Highlights

Revisiting the Love hypothesis for introducing dispersion of longitudinal waves in elastic rods

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- Variational derivation through the Love hypothesis leads to the Bishop-Love equation;
- This is not asymptotically equivalent to the Love equation;
- The Love hypothesis naturally emerges from a two-modal kinematics by multiscale analysis;
- This approach provides a correction term of the same order as that in the Love equation;
- The traditional ill-posedness coming from nonstandard boundary conditions is remedied.

Revisiting the Love hypothesis for introducing dispersion of longitudinal waves in elastic rods

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ARTICLE INFO

Keywords:
Love hypothesis
Dispersion
Multiscale
Logitudinal waves in rods

ABSTRACT

We re-examine the Love *equation*, which forms the first historical attempt at improving on the classical wave equation to encompass for dispersion of longitudinal waves in rods. Dispersion is introduced by accounting for lateral inertia through the Love *hypothesis*. Our aim is to provide a rigorous justification of the Love *hypothesis*, which may be generalized to other contexts. We show that the procedure by which the Love equation is traditionally derived is misleading: indeed, proper variational dealing of the Love hypothesis in a two-modal kinematics (the Mindlin-Herrmann system) leads to the Bishop-Love equation instead. The latter is not asymptotically equivalent to the Love equation, which is in fact a long wave low frequency approximation of the Pochhammer-Chree solution. However, the Love hypothesis may still be retrieved from the Mindlin-Herrmann system, by a slow-time perturbation process. In so doing, the linear KdV equation is retrieved. Besides, consistent approximation demands that a correction term be added to the classical Love *hypothesis*. Surprisingly, in the very special case of isotropic linear elasticity, this correction term produces no effect in the correction term of the Lagrangian, so that, to first order, the same Bishop-Love equation is the Euler-Lagrange equation corresponding to a family of Love-like hypotheses, all being different by the correction term. Remarkably, ill-posedness coming from non-standard (namely non static) natural boundary conditions is now amended.

1. Introduction

The theory describing propagation of longitudinal waves in elastic rods, based on the seminal works of D'Alambert, Bernoulli, Euler and Lagrange (Oliveira et al., 2020), retains great significance, both from the practical as well as from the theoretical standpoint. Modern non-destructive testing procedures are being developed which rely on a deeper understanding on the mechanics of wave propagation, also in connection with the idea of generalized continua (Nobili and Volpini, 2021). As it is well know, the celebrated wave equation represents the prototype for nondispersive phenomena, since it neglects any effect transversal to the direction of wave propagation. In this sense, the wave equation is perfectly unidirectional and it describes a rod with vanishingly thin cross-section. Rayleigh is credited as the first who came to recognize the importance of accounting for transversal effects (Rayleigh, 1894). Shortly later, elaborating on this idea, Love (1927) introduced what is now known as the Rayleigh-Love equation (sometimes simply the Love's equation, or, as in Hutchinson and Percival (1968), Love's modified wave equation), that describes dispersive longitudinal waves in thin elastic rods. This model, which represents the forefather of several successive attempts in the literature, accounts for dispersion through the Love *hypothesis*, which stipulates that *inertial* effects attached to the transversal motion of the cross-section are to be considered. It is important to emphasize that, as Love explicitly points out, only inertial effects are considered, while the elastic response remains unaltered (i.e. totally unidirectional). Indeed, following Hutchinson and Percival (1968), "Love's equation includes the radial inertia of the bar, which adds the effect of dispersion to the description of the wave phenomenon and allows the consideration of shorter wavelengths than does the simple wave equation". The resulting Love equation remains attractive for its simplicity and favorably compares with the exact solution developed by Pochhammer (1876) and, shortly later, independently, by Chree (1889), for a circular cross-section. Besides, Hutchinson and Percival (1968) offer experimental support for the capability of the Love equation to accurately describe the propagation of fundamental modes, while higher modes can be only interpreted through the Pochhammer-Chree solution. Yet, the Love equation cannot accommodate the boundary conditions on the free lateral surface of the rod and, in this sense, it should be regarded as an approximation,

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the limits of which should be well investigated. Nonetheless, the idea behind the Love hypothesis stands at the basis of several refined models of longitudinal wave propagation, a nice account of which may be found in Shatalov et al. (2011). Among these, the Mindlin-Hermann model (Mindlin, 1951) stands out because it represents the first attempt to develop, in a rigorous manner, a refined model starting from a restricted kinematics (here, the Euler-Bernoulli kinematics), through the so-called Kirchhoff method. More recently, the Love hypothesis has been applied outside the linear framework, whereto it properly belongs, to incorporate the lateral motion of the cross-section when developing nonlinear models. As a case in point, Samsonov (1994) considers the Murnaghan model for compressible materials and the Love hypothesis because it is "the first term of a transverse displacement expansion in a power series with respect to the small longitudinal strain [..] and remains valid for long waves, while one should consider the Herrmann and Mindlin model for a possible refinement of the correlation between longitudinal strain and transverse displacement". Accordingly, it is suggested that the Love hypothesis is the leading order term in a small strain expansion, whose refinement is the Mindlin-Herrmann model. Yet, this interpretation does not match the original idea developed by Love, which, instead, calls upon the Love hypothesis only for inertial effects. Besides, as we shall show in this paper, the Love hypothesis is the leading term approximation in a multiscale analysis of the Mindlin-Herrmann system, the latter being asymptotically different from the Love equation in the long wave regime. The same approach by which the Love hypothesis is carried over to the nonlinear framework is undertaken by Dai and Huo (2002), in analogy with Ostrovskii and Sutin (1977), Sørensen et al. (1984) and Clarkson et al. (1986). The Love hypothesis is again retrieved by Dai and Fan (2004) for incompressible elastic materials under finite cylindrical deformations from an asymptotic procedure in the small parameter given by the axial displacement h over the typical wavelength l. The same result is illustrated by Dai and Huo (2000) for compressible materials. As a workaround, Wright (1985) employs the incompressibility constraint to connect transversal and longitudinal motion without the need for the Love hypothesis, the incompatibility of the two assumptions being shown by Amendola and Saccomandi (2021). Furthermore, Samsonov et al. (1998) shows experimental results on soliton formation which support dependence on the cross-section geometry and therefore discourage the adoption of the Love hypothesis. It then appears that it is important to precisely frame the range of validity of the Love hypothesis and clear-cut its origin, so that its adoption and generalization may be rigorously justified. This is precisely the aim of this paper, which revisits the traditional derivation of the Love hypothesis in Sec.2 and then moves, in Sec.3, to illustrate how it also comes from a multiscale analysis of the Mindlin-Herrmann system. The corresponding variational principle is illustrated in Sec.4 and results are finally drawn in Sec.5.

2. Mathematical background

To obtain in a direct way an unimodal (i.e. encompassing a single dependent field) dispersive equation for longitudinal waves in rod, within the framework of linear elasticity, two approaches are possible. The first is connected to the derivation of the wave equation from a discrete lattice: We consider an infinite elastic chain of equidistant particles, with lattice spacing a, in equilibrium and acted upon by linear springs of identical stiffness. In the continuum limit, we justify a Taylor expansion with respect to a and, to second order, we obtain the classical dispersive linear wave equation: the Boussinesq's equation. This approach can be extend to the nonlinear setting in several ways (Maugin, 1999). The second possibility relies on the use the axiomatic theory of continuum mechanics for deducing a dispersive wave equation such as, for example, the Love *hypothesis*, which then leads to the Love *equation*.

The classical derivation of the Love equation (L) is contained in Section 278, Chapter XX, page 428, of the book by A.E.H.Love (1927). Essentially, the same derivation may be found in the book by Graff (Graff, 2012, §2.5.3, p.116) or by Miklowitz (Miklowitz, 1978, §7.1.1.2).

Let us briefly review this derivation, which often goes under the name of the *Love-Rayleigh rod theory*. To obtain the linear Love equation, the first step is to assume the *Navier–Bernouilli* (NB) hypothesis: during deformation, plane cross-sections remain planar and normal to the rod axis (Achenbach, 1973).

Let us consider a rod that, in a reference configuration, is a circular cylinder of radius A and let us introduce cylindrical coordinates in the current configuration $\boldsymbol{x} = r\boldsymbol{e}_r + \theta\boldsymbol{e}_\theta + z\boldsymbol{e}_z$ and, equally, cylindrical coordinates in the reference configuration $\boldsymbol{X} = R\boldsymbol{E}_R + \Theta\boldsymbol{E}_\Theta + Z\boldsymbol{E}_Z$, with $0 \leq R \leq A$. In this framework, the NB hypothesis consists of assuming the following axisymmetric time dependent two-modal motion (Wright, 1981, Eq.(12))

$$r = R + RU(Z,T), \quad \theta = \Theta, \quad z = Z + W(Z,T). \tag{1}$$

Indeed, displacement is described by a two-term powers series expansion in the radial coordinate of the axisymmetric problem. We assume that the determining equation for the functions $U(Z,T)$ and $W(Z,T)$ may be obtained directly as the Euler-Lagrange equations associated with the Lagrangian density $\mathcal{L} = \mathcal{T} - \mathcal{V}$. Here, the kinetic energy density per unit length is given by (Graff, 2012, Eq.(2.5.49))

$$\mathcal{T} = \int_0^A \int_0^{2\pi} \frac{\rho}{2} \left(W_T^2 + R^2 U_T^2 \right) R d\Theta d R, \tag{2}$$

where ρ is the mass density in the reference configuration. It is understood that a coordinate subscript implies differentiation with respect to the relevant variable, i.e. $W_T = \partial W / \partial T$. In the linear framework, the "potential energy" is obtained considering the isotropic strain-energy density per unit volume

$$\mathcal{W} = \mu \,\mathrm{tr}\,(\epsilon^2) + \frac{1}{2} \lambda \,\mathrm{tr}^2(\epsilon),$$

where μ and λ are the usual Lamé parameters and

$$\epsilon = \tfrac{1}{2} \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right], \tag{3}$$

is the infinitesimal strain tensor (by ∇ we denote the two-dimensional gradient operator). Here, $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ clearly denotes the displacement vector. Within the NB assumption, we have that

$$u_r(R, Z, t) = RU(Z, T), \quad u_\theta = 0, \quad u_z(Z, T) = W(Z, T), \tag{4}$$

whence the kinetic energy density is given by

$$\mathcal{T} = \frac{\pi A^2 \rho}{4} \left(2 W_T^2 + A^2 U_T^2 \right). \tag{5}$$

Similarly, we obtain

$$[\epsilon] = \begin{bmatrix} U & 0 & \tfrac{1}{2} R U_Z \\ 0 & U & 0 \\ \tfrac{1}{2} R U_Z & 0 & W_Z \end{bmatrix},$$

whereupon the strain-energy density easily follows

$$\mathcal{W} = \mu \left(2U^2 + W_Z^2 + \tfrac{1}{2} R^2 U_Z^2 \right) + \frac{1}{2} \lambda \left(2U + W_Z \right)^2. \tag{6}$$

Integrating over the cross section

$$\mathcal{V} = \int_0^A \int_0^{2\pi} \mathcal{W} R d\Theta d R, \tag{7}$$

we obtain the potential energy per unit length

$$\mathcal{V} = \pi A^2 \left[2(\mu + \lambda) U^2 + 2\lambda U W_Z + \left(\mu + \tfrac{1}{2} \lambda \right) W_Z^2 + \frac{\mu A^2}{4} U_Z^2 \right]. \tag{8}$$

Two Euler-Lagrange equations naturally emerge

$$\frac{\partial}{\partial T} \frac{\partial \mathcal{L}}{\partial W_T} + \frac{\partial}{\partial Z} \frac{\partial \mathcal{L}}{\partial W_Z} = 0, \quad \frac{\partial}{\partial T} \frac{\partial \mathcal{L}}{\partial U_T} + \frac{\partial}{\partial Z} \frac{\partial \mathcal{L}}{\partial U_Z} - \frac{\partial \mathcal{L}}{\partial U} = 0. \tag{9}$$

namely (see (Shatalov et al., 2011, Eq.(65)) or (Graff, 2012, §8.3.3) where, however, u is our U/A)

$$\begin{aligned} &(\lambda + 2\mu) W_{ZZ} + 2\lambda U_Z = \rho W_{TT}, \\ &\mu A^2 U_{ZZ} - 8(\lambda + \mu) U - 4\lambda W_Z = \rho A^2 U_{TT}. \end{aligned} \tag{10}$$

In the framework of linear elasticity, equations (10) have been first derived by Mindlin and Herrmann (MH) (Mindlin, 1951), whence this system is usually named after them. Once acknowledging for different dimensional reduction, Eqs.(10) correspond to (20) of Wright (1981), to which we refer for a nice discussion in terms of wave propagation. A critique of the MH-equations is presented in many papers, see for example Whiston (1986), on the ground that, for the assumed displacement (1), it is not possible to satisfy the load free condition at the mantle.

The Love (L-) *hypothesis* assumes a linear relationship between the radial displacement and the longitudinal strain i.e

$$U = -\nu_0 W_Z, \tag{11}$$

where

$$\nu_0 = \frac{\lambda}{2(\lambda+\mu)} = \frac{1}{2}\frac{\kappa^2-2}{\kappa^2-1},$$

is the Poisson's ratio and λ and μ are the Lamé parameters of linear elasticity. Here, we let $c_L = \sqrt{(\lambda+2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$, respectively the speed of longitudinal and shear body waves, alongside their ratio $\kappa = c_L/c_S = \sqrt{2(1-\nu_0)/(1-2\nu_0)}$. In the practical range $0 < \nu < \frac{1}{2}$, this ratio is always greater than $\sqrt{2}$ and it becomes unbounded for incompressible materials. Under (11), the dependence of $\mathcal{T}$ and $\mathcal{V}$ with respect to U, U_Z and U_T is replaced by dependence with respect to W_Z, W_{ZZ} and W_{ZT} instead. Hence, it is now $\mathcal{L}^* = \mathcal{L}^*(W_Z, W_T, W_{ZZ}, W_{ZT})$, with namely

$$\mathcal{L}^* = \frac{\pi A^2 \rho}{4}\left(2W_T^2 + A^2\nu_0^2 W_{ZT}^2\right) - \pi A^2\left\{\left[2(\lambda+\mu)\nu_0^2 - 2\lambda\nu_0 + \frac{\lambda}{2} + \mu\right] W_Z^2 + \frac{\mu A^2 \nu_0^2}{4} W_{ZZ}^2\right\}. \tag{12}$$

Carrying out the usual variational procedure we obtain the single partial differential equation describing the dynamics of the rod

$$W_{ZZ} + \frac{\nu_0^2 \mathrm{K}^2}{c_B^2}\left(W_{TTZZ} - c_S^2 W_{ZZZZ}\right) = \frac{W_{TT}}{c_B^2}, \tag{13}$$

where $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$ is Young's modulus, $\mathrm{K}^2 = A^2/2 = I_2/S$ is the (square of the) polar radius of gyration of the cross-section (whose area is S and whose polar moment of inertia is $I_2 = \pi A^4/2$) and we have let the beam longitudinal wavespeed

$$c_B = \sqrt{\frac{E}{\rho}} = c_s\sqrt{\frac{3\kappa^2-4}{\kappa^2-1}}.$$

Equation (13) is sometimes referred to as the Bishop-Love (BL) or the Rayleigh-Bishop equation (Shatalov et al., 2011). Love's equation, as it appears in (Love, 1927, §278) or in (Graff, 2012, Eq.(2.5.61)) or in (Hutchinson and Percival, 1968, Eq.(16)), is given by

$$W_{ZZ} + \frac{\nu_0^2 \mathrm{K}^2}{c_B^2} W_{ZZTT} = \frac{W_{TT}}{c_B^2}. \tag{14}$$

Clearly, with respect to (13), this equation misses the fourth space derivative W_{ZZZZ} and the reason for this discrepancy is that, in the literature, the potential energy considered for developing Love's equation (14) accounts for the elongation term only, as in (Graff, 2012, Eq.(2.5.49)), namely

$$\mathcal{W} = \frac{E}{2} W_Z^2, \tag{15}$$

in contrast to the exact full strain-energy density of linear isotropic elasticity (6). In fact, Love specifically points out, in his book, that transversal deformation is *considered only inasmuch as inertia effects are concerned*, statics being already encompassed by the use of the Young's modulus in (15). It would therefore seem as the Love equation emerges from a very special procedure, that is difficult to generalize.

It is possible to contrast the dispersion relation associated with Love's equation (14), namely

$$\omega^2 = c_B^2 k^2 \left(1 - \frac{\nu_0^2 \mathrm{K}^2}{c_B^2}\omega^2\right), \tag{16}$$

against the exact dispersion relation obtained independently by Pochhammer and Chree (P-C) from the three dimensional linear theory of elasticity, see (Graff, 2012, §8.2.2) or (Shatalov et al., 2011, Eq.(96)), i.e.

$$J_1(A\beta)\left(\frac{2\alpha\left(\beta^2+k^2\right)J_1(A\alpha)}{A\beta}-\frac{\left(k^2-\beta^2\right)^2J_0(A\alpha)}{\beta}\right)-4\alpha k^2J_1(A\alpha)J_0(A\beta)=0, \tag{17}$$

where $J_n(k)$ are Bessel's function of the first kind (Abramowitz and Stegun, 1948, §9). Also, k is the wavenumber along Z, ω is the angular frequency and

$$\alpha=\sqrt{\frac{\omega^2}{c_L^2}-k^2},\quad \beta=\sqrt{\frac{\omega^2}{c_s^2}-k^2},$$

are the wavenumbers in the radial direction (for irrotational and solenoidal waves, respectively). Indeed, the Love equation naturally emerges by taking a regular asymptotic expansion in the radius A of the first branch of the P-C solution (Love, 1927, §201), namely

$$\omega^2=c_B^2\left[k^2+\tfrac{1}{2}\mathrm{K}^2\left(k^4-\frac{\left(6\kappa^4-3\kappa^2-4\right)}{2c_B^2\kappa^2\left(\kappa^2-1\right)}k^2\omega^2+\frac{\left(\kappa^4+\kappa^2-1\right)}{2c_B^2\kappa^2\left(\kappa^2-1\right)c_S^2}\omega^4\right)\right]+O(A^3), \tag{18}$$

and then plugging into the correction term the leading order equation $\omega^2=c_B^2k^2$. Extending the analysis to the first correction in the speed, we get

$$c_B^2k^2=\omega^2\left(1+\tfrac{1}{2}\mathrm{K}^2\frac{\nu_0^2}{c_B^2}\omega^2\right)^2, \tag{19}$$

that reduces to Love's equation provided that $k\sim\omega\ll 1$, that is in the LWLF regime. Furthermore, looking at the eigenform, we get

$$U=-\nu_0W_Z+\frac{\nu_0\mathrm{K}^2}{\kappa^2-1}\left(\frac{\nu_0c_S^2}{c_B^2(c_B^2/c_S^2-1)}+\frac{\kappa^4-2\kappa^2+2}{4\kappa^2}\frac{R^2}{A^2}\right)W_{ZZZ}+O(A^3), \tag{20}$$

where the leading order term is precisely the Love hypothesis, while the $O(A^2)$ correction reveals a $O(R^2)$ contribution which embodies the deviation from the plane cross-section assumption.

Similarly to Figure 2.27 of (Graff, 2012, p.120) or to Figure 1 of (Shatalov et al., 2011, p.208), this comparison is illustrated in Fig.1 in terms of the frequency spectrum, and in Figs.2,3 in terms of the dispersion diagram. Dimensionless wavenumber, frequency and wavespeed have been introduced in analogy with Graff (2012)

$$\bar{k}=(2\pi)^{-1}kA,\quad \bar{\omega}=\omega A/c_s,\quad \bar{c}=\frac{\omega}{kc_B}=\frac{c}{c_B},$$

and so is the parameter value $\nu=0.29$. These Figures show the curves from the P-C (17), Love (14), Bishop-Love (13) and MH (10) models. It is clear that all equations are good low-frequency long-wavelength (LFLW) approximations of the first branch of the P-C solution, which, however, fail already beyond small wavenumbers. Besides, Fig.3 reveals that, as anticipated, the Love model best captures the LWLF regime. In contrast, the two mode MH-system and the BL model provide a qualitatively accurate picture for large wavenumbers, given that they both plateauGraff (2012). Of course, all such models are doomed to fail for it is known that, to obtain a good approximation of the exact solution, at least four modes in the Taylor expansion of the axisymmetric deformation field needs to be considered Shatalov et al. (2011).

Despite its shortcomings, the Love hypothesis is widely used and not only within the framework of the linear theory of elasticity, whereto it properly belongs, but also in the nonlinear setting, see, for instance Dai and Fan (2004), and references therein, or Ostrovskii and Sutin (1977), Sørensen et al. (1984), where, only apparently, a more general approach is taken.

The aim of this paper is to provide a rigorous re-examination and justification of the Love hypothesis, moving from the Navier-Bernouilli approximation, which may be easily extended to the nonlinear regime. This process will lead to a *refined* Love hypothesis, whose merits will be apparent.

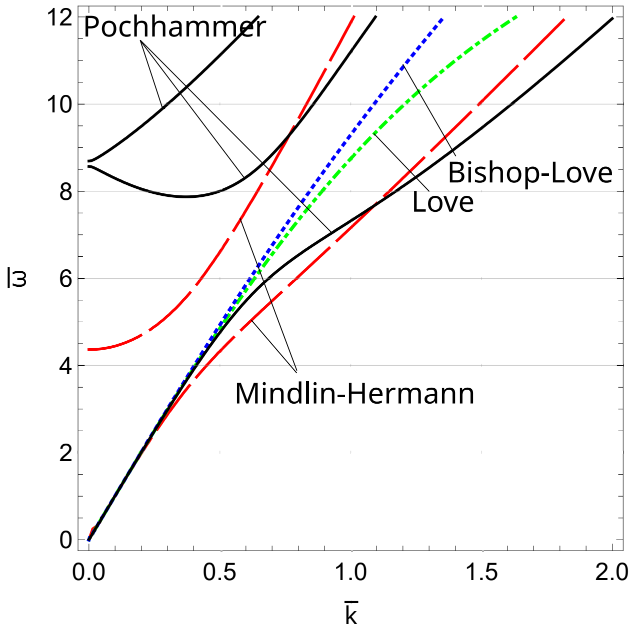

Figure 1: Dimensionless P-C frequency spectrum (black, solid) compared to that obtained from the Love (green, dash-dot), Bishop-Love (blue, dotted) and Mindlin-Herrmann (red, dashed) models ($\nu = 0.29$, $\delta = 0.2$).

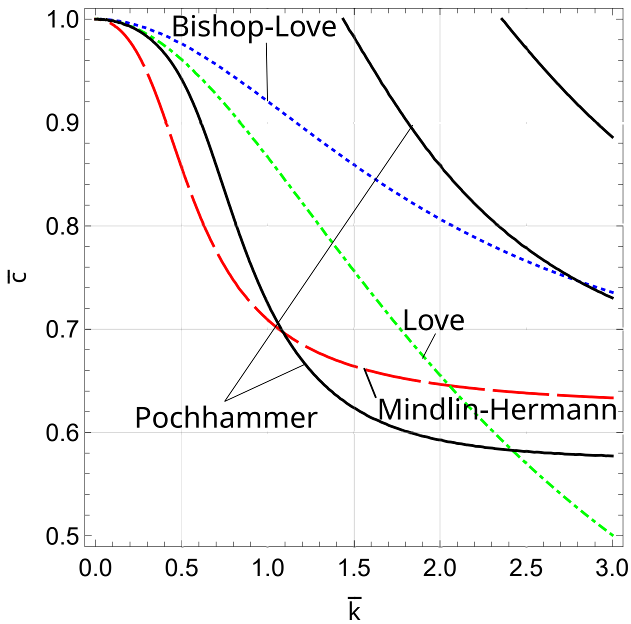

Figure 2: Dimensionless P-C dispersion diagram (black, solid) compared to that obtained from the Love (green, dash-dot), Bishop-Love (blue, dotted) and Mindlin-Herrmann (red, dashed) model ($\nu = 0.29$, $\delta = 0.2$).

3. From the Mindlin-Herrmann system to Love equation

We begin by considering the relationship among the L-equation and the MH-system. For this, we introduce the dimensionless coordinate $\zeta = Z/l$ and the time scale $\mathfrak{T} = l/c$, where l is any characteristic length, such as the rod length, and c is any speed, which we choose to be $c = c_S$ for convenience. The dimensionless time is therefore

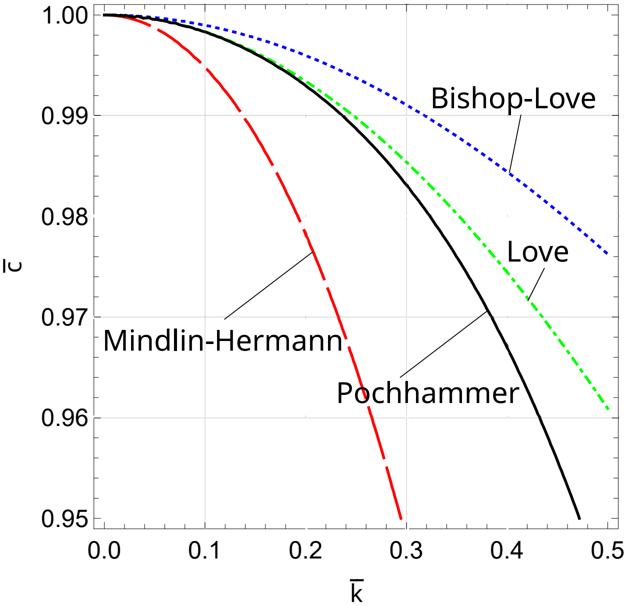

Figure 3: Low-frequency long-wave range in the P-C dispersion diagram (black, solid) compared to that obtained from the Love (green, dash-dot), Bishop-Love (blue, dotted) and Mindlin-Herrmann (red, dashed) model ($\nu = 0.29$, $\delta = 0.2$).

$t = T/\mathfrak{T}$. We also set $\delta = A^2/l^2 \ll 1$ and $(U, W) = (u, lw)$. In dimensionless form, equations (10) read

$$4\left(\kappa^2 - 1\right) u + 2\left(\kappa^2 - 2\right) w_\zeta - \tfrac{1}{2}\delta\left(u_{\zeta\zeta} - u_{tt}\right) = 0, \tag{21a}$$

$$2\left(\kappa^2 - 2\right) u_\zeta + \kappa^2 w_{\zeta\zeta} - w_{tt} = 0. \tag{21b}$$

By solving (21b), we obtain

$$u_\zeta = -\frac{\kappa^2}{2(\kappa^2 - 2)} w_{\zeta\zeta} + \frac{1}{2(\kappa^2 - 2)} w_{tt}, \tag{22}$$

and plugging this into Eq.(21b), differentiated with respect to ζ, we obtain a single partial differential equation in terms of w

$$\frac{c_B^2}{c_S^2} w_{\zeta\zeta} - \frac{\delta}{8(\kappa^2 - 1)}\left(\partial_{\zeta\zeta} - \partial_{tt}\right)\left(\kappa^2 \partial_{\zeta\zeta} - \partial_{tt}\right) w = w_{tt}, \tag{23}$$

where c_B is the speed of elongation waves in rods. Eq.(23) may be recast in terms of dimensional variables

$$W_{ZZ} - \frac{\mathrm{K}^2 c_S^2}{4c_B^2(\kappa^2 - 1)}\left(\partial_{ZZ} - c_S^{-2}\partial_{TT}\right)\left(\kappa^2 \partial_{ZZ} - c_S^{-2}\partial_{TT}\right) W = c_B^{-2} W_{TT}. \tag{24}$$

This equation has already been determined in Wright (1981) and it may be rewritten as

$$W_{ZZ} + \frac{\kappa^4 - 1}{(\kappa^2 - 2)^2}\frac{\nu_0^2 \mathrm{K}^2}{c_B^2} W_{ZZTT} - \frac{\mathrm{K}^2 c_S^2}{4c_B^2(\kappa^2 - 1)}\left(\kappa^2 W_{ZZZZ} + c_S^{-4} W_{TTTT}\right) = c_B^{-2} W_{TT}, \tag{25}$$

so that it can be easily compared with Love's equation (13) and with the Bishop-Love equation (14). It is easily seen that they do not correspond, even in the static case. It is concluded that the unimodal problem emerging from the MH model does not reduce to either Love or Bishop-Love model. However, a hunch to a possible connection is obtained by solving (21a), whence

$$u = -\nu_0 w_\zeta - \delta \frac{1}{8(\kappa^2 - 1)}\left(u_{tt} - u_{\zeta\zeta}\right). \tag{26}$$

To leading order, this is indeed Love's hypothesis (11), which is then supplemented by a $O(\delta)$ correction term proportional to the deviation from the shear wave speed. Indeed, such refined Love hypothesis will come naturally in the next Section from a multiscale analysis.

3.1. Slow-time perturbation

In a multiscale approach, we introduce the slow time $\tau = \delta t$ and the moving coordinate $\xi = \zeta - \hat{c}t$, hence the dimensionless speed of the moving coordinate system is $\hat{c}$,

$$u(\zeta, t) = \phi(\xi, \tau), \quad w(\zeta, t) = \psi(\xi, \tau).$$

Thus, the MH-system (21) becomes

$$-4\left(\kappa^2 - 1\right)\phi + \left(4 - 2\kappa^2\right)\psi_\xi - \tfrac{1}{2}\left(\hat{c}^2 - 1\right)\delta\phi_{\xi\xi} + \hat{c}\delta^2\phi_{\xi\tau} - \tfrac{1}{2}\delta^3\phi_{\tau\tau} = 0, \tag{27a}$$

$$\left(\kappa^2 - \hat{c}^2\right)\psi_{\xi\xi} + 2\left(\kappa^2 - 2\right)\phi_\xi + 2\hat{c}\delta\psi_{\xi\tau} - \delta^2\psi_{\tau\tau} = 0, \tag{27b}$$

whose solution is sought in asymptotic series

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \dots, \quad \psi = \psi_0 + \delta\psi_1 + \delta^2\psi_2 + \dots. \tag{28}$$

To leading order, we obtain the compatibility condition

$$\hat{c} = \pm c_B / c_S, \tag{29}$$

whence the dimensional moving frame speed $\hat{c}c_S$ equals c_B, that is the longitudinal wavespeed in rods. Also, we get

$$\phi_0 = -\nu_0 \psi_{0\,\xi}, \tag{30}$$

which is the leading term in Love's hypothesis (11). Carrying on the analysis, we find that Love's hypothesis is *refined* up to $O(\delta)$ terms through

$$\phi_1 = -\nu_0\left(\psi_{1\xi} - \delta\frac{c_B^2/c_S^2 - 1}{8(\kappa^2 - 1)}\psi_{0\xi\xi\xi}\right) + O(\delta^2). \tag{31}$$

Besides, we obtain the governing equation for the travelling disturbance,

$$\frac{c_B}{c_S}\psi_{0\xi\tau} + \frac{\nu_0^2}{4}\left(\frac{c_B^2}{c_S^2} - 1\right)\psi_{0\xi\xi\xi\xi} = 0. \tag{32}$$

Integrating in ξ and to leading order, Eq.(39) reduces to the well-known linear KdV equation, whose nonlinear form is similarly obtained by Dai and Fan (2004). This equation lends the time evolution of the longitudinal wave profile and, as it is well know, dispersion is introduced by the term W_{ZZZ}. Consequently, dispersion appears through $(1 - c_S^2/c_B^2)$ whereby the wave profile rests unchanged (hence no dispersion), inasmuch as $c_B \approx c_S$, that is bulk shear waves move with a speed close to that of longitudinal waves in the rod. This is never possible and the closer we can get is (for ordinary materials) for $\kappa \to \sqrt{2}$, so that $1 - c_S^2/c_B^2 \to \frac{1}{2}$. In contrast, when κ is extremely large, we get maximum dispersion for $1 - c_S^2/c_B^2 \to 2/3$.Accounting for (32), we can rewrite (30,31) consistently up to $O(\delta)$ in the form

$$\phi_\xi = -\nu_0\psi_{\xi\xi} - \delta\frac{c_B}{c_S(\kappa^2 - 2)}\psi_{\xi\tau} + O(\delta^2), \tag{33}$$

which introduces an inertia-like correction.

Moving back to the original dimensionless variables, we have, to leading order,

$$\partial_{\xi\tau} = \frac{1}{2}\frac{c_B}{c_S\delta}\left(\partial_{\zeta\zeta} - \frac{c_S^2}{c_B^2}\partial_{tt}\right)$$

whence Eq.(32) reads

$$\frac{c_B^2}{c_S^2}w_{\zeta\zeta} - w_{tt} + \tfrac{1}{2}\delta\nu_0^2\left(\frac{c_B^2}{c_S^2} - 1\right)w_{\zeta\zeta\zeta\zeta} = 0, \tag{34}$$

that is resemblant of Eq.(23). Besides, plugging the leading term of (34) into the correction term, two equivalent forms are obtained, namely

$$\frac{c_B^2}{c_S^2} w_{\zeta\zeta} - w_{tt} + \tfrac{1}{2}\delta v_0^2 \left(w_{\zeta\zeta tt} - w_{\zeta\zeta\zeta\zeta} \right) = 0, \tag{35}$$

and also

$$\frac{c_B^2}{c_S^2} w_{\zeta\zeta} - w_{tt} + \tfrac{1}{2}\delta v_0^2 \left(1 - \frac{c_S^2}{c_B^2} \right) w_{\zeta\zeta tt} = 0. \tag{36}$$

Likewise, Eq.(33) becomes

$$u_\zeta = -v_0 w_{\zeta\zeta} - \frac{1}{2(\kappa^2-2)} \left(\frac{c_B^2}{c_S^2} w_{\zeta\zeta} - w_{tt} \right), \tag{37}$$

whose structure resembles that of (26). However, this form is not very attractive because it cannot be integrated with respect to ζ and it fails to bring out the $O(\delta)$ nature of the correction. Naturally, an asymptotically equivalent form is readily obtained through plugging the leading term of (34) into the correction term of (37) and integrating

$$u = -v_0 w_\zeta + \frac{\delta v_0^2}{4(\kappa^2-2)} \left(\frac{c_B^2}{c_S^2} - 1 \right) w_{\zeta\zeta\zeta}. \tag{38}$$

In terms of dimensional variables, Eq.(34) lends

$$W_{ZZ} - c_B^{-2} W_{TT} + v_0^2 \mathrm{K}^2 \left(1 - \frac{c_S^2}{c_B^2} \right) W_{ZZZZ} = 0, \tag{39}$$

while Eqs.(35,36), read, respectively,

$$W_{ZZ} - c_B^{-2} W_{TT} + c_B^{-2} v_0^2 \mathrm{K}^2 \left(W_{TTZZ} - c_S^2 W_{ZZZZ} \right) = 0. \tag{40}$$

and

$$W_{ZZ} - c_B^{-2} W_{TT} + c_B^{-2} v_0^2 \mathrm{K}^2 \left(1 - \frac{c_S^2}{c_B^2} \right) W_{TTZZ} = 0. \tag{41}$$

Clearly, Eq.(40) corresponds to the Bishop-Love model (13), while Eq.(41) is the refined Love equation, the difference with (14) being given by the term in round brackets. As expected, all these models coincide to leading order, and in fact they collapse onto the leading order term in the P-C solution. Furthermore, as it may be physically anticipated, dispersion, regardless of the differential form it takes, always appears as a function of the relative mismatch between the speed of longitudinal and radial waves through the factor $c_B^2/c_S^2 - 1$. This feature is missing from the Love model because of the fulfillment of the boundary conditions in the P-C model, wherefrom it ultimately comes.

Consideration of the refined Love hypothesis (38) in dimensional form gives

$$U = -v_0 W_Z + \frac{\mathrm{K}^2 v_0^2}{2(\kappa^2-2)} \left(\frac{c_B^2}{c_S^2} - 1 \right) W_{ZZZ}, \tag{42}$$

that is similar to the first order eigenform (20), provided that complete correspondence is impossible given that the cubic dependence on the radius is not accessible within the NB kinematics (4). In fact, it is precisely the refined Love hypothesis that allowed us to guess the form of the correction term in the P-C solution as a multiple of W_{ZZZ}.

Also, returning to dimensional variables in the asymptotic series (28), we have

$$W = W_0 + A^2 W_1 + O(A^3), \tag{43a}$$

$$U = -v_0 W_Z + \frac{\mathrm{K}^2 v_0^2}{2(\kappa^2-2)} \left(\frac{c_B^2}{c_S^2} - 1 \right) W_{ZZZ} + O(A^3), \tag{43b}$$

which reminds of the refined assumption introduced in (Porubov and Samsonov, 1993, Eq.(1)), that we rewrite in our symbols,

$$W = W_0 + R^2 W_1 + O(R^3), \tag{44a}$$

$$U = -\nu_0 W_{0Z} + R^2 U_1 + O(R^3). \tag{44b}$$

This structure comes from introducing higher order terms in R and, in fact, it contains three unknown functions of Z and T (i.e. it is tri-modal), two of which, namely W_1 and U_1, may be used to accommodate the zero boundary conditions on the radial stress at the mantle (the zero hoop stress BC being trivially satisfied from the kinematics) Porubov and Samsonov (1993),

$$W_1 = \tfrac{1}{2}\nu_0 W_{0ZZ} \quad U_1 = -\frac{\nu_0^2}{2(3-2\nu_0)} W_{0ZZZ}. \tag{45}$$

However, we have already seen in (20) that this solution form cannot fully represent the P-C solution, because it misses out the correction term in the form $A^2 W_{0ZZZ}$, which is of the same order as (if not bigger than) the $R^2 U_1$ contribution. In this respect, the expression for U_1 in (45) may be seen as complementary to our refined assumption (42), which instead provides only the $O(A^2)$ part of the correction. Yet, we point out that, following the multiscale analysis, the refined Love hypothesis provides an expansion for u that is now consistent up to $O(\delta)$ terms, within the NB kinematics. How this affects the Lagrangian (12), in comparison with the original Love assumption, is now discussed.

4. Unimodal refined variational model

It was shown that the variational procedure by which the Love equation is usually obtained, which makes use of the Love hypothesis, really lends the BL equation instead (and this is because the Love hypothesis is meant for the kinetic term only). The latter is not asymptotically equivalent to the Love equation, for it lacks the factor $1 - c_S^2/c_B^2$ in the correction term. One would therefore be lead to believe that the Lagrangian (12) is accurate only to leading order, given that it was obtained by using the Love hypothesis, which lacks the correction term for U, i.e. U is only correct to $O(1)$. However, it turns out that this is not the case and in fact the Lagrangian (12) is accurate up to $O(\delta)$ terms *regardless of the correction to the Love hypothesis*. In fact, we may say that the Lagrangian (12) accurately represents, up to $O(\delta)$, a *family of Love-like assumptions*, which all differ by the correction term for U. This outcome follows from the fact that, looking at (5,8), we see that the only terms where the correction to U appears are given by the first and by the second term in the potential energy (8). However, it can be easily seen that, for any Love-like assumption, their total contribution vanishes up to $O(\delta)$. Still, it should be emphasized that this cancellation seems entirely accidental and it no longer takes place when, say, nonlinearity is taken into account.

To show that this is in fact the case, in the isotropic linear framework, we introduce the refined Love hypothesis (33) into the system kinetic energy density

$$\mathcal{T} = \tfrac{1}{2}w_t^2 + \tfrac{1}{4}\delta\nu_0^2 w_{\zeta t}^2 - \tfrac{1}{2}\delta^2 \frac{\nu_0^2}{16(\kappa^2-1)}\left(\frac{c_B^2}{c_S^2} - 1\right) w_{\zeta t} w_{\zeta\zeta\zeta t} + O\left(\delta^3\right), \tag{46}$$

as well as into the potential energy,

$$\mathcal{V} = \tfrac{1}{2}\frac{c_B^2}{c_S^2} w_\zeta^2 + \tfrac{1}{4}\delta\nu_0^2 w_{\zeta\zeta}^2 + \delta^2 \frac{\nu_0^2}{16\left(\kappa^2-1\right)}\left(\frac{c_B^2}{c_S^2}-1\right)\left(-w_{\zeta\zeta}w_{\zeta\zeta\zeta\zeta} + \tfrac{1}{2}\left(\frac{c_B^2}{c_S^2}-1\right) w_{\zeta\zeta\zeta}^2\right) + O\left(\delta^3\right). \tag{47}$$

The Lagrangian density immediately follows

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}\frac{c_B^2}{c_S^2} w_\zeta^2 + \frac{w_t^2}{2} + \frac{1}{4}\delta\nu_0^2\left(w_{\zeta t}^2 - w_{\zeta\zeta}^2\right)\\ &+ \delta^2 \frac{\nu_0^2}{16\left(\kappa^2-1\right)}\left(\frac{c_B^2}{c_S^2}-1\right)\left[-w_{\zeta t}w_{\zeta\zeta\zeta t} + w_{\zeta\zeta}w_{\zeta\zeta\zeta\zeta} - \tfrac{1}{2}\left(\frac{c_B^2}{c_S^2}-1\right) w_{\zeta\zeta\zeta}^2\right] + O(\delta^3),\end{aligned} \tag{48}$$

and, as anticipated, it corresponds to Love Lagrangian up to $O(\delta)$ terms. Naturally, the E-L equation is the BL equation up to $O(\delta)$ terms

$$-w_{tt}+\frac{c_B^2}{c_S^2}w_{\zeta\zeta}+\tfrac{1}{2}\delta v_0^2\left(w_{tt}-w_{\zeta\zeta}\right)_{\zeta\zeta}+\delta^2\frac{v_0^2}{16(\kappa^2-1)}\left(\frac{c_B^2}{c_S^2}-1\right)\left[\left(\frac{c_B^2}{c_S^2}-1\right)w_{\zeta\zeta}-2w_{tt}\right]_{\zeta\zeta\zeta\zeta}=0. \tag{49}$$

The natural boundary conditions holding at the rod ends, up to $O(\delta)$, read

$$\left(\frac{c_B^2}{c_S^2}w_\zeta+\tfrac{1}{2}\delta v_0^2 w_{\zeta tt}\right)\mathfrak{d}w+\tfrac{1}{2}\delta v_0^2\left(w_{\zeta\zeta}-w_{\zeta\zeta\zeta}\right)\mathfrak{d}w_\zeta=0, \tag{50}$$

where $\mathfrak{d}$ is the variation symbol, to avoid confusion with the small parameter δ. Such conditions should be compared with (Miklowitz, 1978, Eq.(7.23)) and we point out that the contribution of the strain variation $\mathfrak{d}w_\zeta$ is there missing, owing to the improper dealing of the strain energy. In particular, following Miklowitz (1978) to which we refer for details, we observe that the dual condition on $\mathfrak{d}w$ takes on an unexpected dynamic form which may lead to ill posedness. However, using (34), we can equally write

$$\frac{c_B^2}{c_S^2}\left(w_\zeta+\tfrac{1}{2}\delta v_0^2 w_{\zeta\zeta\zeta}\right)\mathfrak{d}w+\tfrac{1}{2}\delta v_0^2\left(w_{\zeta\zeta}-w_{\zeta\zeta\zeta}\right)\mathfrak{d}w_\zeta=0, \tag{51}$$

which no longer suffers from such drawback.

4.1. Quality of the approximation

As we have already observed, the Love equation provides the best LWLF approximation to P-C and it cannot be surpassed. However, it is interesting to investigate how well the linear KdV, in either of the forms (34), (35) and (36), approximates the problem. Fig.4 shows the frequency spectrum for the P-C solution, alongside the Love, BL, and the KdV (39,41) approximations. As already discussed, from a multiscale perspective, three models are equivalent and correspond to the Love equation only to leading order. The frequency spectra are shown in Fig.5. All models, except MH (not shown) and BL, fail to reproduce the flattening out of the frequency spectrum, that asymptotes to the Rayleigh speed. The KdV (36) appears very similar to the Love model.

5. Concluding Remarks

A seemingly natural way to accommodate for dispersion in the equation for longitudinal waves in thin elastic rods, originally introduced by A.H. Love, consists of accounting for the transversal motion of the cross-section. In particular, the Love hypothesis relates the transversal to the longitudinal strain in the rod through Poisson's ratio. Usually, this hypothesis is introduced in a two-modal kinematics connected to the Navier–Bernoulli assumption of plane cross-sections remaining plane after deformation. The Love equation, encompassing dispersion, is finally arrived at by Hamilton's principle. Although this derivation appears in every classical textbook, its examination reveals that proper dealing with the elastic energy leads instead to a variant of the Love equation, sometimes named the Bishop-Love equation. Indeed, we show that the Love equation is most simply obtained from the regular expansion of the Pochhammer-Chree (P-C) frequency equation for longitudinal waves in cylindrical elastic rods, assuming the cylinder radius A to be small. More specifically, the Love equation is merely a long-wave low-frequency approximation of this solution and it matches the Bishop-Love equation only to leading order. This fact becomes important when dealing with the nonlinear extension of this approach, which equally moves from a two-modal kinematics, in the absence of a general solution like P-C to approximate. In the linear case, this leads to the well-known Mindlin-Herrman system of equations. In the nonlinear case, the resulting system is often very complicated and extra assumptions are needed to make progress. These assumptions often take the form of the Love hypothesis. One therefore wonders if this approach is at least well founded in the linear case. Indeed, we show that the Love hypothesis may also be derived from a slow-time perturbation of the Mindlin-Herrman system. In the process, the governing equation of the longitudinal perturbation is arrived at, namely the linear KdV. These results suggest a rigorous method to generalize the Love hypothesis in more general settings. Besides, it is shown that, already in the linear case, a correction to the Love hypothesis is demanded to achieve consistency (with the accuracy of the longitudinal motion), and this correction is in fact proportional to the

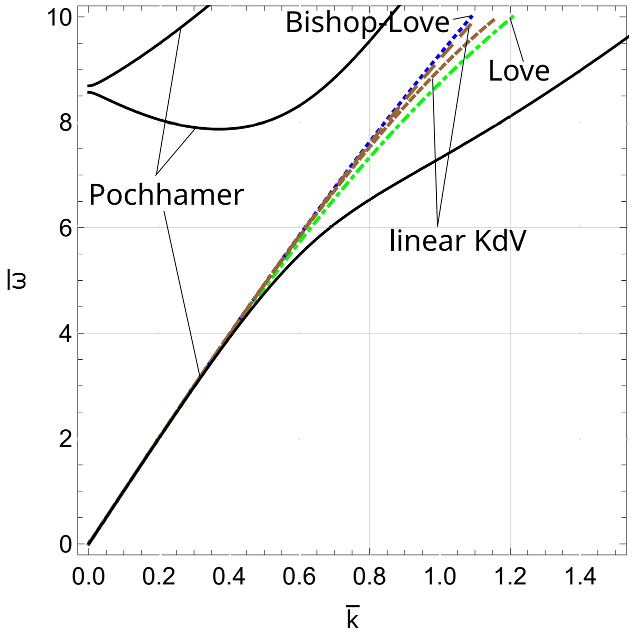

Figure 4: Frequency spectrum for the P-C (black, solid), Love (green, dash-dotted), Bishop-Love (blue, dotted) and the linear KdV (39) (brown, short-dashed) and (41) (brown, long-dashed) for $\nu_0 = 0.29$ and $\delta = 0.2$

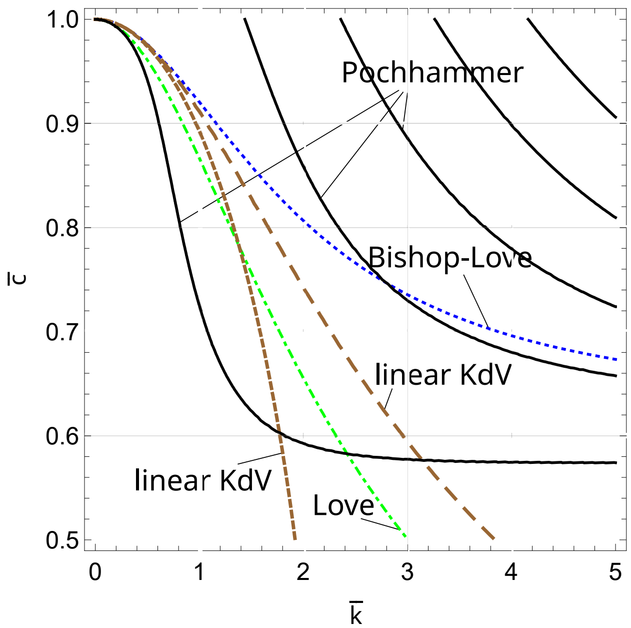

Figure 5: Dispersion diagram for the Pochhammer (black, solid), MH (red, dashed), Love (green, dash-dotted), Bishop-Love (blue) and for the refined model (39) (pink, solid) for $\nu_0 = 0.29$ and $\delta = 0.2$

second derivative of the rod longitudinal strain, i.e. W_{ZZZ}. This dependence of the correction on W_{ZZZ} is also met when expanding, for A small, the P-C eigenform, although this also brings out cubic terms in the radial coordinate $R \leq A$, which are not accessible within our plane cross-section hypothesis. Also, the same form for the correction term already appears in the literature, although for completely different reasons, namely in an attempt to enrich the kinematics to tri-modal and therefore be able to meet two boundary conditions on the mantle, instead of the usual one. Interestingly, in the linear isotropic framework (and in a two-modal kinematics), this correction term for transversal strain may be taken freely, yet retaining a Lagrangian that is consistent to the first correction terms (and likewise for the longitudinal motion). This surprising outcome results from cancellations in the Lagrangian, and in fact we may equally define the Love hypothesis as the assumption through which first order terms in the transversal strain do not affect the Lagrangian first correction. However, in the general case, the original Love hypothesis is not accurate enough, even in the linear case, and should be refined. Besides, slow-time perturbation lends static boundary conditions which no longer cause stress-type problems to be ill-posed. The application of this approach to non-linear scenarios will form the basis for future work.

6. Acknowledgments

This research was supported by the Italian National Group of Mathematical Physics (GNFM), within the Institute of Higher Mathematics (INDAM). AN acknowledges support from the project "Mathematics for Industry 4.0 (Math4I4)", PRIN 2020F3NCPX. He is also grateful for support by the European Union's Horizon 2020 MSCA-RISE-2020, Research and Innovation Staff Exchange programme, grant agreement No 101008140, EffectFact, within the H2020-EU.1.3. - EXCELLENT SCIENCE - Marie Skłodowska-Curie Actions Main Programme. GS is grateful for the support under the project PRIN 2022P5R22A.

Conflict of Interest statement

Authors have no conflict of interest to declare.

Data Availability

This paper makes use of no data.

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