to $SBV(\Omega, \mathbb{S}^2)$;
(4) the jump set of $\nu_u$ is a 1-rectifiable subset of $\Omega$, that is $\mathscr{H}^1(S(\nu_u)) < \infty$;
(5) with $\Phi_u$ the Gauss graph map in (3.12) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and $\xi_u$ the corresponding 2-vector in (3.16), then $\xi_u \in L^1(\Omega, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$.

As already shown, properties (1)–(5) are automatically verified if $\Omega = B^2$ and $u(x) = (x, g(x))$ for some bounded and concave function $g : B^2 \to \mathbb{R}$, provided that condition (2.6) holds, with $\nu = \nu_u$. Moreover, if (1)–(5) hold, the distributional derivative of $\nu = \nu_u$ satisfies the decomposition formula (2.1), where $n = 2$ and $D^C \nu_u = 0$, the approximate gradient $\nabla \nu_u$ belongs to $L^1(\Omega, \mathbb{R}^{3\times 2})$, and the jump component $D^J \nu_u$ satisfies (2.2), where $n = 2$ and $\mathbf{n} = \mathbf{n}_u \in \mathbb{R}^2$ is a given unit normal to $S(\nu_u)$.

For $0 < s < 1$, we denote by $v_{u,s} : U \to \mathbb{R}^3$ the deformation map defined for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$ through the expression (3.4). Since $\nabla \Phi_u \in L^1(\Omega, \mathbb{R}^{6\times 2})$, it turns out that $v_{u,s} \in SBV(U, \mathbb{R}^3)$, so that we can write

$$D v_{u,s} = \nabla v_{u,s}\, \mathscr{L}^3 \llcorner U + D^J v_{u,s}\,.$$

The approximate gradient $\nabla v_{u,s} \in L^1(U, \mathbb{R}^{3\times 3})$ is given by the expression (3.5), for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$, so that the limit (3.6) holds, with $\mathbf{G} \in L^1(\Omega, \mathbb{R}^{3\times 3})$ given $\mathscr{L}^3$-a.e. in $\Omega$ by formula (3.7). Moreover, the jump component satisfies

$$D^J v_{u,s} = s\, x_3\, [\nu_u] \otimes \mathbf{m}_u\, \mathscr{H}^2 \llcorner (S(\nu_u) \times I_{\mathfrak{h}})\,, \quad \mathbf{m}_u := (\mathbf{n}_u, 0)\,.$$

The triplet $(S(\nu_u), \mathbf{n}_u, [\nu_u])$ belongs to the class $\mathscr{PF}$ (see Definition 5.1) and we thus introduce the measure

$$\mu^J_{\nu_u} := \mu_{S(\nu_u),\mathbf{n}_u^\perp,[\nu_u]} = [\nu_u] \otimes \mathbf{n}_u^\perp\, \mathscr{H}^1 \llcorner S(\nu_u)\,. \tag{5.6}$$

According to (5.2), with $\mathbf{n} = \mathbf{n}_u$, by (5.3) we have

$$\langle \mu^J_{\nu_u}, \bar f \rangle = \sum_{j=1}^{3} \sum_{A=1}^{2} \int_{S(\nu_u)} [\nu_u]^j\, \bar f_{jA} t_A \, \mathrm{d}\mathscr{H}^1\,, \quad \forall\, \bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2}).$$

With the latter choice, on account of (5.4), we also have $\bar F_s = D^J v_{u,s}$. Therefore, Theorem 5.1 gives for each $s > 0$ the equation

$$\frac{1}{s} \langle \operatorname{curl} \bar F_s, \bar f \varphi \rangle = \langle [\nu_u] \otimes \mathbf{n}_u^\perp\, \mathscr{H}^1 \llcorner S(\nu_u), \bar f \rangle\,, \quad \bar F_s := D^J v_{u,s}\,, \tag{5.7}$$

for every $\bar f \in C^0_b(\Omega, \mathbb{R}^{3\times 2})$, where, we recall, $\varphi$ is any function in $C^\infty_c(I_{\mathfrak{h}})$ that satisfies $\int_{I_{\mathfrak{h}}} \varphi(x_3)\, \mathrm{d}x_3 = 1$.

Theorem 5.1 allows us to describe the occurrence of possibly irrecoverable sharp folds in the deformed surface.

To this aim, we fix a triplet $(\mathscr{I}, \mathbf{n}, \Theta)$ in $\mathscr{PF}$; based on definitions (5.3) and (5.6), we require the validity of the total variation bound

$$|\mu^J_{\nu_u}|(V) \le |\mu_{\mathscr{I},\mathbf{n}^\perp,\Theta}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}\,. \tag{5.8}$$

By the Borel regularity of the involved measures, property (5.8) implies that the jump set $S(\nu_u)$ is $\mathscr{H}^1$-essentially contained in the 1-rectifiable set $\mathscr{I}$, that is,

$$\mathscr{H}^1(S(\nu_u) \setminus \mathscr{I}) = 0 \,. \tag{5.9}$$

Therefore, without loss of generality we can require that $\mathbf{n} = \mathbf{n}_u$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. It suffices to replace $\Theta$ with $-\Theta$ at points in $S(\nu_u)$ where $\mathbf{n} = -\mathbf{n}_u$.

Since $[\nu_u] = \nu_u^+ - \nu_u^-$, with $\nu_u^\pm \in \mathbb{S}^2$, the angle $\alpha$ between the one-sided normals,

$$\alpha(x) := 2 \ \arcsin \frac{|[\nu_u](x)|}{2},$$

describes the emergence of a wedge on the deformed surface $u(\Omega)$ at $u(x)$, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Since property (5.8) implies that $|[\nu_u]| \leq |\Theta|$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$, we obtain the angle bound:

$$\alpha \leq 2 \ \arcsin \frac{|\Theta|}{2} \leq \frac{\pi}{2}|\Theta|, \quad \mathscr{H}^1\text{-a.e. on } S(\nu_u).$$

The inequality $\mathscr{H}^1(\mathscr{I} \setminus S(\nu_u)) > 0$ may describe the occurrence of irrecoverable sharp folds at previous steps of a process, where at the present stage the deformation is unfolded. No sharp folds appear in the case $\mathscr{H}^1(S(\nu_u)) = 0$, so that $\nu_u \in W^{1,1}(\Omega, \mathbb{S}^2)$, a sufficient condition to its validity being given by equation $\mathscr{H}^1(\mathscr{I}) = 0$.

### 5.3 A Multiplicative Decomposition Formula

For the sake of completeness, even if we shall not pursue the pertinent path, we finally point out a multiplicative decomposition formula (see (5.10)).

Assume that $u \in C(\overline{\Omega}, \mathbb{R}^3)$ satisfies properties (1)–(5). Then, $\det \nabla v_{u,s}(\mathbf{x})$ is given for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$ by formula (3.8), with $\lambda_u^{(k)}(x)$ given for $\mathscr{L}^2$-a.e. $x \in \Omega$ by (3.9), for $k = 1, 2$. Therefore, $\det \nabla v_{u,s} \in L^1(U)$ provided that $\lambda_u^{(k)} \in L^1(\Omega)$ for $k = 1, 2$. More generally, it turns out that the pointwise inequality (3.23) holds true $\mathscr{L}^3$-a.e. on $U$, where $c > 0$ is some absolute real constant. Therefore, besides the validity of the limit (3.10), property (5) gives the summability of all minors of the approximate gradient $\nabla v_{u,s}$.

Assume now that the gradient matrix $\nabla v_{u,s}$ is invertible $\mathscr{L}^3$-a.e. on $U$, for $s > 0$ small. We have already seen that this property holds true if $u$ is smooth. In this case (compare (Mariano and Mucci 2023a)), we can write the distributional derivative measure $F_s = Dv_{u,s}$ through a multiplicative decomposition:

$$F_s = F_s^e F_s^p \,, \quad F_s^e := G_s \mathscr{L}^3 \llcorner U \,. \tag{5.10}$$

The vector field $G_s \in L^1(U, \mathbb{R}^{3\times 3})$ appearing in the factor $F_s^e$ satisfies $G_s(\mathbf{x}) = \nabla v_{u,s}(\mathbf{x})$ for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U \setminus \mathscr{N}$, with $\mathscr{N} := S(\nu_u) \times I_{\mathfrak{h}}$, and $G_s(\mathbf{x}) = \mathbf{I}_3$ if $\mathbf{x} \in \mathscr{N}$.

The plastic factor is given by the sum

$$F_s^p = (\nabla v_{u,s})^{-1} \mathscr{L}^3 \llcorner U + \bar{F}_s \,, \tag{5.11}$$

with $\bar{F}_s$ the measure in $\mathscr{M}_b(U, \mathbb{R}^{3\times 3})$ given by (5.4), where $\Theta := [\nu_u]$ and $\mathbf{m} := \mathbf{m}_u$. Therefore, in general $\operatorname{curl} F_s = 0$, but equation (5.7) implies that $\operatorname{curl} F_s^e \neq 0$ and $\operatorname{curl} F_s^p \neq 0$, if $\mathscr{H}^1(S(\nu_u)) > 0$, that is, if $\nu_u \notin W^{1,1}(\Omega, \mathbb{S}^2)$. In this sense, from a purely kinematic perspective, in the presence of (possibly) irrecoverable sharp folds, we are reasonably entitled to call $F_s^e$ and $F_s^p$ the *elastic* and *plastic* components of deformation gradient.

## 6 Cusps

The occurrence of cusps in the deformed shell $\mathscr{M}_u = u(\Omega)$ can be described in terms of atomic measures. Our starting point is the assumption that any physically plausible configuration is obtained as the limit of a sequence of smooth deformations. Through a minimum energy principle, we then describe the situation when the deformed shell has *folds without wrinkles*. We also require that the folding in the deformed shell *does not degenerate*. Under such ansatz, the theory of currents carried by Gauss graphs implies that there is an essentially unique way to "fill in the jumps" of the unit normal $\nu_u$.

As an example, we analyze concentration phenomena of the Gauss curvature, in the case of convex deformations. The basic idea is that around a cusp the deformation is locally convex.

### 6.1 Deformed Shells as Weak Limits of Gauss Graphs

Let $\psi : ]0, +\infty[ \to \mathbb{R}^+$ be a positive and convex function with linear growth at $+\infty$ and such that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$, as, for example, $\psi(\rho) = \rho^{-1} \vee \rho$. For any smooth map $u : \Omega \to \mathbb{R}^3$, we correspondingly define

$$\mathscr{F}_p(u) := \int_\Omega \big( |\nabla u|^p + \psi(|\partial_1 u \times \partial_2 u|) \big) \, \mathrm{d}x \,, \quad p > 2 \,. \tag{6.1}$$

Moreover, Dirichlet-type conditions are established through the choice of a smooth and injective function $\widetilde{u} \in C^2(\widetilde{\Omega}, \mathbb{R}^3)$ defined in an open set $\widetilde{\Omega} \subset \mathbb{R}^2$ such that the closure of the domain $\Omega$ is well contained in $\widetilde{\Omega}$. As usual, $\|u\|_\infty$ denotes the $L^\infty$ norm of $u$ in $\Omega$.

**Definition 6.1** Let $p > 2$. A function $u : \Omega \to \mathbb{R}^3$ is said to belong to the class $\mathscr{A}_p(\Omega)$, if $u$ is the pointwise $\mathscr{L}^2$-a.e. limit of a sequence $\{u_h\} \subset C^2(\Omega, \mathbb{R}^3)$ of smooth functions that satisfy the following properties:

- every function $u_h$ can be smoothly extended to $\widetilde{\Omega}$ so that $u_h = \widetilde{u}$ on $\widetilde{\Omega} \setminus \Omega$;
- every function $u_h$ satisfies (3.1) for all $x \in \Omega$;

- on account of (3.19) and (6.1),

$$\sup_h \Big( \|u_h\|_\infty + \mathscr{F}_p(u_h) + \mathscr{E}(u_h) + |D\nu_{u_h}|(\Omega) \Big) < \infty;$$

- for each $h$, there exists $s_h \in (0,1)$ such that the deformation map $v_{u_h,s_h}$ given by (3.4) satisfies the global invertibility condition (3.3), for every $f \in C_c^\infty(U \times \mathbb{R}^3, [0,+\infty))$.

Arguments taken from Mariano and Mucci (2023c) imply what follows:

**Proposition 6.1** *Take $u \in \mathscr{A}_p(\Omega)$, with $p > 2$. Then,*

- *$u \in W^{1,p}(\Omega, \mathbb{R}^3)$, and the property (3.1) holds true for $\mathscr{L}^2$-a.e. $x \in \Omega$;*
- *the unit normal $\nu_u$ given by (3.2) belongs to $BV(\Omega, \mathbb{S}^2)$;*
- *with $\Phi_u$ the Gauss graph map in (3.12) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and $\xi_u$ the corresponding 2-vector in (3.16), we have $\xi_u \in L^1(\Omega, \bigwedge^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3))$.*

By Proposition 6.1, for every $u \in \mathscr{A}_p(\Omega)$ we can define the energy $\mathscr{F}_p(u)$ by (6.1), but in terms of the approximate gradient. Moreover, if $\{u_h\}$ is any smooth sequence from Definition 6.1, arguing as in Mariano and Mucci (2023c) we also obtain a lower semicontinuity inequality:

$$\begin{aligned} &\|u\|_\infty + \mathscr{F}_p(u) + \mathscr{E}(u) + |D\nu_u|(\Omega) \le \\ &\qquad \liminf_{h\to\infty} \Big( \|u_h\|_\infty + \mathscr{F}_p(u_h) + \mathscr{E}(u_h) + |D\nu_{u_h}|(\Omega) \Big) < \infty\,. \end{aligned} \tag{6.2}$$

If $u \in \mathscr{A}_p(\Omega)$ and $\{u_h\}$ is taken as above, on account of the definition (3.4) of the deformation map $v_{u,s}$, we could assume in addition that

$$\sup_h \int_U |\det \nabla v_{u_h,s_h}|^{-r}\, \mathrm{d}\mathbf{x}, \quad r > 0,$$

for some sequence $\{s_h\} \subset (0,1)$ such that $s_h \ge \bar{s} > 0$ for every $h$. We thus obtain $\det \nabla v_{u,\bar{s}}(\mathbf{x}) > 0$ for $\mathscr{L}^3$-a.e. $\mathbf{x} \in U$. However, since in general $\nu_u$ fails to be a Sobolev map, unlike the case analyzed in Mariano and Mucci (2023c), we cannot conclude that $v_{u,\bar{s}}$ satisfies the global invertibility condition (3.3).

The approximation process described by Definition 6.1 admits the occurrence of wrinkled folds: a concentration excess of total variation of the unit normal along its jump set is allowed, indeed. Therefore, folds without wrinkles are obtained by requiring energy minimality.

To describe this aspect, for a given map $\nu \in BV(\Omega, \mathbb{S}^2)$ that admits the decomposition formulae (2.1) and (2.2), with $n = 2$, we consider the variation of $\nu$ with respect to the geodesic distance $\mathrm{dist}_{\mathbb{S}^2}$ in $\mathbb{S}^2$, that is,

$$\mathrm{Var}_{\mathbb{S}^2}(\nu) := \int_\Omega |\nabla \nu|\, dx + |D^C \nu|(\Omega) + \int_{S(\nu)} \mathrm{dist}_{\mathbb{S}^2}(\nu^-(x), \nu^+(x))\, \mathrm{d}\mathscr{H}^1(x)\,.$$

By (2.3), we have

$$\frac{2}{\pi}\,\mathrm{Var}_{\mathbb{S}^2}(\nu) \le |D\nu|(\Omega) \le \mathrm{Var}_{\mathbb{S}^2}(\nu) \quad \forall\,\nu \in BV(\Omega,\mathbb{S}^2)$$

and the strict inequality $|D\nu|(\Omega) < \mathrm{Var}_{\mathbb{S}^2}(\nu)$ holds as soon as the jump component $D^J\nu$ is non-trivial.

**Definition 6.2** Let $p > 2$. We denote by $\mathscr{A}_p^\infty(\Omega)$ the class of maps $u$ in $\mathscr{A}_p(\Omega)$ for which we can find a sequence $\{u_h\} \subset C^2(\Omega,\mathbb{R}^3)$ as in Definition 6.1 that satisfies in addition the identity

$$\lim_{h\to\infty}\int_\Omega |\nabla \nu_{u_h}|\,\mathrm{d}x = \mathrm{Var}_{\mathbb{S}^2}(\nu_u)\,. \tag{6.3}$$

### 6.2 Weak Limit Currents

Take $u \in \mathscr{A}_p^\infty(\Omega)$ and a smooth sequence $\{u_h\}$ as in Definition 6.2. By Federer and Fleming's closure theorem, we find a (not relabeled) subsequence such that $G_{\nu_{u_h}} \rightharpoonup T$ weakly in $\mathscr{D}_2(\Omega\times\mathbb{S}^2)$ to some *Cartesian current* $T$ in $\mathrm{cart}(\Omega\times\mathbb{S}^2)$, with underlying function equal to $\nu_u$ (see (Giaquinta et al. 1998a)). Denote by $\mathscr{T}_{\nu_u}$ the class of currents in $\mathrm{cart}(\Omega\times\mathbb{S}^2)$ obtained in this way. Additional notations are instrumental for analyzing the structure of currents $T$ in $\mathscr{T}_{\nu_u}$. They are here in order.

Any 2-form $\omega \in \mathscr{D}^2(\Omega\times\mathbb{S}^2)$ is decomposed into the sum $\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)}$ according to the number of differentials $\mathrm{d}z^j$. Then, a generic current $T \in \mathscr{D}_2(\Omega\times\mathbb{S}^2)$ is equal to the sum $T = T_{(0)} + T_{(1)} + T_{(2)}$, where we define

$$\langle T_{(k)},\omega\rangle := \langle T,\omega^{(k)}\rangle, \quad \forall\,\omega \in \mathscr{D}^2(\Omega\times\mathbb{S}^2),$$

for $k = 0, 1, 2$. We also recall that $\widehat{\mathrm{d}x^i}$ is the covector in $\mathbb{R}^2$ such that $\widehat{\mathrm{d}x^i}\wedge \mathrm{d}x^i = (-1)^i\mathrm{d}x$, for $i = 1, 2$, where $\mathrm{d}x := \mathrm{d}x^1\wedge\mathrm{d}x^2$.

If $\nu_u$ is the unit normal to a given map $u \in \mathscr{A}_p^\infty(\Omega)$, we have denoted by $G_{\nu_u}$ the current in $\mathscr{R}_2(\Omega\times\mathbb{S}^2)$ carried by the rectifiable graph of $\nu_u$ (see Sect. 2.3). Moreover, a current $T^C_{\nu_u}$ in $\mathscr{D}_2(\Omega\times\mathbb{S}^2)$ is naturally associated with the Cantor component $D^C\nu_u$ of the derivative of $\nu_u$. It satisfies $T^C_{\nu_u\,(0)} = 0$, $T^C_{\nu_u\,(2)} = 0$ and

$$\langle T^C_{\nu_u}, \psi(x,z)\,\widehat{\mathrm{d}x^i}\wedge\mathrm{d}z^j\rangle := (-1)^i\int_\Omega \psi(x,\nu_u(x))\,\mathrm{d}(D^C\nu_u)^j_i\,, \tag{6.4}$$

for any test function $\psi \in C_c^\infty(\Omega\times\mathbb{R}^3)$, $i = 1, 2$, $j = 1, 2, 3$, where $\nu_u(x)$ is a precise representative. Indeed, we have

$$\mathbf{M}(T^C_{\nu_u}) = |D^C\nu_u|(\Omega) < \infty\,. \tag{6.5}$$

Finally, concerning the jump component $D^J\nu_u$, we denote by $T^J$ a generic current in $\mathscr{D}_2(\Omega\times\mathbb{S}^2)$ such that $T^J_{(0)} = 0$, $T^J_{(2)} = 0$, and for any choice of $\psi$, $i$ and $j$ as above we write

$$\langle T^J, \psi(x,z)\,\widehat{\mathrm{d}x^i} \wedge \mathrm{d}z^j\rangle = -\int_{S(\nu_u)}\Big(\int_{\Gamma_x} \psi(x,z)\,\mathrm{d}z^j\Big)\,\widehat{t_i}(x)\,\mathrm{d}\mathscr{H}^1(x)\,, \tag{6.6}$$

where $\Gamma_x$ is a rectifiable and oriented arc in $\mathbb{S}^2$ from $\nu_u^-(x)$ to $\nu_u^+(x)$, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. In formula (6.6), referring to (2.1) and (5.2), we considered $\widehat{t_1} = t_2 = n_1$ and $\widehat{t_2} = t_1 = -n_2$. We thus obtain

$$\mathbf{M}(T^J) = \int_{S(\nu_u)} \mathscr{H}^1(\Gamma_x)\,\mathrm{d}\mathscr{H}^1(x)\,. \tag{6.7}$$

The following result motivates the previous notation.

**Lemma 6.1** *Let $\bar{T}$ be a current of the form $\bar{T} = G_{\nu_u} + T^C_{\nu_u} + T^J$. Then, for every $\eta \in \mathscr{D}^1(\Omega \times \mathbb{S}^2)$ such that $\eta = \eta^{(0)}$ we have $\langle \partial\bar{T}, \eta\rangle = 0$.*

***Proof*** We show that $\bar{T}(\mathrm{d}\eta) = 0$ when $\eta = (-1)^{i-1}\varphi(x,y)\,\widehat{\mathrm{d}x^i}$, where $i = 1, 2$ and $\varphi \in C_c^\infty(\Omega \times \mathbb{S}^2)$. The assertion follows by linearity. Since

$$\mathrm{d}\eta = \partial_{x_i}\varphi(x,y)\,\mathrm{d}x + (-1)^i \sum_{j=1}^3 \partial_{y_j}\varphi(x,y)\,\widehat{\mathrm{d}x^i} \wedge \mathrm{d}y^j\,,$$

using that for $\mathscr{L}^2$-a.e. $x \in \Omega$

$$\partial_{x_i}\,[\varphi(x,\nu_u(x))] = \partial_{x_i}\varphi(x,\nu_u(x)) + \sum_{j=1}^3 \partial_{y_i}\varphi(x,\nu_u(x))\,\partial_i \nu_u^j(x)$$

and by the chain rule formula

$$(D^C[\varphi(x,\nu_u(x))])_i = \sum_{j=1}^3 \partial_{y_i}\varphi(x,\nu_u(x))\,(D^C\nu_u)_i^j\,,$$

whereas for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$

$$\int_{\Gamma_x} \sum_{j=1}^3 \partial_{y_j}\varphi(x,y)\,\mathrm{d}y^j = \varphi(x,\nu_u^+(x)) - \varphi(x,\nu_u^-(x)),$$

in terms of the $BV$ function $v(x) := \varphi(x,\nu_u(x))$ we obtain

$$\langle \bar{T}, \mathrm{d}\eta\rangle = \int_\Omega \partial_{x_i}\,v(x)\,\mathrm{d}x + \int_\Omega \mathrm{d}\,(D^C v)_i + \int_{S(v)} \big(v^+(x) - v^-(x)\big)\,n_i\,\mathrm{d}\mathscr{H}^1(x)$$

and the assertion follows from the chain rule formula of $v$. $\square$

The following structure result holds:

**Theorem 6.1** *Assume $T \in \mathscr{T}_{\nu_u}$ for some $u \in \mathscr{A}_p^\infty(\Omega)$ (see (6.12)). Then, $T$ is an i.m. rectifiable current in $\mathscr{R}_2(\Omega \times \mathbb{S}^2)$ that satisfies the null-boundary condition*

$$\langle \partial T, \eta \rangle = 0 \quad \forall\, \eta \in \mathscr{D}^1(\Omega \times \mathbb{S}^2)\,. \tag{6.8}$$

*Moreover, we can decompose $T$ as $T = G_{\nu_u} + T^C_{\nu_u} + T^J + S_T$ so that*

$$\mathbf{M}(T) = \mathbf{M}(G_{\nu_u}) + \mathbf{M}(T^C_{\nu_u}) + \mathbf{M}(T^J) + \mathbf{M}(S_T) < \infty\,, \quad \textit{where} \tag{6.9}$$

- *$G_{\nu_u}$ is the current carried by the graph of $\nu_u$;*
- *$T^C_{\nu_u}$ is the Cantor component defined as above, see (6.4);*
- *$T^J$ is the jump component given as above, where in (6.6) we have*

$$\Gamma_x \textit{ is a geodesic arc from } \nu^-(x) \textit{ to } \nu^+(x) \textit{ for } \mathscr{H}^1\textit{-a.e. } x \in S(\nu)\,; \tag{6.10}$$

- *$S_T \in \mathscr{D}_2(\Omega \times \mathbb{R}^2)$ is completely vertical, that is, $S_{T\,(0)} + S_{T\,(1)} = 0$.*

***Proof*** For a given current $R \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$, we let

$$\mathscr{F}_1(R) := \sup_{\omega \in \mathscr{F}^1} \langle R, \omega \rangle$$

where $\mathscr{F}^1$ denotes the class of 2-forms $\omega \in \mathscr{D}^2(\Omega \times \mathbb{S}^2)$ such that $\omega = \omega^{(1)}$ and $\|\omega\| \le 1$. We thus equivalently have $\mathscr{F}_1(R) = \mathbf{M}(R_{(1)})$.

By the properties of the class $\operatorname{cart}(\Omega \times \mathbb{S}^2)$ (Giaquinta et al. 1998b), we infer that $T \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$, the null-boundary condition (6.8) holds and there exists $\Sigma_T \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ such that $\Sigma_{T\,(0)} = 0$ and

$$T = G_{\nu_u} + \Sigma_T\,, \quad \text{where} \quad \mathbf{M}(T) = \mathbf{M}(G_{\nu_u}) + \mathbf{M}(\Sigma_T) < \infty\,.$$

Let $\{u_h\} \subset C^2(\Omega, \mathbb{R}^3)$ be the approximating sequence from Definition 6.2. Possibly passing to a not relabeled subsequence, we can assume that $G_{\nu_{u_h}} \rightharpoonup T$ and that the strict convergence (6.3) holds. Since $\nu_u$ is a $BV$ function, by the structure properties of the class $\operatorname{cart}(\Omega \times \mathbb{S}^2)$ we can decompose $\Sigma_T$ into the sum of three terms, namely

$$\Sigma_T = T^C_{\nu_u} + T^J + S_T\,,$$

where $T^C_{\nu_u}$ and $T^J$ are given as above. With the previous notation, we thus have

$$\mathscr{F}_1(G_{\nu_u}) = \int_\Omega |\nabla \nu_u|\, \mathrm{d}x\,, \quad \mathscr{F}_1(T^C_{\nu_u}) = \mathbf{M}(T^C_{\nu_u})\,, \quad \mathscr{F}_1(T^J) = \mathbf{M}(T^J)\,,$$

where all terms are finite. Moreover, the null-boundary condition in Lemma 6.1 implies that

$$\mathscr{F}_1(T^C_{\nu_u} + T^J) + \mathscr{F}_1(S_T) \le \mathscr{F}_1(\Sigma_T).$$

In addition, according to (6.5) and (6.7), by the mutual singularity of the measures $D^C \nu_u$ and $D^J \nu_u$ we obtain the decomposition

$$\mathscr{F}_1(T^C_{\nu_u} + T^J) = \mathscr{F}_1(T^C_{\nu_u}) + \mathscr{F}_1(T^J) < \infty,$$

where by the definition of the component $T^j$ we clearly have

$$\int_{S(\nu_u)} \operatorname{dist}_{\mathbb{S}^2}(\nu_u^-, \nu_u^+)\, \mathrm{d}\mathscr{H}^1 \le \mathscr{F}_1(S_T)\,. \tag{6.11}$$

Since $G_{\nu_{u_h}} \rightharpoonup T$, by lower semicontinuity we have

$$\mathscr{F}_1(G_{\nu_u}) + \mathscr{F}_1(\Sigma_T) = \mathscr{F}_1(T) \le \liminf_{h\to\infty} \mathscr{F}_1(G_{\nu_{u_h}}),$$

where the first equality follows from the mutual singularity of the measures associated with the components $G_{\nu_u}$ and $\Sigma_T$ of $T$. Moreover, since

$$\mathscr{F}_1(G_{\nu_{u_h}}) = \int_\Omega |\nabla \nu_{u_h}|\, \mathrm{d}x \quad \forall\, h,$$

the strict convergence in (6.3) is equivalent to the limit

$$\lim_{h\to\infty} \mathscr{F}_1(G_{\nu_{u_h}}) = \mathscr{F}_1(G_{\nu_u}) + \mathscr{F}_1(T^C_{\nu_u}) + \int_{S(\nu_u)} \operatorname{dist}_{\mathbb{S}^2}(\nu_u^-, \nu_u^+)\, \mathrm{d}\mathscr{H}^1.$$

Since $\mathscr{F}_1(G_{\nu_u}) < \infty$, by the previous properties we obtain the inequalities

$$\mathscr{F}_1(T^C_{\nu_u}) + \mathscr{F}_1(T^J) + \mathscr{F}_1(S_T) \le \mathscr{F}_1(\Sigma_T) \le \mathscr{F}_1(T^C_{\nu_u}) + \mathscr{F}_1(T^J) < \infty\,,$$

and hence $\mathscr{F}_1(S_T) = 0$. Recalling that $S_{T\,(0)} = 0$, the identity $\mathscr{F}_1(S_T) = 0$ gives $S_{T\,(1)} = 0$ and hence $S_T = S_{T\,(2)}$. As a consequence, the property (6.9) holds true. Finally, since $\mathscr{F}_1(T^C_{\nu_u}) < \infty$, the equality sign holds true in (6.11). Therefore, we have $\mathscr{H}^1(\Gamma_x) = \operatorname{dist}_{\mathbb{S}^2}(\nu_u^-(x), \nu_u^+(x))$ for $\mathscr{H}^2$-a.e. $x \in S(\nu_u)$. Recalling that $\Gamma_x$ is an oriented arc in $\mathbb{S}^2$ from $\nu_u^-(x)$ to $\nu_u^+(x)$, the property (6.10) holds and the proof is complete. ◻

### 6.3 Mass Minimizing Currents

Take $u \in \mathscr{A}_p^\infty(\Omega)$. By Federer and Fleming's closure theorem and Theorem 6.1, there exists $T_{\min} \in \mathscr{T}_{\nu_u}$ such that

$$\mathbf{M}(T_{\min}) = \min\left\{\mathbf{M}(T) \mid T \in \mathscr{T}_{\nu_u}\right\} . \tag{6.12}$$

Localized possibly irrecoverable strain in a small neighborhood of discrete points and cusps will be described in terms of a mass minimizing current in $\mathscr{T}_{\nu_u}$. However, we now see that it fails to be unique, in general, even in the case when the unit normal $\nu_u$ is a Sobolev map.

***Example 6.1*** Let $\Omega = B^2$ and $u$ be such that the deformed shell $u(B^2)$ forms a cusp, i.e., $u(x) = (x_1, x_2, f(|x|))$, where

$$f(\rho) := 1 - \sqrt{2\rho - \rho^2}\,, \quad \rho \in [0, 1]\,, \tag{6.13}$$

in such a way that with $\rho := |x| > 0$ we have

$$\nu_u(x) = \frac{1}{\sqrt{1 + f'(\rho)^2}}\left(-f'(\rho)\, x_1/\rho,\ -f'(\rho)\, x_2/\rho,\ 1\right) .$$

The unit normal $\nu_u$ belongs to the class $W^{1,q}(B^2, \mathbb{S}^2)$ for every $q < 2$, and around the origin $0_{\mathbb{R}^2}$ it describes the circle $\mathbb{S}^1 := \{z \in \mathbb{S}^2 \mid z_3 = 0\}$, which can be oriented in such a way that

$$\partial G_{\nu_u} \llcorner B^2 \times \mathbb{S}^2 = -\delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^1 \,]\!].$$

Denote by $\mathbb{S}^2_\pm := \{z \in \mathbb{S}^2 \mid \pm z_3 > 0\}$ the naturally oriented upper and lower half spheres, so that

$$\partial [\![\, \mathbb{S}^2_+ \,]\!] = [\![\, \mathbb{S}^1 \,]\!]\,, \quad \partial [\![\, \mathbb{S}^2_- \,]\!] = -[\![\, \mathbb{S}^1 \,]\!]\,.$$

Then, both the currents

$$T_+ := G_{\nu_u} + \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2_+ \,]\!]\,, \quad T_- := G_{\nu_u} - \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2_- \,]\!]$$

are mass minimizers in the class $\mathscr{T}_{\nu_u}$.

We have (see Theorem 6.2)

$$T_+ - T_- = \delta_{0_{\mathbb{R}^2}} \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Moreover, if $\Theta_2$ the 2-form in $\mathbb{S}^2$ given by (3.21), we obtain

$$\langle T_\pm, \varphi(x)\, \Theta_2(z)\rangle = \pm\langle [\![\, \mathbb{S}^2_\pm \,]\!], \Theta_2\rangle \cdot \langle \delta_{0_{\mathbb{R}^2}}, \varphi\rangle = \pm\pi\, \varphi(0_{\mathbb{R}^2}) \tag{6.14}$$

for every $\varphi \in C_c^\infty(B^2)$—see definition (6.18).

***Example 6.2*** Let $\Omega = B^2$ and $u$ be such that the deformed shell $u(B^2)$ forms a sharp fold, i.e., $u(x) = (x_1, x_2, f(|x_1|))$, with $f$ given by (6.13), so that

$$\nu_u(x) = \frac{1}{\sqrt{1 + f'(|x_1|)^2}} \left(-f'(|x_1|)\, x_1/|x_1|,\ 0,\ 1\right), \quad x_1 \neq 0 .$$

The unit normal $\nu_u$ belongs to the class $SBV(B^2, \mathbb{S}^2)$, with jump set $S(\nu_u) = B^2 \cap \{x_1 = 0\}$ and one-sided limits $\nu_u^{\pm} \equiv (\pm 1, 0, 0)$, if we choose $\mathbf{n} \equiv (1, 0)$. Any mass minimizer in $\mathscr{T}_{\nu_u}$ is equal to $T = G_{\nu_u} + T^J$, where in the jump component we can take $\Gamma_x$ equal to any fixed geodesic arc of $\mathbb{S}^2$ from $\nu_u^-$ to $\nu_u^+$.

However, in Theorem 6.2 we shall see that the arc $\Gamma_x$ must be orthogonal to the orienting unit vector to the fold $u(S(\nu_u))$ of the deformed shell at $u(x)$, and definitely it is parameterized by $\Phi(t) = (-\cos t, 0, \sin t)$, $t \in [0, \pi]$, for every $x \in S(\nu_u)$.

The latter property applies when the sharp fold does not degenerate. An opposite circumstance is described in the following example.

***Example 6.3*** Let $\Omega = B^2$ and $u \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ be given by

$$u(x) = \left(\frac{x_1|x_2|}{|x|}, \frac{x_2|x_2|}{|x|}, 0\right), \quad x = (x_1, x_2), \quad x_2 \neq 0 ,$$

so that the continuous representative satisfies $u(x_1, 0) \equiv 0_{\mathbb{R}^3}$. Thus, we have $\partial_1 u \times \partial_2 u = (0, 0, x_2|x_2|/|x|^2)$ for $x_2 \neq 0$, whence $\nu_u(x) = (0, 0, 1)$ if $x_2 > 0$, and $\nu_u(x) = (0, 0, -1)$ if $x_2 < 0$, so that $S(\nu_u) = B^2 \cap \{x_2 = 0\}$. Also, $u(S(\nu_u)) = \{0_{\mathbb{R}^3}\}$ and $u$ maps the half disks $B^2 \cap \{\pm x_2 > 0\}$ onto the disk

$$D = \{y \in \mathbb{R}^3 \mid {y_1}^2 + (y_2 - 1/2)^2 < 1/4,\ \ y_3 = 0\},$$

but with opposite orientation. In particular, with $\mathbf{t} = (1, 0)$ the unit tangent vector to $S(\nu_u)$, we have $\partial_{\mathbf{t}} u(x) = 0$ for every $x \in S(\nu_u)$.

We aim at excluding situations like the previous one. So, for $u \in \mathscr{A}_p^\infty(\Omega)$, we denote by $G_{\Phi_u}$ the current carried by the graph of the map $\Phi_u$ given by (3.12), and we notice that if it is endowed with finite mass, $G_{\Phi_u} \in \mathscr{R}_2(U)$, where $U := \Omega \times \mathbb{R}^3 \times \mathbb{S}^2$. We recall that the unit vector $\mathbf{t} = \mathbf{t}(x) = \mathbf{n}_u(x)^\perp$ fixes the orientation to the approximate tangent 1-space to the jump set $S(\nu_u)$ at $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$.

**Theorem 6.2** *Let $u \in \mathscr{A}_p^\infty(\Omega)$ be such that*

$$\mathbf{M}(G_{\Phi_u}) + \mathbf{M}(\partial G_{\Phi_u}) < \infty . \tag{6.15}$$

*Then, $u$ is approximately differentiable $\mathscr{H}^1$-a.e. on $S(\nu_u)$. Assume in addition that the tangential gradient $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Then, for any couple $T_1$, $T_2$ of mass minimizing currents in $\mathscr{T}_{\nu_u}$, we can find a current $L \in \mathscr{R}_0(\Omega)$ such that*

$$T_1 - T_2 = L \times [\![\, \mathbb{S}^2 \,]\!] . \tag{6.16}$$

***Proof*** Setting $S := T_1 - T_2$, it suffices to show that $S$ is a completely vertical integral 2-cycle; that is, $S \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ satisfies $\partial S = 0$ and $S = S_{(2)}$. Indeed, following (Giaquinta et al. 1998b), any completely vertical 2-form $\omega = \omega^{(2)} \in \mathscr{D}^2(\Omega \times \mathbb{S}^2)$ can be written as

$$\omega^{(2)}(x, z) = \phi(x)\Theta_2(z) + \mathrm{d}_z \eta(x, z),$$

where $\phi \in C_c^\infty(\Omega)$, $\Theta_2$ is the volume 2-form in (3.21) and $\eta \in \mathscr{D}^1(\Omega \times \mathbb{S}^2)$ is of the type $\eta = \eta^{(1)}$. Writing $\mathrm{d}\eta = \mathrm{d}_x\eta + \mathrm{d}_z\eta$, we have $(\mathrm{d}_x\eta)^{(2)} = 0$ and hence $\langle S, \mathrm{d}_x\eta\rangle = 0$. Therefore, we obtain:

$$\langle S, \mathrm{d}_z\eta\rangle = \langle \partial S, \eta\rangle - \langle S, \mathrm{d}_x\eta\rangle = 0,$$

whence the action of $S$ is possibly nonzero only on the term $\phi\Theta_2$, and definitely (6.16) holds.

By Theorem 6.1, for $\ell = 1, 2$ we can write $T_\ell = G_{\nu_u} + T^C_{\nu_u} + T^J_\ell + S_{T_\ell}$, so that $S = S_{T_1} - S_{T_2} + T_1^J - T_2^J$. Therefore, if we show that the jump components are equal, that is,

$$T_1^J = T_2^J\,, \tag{6.17}$$

we obtain that $S \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ satisfies $\partial S = 0$ and $S = S_{(2)}$.

The property (6.17) is trivial if $\nu_u \in W^{1,1}(\Omega, \mathbb{S}^2)$. More generally, if (6.15) holds, a result by Ambrosio et al. (1998) implies that $\Phi_u$ is a function in $SBV$ whose trace functions are approximately differentiable $\mathscr{H}^1$-almost everywhere on the jump set of $\Phi_u$. Since $u \in W^{1,p}(\Omega, \mathbb{R}^2)$, with $p > 2$, the first assertion follows.

Assume that $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Then, the unit vector $\mathbf{T}(x) := \partial_{\mathbf{t}} u(x)/|\partial_{\mathbf{t}} u(x)|$ provides an orientation to the fold $u(S(\nu_u))$ of the deformed shell $u(\Omega)$ at $u(x)$, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$.

By the properties of weak limits of the Gauss graphs (see (Anzellotti et al. 1990; Mariano and Mucci 2023a)), it turns out that, for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$, the arc $\Gamma_x$ appearing in the definition (6.6), for $T = T_\ell$, must belong to the great circle obtained by the intersection of the Gauss sphere $\mathbb{S}^2$ with the 2-space $\Pi_x^\perp$ orthogonal to the vector $\mathbf{T}(x)$, and the orientation of $\Gamma_x$ is consistent with the one induced by $\mathbf{T}(x)$ on $\Pi_x^\perp$. Indeed, when $\nu_u^-(x) \neq -\nu_u^+(x)$ we have $\mathbf{T}(x) = \lambda(\nu_u^-(x) \times \nu_u^+(x))$ for some $\lambda > 0$. Instead, when $\nu_u^-(x) = -\nu_u^+(x)$, up to a rotation in $\mathbb{R}^3$ we may assume that $\nu_u^-(x) = (-1, 0, 0)$, $\nu_u^+(x) = (1, 0, 0)$, $\mathbf{T}(x) = (0, 1, 0)$ and the oriented arc $\Gamma_x$ is parameterized by $\Phi(t) = (-\cos t, 0, \sin t)$, $t \in [0, \pi]$. As a consequence, the geodesic arcs $\Gamma_x$ only depend on the deformation map $u$ and on its normal $\nu_u$. We thus definitely obtain equation (6.17), and the proof is complete. □

### 6.4 Need for Atomic Measures

Let $T$ be a mass minimizer in the class $\mathscr{T}_{\nu_u}$ for some $u \in \mathscr{A}_p^\infty(\Omega)$. Recalling that $\Theta_2$ is the volume 2-form in (3.21), we denote by $\boldsymbol{\mu}_T$ the Borel regular signed measure on $\Omega$ generated by the finite mass distribution

$$\boldsymbol{\mu}_T(\phi) := \langle T, \phi(x)\,\Theta_2(z)\rangle\,, \quad \phi \in C_c^\infty(\Omega)\,. \tag{6.18}$$

If the minimizer $T$ in $\mathscr{T}_{\nu_u}$ is unique, we shall write $\boldsymbol{\mu}_u := \boldsymbol{\mu}_T$. In general, we decompose $\boldsymbol{\mu}_T$ into mutually singular components, namely, $\boldsymbol{\mu}_T = \boldsymbol{\mu}_T^a + \boldsymbol{\mu}_T^s$, where $\boldsymbol{\mu}_T^a$ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$. Hence, we have

$$\boldsymbol{\mu}_T^a(\phi) = \langle G_{\nu_u}, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

Since it only depends on the unit normal $\nu_u$ to $u$, we shall write $\boldsymbol{\mu}_u^a = \boldsymbol{\mu}_T^a$. On account of (3.22) and the area formula, we have

$$\boldsymbol{\mu}_u^a(\phi) = \int_\Omega \phi(x)\,\sqrt{g_u(x)}\,\mathbf{K}_u(x)\,\mathrm{d}x$$

with $\mathbf{K}_u(x)$ given by (3.15) for $\mathscr{L}^2$-a.e. $x \in \Omega$, and we may say that the measure $\boldsymbol{\mu}_T$ describes concentration of the Gauss curvature energy of the shell surface. Indeed, in the smooth case we have $\boldsymbol{\mu}_T = \boldsymbol{\mu}_u^a$, so that since $[\![\,\mathscr{G}\mathscr{M}_u\,]\!] = \Phi_{u\#}[\![\,\Omega\,]\!]$, we obtain

$$\int_{\mathscr{M}_u} \mathbf{K}_u\,\mathrm{d}\mathscr{H}^2 = \int_\Omega \sqrt{g_u}\,\mathbf{K}_u \mathrm{d}x = \int_\Omega {\Phi_u}^{\#}\Theta_2 = \langle [\![\,\mathscr{G}\mathscr{M}_u\,]\!], \Theta_2\rangle.$$

On account of Theorem 6.1, the singular part satisfies

$$\boldsymbol{\mu}_T^s(\phi) = \langle \Sigma_T, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

We finally denote by $\boldsymbol{\mu}_T^{at}$ the *atomic component* of $\boldsymbol{\mu}_T$. Since the measure $\boldsymbol{\mu}_T$ has finite total variation, the Schwartz theorem implies that $\boldsymbol{\mu}_T^{at}$ is concentrated on a countable set of points.

To avoid deformed surfaces with spread wrinkles, we require that the singular component $\boldsymbol{\mu}_T^s$ be concentrated on a 1-rectifiable set of $\Omega$. However, in general this is not the case even for convex shells. Indeed, as in case of the map $u$ from Example 4.1, with $T = G_{\nu_u} + T_{\nu_u}^C$, it turns out that $\boldsymbol{\mu}_T^s$ is concentrated on the set $\Sigma = \{x \in B^2 : |x| \in C\}$, where $C$ is the Cantor middle-thirds set, whence $\Sigma$ has Hausdorff dimension $1 + \log_3 2$. Therefore, we shall work with deformation maps whose normal $\nu_u$ is a special function of bounded variation.

By Theorem 6.1, we have

$$\boldsymbol{\mu}_T^{at}(\phi) = \langle S_T, \phi(x)\,\Theta_2(z)\rangle \quad \forall\,\phi \in C_c^\infty(\Omega)\,.$$

Therefore, if $\nu_u \in SBV(\Omega, \mathbb{S}^2)$, under the hypotheses of Theorem 6.2, by (6.17) and (6.16), since $\mathbf{M}(T_1) = \mathbf{M}(T_2)$, we obtain what follows:

- the component $\boldsymbol{\mu}_T^s - \boldsymbol{\mu}_T^{at}$ does not depend on the choice of the mass minimizing current $T$ in $\mathscr{T}_{\nu_u}$, and it is concentrated on the jump set $S(\nu_u)$;
- the total variation $|\boldsymbol{\mu}_T^{at}|$ of the atomic component does not depend on the choice of the mass minimizing current $T$ in $\mathscr{T}_{\nu_u}$—see eq. (6.14).

As a consequence, with an abuse of notation we shall write $\boldsymbol{\mu}_u^{at} = \boldsymbol{\mu}_T^{at}$, independently of the choice of $T = T_{\min}$ in (6.12). Indeed, if $T_1$ and $T_2$ are mass minimizers in $\mathscr{T}_{\nu_u}$, we have

$$|\boldsymbol{\mu}_{T_1}|(V) = |\boldsymbol{\mu}_{T_2}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}.$$

We thus introduce a nonnegative and finite Borel measure $\mathbf{m}_p$ whose atomic part $\mathbf{m}_p^{at}$ is concentrated in a given countable set of points in $\Omega$, in such a way that the atomic part of the measure $\boldsymbol{\mu}_u$ is concentrated in the support of $\mathbf{m}_p^{at}$, namely

$$|\boldsymbol{\mu}_u|(V) \le \mathbf{m}_p(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}\,. \tag{6.19}$$

We call $\mathbf{m}_p^{at}$ *the atomic component of a deformation implying possibly irrecoverable strain*. We have

$$|\boldsymbol{\mu}_u^{at}|(\{x\}) := \lim_{r \to 0^+} |\boldsymbol{\mu}_u|(B^2(x,r)) = c > 0 \quad \Longrightarrow \quad \mathbf{m}_p^{at}(\{x\}) \ge c\,. \tag{6.20}$$

At some $x \in \Omega$, we could have $|\boldsymbol{\mu}_u^{at}|(\{x\}) = 0$ but $\mathbf{m}_p^{at}(\{x\}) > 0$. In that case, $u(x)$ may be interpreted as a plastic nucleation point where the deformed surface presents no cusps.

### 6.5 Examples

In all the examples below, we have $\Omega = B^2$ and $u(x) = (x, g(x))$ for some bounded and concave function $g : B^2 \to \mathbb{R}$ such that $\nu_u \in SBV(B^2, \mathbb{S}^2)$. Moreover, the mass minimizer in $\mathscr{T}_{\nu_u}$ is unique and will be indicated by $T_u$. Qualitatively different behaviors of the singular measure $\boldsymbol{\mu}_u^s$ appear.

In the first example, the measure $\boldsymbol{\mu}_u^s$ is a Dirac mass at the vertex of a cone, with a weight equal to the angle defect (that is, to the area of the geodesically convex set of $\mathbb{S}^2$ determined by the solid tangent cone at the vertex; see Fig. 2).

***Example 6.4*** Let $g$ be of the form $g(x) = -m\rho$, where $m > 0$ and $\rho := |x|$, so that for $\rho > 0$ we have $\nu_u(x) = (mx_1/\rho, mx_2/\rho, 1)/\sqrt{1+m^2}$. Therefore, the image of $\nu_u$ is a one-dimensional set given by the parallel $\Gamma_m := \{z \in \mathbb{S}^2 \mid z_3 = (1+m^2)^{-1/2}\}$. Moreover, since as currents in $\mathscr{R}_2(B^2 \times \mathbb{S}^2)$

$$\partial G_{\nu_u} = -\delta_{0_{\mathbb{R}^2}} \times [\![\, \Gamma_m \,]\!],$$

with $\Sigma_m$ the set defined by $\Sigma_m := \{z \in \mathbb{S}^2 \mid z_3 > (1+m^2)^{-1/2}\}$, using the identity $[\![\, \Gamma_m \,]\!] = \partial [\![\, \Sigma_m \,]\!]$ we get $T_u = G_{\nu_u} + S_u$, where

$$S_u = \delta_{0_{\mathbb{R}^2}} \times [\![\, \Sigma_m \,]\!].$$

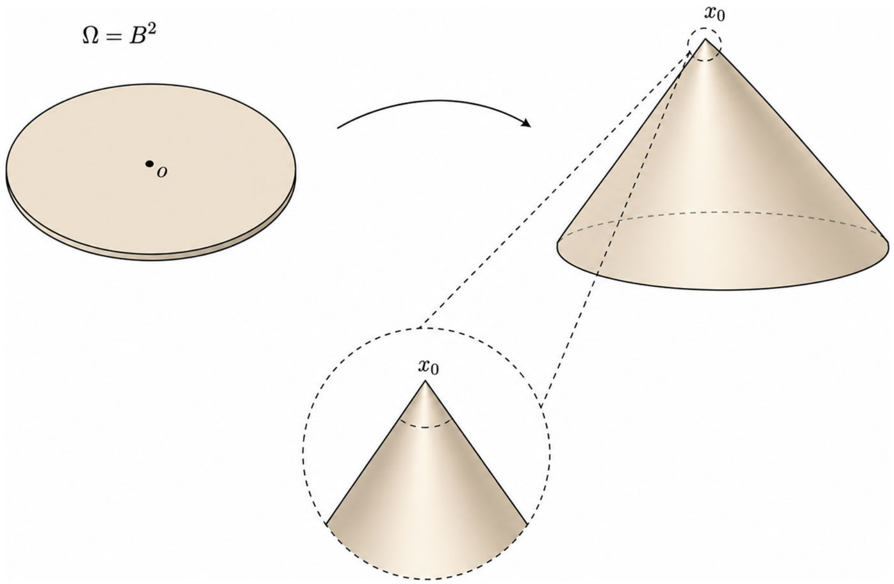


**Fig. 2** Conical singularity and atomic Gauss curvature concentration

Hence, we have

$$\langle \boldsymbol{\mu}_u^s, \varphi \rangle = \varphi(0_{\mathbb{R}^2}) \cdot \alpha_m \,, \quad \alpha_m = 2\pi \left( 1 - \frac{1}{\sqrt{1+m^2}} \right).$$

We thus have $\boldsymbol{\mu}_u^s = \boldsymbol{\mu}_u^{at} = \alpha_m \, \delta_{0_{\mathbb{R}^2}}$. The deformed shell $u(B^2)$ has no sharp folds, since $\nu_u$ is a Sobolev map in $W^{1,p}(B^2, \mathbb{S}^2)$ for every $p < 2$.

The second example describes a developable surface. We have $\boldsymbol{\mu}_u = 0$, even if the jump component of the distributional derivative of $\nu_u$ is nonzero. A sharp fold in the deformed shell occurs; it is a straight segment (see Fig. 3).

***Example 6.5*** Let $g(x) = -m|x_1|$, where $m > 0$. Then, we have $\nu_u \equiv a_m^+$ if $x_1 > 0$ and $\nu_u \equiv a_m^-$ if $x_1 < 0$, where $a_m^\pm := (\pm m, 0, 1)/\sqrt{1+m^2}$. The deformed surface $u(B^2)$ has a sharp fold given by the straight segment $I \times \{0\}$ in $\mathbb{R}^3$, where $I := \{x \in B^2 \mid x_1 = 0\}$. Moreover, since

$$\partial G_{\nu_u} = [\![ \, I \, ]\!] \times \left( \delta_{a_m^-} - \delta_{a_m^+} \right),$$

we can write $T_u = G_{\nu_u} + S_u$ with

$$S_u = -[\![ \, I \, ]\!] \times \Gamma,$$

where $\Gamma \in \mathscr{R}_1(\mathbb{S}^2)$ is associated with the oriented geodesic arc in $\mathbb{S}^2$ from $a_m^-$ to $a_m^+$, whence $\boldsymbol{\mu}_u = 0$.

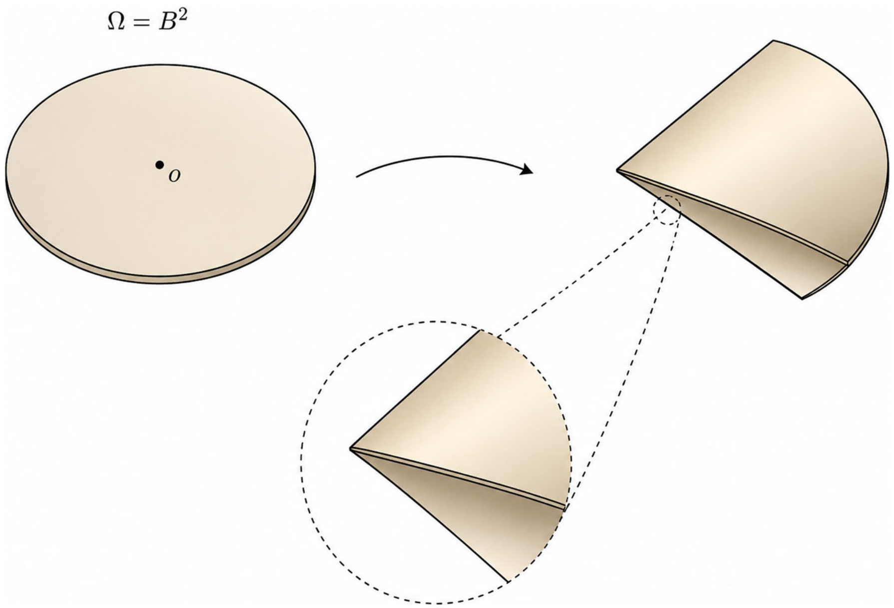


**Fig. 3** Developable fold with vanishing Gauss curvature measure

The third example shows that when $D^J \nu_u \neq 0$, it may happen that the singular measure $\boldsymbol{\mu}_u^s$ is concentrated on a one-dimensional set $\mathscr{I}$. In this case, the sharp fold pertaining to the deformed shell is a line of $\mathbb{R}^3$ with nonzero curvature (see Fig. 4).

***Example 6.6*** Let $g$ be of the form $g(x) = -|x_1| - x_2^2/2$, so that

$$\nu_u(x) = \frac{(\operatorname{sgn}(x_1), x_2, 1)}{\sqrt{2 + x_2^2}}, \quad x_1 \neq 0.$$

The deformed surface $u(B^2)$ has a sharp fold given by the support of the curve $\gamma(t) = (0, t, -t^2/2)$, where $t \in [-1, 1]$, whose curvature is nonzero at any point. We have:

$$\partial G_{\nu_u} = \Psi^-_{\#} [\![ -1, 1 ]\!] - \Psi^+_{\#} [\![ -1, 1 ]\!],$$

where $\Psi^\pm : (-1, 1) \to B^2 \times \mathbb{S}^2$ is given by

$$\Psi^\pm(t) := \big(0, t, \psi_\pm(t)\big), \quad \psi_\pm(t) := \frac{(\pm 1, t, 1)}{\sqrt{2 + t^2}}.$$

Therefore, we infer that $T_u = G_{\nu_u} + S_u$ where

$$S_u = F_{\#}[\![ [-1, 1]^2 ]\!], \quad F(t, \lambda) := \left(0, t, \frac{(\lambda, t, 1)}{\sqrt{\lambda^2 + t^2 + 1}}\right).$$

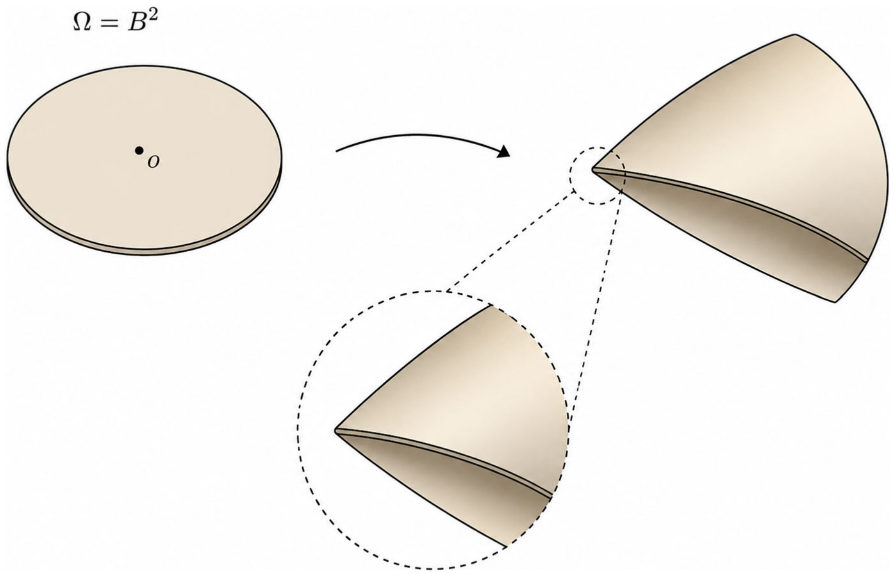


**Fig. 4** Curved fold carrying a one-dimensional singular measure

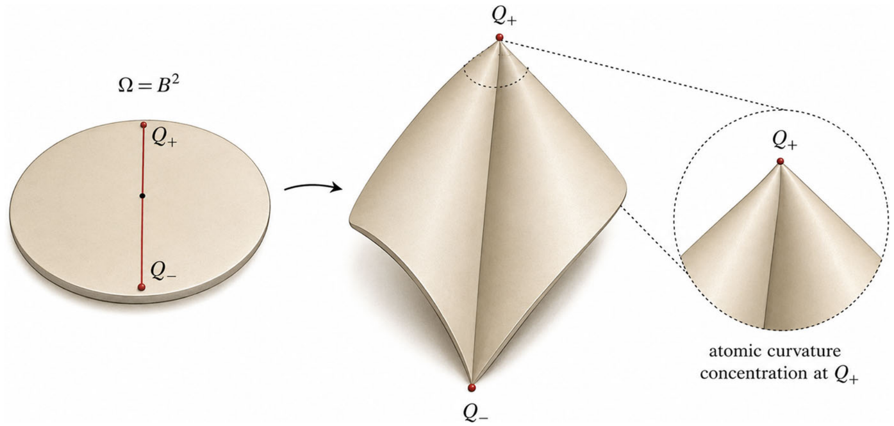


**Fig. 5** Combined fold-cusp configuration with atomic concentration

As a consequence, the singular measure $\boldsymbol{\mu}_u^s$ is non-trivial; it is concentrated on the segment $I := \{x \in B^2 \mid x_1 = 0\}$, and we obtain $\boldsymbol{\mu}_u^{at} = 0$ whereas

$$\boldsymbol{\mu}_u^s(B^2) = \mathscr{H}^2(F([-1, 1]^2)) .$$

In the last example, both the measures $\boldsymbol{\mu}_u^{at}$ and $\boldsymbol{\mu}_u^s - \boldsymbol{\mu}_u^{at}$ are non-trivial (see Fig. 5 for a situation falling in the same class of the following example.).

***Example 6.7*** Let $g$ be a function given by

$$g(x) = \begin{cases} -\sqrt{x_1^2 + (x_2 - 1/2)^2} & \text{if } \ x_2 \geq 1/2 \\ -|x_1| & \text{if } \ |x_2| \leq 1/2 \\ -\sqrt{x_1^2 + (x_2 + 1/2)^2} & \text{if } \ x_2 \leq -1/2 \end{cases} .$$

For $x_1 \neq 0$, we have

$$\nu_u(x) = \begin{cases} \frac{1}{\sqrt{2}} \Big( \frac{x_1}{\sqrt{x_1^2 + (x_2 - 1/2)^2}}, \frac{x_2 - 1/2}{\sqrt{x_1^2 + (x_2 - 1/2)^2}}, 1 \Big) & \text{if } \ x_2 \geq 1/2 \\ \frac{1}{\sqrt{2}} \big( \operatorname{sgn}(x_1), 0, 1 \big) & \text{if } \ |x_2| \leq 1/2 \\ \frac{1}{\sqrt{2}} \Big( \frac{x_1}{\sqrt{x_1^2 + (x_2 + 1/2)^2}}, \frac{x_2 + 1/2}{\sqrt{x_1^2 + (x_2 + 1/2)^2}}, 1 \Big) & \text{if } \ x_2 \leq -1/2 \end{cases} ,$$

and hence $\nu_u \in SBV(B^2, \mathbb{S}^2)$. Moreover, setting $a_\pm := (\pm 1, 0, 1)/\sqrt{2}$, $I := \{x \in B^2 \mid x_1 = 0, |x_2| \leq 1/2\}$, $Q_\pm := (0, \pm 1/2)$, $\gamma^+(\theta) := (\cos\theta, \sin\theta, 1)/\sqrt{2}$, and $\gamma^-(\theta) := (\cos(\theta + \pi), \sin(\theta + \pi), 1)/\sqrt{2}$, we get

$$\partial G_{\nu_u} = [\![\, I \,]\!] \times \big( \delta_{a_-} - \delta_{a_+} \big) - \delta_{Q_+} \times \gamma^+_\# [\![\, 0, \pi \,]\!] - \delta_{Q_-} \times \gamma^-_\# [\![\, 0, \pi \,]\!]$$

where $\partial [\![\, I \,]\!] = \delta_{Q_+} - \delta_{Q_-}$ and $\partial \gamma^+_\# [\![\, 0, \pi \,]\!] = \delta_{a_-} - \delta_{a_+} = -\partial \gamma^-_\# [\![\, 0, \pi \,]\!]$. Therefore, we have $T_u = G_{\nu_u} + S_u$ with

$$S_u = -[\![\, I \,]\!] \times [\![\, J \,]\!] + \delta_{Q_+} \times [\![\, D_+ \,]\!] + \delta_{Q_-} \times [\![\, D_- \,]\!]$$

where $J$ is the oriented geodesic arc in $\mathbb{S}^2$ from $a_-$ to $a_+$, whereas $D_\pm$ is the geodesically convex subset of $\mathbb{S}^2$ enclosed by the curves $J$ and $\gamma^\pm([0, \pi])$, that is, $D_\pm := \{z \in \mathbb{S}^2 \mid z_3 > 1/\sqrt{2}, \pm z_2 \geq 0\}$. As a consequence, the singular measure $\boldsymbol{\mu}_u^s$ has a 1-dimensional component $\boldsymbol{\mu}_u^s - \boldsymbol{\mu}_u^{at}$ concentrated on the segment $I$, and an atomic part given by

$$\boldsymbol{\mu}_u^{at} = \mathscr{H}^2(D_+)\, \delta_{Q_+} + \mathscr{H}^2(D_-) \times \delta_{Q_-} .$$

Further combined fold-cusp configurations can be considered, as those in Fig. 6.

## 7 An Existence Result

Here, we consider shells equilibrium configurations free of spread wrinkles but with possibly irrecoverable sharp folds and cusps. So, we discuss energy minimization. To this aim we require that the unit normal $\nu_u$ field over the deformed shell is free of the Cantor component. Thus, on the basis of Theorem 6.2, along energy minimizing

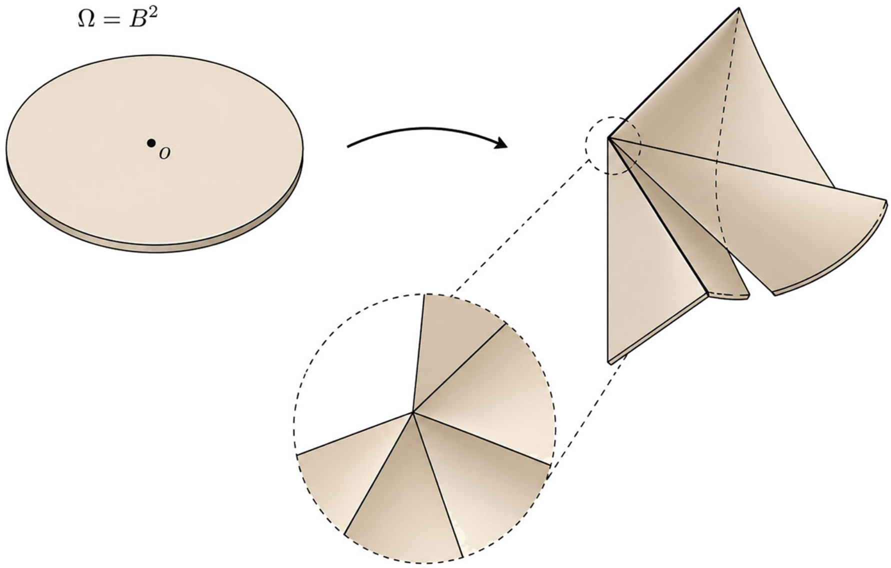


**Fig. 6** Combined fold-cusp configuration with atomic concentration

sequences we need to have weak convergence in $L^1$ of all the minors of approximate gradient $\nabla\Phi_u$ of the Gauss graph map $\Phi_u$ in (3.12). Also, we need to apply the closure-compactness theorem in $SBV(\Omega, \mathbb{S}^2)$. We thus introduce a further subclass of competitors in $\mathscr{A}_p^\infty(\Omega)$.

**Definition 7.1** For $p > 2$, we denote by $\widehat{\mathscr{A}_p^\infty}(\Omega)$ the subclass of maps $u \in \mathscr{A}_p^\infty(\Omega)$—with $\mathscr{A}_p^\infty(\Omega)$ introduced in Definition 6.2—such that

- the approximate gradient $\nabla\nu_u$ belongs to the Lebesgue space $L^p(\Omega, \mathbb{R}^{3\times 2})$;
- the boundary of the graph current $G_{\Phi_u}$ has finite mass, $\mathbf{M}(\partial G_{\Phi_u}) < \infty$;
- for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$, the approximate tangential gradient is non-trivial, that is, $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$.

By the parallelogram and by Hölder inequalities, a deformation map $u$ in the class $\widehat{\mathscr{A}_p^\infty}(\Omega)$ satisfies $\mathscr{E}(u) < \infty$, where the energy $\mathscr{E}(u)$ is given by (3.19)–(3.20), and also the mass bound $\mathbf{M}(G_{\Phi_u}) < \infty$, so that Theorem 6.2 guarantees the existence of $\partial_{\mathbf{t}} u$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_u)$. Finally, the bound $\mathbf{M}(\partial G_{\Phi_u}) < \infty$ implies that $\mathbf{M}(\partial G_{\nu_u}) < \infty$, whence $\nu_u \in SBV(\Omega, \mathbb{S}^2)$ and $\mathscr{H}^1(S(\nu_u)) < \infty$.

Four basic ingredients are chosen:

- a class $\widehat{\mathscr{A}_p^\infty}(\Omega)$ as in Definition 7.1, for some exponent $p > 2$;
- a real constant $K > 0$ yielding a uniform bound of the deformation maps;
- a triplet $(\mathscr{I}, \mathbf{n}, \Theta)$ in the class $\mathscr{PF}$, according to Definition 5.1: It accounts for the potential occurrence of possibly irrecoverable sharp folds;
- a nonnegative and finite regular measure $\mathbf{m}_p$ whose atomic component $\mathbf{m}_p^{at}$ is concentrated in a countable subset of $\Omega$: it accounts for the potential occurrence of possibly irrecoverable localized strains in 'small' neighborhoods of isolated points.

**Definition 7.2** We denote by $\widetilde{\mathscr{A}_p}(\Omega)$ the class of functions $u$ in $\widehat{\mathscr{A}_p^\infty}(\Omega)$ satisfying the following properties:

- according to the notation in (5.3) and (5.6), the total variation bound (5.8) holds, so that inequality (5.9) is satisfied;
- the measure $\boldsymbol{\mu}_u$ satisfies the local bound (6.19) with respect to the measure $\mathbf{m}_p$, so that the relation (6.20) concerning the atomic measures $\boldsymbol{\mu}_u^{at}$ and $\mathbf{m}_p^{at}$ is satisfied;
- $\|u\|_\infty \le K < +\infty$.

A class of energies that agree with the peculiarities of deformations in $\widetilde{\mathscr{A}_p}(\Omega)$, to within dimensional constants, is given by

$$\mathscr{E}_p(u) := \mathscr{F}_p(u) + \int_\Omega |\nabla \nu_u|^p \, \mathrm{d}x + \mathbf{M}(\partial G_{\Phi_u}) + \int_{S(\nu_u)} \psi(|\partial_{\mathbf{t}} u|) \, \mathrm{d}\mathscr{H}^1 + \int_\Omega e(u(x)) \, \mathrm{d}x \tag{7.1}$$

where, we recall, the function $\psi : ]0, +\infty[ \to \mathbb{R}^+$ is convex, nonnegative, with linear growth at $+\infty$, and such that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$. The term $\mathscr{F}_p(u)$ is given by (6.1); it is an adaptation to shells of an energy proposed by Hadamard (1903). The fourth addendum guarantees that the non-degeneracy assumption $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ is preserved in the minimization process. Finally, $e : \mathbb{R}^3 \to \mathbb{R}$ is assumed to be a continuous density function, so that the last addendum represents (in a sense not specified here) the action of external bulk forces. By construction, according to Definitions 7.1 and 7.2, the class $\widetilde{\mathscr{A}_p}(\Omega)$ always contains functions $u$ with finite energy, namely those such that $\mathscr{E}_p(u) < \infty$.

**Theorem 7.1** *With the previous notation, the infimum of the problem*

$$\inf\{\mathscr{E}_p(u) \mid u \in \widetilde{\mathscr{A}_p}(\Omega)\}$$

*is attained.*

***Proof*** We repeatedly pass to not relabeled subsequences. In fact, we recall that the weak convergence of i.m. rectifiable currents with support contained in a compact set is metrizable.

Let $\{u_k\} \subset \widetilde{\mathscr{A}_p}(\Omega)$ be an energy minimizing sequence. By applying Proposition 6.1 to each $u_k$, and on account of the lower semicontinuity inequality (6.2), we find the existence of a function $u \in \mathscr{A}_p^\infty(\Omega)$ such that $\|u\|_\infty \le K$ and

$$\mathscr{F}_p(u) \le \liminf_{k\to\infty} \mathscr{F}_p(u_k) < \infty \,.$$

Moreover, since for every $k$

$$\mathscr{H}^1(S(\nu_{u_k})) \le 2\,\mathbf{M}(\partial G_{\nu_{u_k}}) \le 2\,\mathbf{M}(\partial G_{\Phi_{u_k}}) \le C < \infty,$$

we have

$$\sup_k \Big( \|\nu_{u_k}\|_\infty + \int_\Omega |\nabla \nu_{u_k}|^p \, \mathrm{d}x + \mathscr{H}^1(S(\nu_{u_k})) \Big) < \infty \,,$$

and the compactness theorem in SBV applies, see (Ambrosio et al. 2000). It implies that

$$\int_\Omega |\nabla v_u|^p \, \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega |\nabla v_{u_k}|^p \, \mathrm{d}x$$

and that

$$|\mu^J_{v_u}|(V) \le \liminf_{k\to\infty} |\mu^J_{v_{u_k}}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open}$$

so that property (5.8) is preserved in the limit process.

Furthermore, using that $p > 2$, by the parallelogram and Hölder inequalities we obtain the weak convergence in $L^1(\Omega)$ of all the minors of the gradient $6 \times 2$ matrix $\nabla\Phi_{u_k}$ to the corresponding minor of $\nabla\Phi_u$. Therefore, using that

$$\sup_k \Big(\mathbf{M}(G_{\phi_{u_k}}) + \mathbf{M}(\partial G_{\phi_{u_k}})\Big) < \infty,$$

by Federer and Fleming's closure theorem we infer that $G_{\phi_{u_k}} \rightharpoonup G_{\phi_u}$ weakly as currents in $\mathscr{D}_2(U)$, where $U := \Omega \times \mathbb{R}^3 \times \mathbb{S}^2$, and by lower semicontinuity of the mass we have

$$\mathbf{M}(G_{\phi_u}) \le \liminf_{k\to\infty} \mathbf{M}(G_{\phi_{u_k}}) < \infty\,, \quad \mathbf{M}(\partial G_{\phi_u}) \le \liminf_{k\to\infty} \mathbf{M}(\partial G_{\phi_{u_k}}) < \infty\,.$$

As a consequence, by the first assertion in Theorem 6.2 we infer that $u$ is approximately differentiable $\mathscr{H}^1$-a.e. on $S(v_u)$.

Furthermore, denoting by $\partial_{\mathbf{t}_k} u_k$ the approximate tangential gradient of $u_k$ to $S(v_{u_k})$, the weak convergence $\partial G_{\phi_{u_k}} \rightharpoonup \partial G_{\phi_u}$ in $\mathscr{D}_1(U)$ implies the weak convergence in the sense of measures

$$|\partial_{\mathbf{t}_k} u_k|\, \mathscr{H}^1 \llcorner S(v_{u_k}) \rightharpoonup \mu$$

to some nonnegative Borel measure $\mu$ greater than $|\partial_{\mathbf{t}} u|\, \mathscr{H}^1 \llcorner S(v_k)$. Therefore, using a lower semicontinuity result due to Reshetnyak (1968), we have

$$\int_{S(v_u)} \psi(|\partial_{\mathbf{t}} u|)\, \mathrm{d}\mathscr{H}^1 \le \liminf_{k\to\infty} \int_{S(v_{u_k})} \psi(|\partial_{\mathbf{t}_k} u_k|)\, \mathrm{d}\mathscr{H}^1 < \infty.$$

Recalling that $\psi(\rho) \to +\infty$ when $\rho \to 0^+$, we thus obtain that $\partial_{\mathbf{t}} u \neq 0_{\mathbb{R}^3}$ for $\mathscr{H}^1$-a.e. $x \in S(v_u)$, and hence, $u \in \widehat{\mathscr{A}_p^\infty}(\Omega)$. In particular, the second part of Theorem 6.2 applies to $u$.

Let now $T_k$ be a minimizing current in $\mathscr{T}_{v_{u_k}}$, so that $|\boldsymbol{\mu}_{u_k}| = |\boldsymbol{\mu}_{T_k}|$ for every $k$. Again by Federer and Fleming's closure theorem, a subsequence of $\{T_k\}$ weakly converges to a Cartesian current $\bar{T}$ in the class $\mathscr{T}_u$. Therefore, denoting by $T$ a minimizing current in $\mathscr{T}_u$, we obtain

$$|\boldsymbol{\mu}_u|(V) = |\boldsymbol{\mu}_T|(V) \le |\boldsymbol{\mu}_{\bar{T}}|(V) \le \liminf_{k\to\infty} |\boldsymbol{\mu}_{T_k}|(V) \quad \forall\, V \subset \Omega\,, \quad V \text{ open.}$$

As a consequence, property (6.19) is preserved and definitely $u \in \widetilde{\mathscr{A}_p}(\Omega)$. Finally, by putting the energy terms together, we obtain the lower semicontinuity inequality

$$\mathscr{E}_p(u) \le \liminf_{k\to\infty} \mathscr{E}_p(u_k) < \infty\,,$$

the term $u \mapsto \int_\Omega e(u(x))\,\mathrm{d}x$ being continuous. $\square$

## 8 Concluding Remarks

We have provided a functional characterization of shell deformations admitting possibly irrecoverable sharp folds and cusps while maintaining compactness. In this functional setting, we have considered a class of energies accounting for the peculiar characteristics of these singular behavior, proving for it the existence of minimizers. Our results can be generalized to shells with additional through-the-thickness microstructure described by phase fields taking values on a finite-dimensional geodetic complete differentiable manifold not embedded in any linear space. Besides this potential extension, the results collected here underline once again that the selection of a function space for an admissible class of deformations is per se a constitutive choice.

**Acknowledgements** This work has been developed within programs of the research group in 'Theoretical Mechanics' of the 'Centro di Ricerca Matematica Ennio De Giorgi' of the Scuola Normale Superiore Pisa. The Italian groups of Mathematical Physics (GNFM-INDAM) and Analysis, Probability and Applications (GNAMPA-INDAM) are acknowledged.

**Author Contributions** The authors, Prof. Paolo Maria Mariano and Prof. Domenico Mucci, equally share the responsibility of this manuscript.

**Funding** Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement.

**Data Availability** No datasets were generated or analyzed during the current study.

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest regarding this manuscript.

## References

Acerbi, E., Percivale, D.: Reinforcement of plates in Hencky's plasticity. Comm. Partial Diff. Eq. **17**, 2127–2136 (1992)

Ambrosio, L., Braides, A., Garroni, A.: Special functions with bounded variation and with weakly differentiable traces on the jump set. Nonlinear Diff. Equations Appl. **5**, 219–243 (1998)

Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. Oxford University Press, Oxford (2000)

Antman, S.S.: Nonlinear Problems of Elasticity. Springer, Berlin (1995)

Antman, S.S., Marlow, R.S.: Material constraints, Lagrange multipliers, and compatibility. Applications to rod and shell theories. Arch. Rational Mech. Anal. **116**, 257–299 (1991)

Anzellotti, G.: Functionals depending on curvatures, in *Proc. of the Conference on Partial Differential Equations and Geometry* (Torino, 1988), *Rend. Sem. Mat. Univ. Politec. Torino* 1989, Special Issue, 47-62 (1990)

Anzellotti, G., Baldo, S., Percivale, D.: Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity. Asympt. Anal. **9**, 61–100 (1994)

Anzellotti, G., Ossanna, E.: Singular sets of convex bodies and surfaces with generalized curvatures. Manuscr. Math. **86**, 417–433 (1995)

Anzellotti, G., Serapioni, R., Tamanini, I.: Curvatures, functionals, currents, Indiana Univ. Math. J. **39**, 617–669 (1990)

Bhattacharya, K., James, R.D.: A theory of thin films of martensitic materials with applications to microactuators. J. Mech. Phys. Solids **47**, 531–576 (1999)

Carriero, M., Leaci, A., Tomarelli, F.: Plastic free discontinuities and special bounded Hessian. C. R. Acad. Sci. Paris **314**, 595–600 (1992)

Carriero, M., Leaci, A., Tomarelli, F.: Strong solution for an elastic-plastic plate. Calc. Var. **2**, 219–240 (1994)

Ciarlet, P.G.: Mathematical Elasticity II – Theory of Plates. Elsevier, Amsterdam (1997)

Ciarlet, P.G.: Mathematical modelling of linearly elastic shells. Acta Numer **10**, 103–214 (2001)

Ciarlet, P.G., Mardare, C.: Asymptotic justification of the intrinsic equations of Koiter's model of a linearly elastic shell [Justification asymptotique des équations intrinsèques du modèle de coques linéairement élastiques de Koiter]. Comp. Rendus Math. **357**, 99–110 (2019)

Ciarlet, P.G., Nečas, J.: Injectivity and self-contact in nonlinear elasticity. Arch. Rational Mech. Anal. **97**, 171–188 (1987)

Chinchaladze, N., Jaiani, G., Maistrenko, B., Podio-Guidugli, P.: Concentrated contact interactions in cuspidate prismatic shell-like bodies. Arch. Appl. Mech. **81**, 1487–1505 (2011)

Demengel, F.: Fonctions a Hessian Borné. Ann. Inst. Fourier **34**, 155–190 (1984)

Demengel, F.: Compactness theorem for spaces of functions with bounded derivatives and applications to limit analysis problems in plasticity. Arch. Rational Mech. Anal. **105**, 123–161 (1989)

Di Carlo, A., Podio-Guidugli, P., Williams, W.O.: Shells with thickness distension. Int. J. Solids Struct. **38**, 1201–1225 (2001)

Ericksen, J.L., Truesdell, C.A.: Exact theory of stress and strain in rods and shells. Arch. Rational Mech. Anal. **1**, 295–323 (1958)

Friesecke, G., James, R.D.: A scheme for the passage from atomic to continuum theory for thin films, nanotubes and nanorods. J. Mech. Phys. Solids **48**, 1519–1540 (2000)

Friesecke, G., James, R.D., Müller, S.: A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence. Arch. Rational Mech. Anal. **180**, 183–236 (2006)

Friesecke, G., James, R.D., Mora, M.G., Müller, S.: Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence. Compt. Rend. Math. **336**, 697–702 (2003)

Giaquinta, M., Modica, G., Souček, J.: Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.*, **106**, 97-159. [Erratum and addendum, *Arch. Rational Mech. Anal.*, (1990) **109**, 385-392.] (1989)

Giaquinta, M., Modica, G., Souček, J.: Cartesian Currents in the Calculus of Variations, vol. I. Springer-Verlag, Berlin (1998)

Giaquinta, M., Modica, G., Souček, J.: Cartesian Currents in the Calculus of Variations, vol. II. Springer-Verlag, Berlin (1998)

Hadamard, J.: Lecons sur la Propagation des Ondes et les Équations de l'Hydrodynamique. Hermann, Paris (1903)

Koiter, W.T.: A consistent first approximation in the general theory of thin elastic shells, in Proc. In: Koiter, W.T.E. (ed.) IUTAM Symposium on the Theory of Thin Elastic Shells, pp. 12–33. Delft, North-Holland, Amsterdam (1960)

Koiter, W.T.: On the nonlinear theory of thin elastic shells. Proc. K. Ned. Akad. Wet. B **69**, 1–54 (1966)

Le Dret, H., Raoult, A.: The membrane shell model in nonlinear elasticity: a variational asymptotic derivation. J. Nonlinear Sci. **5**, 48–73 (1996)

Le Dret, H., Raoult, A.: Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results. Arch. Rational Mech. Anal. **154**, 101–134 (2000)

Lembo, M., Podio-Guidugli, P.: How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body. Int. J. Solids Struct. **44**, 1337–1369 (2007)

Lewicka, M.: Quantitative immersability of Riemann metrics and the infinite hierarchy of prestrained shell models. Arch. Rational Mech. Anal. **236**, 1677–1707 (2020)

Lewicka, M., Müller, S.: The uniform Korn-Poincaré inequality in thin domains. Linégalité de Korn-Poincaré dans les domaines minces. Ann. Henri Poincaré C: Anal. Nonlineaire **38**, 443–469 (2011)

Mariano, P.M.: Equilibrium configurations of mixture thin films undergoing large strains. Math. Meth. Appl. Sci. **41**, 479–489 (2018)

Mariano, P.M., Mucci, D.: Micro-slip-induced multiplicative plasticity: existence of energy minimizers. Arch. Rational Mech. Anal. **43**(247) (2023)

Mariano, P.M., Mucci, D.: Crack nucleation in shells with through-the-thickness microstructure, SIAM. J. Math. Anal. **55**, 4977–4999 (2023)

Mariano, P.M., Mucci, D.: Equilibrium of thin shells under large strains without through-the-thickness shear and self-penetration of matter. Math. Eng. **5**(6), 1–21 (2023)

Mucci, D.: On the curvature energy of Cartesian surfaces. J. Geom. Anal. **31**, 8460–8519 (2021)

Mucci, D., Saracco, A.: Conical points of convex surfaces: generalized unit normal and angle defect. J. Convex Anal., (2024). (in print)

Müller, S., Olbermann, H.: Conical singularities in thin elastic sheets. Calc. Var. **49**, 1177–1186 (2014)

Neff, P.: $\Gamma$-convergence for a geometrically exact Cosserat shell-model of defective elastic crystals. CISM Courses and Lectures **516**, 301–361 (2010)

Olbermann, H.: Energy scaling law for the regular cone. J. Nonlinear Sci. **26**, 287–314 (2016)

Olbermann, H.: Energy scaling law for a single disclination in a thin elastic sheet. Arch. Rational Mech. Anal. **224**, 985–1019 (2017)

Olbermann, H.: The shape of low energy configurations of a thin elastic sheet with a single disclination. Anal. PDE **11**, 1285–1302 (2018)

Percivale, D.: Perfectly plastic plates: A variational definition. J. Reine Angew. Math. **411**, 39–50 (1990)

Percivale, D., Tomarelli, F.: From SBD to SBH: the elastic-plastic plate. Interf. Free Bound. **4**, 137–165 (2002)

Podio-Guidugli, P.: An exact derivation of the thin plate equation. J. Elasticity **22**, 121–133 (1989)

Reshetnyak, Y.G.: Weak convergence of completely additive vector measures on a set. Sibirsk. Mat. Zh. **9**, 1386–1394 (1968)

Saem, M.M., Ghiba, I.-D., Neff, P.: A geometrically nonlinear Cosserat (micropolar) curvy shell model via Gamma convergence. J. Nonlinear Sci. **33**, 70 (2023)

Saem, M.M., Bulgariu, E., Ghiba, I.-D., Neff, P.: Explicit formula for the $\Gamma$-convergence homogenised quadratic curvature energy in isotropic Cosserat shell models. Zeit. Angew. Math. Phys. - ZAMP **76**, 76 (2025)

Schmidt, B.: A derivation of continuum nonlinear plate theory from atomistic models, Multiscale Mod. Sim. **5**, 664–694 (2006)

Schmidt, B.: Plate theory for stressed heterogeneous multilayers of finite bending energy. J. Math. Pures Appl. **88**(107), 122 (2007)

Schmidt, B.: On the passage from atomic to continuum theory for thin films. Arch. Rational Mech. Anal. **190**, 1–55 (2008)

Simo, J.C., Fox, D.D.: On a stress resultant geometrically exact shell model. I. Formulation and optimal parametrization. Comput. Methods Appl. Mech. Engrg. **72**, 267–304 (1989)

Simo, J.C., Marsden, J.E., Krishnaprasad, P.S.: The Hamiltonian structure of nonlinear elasticity: The material and convective representation of solids, rods and plates. Arch. Rational Mech. Anal. **104**, 125–183 (1988)

Steigmann, D.J.: On the relationship between the Cosserat and Kirchhoff-Love theories of elastic shells. Math. Mech. Solids **4**, 275–288 (1999)

Steigmann, D.J.: Koiter's shell theory from the perspective of three-dimensional nonlinear elasticity. J. Elasticity **111**, 91–107 (2013)

Steigmann, D.J.: Equilibrium of elastic lattice shells. J. Eng. Math. **109**, 47–61 (2018)

Tomarelli, F.: (1992) Special Bounded Hessian and partial regularity of equilibrium for a plastic plate, in *Developments in PDE and Applications to Mathematical Physics*, Buttazzo, G., Galdi, G.P., Zanghirati, L. (Edt.s), Plenum Press, New York, 235-240

Villaggio, P.: Mathematical Models of Elastic Structures. Cambridge University Press, Cambridge, UK (1997)