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On Generalized Varga Materials

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Abstract

We study the existence and regularity of minimizers of an energy functional which in the physical 3D dimension corresponds to the so–called generalized Varga materials and includes an additional term accounting for surface tension. Due to the linear growth of the strain energy, we relax the problem in a suitable class of extended graphs of radially symmetric functions of bounded variations. Besides cavitation at the origin, a new phenomenon due to the occurrence of a spherical fracture inside the body is observed.

Keywords Generalized Varga materials · Radially symmetric deformations · Functions of bounded variations · Cavitation and fractures

Mathematics Subject Classification 49J10 · 49J45 · 74G65

1 Introduction

In this paper we provide an analytic justification for the occurrence of nucleation of radially symmetric minimizers of materials subject to a strain energy that in the physical dimension $N = 3$ describes the so called generalized Varga materials.

Our analysis shows also that these materials may exhibit a new phenomenon: due to the linear growth of the volume term, besides nucleation, energy minimizers may present a fracture at a positive radius.

The general framework we consider deals with the minimization of the energy

$$E(u) = \int_{B^N} W(\nabla u(x))\, dx \tag{1}$$

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among (sufficiently smooth) radially symmetric deformation $u\colon B^N \to \mathbb{R}^N$

$$u(x) = v(|x|)\,\frac{x}{|x|}, \qquad 0 < |x| < 1, \tag{2}$$

of the reference configuration B^N, the open unit ball of $\mathbb{R}^N$ centered at the origin, subject to the boundary displacement condition $u(x) = \lambda x$ for $|x| = 1$ with $\lambda \geq 1$.

The radially symmetric deformation u given by (2) is uniquely determined by its *profile function* $v\colon (0, 1) \to \mathbb{R}$ which is assumed to be strictly increasing and to satisfy $0 \leq v(0) < v(1) = \lambda$ where, here and in the sequel, $v(0) = v(0^+)$ and $v(1) = v(1^-)$ are the right and left limits at the endpoints of the interval $(0, 1)$.

The existence of radial minimizers u of E with profile function v such that $v(0) > 0$ for large enough displacement λ corresponds to the occurrence of a spherical fracture – a *cavity* – inside the body. This behaviour is actually observed in experiments with elastomers, see for instance [10, 15] and [16], and the existence of minimizers u with cavity has been widely studied starting from J. Ball's seminal paper [3], see also [17, 20, 22, 23] and more recently [19]. See also [7] for an analysis of cavitation in the nonconvex setting of the so–called Blatz–Ko elastic materials.

Throughout the paper we assume that the strain energy density is an isotropic function $W\colon \mathbb{M}^{N\times N}_+ \to \mathbb{R}$ defined on the set $\mathbb{M}^{N\times N}_+$ of matrices with positive determinant such that $W(A) \to +\infty$ as $\det A \to 0^+$ or $\det A \to +\infty$. Therefore, W can be written as a symmetric function of the singular values $\lambda_n(A)$ $(n = 1, \ldots, N)$ of A, i.e.

$$W(A) = \Phi(\lambda_1(A), \cdots, \lambda_N(A)), \qquad A \in \mathbb{M}^{N\times N}_+,$$

where Φ is a real–valued function invariant under interchange of the entries and defined on the set of vectors of $\mathbb{R}^N$ with positive components and, for profile functions v as above, the corresponding deformations u in (2) are differentiable almost everywhere in B^N with gradients $\nabla u(x)$ given by

$$\nabla u(x) = \frac{v(|x|)}{|x|}\mathbb{I}_N + \left(v'(|x|) - \frac{v(|x|)}{|x|}\right)\frac{x \otimes x}{|x|^2} \qquad \text{for a.e. } x \in B^N$$

(see [3] or [23]) where $\mathbb{I}_N$ is the $N \times N$ identity matrix and with singular values or principal stretches of $\nabla u(x)$ given for a.e. $x \in B^N$ by

$$\lambda_1(\nabla u(x)) = v'(|x|) \quad \text{and} \quad \lambda_n(\nabla u(x)) = \frac{v(|x|)}{|x|}, \qquad n = 2, \ldots, N, \tag{3}$$

so that

$$W(\nabla u(x)) = \Phi\left(v'(x), v(|x|)/|x|, \cdots, v(|x|)/|x|\right) \qquad \text{for a.e. } x \in B^N.$$

In the so called *generalized Varga materials* that we are going to consider here, the strain energy density features a special, simple structure for which a closed form of solutions of the equilibrium equation can be found, at least formally. More precisely, in view of (3), assuming $N = 3$, setting $\xi := \lambda_1$ and $\eta := \lambda_2 = \lambda_3$ and writing

$$\Phi(\xi, \eta) := \Phi(\xi, \eta, \eta), \qquad \xi, \eta > 0,$$

for the ease of notation, the function Φ associated with the strain energy of generalized Varga materials is given by

$$\Phi(\xi, \eta) := c_1(\xi + 2\eta) + c_2(2\xi\eta + \eta^2) + w(\xi\eta^2), \qquad \xi, \eta > 0, \tag{4}$$

for suitable coefficients $c_1, c_2 \geq 0$ and some nonnegative, smooth and strictly convex function $w\colon (0, +\infty) \to \mathbb{R}$ satisfying $w(J) \to +\infty$ as $J \to 0^+$ and as $J \to +\infty$. The recession at infinity of w is defined by

$$w^\infty := \lim_{J \to +\infty} \frac{w(J)}{J} \in (0, +\infty] \tag{5}$$

and the cases $0 < w^\infty < +\infty$ and $w^\infty = +\infty$ correspond to the commonly named cases of linear and superlinear growth of the volume term at infinity respectively.

In the superlinear case, radially symmetric minimizers of E exist and turn out to be smooth with a profile v satisfying

$$\left(\frac{v(r)}{r}\right)^2 v'(r) = c, \qquad 0 < r < 1.$$

i.e. $\det \nabla u(x) = c$ for every $x \in B^3$, $x \neq 0$. This is consistent with the analysis in [15] where the existence of smooth, radially symmetric critical points of E is taken for granted and their properties are investigated.

1.1 The Surface–Tension Term

In this paper, we focus on the existence and qualitative properties of minimizers of the energy with density Φ as in (4) in the case when the volume term w has linear growth at infinity and satisfies

$$w(J) \geq w^\infty J + w_0, \qquad J > 0, \tag{6}$$

for some constant $w_0 \in \mathbb{R}$. Hence, due to the linear growth of Φ with respect to $|\nabla u|$, the natural functional setting for the minimization of E is the set of (radially symmetric) BV functions.

The materials described by this choice of Φ and w can be named elasto–plastic brittle materials. Assume in fact that $u_h\colon B^3 \to \mathbb{R}^3$ ($h \geq 1$) is a minimizing sequence of E among radially symmetric smooth maps, subject to the boundary displacement condition. Due to the linear growth, the minimizing sequence in general has a subsequence which converges in $L^1(B^3, \mathbb{R}^3)$ only and the limit function u is a radially symmetric map with an increasing profile function $v\colon (0, 1) \to \mathbb{R}$ satisfying

$0 \le v(0) \le v(1) \le \lambda$. Therefore, besides the occurrence of cavitation $v(0) > 0$ analyzed in [15] in the superlinear case $w^\infty = +\infty$, in the linear case $0 < w^\infty < +\infty$ considered here these materials may present one of the following three behaviours or a combination thereof:

- *fractured body*: the profile function $v(r)$ fails to be continuous in $(0, 1)$, but no energy concentration appears at the discontinuity points;
- *plasticity*: energy concentration may appear at $R = 0$, if $v(0) > 0$, or at some shell $|x| = R$, where $0 < R < 1$;
- *diffuse fractures*: it cannot be excluded a priori that $v(r)$ behaves like e.g. $v(r) = (\lambda - 1)\, r + v_C(r)$ for $0 < r < 1$ $(\lambda > 1)$ where v_C is the usual Cantor–Vitali function.

Roughly speaking, in the linear growth case considered here, as the parameter λ of the boundary displacement condition $u(x) = \lambda x$ grows from $\lambda = 1$ to infinity in time, elasto–dynamics suggests that a threshold λ_0 exists such that, for $\lambda \ge \lambda_0$, the trivial configuration $u(x) = \lambda\, x$ fails to be energy favorable and the body brakes in two possible ways: at the origin, so that cavitation appears, or at a shell $|x| = R$ for some positive radius $R \in (0, 1)$.

In order to exclude the other phenomena, that we call plasticity and diffuse fractures, we postulate that the body under scrutiny is subject to a positive *surface tension*. From the mathematical point of view this physical assumption is described by adding to the energy functional (1) a surface–like term proportional through a positive constant μ to the total area of the boundary of the graph

$$\mathcal{G}_u = \Big\{(x, u(x)) \mid x \in B^3 \text{ and } x \neq 0\Big\}$$

of the deformation map u. Assuming that the profile function v associated with u is continuous and $0 \le v(0) < v(1) = \lambda$, it is given by the term

$$\mu\, 4\pi \left[(1 + \lambda^2) + v(0)^2\right],$$

where $\mu > 0$ is a constant which depends on the material. If instead the increasing profile function $v(r)$ is discontinuous at some point $R \in (0, 1)$ and $v(R^\pm)$ denote the left and right limits of v at $r = R$, the boundary of the graph of the corresponding deformation u contains two further $SO(3) \times SO(3)$ invariant surfaces and the total area of the graph of u features an additional term given by

$$\mu\, 4\pi \left[\Big(R^2 + v(R^-)^2\Big) + \Big(R^2 + v(R^+)^2\Big)\right]$$

for every discontinuity point $R \in (0, 1)$ of the profile function v.

The very same surface term is considered in [18], where the minimization is not restricted to radial deformations. However, the results of [18] apply to the so–called *neohookean materials*, i.e. require that the volume term w be superlinear and that the energy density control the L^p–norm of the gradient with $p > N - 1$ and hence do not apply to the case of generalized Varga materials considered here. See also [9] and [14] for related results in the limiting case $p = N - 1$ and also [12, 13] and [5] for further results. We also mention [2] for a regularization approach to problems with linear growth.

1.2 Admissible Profiles and the Energy Functional

On the ground of the previous arguments, we consider deformations u as in (2) with possibly discontinuous profile functions v. Since the fracture that may appear inside the body is conceivably unique, from now on we assume that all profile functions may have at most one point of discontinuity $r = R$ and, in addition, that this discontinuity point is located in the interval $(0, R_0]$, i.e. $R \in (0, R_0]$, for some positive parameter $R_0 \in (0, 1)$. This latter mathematical assumption is physically justified by the observation that, in any possible experiment, the specimen must be clamped in some region $(R_0, 1)$ and no fractures can appear in the clamped portion of the material. Then, we define the set of admissible profiles v as follows.

Definition 1.1 Let $\lambda \geq 1$. The *admissible profiles* are the functions $v\colon I \to \mathbb{R}$ ($I = (0, 1)$) with the following properties:

- v is nonnegative, strictly increasing and such that $v(1) = \lambda$;
- either $v \in W^{1,1}(I)$ or $v \in W^{1,1}(I \setminus \{R\})$ for some $R \in (0, R_0]$ and $v(0) = 0$;

and the set of all such functions is denoted by $\mathcal{A}(\lambda)$. □

Therefore, a profile function v in $\mathcal{A}(\lambda)$ is either a Sobolev function in $W^{1,1}(I)$ whose value at $r = 0$ may be $v(0) = 0$ or $v(0) > 0$ or v has a unique discontinuity point $R \in (0, R_0]$, i.e. $v(R^-) < v(R^+)$, and is a Sobolev function in each interval $(0, R)$ and $(R, 1)$ with $v(0) = 0$. The condition $v(1) = \lambda$ encodes for the profile function v the Dirichlet boundary condition $u(x) = \lambda x$ for $|x| = 1$.

For generalized Varga materials in the physical dimension $N = 3$, the energy of a radially symmetric deformation u with an admissible profile function $v \in \mathcal{A}(\lambda)$ is thus given by

$$E_\mu(u) = \int_{B^3} W(\nabla u(x))\, dx + \mu \mathcal{H}^2(\partial \mathcal{G}_u), \tag{7}$$

where W is associated with Φ given by (4) and $\mathcal{H}^2$ denotes the 2−dimensional Hausdorff measure. The energy E_μ can be rewritten in terms of the profile function v by means of the functional

$$E_\mu(u) = F_\mu(v) := F(v) + \mu\, \Sigma(v), \qquad v \in \mathcal{A}(\lambda), \tag{8}$$

where, for any function $v \in \mathcal{A}(\lambda)$, we have set

$$\begin{aligned} F(v) := 4\pi \int_0^1 r^2 \Bigg\{ & c_1 \left[v'(r) + 2\frac{v(r)}{r} \right] + \\ & + c_2 \left[2v'(r)\frac{v(r)}{r} + \left(\frac{v(r)}{r}\right)^2 \right] + w\left(v'(r) \left(\frac{v(r)}{r}\right)^2 \right) \Bigg\}\, dr \end{aligned} \tag{9}$$

and, recalling the definition of v in $\mathcal{A}(\lambda)$, for the area term $\Sigma(v)$ we have set

$$\Sigma(v) := 4\pi \left\{ [v(0)]^2 + \left(1+\lambda^2\right) \right\}$$

when $v \in \mathcal{A}(\lambda)$ is continuous and

$$\Sigma(v) = 4\pi \left\{ \left(R^2 + [v(R^-)]^2\right) + \left(R^2 + [v(R^+)]^2\right) + \left(1+\lambda^2\right) \right\}$$

when $v \in \mathcal{A}(\lambda)$ has a discontinuity point at $r = R \in (0, R_0]$.

1.3 Hypotheses on *w* and Normalization Conditions

In the physical dimension $N = 3$, let $\Phi(\xi, \eta)$ be the strain energy of generalized Varga materials defined in (4).

For the volume term, we assume that the function $w\colon (0, +\infty) \to \mathbb{R}$ satisfies the following hypotheses:

(H1) $w \in C^2(0, +\infty)$;

(H2) w is strictly convex in $(0, +\infty)$ and $w''(J) > 0$ for every $J > 0$;

(H3) $\lim_{J\to 0^+} w(J) = +\infty$;

(H4) $\lim_{J\to +\infty} w(J)/J = w^\infty \in (0, +\infty)$ and $w(J) \geq w^\infty J + w_0$ for every $J > 0$;

(H5) there exist $0 < \delta < 1$ and $C \geq 0$ such that

$$|t-1| \leq \delta \qquad \Longrightarrow \qquad (tJ)|w'(tJ)| \leq C\left[w(J)+1\right] \quad \forall J > 0.$$

We notice that hypotheses (H2) and (H3) imply that $w'(J) \to -\infty$ as $J \to 0^+$ and that hypothesis (H4) expresses the property that w has linear growth at infinity and rules out convex functions as $J \in (0, +\infty) \mapsto J - \log J$. Moreover, in view of the strict convexity of w and (H4), we have $w'(J) < w^\infty$ for every $J > 0$ and it is not restrictive to assume in addition that

$$w_0 \geq 0 \tag{10}$$

and to extend the definition of w to the whole real line by setting

$$w(J) = +\infty, \qquad J \leq 0.$$

As regards the hypothesis (H5), it is a structure hypothesis on w which is satisfied for instance by suitable perturbations of the model case $w(J) = aJ + b/J^\alpha$, $J > 0$ ($a, b > 0$ and $\alpha > 0$), see [4]. It implies in particular that

$$0 \leq J|w'(J)| \leq C\left[w(J)+1\right], \qquad J > 0. \tag{11}$$

As to the coefficients $c_1, c_2 \geq 0$ in (4) which are so far undetermined, we choose their values by imposing for physical reasons that the profile function $v(r) = r$ of the undeformed configuration $u(x) = x$ is stable with respect to outer variations $v + \delta\varphi$ with smooth test functions φ not necessarily compactly supported in I. As we shall see in Sec. 5, for generalized Varga materials in dimension $N = 3$ this gives the equation

$$c_1 + 2c_2 + w'(1) + 2\mu = 0 \tag{12}$$

which corresponds to the first normalization condition assumed in [15], when $\mu = 0$. Here, since $\mu > 0$ and the coefficients c_1 and c_2 are nonnegative, we must have $w'(1) < 0$ which means that the unique critical point $J_{\rm cr}$ of the function w is strictly greater than 1.

We notice that in [15] it is further assumed that the energy density vanishes in the undeformed configuration. Here, as a consequence of (H4) and (10), we assume instead that $w(J) \geq 0$ for every $J > 0$. From the point of view of minimization of F_μ this difference from [15] is immaterial.

As a model example, for any choice of the coefficient $\mu > 0$ of the surface–like term, we consider the function

$$w(J) = aJ + \frac{b}{J}, \qquad J > 0, \tag{13}$$

where a, b are positive numbers, Then, this function w satisfies the hypotheses (H1),…,(H5) with $w^\infty = a > 0$ and, since $c_1, c_2 \geq 0$ and $\mu > 0$, the normalization condition (12) requires that $w'(1) = a - b < 0$, i.e. $b > a > 0$ and this is fulfilled by taking e.g.

$$c_1 = c_2 = (b - a - 2\mu)/3 \qquad \text{and} \qquad b - a \geq 2\mu > 0.$$

For this choice of the parameters, the strain energy density corresponding to generalized Varga materials in dimension $N = 3$ is

$$\Phi(\xi, \eta) := \frac{b - a - 2\mu}{3} \Big[(\xi + 2\eta) + (2\xi\eta + \eta^2)\Big] + a\,\xi\eta^2 + b\,\frac{1}{\xi\eta^2} \tag{14}$$

for every $\xi, \eta > 0$ and the lower bound (6) holds with $w_0 = 0$.

1.4 Main Result: Existence of Minimizers

In the next sections, we extend the definition of generalized Varga materials to any dimension $N \geq 3$ and we prove that for every $N \geq 3$ and for every choice of the parameters $\lambda \geq 1$ and $\mu > 0$ the minimum of the energy of generalized Varga materials

$$E_\mu(u) = E(u) + \mu\,\mathcal{H}^{N-1}(\partial\mathcal{G}_u)$$

among radially symmetric maps u with a profile function $v \in \mathcal{A}(\lambda)$ is attained and that a complete description of minimizers u is available (Theorem 4.4).

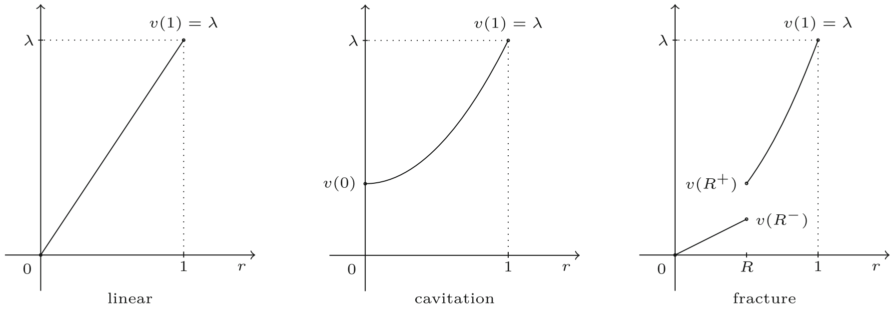

Fig. 1 Possible minimizers $v \in \mathcal{A}(\lambda)$ of F_μ in Theorem 1.1

Since we have not given the definition of generalized Varga materials in arbitrary dimension yet, here we state the result for the case $N = 3$ and we postpone the case of arbitrary dimension $N \geq 3$ to Theorem 4.4.

Throughout the paper, all results are stated in terms of the equivalent energy F_μ in (8) as a function of the profile functions v.

Theorem 1.1 *Let F_μ be the energy functional defined by* (8) *where $c_1, c_2 \geq 0$ and w satisfies hypotheses* (H1),…,(H5)*. Then, for every $\lambda \geq 1$ and $\mu > 0$ the minimum problem*

$$\inf \left\{ F_\mu(\bar{v}) \mid \bar{v} \in \mathcal{A}(\lambda) \right\}$$

has a solution $v \in \mathcal{A}(\lambda)$. Moreover, the minimizer $v(r)$, $r \in I$, is of one of the following three types:

(a) *linear: $v(r) = \lambda r$;*
(b) *cavitation: $v(r) = \left(Jr^3 + k\right)^{1/3}$ with $v(0) = k > 0$ and $J = \lambda^3 - k > 0$;*
(c) *fracture:*

$$v(r) = \begin{cases} Kr & \text{if } 0 < r < R \\ \left(Jr^3 + k\right)^{1/3} & \text{if } R < r < 1 \end{cases}$$

with $R \in (0, R_0]$, $K > 0$, $JR^3 + k > (KR)^3$ and $J = \lambda^3 - k > 0$.

The possible three types of solutions are represented in Figure 1.
This theorem is proved in Sec. 4 for arbitrary dimension $N \geq 3$ and in the last Sec. 5 we analyze the occurrence of cavitations and fractures in the model case (14) and we compare our results with those obtained by Horgan in [15].

2 Notation and Preliminary Results

In this section, before defining the energy functional for generalized Varga materials in any dimension $N \geq 3$, we gather notation and preliminary results. In particular, we review some basic facts on one–dimensional functions of bounded variations and lower semicontinuity of integral functionals in the same setting.

Throughout the paper we use standard measure and functional analytic notation; in particular, we denote the Lebesgue and Borel σ – algebras of $\mathbb{R}$ and I by $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}(I)$ respectively and we denote the Lebesgue measure in $\mathbb{R}$ by $\mathcal{L}^1$ and the α – dimensional Hausdorff measure by $\mathcal{H}^\alpha$ respectively.

2.1 Functions of Bounded Variation and Lower Semicontinuity in *BV*

The Banach space of real–valued, bounded Borel measures $\mu\colon \mathcal{B}(I) \to \mathbb{R}$ in I endowed with the total variation norm $\|\mu\|_{\mathcal{M}(I)} = |\mu|(I)$ is the dual space of $C_0(I)$ and is denoted by $\mathcal{M}(I)$. If μ is any such measure, its Lebesgue decomposition into the (mutually singular) absolutely continuous μ^{ac} and singular part μ^s with respect to $\mathcal{L}^1$ is

$$\mu = \mu^{ac} + \mu^s = \theta\mathcal{L}^1 + \mu^s,$$

where $\theta \in L^1(I)$ is the Radon–Nikodym derivative of μ with respect to $\mathcal{L}^1$. We denote also by $BV(I)$ the Banach space of *functions of bounded variation*, i.e. those functions $v \in L^\infty(I)$ whose distributional derivative Dv is a real–valued, bounded Borel measure in I:

$$\int_0^1 v(r)\varphi'(r)\,dr = -\int_0^1 \varphi(r)\,d(Dv)(r), \qquad \forall\,\varphi \in C_c^\infty(I),$$

endowed with the norm $\|v\|_{BV(I)} = \|v\|_{L^1(I)} + \|Dv\|_{\mathcal{M}(I)}$.

In this one–dimensional setting, the precise representative

$$v(r) = \lim_{h\to 0^+} \frac{1}{2h}\int_{r-h}^{r+h} v(\rho)\,d\rho, \qquad r \in I,$$

of a function $v \in BV(I)$ is defined at every point in I and in the sequel we shall always use the precise representative when dealing with functions in $BV(I)$. With this convention, every function in $BV(I)$ is the difference of two increasing and bounded functions, the left and right limits $v(r^\pm)$ of v exist at every point $r \in I$ and the discontinuity set $S(v)$ of v is countable. Moreover, v is (classically) differentiable at $\mathcal{L}^1$–a.e. point in I with (classical) derivative $v' \in L^1(I)$ and the Lebesgue decomposition of the distributional derivative Dv is

$$Dv = D^{ac}v + D^s v = v'\mathcal{L}^1 + D^s v$$

and the singular part $D^s v$ can be further decomposed into the (mutually singular) Cantor and jump parts

$$D^s v = D^c v + D^j v = D^c v + [v(r^+) - v(r^-)]\,\mathcal{H}^0 \llcorner S(v),$$

where $D^c v(B) = 0$ for every countable set $B \subset I$.

Finally, we recall that, if $\{v_n\}_n \subset BV(I)$ is a bounded sequence in $BV(I)$, i.e. $\|v_n\|_{BV(I)} \le C$ for every n, by Helly's selection theorem there exist a subsequence $v_m = v_{n_m}$ $(m \ge 1)$ and a function $v \in BV(I)$ such that

- $v_m(t) \to v(t)$ as $m \to +\infty$ at $\mathcal{L}^1$–a.e. point $t \in I$;
- $Dv_m \rightharpoonup Dv$ in w^*–$\mathcal{M}(I)$ as $m \to +\infty$.

Moreover, $\|v_n\|_{L^\infty(I)} \le C$ for every n and $v_m \to v$ in $L^1(I)$. In this case, we say that $v_m \rightharpoonup v$ weakly* in $BV(I)$ as $m \to +\infty$. This remark applies in particular to sequences of functions in $\mathcal{A}(\lambda)$ because we have

$$|Dv|(I) = Dv(I) = v(1) - v(0) \le \lambda$$

for any such function v. We refer to [1] and [11], Sec. 4.4.1, for the other properties of BV functions that will be used in the sequel.

Next, we turn to the (sequential) lower semicontinuity of integral functionals with respect to the weak*–BV convergence. To this aim, consider a function $f : I \times \mathbb{R} \to [0, +\infty]$ such that

(F1) f is lower semicontinuous in $I \times \mathbb{R}$;
(F2) the function $p \in \mathbb{R} \mapsto f(r, p) \in [0, +\infty]$ is convex for every $r \in I$;
(F3) there exists $p_0 \in L^\infty(I)$ such that $r \in I \mapsto f(r, p_0(r))$ is in $L^\infty(I)$;

and let $f^\infty : I \times \mathbb{R} \to [0, +\infty]$ defined by

$$f^\infty(r, p) = \lim_{t \to +\infty} \frac{f(r, p_0(r) + tp)}{t}, \qquad (r, p) \in I \times \mathbb{R},$$

be the *recession function of f* (see [21]). The recession function f^∞ is clearly $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R})$ measurable and its definition does not depend on the choice of p_0.

Then, we recall the following Goffman–Serrin's type (sequential) lower semicontinuity result in weak*–BV ([6], Theorem 3.4.1 and Corollary 3.4.2).

Theorem 2.1 *Let $f : I \times \mathbb{R} \to [0, +\infty]$ be a function such that the hypotheses* (F1), (F2) and (F3) *hold. Then, the functional*

$$v \in BV(I) \mapsto \int_I f(r, v'(r))\, dr + \int_I f^\infty\left(r, \frac{\mathrm{d}D^s v}{\mathrm{d}|D^s v|}(r)\right) d|D^s v|(r)$$

is (sequentially) lower semicontinuous in weak–$BV(I)$.*

2.2 Generalized Varga Type Integrals in Higher Dimension

In any dimension $N \ge 3$ the principal invariants $I_1, \ldots, I_N$ of a matrix with positive determinant $A \in \mathbb{M}^{N \times N}_+$ are the symmetric and n–homogeneous functions $I_n = I_n(\lambda_1, \ldots, \lambda_N)$ $(n = 1, \ldots, N)$ of the singular values $\lambda_1, \ldots, \lambda_N$ of A. In the sequel, we shall adopt the shorthand notation

$$J = I_N(\lambda_1, \ldots, \lambda_N) := \lambda_1 \cdots \lambda_N$$

for the determinant. For matrices $A \in \mathbb{M}_+^{N\times N}$ whose singular values are $\xi := \lambda_1$ and $\eta := \lambda_2 = \ldots = \lambda_N$ as in (3), the principal invariants are given by the formulas

$$I_n = I_n(\xi,\eta) = \binom{N-1}{n-1}\xi\eta^{n-1} + \binom{N-1}{n}\eta^n; \quad J = J(\xi,\eta) = \xi\eta^{N-1}; \quad (15)$$

for $n = 1,\ldots,N-1$. In particular, for $N = 3$ we get $I_1 = \xi + 2\eta$, $I_2 = 2\xi\eta + \eta^2$ and $J = \xi\eta^2$ as in the definition (4).

Therefore, the general form of the strain energy density corresponding to generalized Varga materials in dimension N becomes

$$\Phi(\xi,\eta) := \sum_{n=1}^{N-1} c_n\, I_n(\xi,\eta) + w(J(\xi,\eta)), \qquad \xi,\eta > 0,$$

for non negative coefficients c_n ($n = 1,\ldots,N-1$) and for a smooth, nonnegative and strictly convex function $w\colon (0,+\infty) \to [0,+\infty)$ satisfying hypotheses (H1),…,(H5). In particular, for $N = 3$ we recover formula (4).

For any radially symmetric map $u\colon B^N \to \mathbb{R}^N$ with profile function $v \in \mathcal{A}(\lambda)$, the corresponding energy functional (7) can be written as the sum of two terms as in (8) and the first term F reduces to

$$F(v) := \sigma_{N-1}\int_0^1 r^{N-1}\left[\sum_{n=1}^{N-1} c_n I_n\big(v'(r), v(r)/r\big) + w\left(J(v'(r), v(r)/r)\right)\right]dr, \quad (16)$$

where σ_{N-1} denotes the $\mathcal{H}^{N-1}$–measure of $\mathbb{S}^{N-1}$, the unit $(N-1)$–sphere centered at the origin in $\mathbb{R}^N$. For $N = 3$, recalling that $I_1 = \xi + 2\eta$, $I_2 = 2\xi\eta + \eta^2$ and $J = \xi\eta^2$, we recover the definition of $F(v)$ in (9).

As to the other term Σ appearing in (8), we notice that, if the deformation $u\colon B^N \to \mathbb{R}^N$ is the radially symmetric map associated with a profile function v in the class $\mathcal{A}(\lambda)$ the graph $\mathcal{G}_u$ of u is connected to the graph $\mathcal{G}_v$ of v by the formula

$$\mathcal{G}_u = h(\mathcal{G}_v \times \mathbb{S}^{N-1}),$$

where $h\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N\times\mathbb{R}^N$ is the map defined by

$$h(r,\rho,z) := (rz, \rho z), \qquad (r,\rho,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N. \quad (17)$$

In a similar way, the boundary $\partial\mathcal{G}_u$ of the graph of u is the $SO(N)\times SO(N)$ invariant $(N-1)$–surface given by

$$\partial\mathcal{G}_u = h(\partial\mathcal{G}_v \times \mathbb{S}^{N-1}),$$

where, recalling the definition of v in $\mathcal{A}(\lambda)$, the boundary of the graph $\mathcal{G}_v$ is given by

$$\partial \mathcal{G}_v = \begin{cases} \{(0, v(0)), (1, \lambda)\} & \text{if } v \text{ is continuous} \\ \left\{(0, 0), (1, \lambda), (r, v(R^-)), (r, v(R^+))\right\} & \text{if } S(v) = \{R\}. \end{cases}$$

As a consequence, the $(N-1)$–dimensional measure of the boundary of the graph $\mathcal{G}_u$ is given by $\mathcal{H}^{N-1}(\partial \mathcal{G}_u) = \Sigma(v)$ where $\Sigma(v)$ is either given by

$$\Sigma(v) := \sigma_{N-1} \left\{ [v(0)]^{N-1} + \left(1 + \lambda^2\right)^{\frac{N-1}{2}} \right\} \tag{18}$$

when $v \in \mathcal{A}(\lambda)$ is continuous or it is given by

$$\Sigma(v) := \sigma_{N-1} \left\{ \left(R^2 + [v(R^-)]^2\right)^{\frac{N-1}{2}} + \left(R^2 + [v(R^+)]^2\right)^{\frac{N-1}{2}} + \left(1 + \lambda^2\right)^{\frac{N-1}{2}} \right\} \tag{19}$$

when $v \in \mathcal{A}(\lambda)$ has a discontinuity point at $r = R \in (0, R_0]$, since in such case we have $v(0) = 0$ by definition.

Therefore, for generalized Varga materials in any dimension $N \geq 3$, we define the energy $F_\mu(v)$ of a radially symmetric deformation with an admissible profile function $v \in \mathcal{A}(\lambda)$ as follows.

Definition 2.1 Let $\lambda \geq 1$ and $\mu > 0$. The energy $F_\mu \colon \mathcal{A}(\lambda) \to [0, +\infty]$ is defined by

$$F_\mu(v) := F(v) + \mu\, \Sigma, \qquad v \in \mathcal{A}(\lambda),$$

where $F(v)$ and $\Sigma(v)$ are defined by (16) and by either (18) or (19) respectively. □

For $N = 3$, the energy F_μ defined above reduces to the definition given in (8). We also remark that, by setting $w(0) = +\infty$, the functional F is actually well defined for every non negative and increasing function $v \colon I \to \mathbb{R}$, regardless of v enjoying all the properties stated in the definition of $\mathcal{A}(\lambda)$.

Finally, we assume also that w satisfies the normalization condition which ensures the stability of the undeformed state $v(r) = r$ with respect to outer variations as discussed in the Introduction in the case $N = 3$. In Sec. 5, in dimension $N \geq 3$, we obtain

$$\sum_{n=1}^{N-1} \binom{N-1}{n-1} c_n + w'(1) + (N-1)\, 2^{(N-3)/2} \mu = 0 \tag{20}$$

which reduces to (12) for $N = 3$.

2.3 Extended Integrands and Derivative Measures

By elementary computations, for every nonnegative and increasing function $v\colon I \to \mathbb{R}$ the principal invariants I_n given in (15) satisfy

$$r^{N-1} I_n\left(v'(r), v(r)/r\right) = \frac{1}{N}\binom{N}{n}\frac{\mathrm{d}}{\mathrm{d}r}r\left(r^{N-n}[v(r)]^n\right) \tag{21}$$

for a.e. $r \in I$ and every n and hence, writing $r^{N-n}v^n$ as a shorthand for the function $r \in I \mapsto r^{N-n}[v(r)]^n$ ($n = 1, \ldots, N$), the functional F in (16) can be written as

$$F(v) = \sigma_{N-1} \sum_{1\le n\le N} \int_0^1 f_n\left(r, (r^{N-n}v^n)'\right) dr, \qquad v \in \mathcal{A}(\lambda), \tag{22}$$

where the integrands $f_n\colon I \times \mathbb{R} \to [0, +\infty]$ are defined by

$$f_n(r, p) = \begin{cases} 0 & \text{if } r \in I \text{ and } p \le 0 \\ \dfrac{c_n}{N}\dbinom{N}{n} p & \text{if } r \in I \text{ and } p > 0 \end{cases} \qquad (n = 1, \ldots, N-1)$$

and

$$f_N(r, p) = \begin{cases} +\infty & \text{if } r \in I \text{ and } p \le 0 \\ r^{N-1} w\left(p/(N r^{N-1})\right) & \text{if } r \in I \text{ and } p > 0. \end{cases}$$

These functions f_n satisfy hypotheses (F1), (F2) and (F3) with recession functions $(r, p) \in I \times \mathbb{R} \mapsto f_n^\infty p$ with coefficients f_n^∞ given by

$$f_n^\infty = \begin{cases} 0 & \text{for } p \le 0 \\ \dfrac{c_n}{N}\dbinom{N}{n} & \text{for } p > 0; \end{cases} \qquad f_N^\infty = \begin{cases} 0 & \text{for } p \le 0 \\ \dfrac{w^\infty}{N} & \text{for } p > 0; \end{cases} \tag{23}$$

for $n = 1, \ldots, N-1$ and $n = N$ respectively and for every $r \in I$. In this way, the functional F in formula (22) is actually well defined for every function v in $BV(I)$ and the extended integral functional

$$\overline{F}(v) := \sigma_{N-1} \sum_{1\le n\le N} \left\{\int_0^1 f_n\left(r, (r^{N-n}v^n)'\right) dr + f_n^\infty D^s(r^{N-n}v^n)(I)\right\} \tag{24}$$

is (sequentially) lower semicontinuous with respect to weak*–$BV(I)$ convergence (Theorem 2.1).

For functions $v \in \mathcal{A}(\lambda)$ the original functional F defined by either (16) or (22) and the extended integral (24) are connected by

$$F(v) = \overline{F}(v) - F_{\text{hole}}(v), \qquad v \in \mathcal{A}(\lambda), \tag{25}$$

where the additional term F_{hole} is defined by

$$F_{\text{hole}}(v) = \begin{cases} 0 & \text{if } S(v) = \varnothing \\ \sum_{1 \le n \le N} f_n^\infty R^{N-n} \left\{ [v(R^+)]^n - [v(R^-)]^n \right\} & \text{if } S(v) = \{R\} \end{cases} \tag{26}$$

and accounts for the possible (negative) energy contribution due to the hole in the graph of v.

Finally, we notice that the action of the extended integral functional $\overline{F}$ can be described in terms of the total variation of measures.

In fact, for a given nonnegative and increasing function $v \colon I \to \mathbb{R}$ such that $v(1) \le \lambda$ or more generally for every function $v \in BV(I)$, we can decompose the derivative measure $D(r^{N-n}v^n)$ ($n = 1, \ldots, N$) into its absolutely continuous and singular components, i.e.

$$D(r^{N-n}v^n) = D^{ac}(r^{N-n}v^n) + D^s(r^{N-n}v^n).$$

For every function v as above, from formula (21) we get

$$\int_0^1 f_n\left(r, (r^{N-n}v^n)'\right) dr = f_n^\infty D^{ac}(r^{N-n}v^n)(I), \qquad n = 1, \ldots, N-1, \tag{27}$$

for the absolutely continuous component and, writing here and in the sequel for the sake of brevity

$$Jv(r) := v'(r) \left(\frac{v(r)}{r}\right)^{N-1} \qquad \text{for a.e. } r \in I \tag{28}$$

for the Jacobian and noticing that we obviously have

$$\int_0^1 r^{N-1} Jv(r)\, dr \le \frac{[v(1)]^N}{N} \le \frac{\lambda^N}{N},$$

we can prove the following result.

Proposition 2.1 *Let $v \colon I \to \mathbb{R}$ be a nonnegative and increasing function such that $v(1) \le \lambda$. Then,*

$$\overline{F}(v) = \sigma_{N-1} \sum_{1 \le n \le N} f_n^\infty D(r^{N-n}v^n)(I) + \overline{F}_{\text{vol}}(v), \tag{29}$$

where the volume term $\overline{F}_{\text{vol}}$ *is given by*

$$\overline{F}_{\text{vol}}(v) := \sigma_{N-1} \int_0^1 r^{N-1} \left[w(Jv(r)) - w^\infty Jv(r) \right] dr. \tag{30}$$

Proof In view of the definition of $\overline{F}$ in (24) and (27), we have

$$\overline{F}(v) = \sigma_{N-1} \sum_{1 \leq n \leq N-1} f_n^\infty D(r^{N-n} v^n)(I) + \\ + \sigma_{N-1} \left\{ \int_0^1 f_N \left(r, (v^N)' \right) dr + f_N^\infty D^s(v^N)(I) \right\}$$

and, for the last two summands, from the definitions of f_N and f_N^∞ in (23) and from the definition of $Jv(r)$ in (28) we have

$$\int_0^1 f_N \left(r, (v^N)' \right) dr + f_N^\infty D^s(v^N)(I) = \\ = \int_0^1 r^{N-1} w(Jv(r))\, dr + \frac{w^\infty}{N} D^s(v^N)(I) = \\ = f_N^\infty D(v^N)(I) + \int_0^1 \left[w(Jv(r)) - w^\infty Jv(r) \right] dr$$

whence the conclusion follows. □

At last, we notice that, if $\overline{F}_{\text{vol}}(v) < +\infty$, in view of (H3), v must be strictly increasing and, since

$$D(v^N)(I) = [v(1)]^N - [v(0)]^N \qquad \text{and} \qquad D(r^{N-n} v^n)(I) = [v(1)]^n$$

for $n = 1, \ldots, N-1$, if the boundary condition $v(1) = \lambda$ is satisfied, we obtain

$$\overline{F}(v) = C(N, \lambda) + \overline{F}_{\text{vol}}(v) - w^\infty \frac{\sigma_{N-1}}{N} [v(0)]^N,$$

where $C(N, \lambda)$ is a positive constant depending only on N and λ.

3 Relaxation and Existence of Generalized Minimizers

In this section we consider the minimum problem

$$\inf \left\{ F_\mu(v) \mid v \in \mathcal{A}(\lambda) \right\}, \tag{31}$$

where F_μ ($\mu > 0$) is the energy functional defined in Definition 2.1 for generalized Varga materials in dimension N with coefficients $c_n \geq 0$ ($n = 1, \ldots, N-1$) and volume term w satisfying hypotheses (H1),…,(H5).

Our goal is to apply the direct method of the Calculus of Variations and we recall that, due to the monotonicity and the constraint on the values, the set of admissible functions $\mathcal{A}(\lambda)$ is bounded in $BV(I)$ and therefore every minimizing sequence $\{v_h\}_h \subset \mathcal{A}(\lambda)$ has a subsequence which converges in weak*– $BV(I)$ to a nonnegative and increasing function v such that $v(1) \leq \lambda$. Yet, due to the linear growth of the strain energy density with respect to the principal invariants, nothing ensures that the limit function is in $\mathcal{A}(\lambda)$: the limit function v need not satisfy the boundary condition $v(1) = \lambda$, neither be strictly increasing nor have at most a unique point of discontinuity and be Sobolev off that point. In principle, the limit function v might have a countable dense set of points of discontinuity and have a nontrivial Cantor part as well. Moreover, even if $\{v_h\}_h$ is a minimizing sequence such that $v_h \to v$ in weak*– $BV(I)$ with $v \in \mathcal{A}(\lambda)$, it can be proved that $F(v) \leq \liminf_h F(v_h)$ but the functional F_μ fails to be lower semicontinuous along the sequence $\{v_h\}_h$ due to the presence of the area term Σ (Example 3.2).

In order to overcome these difficulties, we proceed by relaxation: we extend the functional F_μ to a functional $\mathcal{F}_\mu$ defined on more general objects that we call *generalized graphs*. Technically speaking, these generalized graphs are (integer multiplicity) rectifiable 1–currents in $\mathbb{R} \times \mathbb{R}$. The relaxed functional $\mathcal{F}_\mu$ turns out to be lower semicontinuous with respect to a suitable notion of convergence of generalized graphs (Theorem 3.1) and has a minimizer among generalized graphs (Theorem 3.2). Finally, we prove that this minimizer is the graph of a function $v \in \mathcal{A}(\lambda)$ as in Theorem 1.1 or in Theorem 4.4, depending on the value of $N \geq 3$.

Before introducing the generalized graphs and the relaxed functional $\mathcal{F}_\mu$, we first give an elementary example in the physical dimension $N = 3$ which describes the onset of fractures and the consequent energy concentration in the weak limit process. The same example shows that the energy functional F_μ fails to be lower semicontinuous on $\mathcal{A}(\lambda)$ with respect to weak*– $BV(I)$ convergence.

3.1 Energy Concentration and Lack of Lower Semicontinuity

The following example shows that energy concentration may occur at discontinuity points: if the admissible profile $v \in \mathcal{A}(\lambda)$ has a discontinuity point at $r = R$, the limit of the energies $F(v_\varepsilon)$ of smooth approximating profiles v_ε may be strictly larger than $F(v)$.

Example 3.1 Let $N = 3$ and, for $0 < R \leq R_0$, $0 < \varepsilon < 1 - R$ and $0 < \alpha < \beta < \lambda$, let $v_\varepsilon \in \mathcal{A}(\lambda)$ be the continuous and strictly increasing function which is linear on each interval $(0, R)$, $(R, R+\varepsilon)$ and $(R+\varepsilon, 1)$ and is such that $v_\varepsilon(0) = 0$, $v_\varepsilon(R) = \alpha$, $v_\varepsilon(R+\varepsilon) = \beta$, $v_\varepsilon(1) = \lambda$. Clearly, $\{v_\varepsilon\}_\varepsilon$ converges in weak*– $BV(I)$ as $\varepsilon \to 0^+$ to the discontinuous and strictly increasing function $v \in \mathcal{A}(\lambda)$ which is linear on each interval $(0, R)$ and $(R, 1)$ and is such that $v(0) = 0$, $v(R^-) = \alpha$, $v(R) = (\alpha+\beta)/2$, $v(R^+) = \beta$ and $v(1) = \lambda$.

Setting for the sake of brevity

$$v_\varepsilon'(r) = \xi_\varepsilon(r) := \frac{\beta - \alpha}{\varepsilon} \quad \text{and} \quad \frac{v_\varepsilon(r)}{r} = \eta_\varepsilon(r) := \frac{1}{\varepsilon\, r}\,[(\beta-\alpha)(r-R) + \alpha\varepsilon]$$

for $R < r < R + \varepsilon$, by elementary computations we obtain

$$\lim_{\varepsilon \to 0^+} 4\pi \int_R^{R+\varepsilon} r^2 \left[\xi_\varepsilon(r) + 2\eta_\varepsilon(r)\right] dr = 4\pi\, R^2(\beta - \alpha);$$
$$\lim_{\varepsilon \to 0^+} 4\pi \int_R^{R+\varepsilon} r^2 \left[2\xi_\varepsilon(r)\eta_\varepsilon(r) + [\eta_\varepsilon(r)]^2\right] dr = 4\pi\, R(\beta^2 - \alpha^2);$$

and, recalling that w^∞ is the recession of the integrand w, we also get

$$\lim_{\varepsilon \to 0^+} 4\pi \int_R^{R+\varepsilon} r^2 w\left(\xi_\varepsilon(r)[\eta_\varepsilon(r)]^2\right) dr = \frac{4\pi}{3} w^\infty(\beta^3 - \alpha^3)\,.$$

Therefore, we have

$$\lim_{\varepsilon \to 0^+} F(v_\varepsilon) = F(v) + 4\pi \left\{c_1\, R^2(\beta - \alpha) + c_2\, R(\beta^2 - \alpha^2) + \frac{w^\infty}{3}(\beta^3 - \alpha^3)\right\}. \quad (32)$$

The terms $R^{3-n}(\beta^n - \alpha^n)$ which are the contributions to the total energy of the singular part of $D(r^{N-n}v^n)$ $(n = 1, 2, 3)$ account for the energy concentration phenomenon occurring in the limit process. □

The functional F is lower semicontinuous along weak*–$BV(I)$ convergent sequences in $\mathcal{A}(\lambda)$. However, this is not the case for the area term Σ.

Example 3.2 Let $N = 3$ and let v and v_ε be as in the previous example. For the area term given by (18) and (19), we clearly have

$$\Sigma(v_\varepsilon) = 4\pi(1 + \lambda^2), \qquad \varepsilon > 0;$$
$$\Sigma(v) = 4\pi\left[(1 + \lambda^2) + (R^2 + \alpha^2) + (R^2 + \beta^2)\right];$$

since v is discontinuous at $r = R$ with $\alpha = v(R^-) < v(R^+) = \beta$. Therefore, we have

$$\Sigma(v) > \lim_{\varepsilon \to 0^+} \Sigma(v_\varepsilon)$$

so that the functional $v \mapsto \Sigma(v)$ fails to be lower semicontinuous along weak*–$BV(I)$ convergent sequences of functions in $\mathcal{A}(\lambda)$ and this affects the lower semicontinuity of F_μ as well. In fact, from (32) we have

$$\lim_{\varepsilon \to 0^+} F_\mu(v_\varepsilon) = F(v) + $$
$$+ 4\pi \left\{c_1\, R^2(\beta - \alpha) + c_2\, R(\beta^2 - \alpha^2) + \frac{1}{3} w^\infty(\beta^3 - \alpha^3)\right\} + 4\pi\mu(1 + \lambda^2),$$

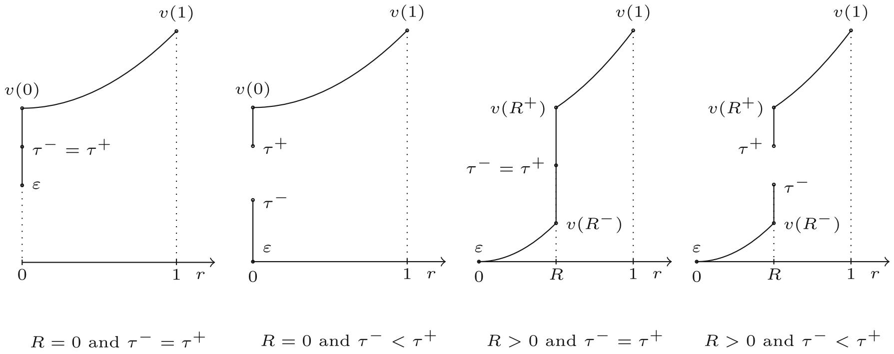

Fig. 2 Prototypical generalized graphs $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$

whereas $F_\mu(v)$ is given by

$$F_\mu(v) = F(v) + 4\pi\mu\left[(R^2+\alpha^2) + (R^2+\beta^2)\right] + 4\pi\mu(1+\lambda^2)$$

and, taking $v_h = v_{\varepsilon_h}$ with $\varepsilon_h \to 0^+$, we see that the functional F_μ fails to be lower semicontinuous for suitable choices of α and β. □

3.2 Relaxation and Existence of Generalized Minimizers

In the minimization problem (31) we have not taken into consideration the contribution of the area term at $r = 0$ which plays a fundamental role in the cavitation phenomenon. Moreover, since we only have weak*$-BV(I)$ convergence, the Dirichlet condition $v(1) = \lambda$ is not preserved in the minimization process.

For these reasons and on account of the lack of lower semicontinuity of F_μ, we now extend the energy functional F_μ to a new energy functional $\mathcal{F}_\mu$ defined on more general objects which takes into account the contributions to the total energy at $r = 0$ and $r = 1$. These more general objects are defined as follows.

Definition 3.1 The *generalized graphs* are elements of the form $(v, \varepsilon, R, \tau^{\pm})$ where $v\colon I \to \mathbb{R}$ is a non negative and increasing function with $v(1) \le \lambda$ and the parameters $R \in [0, R_0]$ and $\varepsilon, \tau^{\pm} \in [0, \lambda]$ satisfy one of the following alternative conditions, depending on the value of R:

- $R = 0$ and either $0 \le \varepsilon \le \tau^- = \tau^+ \le v(0)$ or $0 = \varepsilon \le \tau^- < \tau^+ \le v(0)$;
- $R \in (0, R_0]$, $\varepsilon = 0$ and $v(R^-) \le \tau^- \le \tau^+ \le v(R^+)$;

and the set of all such elements is denoted by $\Gamma(\lambda)$. □

The four prototypical generalized graphs are represented in Figure 2.
We emphasize that the non negative and increasing function v of a generalized graph $(v, \varepsilon, R, \tau^{\pm})$ has no further regularity other than monotonicity: it may have countably many discontinuity points besides the possible discontinuity at $r = R$ and may also have a non trivial Cantor part.

Remark 3.1 We associate with the generalized graph $(v, \varepsilon, R, \tau^{\pm})$ the rectifiable set γ^* defined by

$$\{(\{0\} \times [\varepsilon, v(0)]) \cup (\partial SG_v \cap (I \times \mathbb{R})) \cup (\{1\} \times [v(1), \lambda])\} \setminus \left(\{R\} \times [\tau^-, \tau^+]\right)$$

where SG_v denotes the subgraph of v, i.e. $SG_v = \{(r, \rho) \in I \times \mathbb{R} \mid \rho < v(r)\}$.

The set γ^* may be seen as the support of an oriented curve γ connecting the point $(0, \varepsilon)$ to $(1, \lambda)$. If v is smooth with $v(0) = \varepsilon$ and $v(1) = \lambda$, it agrees with the cartesian curve $r \in I \mapsto (r, v(r))$. When $\tau^- < \tau^+$, the last term in the definition of γ^* represents a "hole" in the support of the curve γ. □

We define the functional $\mathcal{F}\colon \Gamma(\lambda) \to [0, +\infty]$ by the formula

$$\mathcal{F}(v, \varepsilon, R, \tau^{\pm}) := \overline{F}(v) - F_{\text{hole}}(R, \tau^{\pm}) + F_0(v(0), \varepsilon) + F_1(v(1)) \tag{33}$$

for every generalized graph $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$ where $\overline{F}(v)$ is given by (24) and we have set

$$F_{\text{hole}}(R, \tau^{\pm}) := \sigma_{N-1} \sum_{n=1}^{N} f_n^{\infty} R^{N-n} \left([\tau^+]^n - [\tau^-]^n\right);$$

$$F_0(v(0), \varepsilon) := \sigma_{N-1} f_N^{\infty} \left([v(0)]^N - \varepsilon^N\right);$$

$$F_1(v(1)) := \sigma_{N-1} \sum_{n=1}^{N} f_n^{\infty} \left(\lambda^n - [v(1)]^n\right);$$

with coefficients f_n^{∞} defined in (23). In this way, in view of Remark 3.1, the term F_{hole} represents the (negative) energy contribution of the possible "hole" in the support of the curve γ above the point at $r = R$ (see Remark 3.1) whereas the term $F_0(v(0), \varepsilon)$ accounts for the possible contribution of the term at $r = 0$, when $v(0) > \varepsilon$, and the term $F_1(v(1))$ contains the relevant information of the boundary condition at $r = 1$, when $v(1) < \lambda$.

Similarly, we extend the area term Σ to generalized graphs $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$ by setting

$$\mathcal{S}(v, \varepsilon, R, \tau^{\pm}) := \sigma_{N-1} \left\{ \varepsilon^{N-1} + \left(1 + \lambda^2\right)^{(N-1)/2} \right\} \tag{34}$$

if $\tau^- = \tau^+$ and, in case $\tau^- < \tau^+$, by

$$\mathcal{S}(v, \varepsilon, R, \tau^{\pm}) := \begin{cases} \sigma_{N-1} \left\{ [\tau^+]^{N-1} + \left(1 + \lambda^2\right)^{(N-1)/2} \right\} \\ \sigma_{N-1} \left\{ \sum_{\kappa=\pm} \left(R^2 + [\tau^{\kappa}]^2\right)^{(N-1)/2} + \left(1 + \lambda^2\right)^{(N-1)/2} \right\} \end{cases} \tag{35}$$

where the first formula applies when $R = 0$ and $\varepsilon = \tau^-$ and the second one in all other cases.

Finally, we obviously set

$$\mathcal{F}_\mu(v, \varepsilon, R, \tau^\pm) := \mathcal{F}(v, \varepsilon, R, \tau^\pm) + \mu\, \mathcal{S}(v, \varepsilon, R, \tau^\pm), \tag{36}$$

for every $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$.

We notice that to the profile function $v \in \mathcal{A}(\lambda)$ corresponds the generalized graph $(v, \varepsilon, R, \tau^\pm)$ with $R = 0$ and $\varepsilon = \tau^\pm = v(0)$ when v is continuous in I and with $\varepsilon = v(0) = 0$ and $\tau^\pm = v(R^\pm)$ when $R \in (0, R_0]$ is the unique discontinuity point of v. In the latter case, $F_{\text{hole}}(R, \tau^\pm)$ coincides with $F_{\text{hole}}(v)$ defined by (26). Therefore, if $v \in \mathcal{A}(\lambda)$, the equalities

$$\mathcal{F}(v, \varepsilon, R, \tau^\pm) = F(v) \qquad \text{and} \qquad \mathcal{S}(v, \varepsilon, R, \tau^\pm) = \Sigma(v)$$

hold so that we definitely have the equality

$$\mathcal{F}_\mu(v, \varepsilon, R, \tau^\pm) = F_\mu(v), \qquad v \in \mathcal{A}(\lambda),$$

when $(v, \varepsilon, R, \tau^\pm)$ is the generalized graph associated with the admissible profile $v \in \mathcal{A}(\lambda)$.

Remark 3.2 In the language of currents ([11], Sec. 4.1.5) the generalized graph $(v, \varepsilon, R, \tau^\pm)$ is the rectifiable 1–current T in $\mathbb{R} \times \mathbb{R}$ given by integration of compactly supported smooth forms in $\mathbb{R} \times \mathbb{R}$ along the naturally oriented rectifiable set γ^* defined in Remark 3.1. These currents naturally arise as weak limits of sequences of currents carried by the graphs of functions in $\mathcal{A}(\lambda)$. Moreover, the term $\mathcal{S}(v, \varepsilon, R, \tau^\pm)$ is the mass of the push forward through the map h given by (17) of the current $\partial T \times \mathbb{S}^{N-1}$ where ∂T is the rectifiable 0–current given by the boundary of T ([11], Sec. 4.2.6). In fact, we have

$$\partial T = \left(\delta_{(1,\lambda)} - \delta_{(0,\varepsilon)}\right) - \left(\delta_{(R,\tau^+)} - \delta_{(R,\tau^-)}\right), \tag{37}$$

where δ_P denotes the Dirac measure at a point $P \in \mathbb{R} \times \mathbb{R}$. □

We now introduce a notion of convergence of generalized graphs.

Definition 3.2 Let $(v_h, \varepsilon_h, R_h, \tau_h^\pm) \in \Gamma(\lambda)$ $(h \geq 1)$ and $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ be generalized graphs. The sequence $\{(v_h, \varepsilon_h, R_h, \tau_h^\pm)\}_h$ converges weakly to $(v, \varepsilon, R, \tau^\pm)$ in $\Gamma(\lambda)$ as $h \to +\infty$ if the following properties hold:

- $v_h \to v$ in weak*–$BV(I)$ as $h \to +\infty$;
- $\varepsilon_h \to \varepsilon$, $R_h \to R$ and $\tau_h^\pm \to \tau^\pm$ as $h \to +\infty$;

and in that case we write $(v_h, \varepsilon_h, R_h, \tau_h^\pm) \rightharpoonup (v, \varepsilon, R, \tau^\pm)$ weakly as $h \to +\infty$. □

Since the functions v of generalized graphs $(v, \varepsilon, R, \tau^\pm)$ in $\Gamma(\lambda)$ are increasing, non negative and such that $v(1) \leq \lambda$, it follows easily that the set $\Gamma(\lambda)$ is closed with respect to this type of convergence and we notice that, in the special case of generalized graphs associated with functions v_h and v in $\mathcal{A}(\lambda)$, we are thus requiring, besides the weak*–$BV(I)$ convergence of v_h to v, the pointwise convergence of $v_h(0) \to v(0)$ and $v_h(R_h^{\pm}) \to v(R^{\pm})$ too. Therefore, we are considering here a stronger convergence than the usual weak*–BV convergence.

For the functional $\mathcal{F}_\mu$ the following lower semicontinuity property holds.

Theorem 3.1 *Let $\mathcal{F}_\mu : \Gamma(\lambda) \to [0, +\infty]$ be the relaxed functional defined by (36) and let $(v_h, \varepsilon_h, R_h, \tau_h^{\pm})$ $(h \geq 1)$ and $(v, \varepsilon, R, \tau^{\pm})$ be generalized graphs in $\Gamma(\lambda)$ such that*

$$(v_h, \varepsilon_h, R_h, \tau_h^{\pm}) \rightharpoonup (v, \varepsilon, R, \tau^{\pm}) \qquad \textit{weakly as } h \to +\infty.$$

Then,

$$\mathcal{F}_\mu(v, \varepsilon, R, \tau^{\pm}) \leq \liminf_{h\to\infty} \mathcal{F}_\mu(v_h, \varepsilon_h, R_h, \tau_h^{\pm}).$$

Proof From the definition of weak convergence of generalized graphs it is easy to check that for the area term $\mathcal{S}$ defined in either (34) or (35) we have

$$\mathcal{S}(v, \varepsilon, R, \tau^{\pm}) \leq \liminf_{h\to+\infty} \mathcal{S}(v_h, \varepsilon_h, R_h, \tau_h^{\pm})$$

because $\tau^- < \tau^+$ implies $\tau_h^- < \tau_h^+$ eventually. Hence, we only have to prove that

$$\mathcal{F}(v, \varepsilon, R, \tau^{\pm}) \leq \liminf_{h\to+\infty} \mathcal{F}(v_h, \varepsilon_h, R_h, \tau_h^{\pm})$$

and we can assume that the right hand side is finite and, possibly passing to a (not relabeled) subsequence, we can also assume that the lim inf at the right hand side of the previous formula is actually a limit, i.e.

$$\lim_{h\to+\infty} \mathcal{F}(v_h, \varepsilon_h, R_h, \tau_h^{\pm}) \in [0, +\infty).$$

Moreover, recalling the definition of $\mathcal{F}$ in (33), we see that the term F_{hole} is obviously continuous because $R_h \to R$ and $\tau_h^{\pm} \to \tau^{\pm}$ as $h \to +\infty$ and we are left to prove that

$$\begin{aligned} &\overline{F}(v) + F_0(v(0), \varepsilon) + F_1(v(1)) \\ &\qquad \leq \lim_{h\to+\infty} \left\{\overline{F}(v_h) + F_0(v_h(0), \varepsilon_h) + F_1(v_h(1))\right\}. \end{aligned} \tag{38}$$

Possibly passing to a further (not relabeled) subsequence, we have

$$\begin{matrix} v_h(0) \to v_0 \\ v_h(1) \to v_1 \end{matrix} \qquad \text{as } h \to +\infty \text{ and } 0 \leq v_0 \leq v(0) < v(1) \leq v_1 \leq \lambda.$$

Assume first that $v_0 < v(0)$ and $v_1 = v(1)$ so that (38) reduces to

$$\overline{F}(v) + F_0(v(0), \varepsilon) \leq \lim_{h\to+\infty} \left\{\overline{F}(v_h) + F_0(v_h(0), \varepsilon_h)\right\} \tag{39}$$

because of the continuity of F_1. By the almost everywhere convergence $v_h \to v$ as $h \to +\infty$ and by the monotonicity of the functions v_h and v, by possibly passing to a (not relabeled) subsequence we can find numbers $\rho_k \in (0, 1)$ $(k \geq 1)$ such that

- for every $h \geq 1$, the function v_h is continuous at all points $r = \rho_k$ $(k \geq 1)$;
- $\rho_k \neq R_k$ for every k and $\rho_k \to 0^+$ as $k \to +\infty$;
- $v_h(\rho_h) \to v(0)$ as $h \to +\infty$.

For every h we define the increasing function $\overline{v}_h \colon I \to \mathbb{R}$ by setting

$$\overline{v}_h(r) := \begin{cases} \left(J_h r^N + \varepsilon_h^N\right)^{1/N} & \text{for } 0 < r \leq \rho_h \\ v_h(r) & \text{for } \rho_h \leq r < 1 \end{cases}$$

where

$$J_h := \frac{[v_h(\rho_h)]^N - \varepsilon_h^N}{\rho_h^N}.$$

Then, $\overline{v}_h(0) = \varepsilon_h$ for every h so that for these modified functions $\overline{v}_h$ we obviously have

$$F_0(\overline{v}_h(0), \varepsilon_h) = 0.$$

Moreover, $J\overline{v}_h(r) = J_h$ for every $r \in (0, \rho_h)$ and for every h and, since $\varepsilon_h \to \varepsilon$ and $v_h(\rho_h) \to v(0)$ as $h \to +\infty$ with $v_0 < v(0)$, we conclude that $J_h \to +\infty$ as $h \to +\infty$.

Since $\overline{v}_h$ is smooth in the half open interval $(0, \rho_h]$ and $\overline{v}_h = v_h$ in the open interval $(\rho_h, 1)$, the equality

$$D^s(r^{N-n}\overline{v}_h)(I) = D^s(r^{N-n}v_h)(\rho_h, 1)$$

holds for every h and, recalling the definition of $\overline{F}$ in (24), we can write $\overline{F}(\overline{v}_h)$ as

$$\overline{F}(\overline{v}_h) = \sigma_{N-1} \sum_{1 \leq n \leq N} \left(A_{n,h} + B_{n,h}\right), \qquad h \geq 1,$$

where we have set

$$\begin{aligned} A_{n,h} &= \int_0^{\rho_h} f_n\left(r, (r^{N-n}\overline{v}_h^n)'\right) dr \\ B_{n,h} &= \int_{\rho_h}^1 f_n\left(r, (r^{N-n}v_h^n)'\right) dr + f_n^\infty D^s(r^{N-n}v_h^n)(\rho_h, 1) \end{aligned} \qquad n = 1, \ldots, N,$$

for every h and we separately compute the limit of $A_{n,h}$ and $B_{n,h}$ as $h \to +\infty$. For $n = 1, \ldots, N-1$ we have

$$A_{n,h} = \int_0^{\rho_h} f_n\left(r, (r^{N-n}\overline{v}_h^n)'\right) dr = f_n^\infty \rho_h^{N-n}[\overline{v}_h(\rho_h)]^n = f_n^\infty \rho_h^{N-n}[v_h(\rho_h)]^n \to 0$$

as $h \to +\infty$ since $N-n \geq 1$ and $[v_h(\rho_h)]^n \leq \lambda^n$ for every h whereas for the remaining term $A_{N,h}$, recalling that $J\overline{v}_h = J_h$ in the interval $(0, \rho_h)$ and that $J_h \to +\infty$ as $h \to +\infty$, we get

$$A_{N,h} = \int_0^{\rho_h} r^{N-1} w(J_h)\,dr = \frac{1}{N}\rho_h^N J_h \frac{w(J_h)}{J_h} =$$
$$= \frac{1}{N}\left\{[v_h(\rho_h)]^N - \varepsilon_h^N\right\}\frac{w(J_h)}{J_h} \to \frac{w^\infty}{N}\left\{[v(0)]^N - \varepsilon^N\right\}$$

as $h \to +\infty$. We have thus proved that

$$\lim_{h\to+\infty} \sigma_{N-1} \sum_{1\leq n\leq N} A_{n,h} = F_0(v(0), \varepsilon). \tag{40}$$

We then turn to the terms $B_{n,h}$. For fixed $0 < \eta < 1$ we have $0 < \rho_h < \eta$ eventually and hence the inequality

$$B_{n,h} \geq \int_\eta^1 f_n\left(r, (r^{N-n}v_h^n)'\right) dr + f_n^\infty D^s(r^{N-n}v_h^n)(\eta, 1)$$

holds for the same h. Letting $h \to +\infty$, from Theorem 2.1 we get

$$\liminf_{h\to+\infty} B_{n,h} \geq \int_\eta^1 f_n\left(r, (r^{N-n}v^n)'\right) dr + f_n^\infty D^s(r^{N-n}v^n)(\eta, 1)$$

and then, letting $\eta \to 0^+$, we obtain

$$\liminf_{h\to+\infty} \sigma_{N-1} \sum_{1\leq n\leq N} B_{n,h} \geq \overline{F}(v)$$

and, combining the previous inequality and (40), we conclude that

$$\overline{F}(v) + F_0(v(0), \varepsilon) \leq \liminf_{h\to+\infty} \overline{F}(\overline{v}_h). \tag{41}$$

Finally, we claim that

$$\limsup_{h\to+\infty} \left[\overline{F}(\overline{v}_h) - \overline{F}(v_h)\right] \leq \lim_{h\to+\infty} F_0(v_h(0), \varepsilon_h) \tag{42}$$

so that the lower semicontinuity inequality (39) follows from (41).

In order to prove (42), we notice that, since $\overline{v}_h = v_h$ in the interval $(\rho_h, 1)$ and $\overline{v}_h$ is smooth in the interval $(0, \rho_h)$ for every h, the difference $\overline{F}(\overline{v}_h) - \overline{F}(v_h)$ reduces to

$$\overline{F}(\overline{v}_h) - \overline{F}(v_h) = \sigma_{N-1}\left(\Delta_{1,h} + \ldots + \Delta_{N,h}\right),$$

where for $n = 1, \ldots, N$

$$\Delta_{n,h} = \int_0^{\rho_h} f_n\left(r, (r^{N-n}\overline{v}_h^n)'\right) dr + \\ -\left\{\int_0^{\rho_h} f_n\left(r, (r^{N-n}v_h^n)'\right) dr + f_n^\infty D^s(r^{N-n}v_h^n)(0, \rho_h)\right\}.$$

For $n = 1, \ldots, N-1$ we have

$$\int_0^{\rho_h} f_n\left(r, (r^{N-n}\overline{v}_h^n)'\right) dr = f_n^\infty \rho_h^{N-n}[v(\rho_h)]^n = \\ = \int_0^{\rho_h} f_n\left(r, (r^{N-n}v_h^n)'\right) dr + f_n^\infty D^s(r^{N-n}v_h^n)(0, \rho_h)$$

for every h since $N-n \geq 1$ and $\overline{v}_h(\rho_h) = v_h(\rho_h)$ whence we conclude that $\Delta_{n,h} = 0$ for such n for every h whereas for $\Delta_{N,h}$, recalling that $r \in I \mapsto r^{N-1}Jv_h(r)$ is integrable and that $\overline{v}_h(0) = \varepsilon_h$ and arguing as in the computation of $A_{N,h}$, we find

$$\Delta_{N,h} = \int_0^{\rho_h} r^{N-1}\left[w(J_h) - w(Jv_h)\right] dr - \frac{w^\infty}{N} D^s(v_h^N)(0, \rho_h) = \\ = \int_0^{\rho_h} r^{N-1} w(J_h)\, dr - \int_0^{\rho_h} r^{N-1}\left[w(Jv_h) - w^\infty Jv_h\right] dr + \\ - w^\infty D(v_h^N)(0, \rho_h) = \\ = \frac{[v_h(\rho_h)]^N - \varepsilon_h^N}{N} \cdot \frac{w(J_h)}{J_h} - \int_0^{\rho_h} r^{N-1}\left[w(Jv_h) - w^\infty Jv_h\right] dr + \\ - \frac{w^\infty}{N}\left\{[v_h(\rho_h)]^N - [v_h(0)]^N\right\}$$

for every h. In view of (H4) and (10), we have $w(Jv_h) - w^\infty Jv_h \geq w_0 \geq 0$ and hence the last two summands are negative. Therefore, recalling that $J_h \to +\infty$ as $h \to +\infty$, we conclude that

$$\Delta_{N,h} \leq w^\infty \frac{\left\{[v_h(\rho_h)]^N - \varepsilon_h^N\right\}}{N} + \left(\frac{w(J_h)}{J_h} - w^\infty\right) \frac{\left\{[v_h(\rho_h)]^N - \varepsilon_h^N\right\}}{N} = \\ = \frac{F_0(v_h(0), \varepsilon_h)}{\sigma_{N-1}} + o(1)$$

as $h \to +\infty$ and this completes the proof of (42).

We have thus proved that $\mathcal{F}$ is lower semicontinuous along weakly convergent sequences of generalized graphs such that $v_0 < v(0)$ and $v_1 = v(1)$. The same argument works for sequences of generalized graphs such that $v_0 = v(0)$ and $v(1) < v_1$ and for the general case as well. □

Since the set of generalized graphs $\Gamma(\lambda)$ is clearly (sequentially) compact with respect to the weak convergence of generalized graphs, we immediately conclude that the relaxed functional $\mathcal{F}_\mu$ has a minimizer $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$. Moreover, since $\mathcal{F}(v) < +\infty$, the corresponding function $v : I \to \mathbb{R}$ is positive and strictly increasing.

Theorem 3.2 *Let $\mathcal{F}_\mu : \Gamma(\lambda) \to [0, +\infty]$ be the relaxed functional defined by (36). Then, there exists a generalized graph $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ such that*

$$\mathcal{F}_\mu(v, \varepsilon, R, \tau^\pm) = \min_{\Gamma(\lambda)} \mathcal{F}_\mu.$$

Moreover, the function $v : I \to \mathbb{R}$ is positive and strictly increasing.

4 Regularity of Generalized Minimizers

In this section, we investigate the regularity properties of generalized minimizers $(v, \varepsilon, R, \tau^\pm)$ in $\Gamma(\lambda)$ of the relaxed energy functional $\mathcal{F}_\mu$ (Theorem 3.2). We prove that every such element $(v, \varepsilon, R, \tau^\pm)$ is given by a function $v \in \mathcal{A}(\lambda)$ with $\varepsilon = v(0)$ and either $\tau^\pm = v(R^\pm)$, if $R \in (0, R_0]$, or $\tau^\pm = \varepsilon = v(0)$, if $R = 0$. In particular, the function v is one of those listed in Theorem 1.1 for $N = 3$.

4.1 Inner Variations and the Euler–Lagrange Equation

In this part we explore the optimality conditions satisfied by the positive and strictly increasing function $v : I \to \mathbb{R}$ associated with a generalized minimizer $(v, \varepsilon, R, \tau^\pm)$ of the relaxed energy functional $\mathcal{F}_\mu$ (Theorem 3.2).

We perform inner variations of the positive and strictly increasing function $v : I \to \mathbb{R}$ of a minimizer $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ of the relaxed energy $\mathcal{F}_\mu$ and we derive the Euler–Lagrange equation satisfied by v in every open interval $(a, b) \subset I$ such that either $R \notin (a, b)$ or $R \in (a, b)$ and $\tau^- = \tau^+$. In the language of currents, see Remark 3.2, this condition means that the rectifiable $1-$current T given by integration along the naturally oriented rectifiable set γ^* defined in Remark 3.1 has no boundary in the vertical strip $(a, b) \times \mathbb{R}$, compare (37).

To this aim, let $v : I \to \mathbb{R}$ be a strictly increasing function with $0 \le v(0) < v(1) \le \lambda$ and let (a, b) be an open interval in I. For every smooth function $\psi \in C_c^\infty(I)$ with support $\operatorname{spt}(\psi) \subset (a, b)$ and for every small enough δ, the function

$$\Psi_\delta(r) = r + \delta\psi(r), \qquad r \in I,$$

is a strictly increasing, smooth diffeomorphism of I onto itself and hence, for every small enough δ, the functions $v_\delta : I \to \mathbb{R}$ defined by

$$v_\delta(\rho) = v\left(\Psi_\delta^{-1}(\rho)\right), \qquad \rho \in I,$$

are strictly increasing in I with $0 \le v_\delta(0) < v_\delta(1) \le \lambda$ as well. We call such functions v_δ *inner variations of v in the interval* (a, b) and we notice that for small enough δ the original function v and the inner variations v_δ coincide in $I \setminus (a, b)$ and also that the equalities $v(a^+) = v_\delta(a^+)$ and $v(b^-) = v_\delta(b^-)$ hold. For the inner variations v_δ of v in the interval (a, b) the following result holds.

Lemma 4.1 *Let $v\colon I \to \mathbb{R}$ be a nonnegative and increasing function such that $v(1) \leq \lambda$ and let v_δ be inner variations of v in the interval $(a, b) \subset I$ for small enough δ. Then,*

(a) $\displaystyle\int_0^1 \rho^{N-1} J v_\delta(\rho)\, d\rho = \int_0^1 r^{N-1} J v(r)\, dr < +\infty$;

(b) *there exists $C \geq 0$ such that*

$$\int_0^1 \rho^{N-1} w(J v_\delta(\rho))\, d\rho \leq C \left[\int_0^1 r^{N-1} w(J v(r))\, dr + 1 \right];$$

for every small enough δ.

The constant $C \geq 1$ in (b) depends on the function ψ which defines Ψ_δ but can be taken to be independent of δ.

Proof Let $\psi \in C_c^\infty(I)$ with $\operatorname{spt}(\psi) \subset (a, b)$ be the smooth function that defines the inner variations v_δ.

The equality in (a) comes from the area formula applied to the functions u and u_δ associated with v and v_δ by formula (2) and from the equality of the images $u(B^N) = u_\delta(B^N)$. In fact, since v is (classically) differentiable almost everywhere in I, in view of the definition of Ψ_δ, by elementary computations we get

$$J v_\delta(\rho)\bigg|_{\rho = \Psi_\delta(r)} = \frac{v'(r)}{\Psi_\delta'(r)} \left[\frac{v(r)}{\Psi_\delta(r)} \right]^{N-1} = J v(r) \left(\frac{1}{1 + \delta\psi(r)/r} \right)^{N-1} \frac{1}{1 + \delta\psi'(r)}$$

for a.e. $r \in I$ and hence, by the change of variable formula with $\rho = \Psi_\delta(r)$, by elementary computations we obtain

$$\int_0^1 \rho^{N-1} J v_\delta(\rho)\, d\rho = \\ = \int_0^1 [\Psi_\delta(r)]^{N-1} J v_\delta\, (\Psi_\delta(r))\, \Psi_\delta'(r)\, dr = \int_0^1 r^{N-1} J v(r)\, dr \leq \lambda^N / N$$

which establishes (a).

As to (b), exploiting again elementary computations based on the binomial and Taylor formula, we obtain

$$\left(\frac{1}{1 + \delta\psi(r)/r} \right)^{N-1} \frac{1}{1 + \delta\psi'(r)} = 1 - \delta\varphi_\delta(r), \qquad r \in I, \tag{43}$$

where $\varphi_\delta \in C_c^\infty(I)$ is the smooth function defined by

$$\varphi_\delta(r) = (N-1)\psi(r)/r + \psi'(r) + \omega_\delta(r), \qquad r \in I, \tag{44}$$

and $\omega_\delta(r) \to 0$ uniformly for $r \in I$ as $|\delta| \to 0^+$. Thus,

$$J v_\delta(\rho)\Big|_{\rho=\Psi_\delta(r)} = J v(r)\left[1 - \delta\varphi_\delta(r)\right] \tag{45}$$

for a.e. $r \in I$ so that, from the intermediate value theorem and (H5), we get

$$\begin{aligned}
&\left| w\left(J v(r)\left[1 - \delta\varphi_\delta(r)\right]\right) - w\left(J v(r)\right) \right| = \\
&\quad = \left|\delta w'\left(J v(r)[1 - \theta\delta\varphi_\delta(r)]\right) J v(r)\varphi_\delta(r)\right| = \\
&\quad = \left|\delta\left(J v(r)[1 - \theta\delta\varphi_\delta(r)]\right) w'\left(J v(r)[1 - \theta\delta\varphi_\delta(r)]\right)\right| \frac{|\varphi_\delta(r)|}{|1 - \theta\delta\varphi_\delta(r)|} \le \\
&\quad \le C|\delta|\left[w(J v(r)) + 1\right] \frac{|\varphi_\delta(r)|}{|1 - \theta\delta\varphi_\delta(r)|}
\end{aligned}$$

for a.e. $r \in I$ and for some $\theta = \theta(r, \delta) \in (0, 1)$ with a constant $C \ge 0$ which depends on w only. Since the function

$$r \in I \mapsto \frac{\varphi_\delta(r)}{1 - \theta\delta\varphi_\delta(r)}$$

is bounded on the interval I uniformly for every small enough δ, we get

$$\left| w\left(J v(r)\left[1 - \delta\varphi_\delta(r)\right]\right) - w\left(J v(r)\right) \right| \le C|\delta|\left[w(J v(r)) + 1\right] \tag{46}$$

for a.e. $r \in I$ with a constant $C \ge 1$ which depends on the smooth function ψ but can be taken to be independent of δ and this obviously yields

$$w\left(J v(r)\left[1 - \delta\varphi_\delta(r)\right]\right) \le C\left[w(J v(r)) + 1\right]$$

for a.e. $r \in I$ with a constant $C \ge 1$ which depends on the same quantities as before. Finally, by the change of variable formula with $\rho = \Psi_\delta(r)$ and the previous inequality, we obtain

$$\begin{aligned}
&\int_0^1 \rho^{N-1} w(J v_\delta(\rho))\, d\rho = \\
&\quad = \int_0^1 \left[r + \delta\psi(r)\right]^{N-1} w\left(J v(r)[1 - \delta\varphi_\delta(r)]\right)\left[1 + \delta\psi'(r)\right] dr \le \\
&\quad \le C\left(\int_0^1 r^{N-1} w(J v(r))\, dr + 1\right),
\end{aligned}$$

where, once more, the constant $C \ge 1$ depends only on the choice of ψ. □

Now, we consider the positive and strictly increasing function $v\colon I \to \mathbb{R}$ associated with a minimizing element $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$ of the relaxed energy functional $\mathcal{F}_{\mu}$ (Theorem 3.2) and we look for the optimality conditions satisfied by v by performing inner variations v_{δ} of v.

Theorem 4.1 *Let $v\colon I \to \mathbb{R}$ be the positive and strictly increasing function of a generalized minimizer $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$ of the relaxed energy functional $\mathcal{F}_{\mu}$ and let $(a, b) \subset I$ be an open interval such that either $R \notin (a, b)$ or $R \in (a, b)$ and $\tau^- = \tau^+$. Then,*

(a) *the function*

$$r \in (a, b) \mapsto r^{N-2}\left[w(Jv(r)) - w'(Jv(r))Jv(r)\right]$$

is in $L^1(a, b)$ if $a > 0$ and in $L^1_{\mathrm{loc}}(a, b)$ if $a = 0$;

(b) *the function*

$$r \in (a, b) \mapsto r^{N-1}\left[w(Jv(r)) - w'(Jv(r))Jv(r)\right]$$

is in $W^{1,1}(a, b)$ if $a > 0$ and in $W^{1,1}_{\mathrm{loc}}(a, b)$ if $a = 0$;

(c) *the Euler–Lagrange equation* (EL)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left[r^{N-1}\left(w(Jv(r)) - w'(Jv(r))Jv(r)\right)\right] &= \\ &= (N-1)r^{N-2}\left[w(Jv(r)) - w'(Jv(r))Jv(r)\right] \end{aligned} \tag{47}$$

holds for a.e. $r \in (a, b)$.

Proof First, we notice that we obviously have $\mathcal{F}_{\mu}(v, \varepsilon, R, \tau^{\pm}) < +\infty$. Then, for small enough $\delta \neq 0$, we let v_{δ} be the inner variations of v associated with the open interval (a, b) and with a smooth function $\psi \in C^{\infty}_{\mathrm{c}}(I)$ such that $\mathrm{spt}\,(\psi) \subset (a, b)$. We assume throughout the rest of the proof that δ is such that the previous condition holds.

The equality $v_{\delta} = v$ out of a compact subset of the interval (a, b) implies that $(v_{\delta}, \varepsilon, R_{\delta}, \tau^{\pm})$ where $R_{\delta} = \Psi_{\delta}(R)$ is an admissible element of $\Gamma(\lambda)$ and we compute the limit as $\delta \to 0$ of the (rescaled by σ_{N-1}) differential quotient

$$\Delta(\delta) = \frac{\mathcal{F}_{\mu}(v_{\delta}, \varepsilon, R_{\delta}, \tau^{\pm}) - \mathcal{F}_{\mu}(v, \varepsilon, R, \tau^{\pm})}{\sigma_{N-1}\delta}, \qquad \delta \neq 0, \tag{48}$$

which must be zero because of the minimality of $(v, \varepsilon, R, \tau^{\pm})$.

Since we have $\tau^+ = \tau^-$ if $R \in (a, b)$, the very same reasoning implies that $\mathcal{S}(v_{\delta}, \varepsilon, R_{\delta}, \tau^{\pm}) = \mathcal{S}(v, \varepsilon, R, \tau^{\pm})$ and that

$$D\left(\rho^{N-n}v_{\delta}^n\right)(I) = D\left(r^{N-n}v^n\right)(I), \qquad n = 1, \ldots, N$$

and hence, exploiting (29) and (a) of Lemma 4.1, the (rescaled) differential quotient $\Delta(\delta)$ reduces to the (rescaled) differential quotient of the volume part $\overline{F}_{\text{vol}}$ defined by (30), i.e.

$$\Delta(\delta) = \frac{\overline{F}_{\text{vol}}(v_\delta) - \overline{F}_{\text{vol}}(v)}{\sigma_{N-1}\delta} = \frac{1}{\delta}\int_0^1 r^{N-1}\left[w(Jv_\delta(r)) - w(Jv(r))\right] dr.$$

Recalling (45) and (43) and arguing as in the proof of Lemma 4.1, we find

$$\begin{aligned}\Delta(\delta) &= \frac{1}{\delta}\int_0^1 r^{N-1}\left\{\frac{w\left(Jv(r)[1-\delta\varphi_\delta(r)]\right)}{1-\delta\varphi_\delta(r)} - w(Jv(r))\right\} dr = \\ &= \frac{1}{\delta}\int_0^1 \frac{r^{N-1}}{1-\delta\varphi_\delta(r)}\left\{w\left(Jv(r)[1-\delta\varphi_\delta(r)]\right) - w(Jv(r))\right\} dr + \\ &\quad + \int_0^1 r^{N-1}\frac{\varphi_\delta(r)}{1-\delta\varphi_\delta(r)} w(J(v(r)))\, dr = \\ &= \Delta_1(\delta) + \Delta_2(\delta)\end{aligned}$$

for (small enough) $\delta \neq 0$ with $\varphi_\delta(r)$ defined by (44).

We consider the term $\Delta_1(\delta)$ first. From the limit of φ_δ as $\delta \to 0$ in (44), we get

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta}\frac{r^{N-1}}{1-\delta\varphi_\delta(r)}\left\{w\left(Jv(r)[1-\delta\varphi_\delta(r)]\right) - w(Jv(r))\right\} = \\ &\quad = -r^{N-1}w'(Jv(r))Jv(r)\left[(n-1)\psi(r)/r + \psi'(r)\right]\end{aligned}$$

for a.e. $r \in I$ and from (46) we have

$$\frac{1}{|\delta|}\frac{r^{N-1}}{|1-\delta\varphi_\delta(r)|}\left|w\left(Jv(r)[1-\delta\varphi_\delta(r)]\right) - w(Jv(r))\right| \le Cr^{N-1}\left[w(Jv(r)) + 1\right]$$

for a.e. $r \in I$ for every small enough δ with a constant C independent of δ. Therefore, since the right hand side of the previous inequality is integrable in I, we can pass to the limit within the integral as $\delta \to 0$ and we find

$$\lim_{\delta\to 0}\Delta_1(\delta) = -\int_0^1 r^{N-1}w'(Jv(r))Jv(r)\left[(N-1)\psi(r)/r + \psi'(r)\right] dr.$$

As to second term $\Delta_2(\delta)$, by the dominated convergence theorem we obviously find

$$\lim_{\delta\to 0}\Delta_2(\delta) = \int_0^1 r^{N-1}w(Jv(r))\left[(N-1)\psi(r)/r + \psi'(r)\right] dr.$$

We have thus proved that the limit of $\Delta(\delta)$ as $\delta \to 0$ exists and then it must vanish because of the minimality of v. Therefore, the equation

$$\int_0^1 r^{N-1} \left\{w(Jv(r)) - w'(Jv(r))Jv(r)\right\} \left[(N-1)\psi(r)/r + \psi'(r)\right] dr = 0$$

holds for every function $\psi \in C_c^\infty(I)$ with $\operatorname{spt}(\psi) \subset (a, b)$ from which, in view of the integrability of the function

$$r \in (0,1) \mapsto r^{N-1} \left[w(Jv(r)) - w'(Jv(r))Jv(r)\right] \tag{49}$$

which follows from the integrability of $r \mapsto r^{N-1} w(Jv(r))$ and from (11), we get the equality

$$\int_0^1 r^{N-1} \left[w(Jv(r)) - w'(Jv(r))Jv(r)\right] \psi'(r)\, dr = \\ = -(N-1) \int_0^1 r^{N-2} \left[w(Jv(r)) - w'(Jv(r))Jv(r)\right] \psi(r)\, dr$$

for every function ψ as before.

If $a > 0$, the integrability of (49) obviously gives (a) which, together with the previous equality, establishes the validity of (b) and (c). If $a = 0$ instead, from the integrability of the function

$$r \in (0, b) \mapsto r^{N-2} \left[w(Jv(r)) - w'(Jv(r))Jv(r)\right] \psi(r)$$

for every $\psi \in C_c^\infty(I)$ with $\operatorname{spt}(\psi) \subset (a, b)$ we can conclude only that the function in (a) is in $L^1_{\mathrm{loc}}(a, b)$ from which (b) and (c) follow as before. □

4.2 Regularity of Generalized Minimizers

The Euler–Lagrange equation (47) implies that the function v associated with a minimizing element $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ of $\mathcal{F}_\mu$ is smooth with constant Jacobian determinant in every interval $(a, b) \subset I$ such that either $R \notin (a, b)$ or $R \in (a, b)$ and $\tau^- = \tau^+$. This is proved in the following theorem.

Theorem 4.2 *Let $v \colon I \to \mathbb{R}$ be the positive and strictly increasing function of a generalized minimizer $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ of the relaxed energy functional $\mathcal{F}_\mu$ and let $(a, b) \subset I$ be an open interval such that either $R \notin (a, b)$ or $R \in (a, b)$ and $\tau^- = \tau^+$. Then, there exist $J > 0$ and $k \geq -Ja^N$ such that*

$$v(r) = \left(Jr^N + k\right)^{1/N}, \qquad r \in (a, b), \tag{50}$$

In particular, $Jv(r) = J$ for every $r \in (a, b)$.

Proof The derivative of the function $t \in (0, +\infty) \mapsto w(t) - w'(t)t$ is strictly negative in $(0, +\infty)$ because of (H2) and hence the function itself is invertible from $(0, +\infty)$ onto $(q, +\infty)$ where

$$q := \lim_{t \to +\infty} \left[w(t) - w'(t)t \right]$$

and its inverse function is in $C^1(q, +\infty)$. By (b) of Theorem (4.1) we know that the function

$$r \in (a, b) \mapsto \left[w(Jv(r)) - w'(Jv(r))Jv(r) \right]$$

is either in $W^{1,1}(a, b)$ or in $W^{1,1}_{\mathrm{loc}}(a, b)$ depending on the value of a and hence the Jacobian $Jv(r)$ can be written as the composition of a $W^{1,1}_{\mathrm{loc}}$ with a C^1 function whence we derive that $Jv \in W^{1,1}_{\mathrm{loc}}(a, b)$. Then, differentiating the (EL) equation (47), we find that

$$-w''(Jv(r))(Jv)'(r)Jv(r) = 0 \qquad \text{for a.e. } r \in (a, b)$$

which, on account of (H2) and $Jv > 0$ almost everywhere in I, gives

$$(Jv)'(r) = 0 \qquad \text{for a.e. } r \in (a, b) \tag{51}$$

and this implies that $Jv(r) = J$ for every $r \in (a, b)$ for some $J > 0$.

Finally, in order to integrate (51) and conclude that v is actually given by formula (50), we must rule out the possibility that v has a singular part, i.e. we have to prove that the singular part $(Dv)^s(a, b)$ vanishes. This can be proved by the following energy argument which exploits the minimality of $(v, \varepsilon, R, \tau^{\pm})$ again. Set

$$v_a := v(a^+) < v(b^-) =: v_b$$

and let $v_* : I \to \mathbb{R}$ be the positive and strictly increasing function which agrees with v in $I \setminus (a, b)$ and is of the form (50) in (a, b) with $v_*(a^+) = v_a$ and $v_*(b^-) = v_b$, i.e.

$$v_*(r) := \begin{cases} v(r) & r \in I \setminus (a, b) \\ \left(J_* r^N + k_*\right)^{1/N} & r \in (a, b) \end{cases},$$

where

$$J_* = \frac{v_b^N - v_a^N}{b^N - a^N} \qquad \text{and} \qquad k_* = \frac{v_a^N b^N - v_b^N a^N}{b^N - a^N}. \tag{52}$$

Then, $Jv(r) = J$ and $Jv_*(r) = J_*$ for every $r \in (a, b)$ and clearly $(v_*, \varepsilon, R, \tau^{\pm})$ is an admissible element of $\Gamma(\lambda)$. We also have $\mathcal{S}(v_*, \varepsilon, R, \tau^{\pm}) = \mathcal{S}(v, \varepsilon, R, \tau^{\pm})$ and hence from the equality of the total variations

$$D(r^{N-n} v^n)(I) = D(r^{N-n} v_*^n)(I)$$

for every n and from (29) we obtain

$$\begin{aligned} \mathcal{F}_\mu(v_*, \varepsilon, R, \tau^{\pm}) - \mathcal{F}_\mu(v, \varepsilon, R, \tau^{\pm}) &= \overline{F}(v_*) - \overline{F}(v) = \\ &= \frac{\sigma_{N-1}}{N}(b^N - a^N)\left\{\left[w(J_*) - w^\infty J_*\right] - \left[w(J) - w^\infty J\right]\right\}. \end{aligned}$$

The minimality of $(v, \varepsilon, R, \tau^{\pm})$ yields $\mathcal{F}_\mu(v, \varepsilon, R, \tau^{\pm}) \le \mathcal{F}_\mu(v_*, \varepsilon, R, \tau^{\pm})$ whence we conclude that $J_* \le J$ because the function $t \in (0, +\infty) \mapsto w(t) - w^\infty t$ is strictly decreasing. On the other hand, from the equality of the total variations $Dv^N(a,b) = Dv_*^N(a,b)$ we have

$$N \int_a^b r^{N-1} Jv(r)\,dr + (Dv^N)^s(a,b) = N \int_a^b r^{N-1} Jv_*(r)\,dr$$

whence, recalling that $Jv(r) = J$ and $Jv_*(r) = J_*$ for every $r \in (a,b)$, we conclude that

$$J(b^N - a^N) + (Dv^N)^s(a,b) = J_*(b^N - a^N).$$

Since $J_* \le J$, it follows that $(Dv^N)^s(a,b) = 0$ which implies that the function v is in $W^{1,1}_{\text{loc}}(a,b)$. Hence, we can integrate (51), thus concluding that v is actually given by (50) for suitable $J > 0$ and $k \ge -Ja^N$. □

From the previous theorem we obtain also that the function v of a generalized minimizer $(v, \varepsilon, R, \tau^{\pm})$ of $\mathcal{F}_\mu$ has the form

$$v(r) = \left(Jr^N + k\right)^{1/N}, \qquad r \in I, \tag{53}$$

for suitable numbers $J > 0$ and $k \ge 0$, when either $R = 0$ or $R \in (0, R_0]$ and $\tau^- = \tau^+$. Otherwise, if $R \in (0, R_0]$ and $\tau^- < \tau^+$, the point $r = R$ is the unique discontinuity point of v and v has the form

$$v(r) = \begin{cases} \left(J_- r^N + k_-\right)^{1/N} & \text{if } r \in (0, R) \\ \left(J_+ r^N + k_+\right)^{1/N} & \text{if } r \in (R, 1) \end{cases} \tag{54}$$

for suitable numbers $J_\pm > 0$ and $k_- \ge 0$ and $k_+ > -J_+ R^N$, provided that $0 < v(R^-) < v(R^+) \le \lambda$. In particular, v has at most one discontinuity point and has no Cantor part.

Now, we analyze the real parameters ε and $\tau^{\pm}$ corresponding to a generalized minimizer $(v, \varepsilon, R, \tau^{\pm})$ of $\mathcal{F}_\mu$.

Theorem 4.3 *Let $(v, \varepsilon, R, \tau^{\pm}) \in \Gamma(\lambda)$ be a generalized minimizer of the relaxed energy functional $\mathcal{F}_\mu$. Then,*

(a) *if $R = 0$, then $(v, \varepsilon_*, 0, \tau_*^{\pm}) \in \Gamma(\lambda)$ where $\varepsilon_* = \tau_*^{\pm} = v(0)$ is also a generalized minimizer of $\mathcal{F}_\mu$;*
(b) *if $R \in (0, R_0]$, then $v(0) = \varepsilon = 0$ and $\tau^{\pm} = v(R^{\pm})$;*
(c) *$v(1) = \lambda$.*

In particular, this implies that the rectifiable set γ^* defined in Remark 3.1 corresponding to the generalized minimizer $(v, \varepsilon, R, \tau^{\pm})$ has no "vertical" parts.

Proof (a) Since $R = 0$, the parameters ε and $\tau^{\pm}$ satisfy either $0 \le \varepsilon \le \tau^- = \tau^+$ or $\varepsilon = 0$ and $\tau^- < \tau^+$ by definition of generalized graph and the function v is of the form (53) by Theorem 4.2 with $J = [v(1)]^N - [v(0)]^N$ and $k = [v(0)]^N$.

First, we suppose that the equality $\tau^- = \tau^+$ holds and we assume by contradiction that $\varepsilon < v(0)$. For $v_0 \in [\varepsilon, v(0)]$, we consider the comparison function

$$v_*(r) = \left(J_* r^N + k_*\right)^{1/N}, \qquad r \in I,$$

where J_* and k_* are given by (52) with $v_a = v_0$, $v_b = v(1)$ and $a = 0$ and $b = 1$. Then, $v_*(0) = v_0$ and $J v_*(r) = J_*$ for every $r \in I$ with $J_* > J$. Setting $\varepsilon_* = \tau_*^{\pm} = \varepsilon$, the corresponding element $(v, \varepsilon_*, 0, \tau_*^{\pm})$ is a generalized graph in $\Gamma(\lambda)$ such that the terms F_{hole}, F_1 and $\mathcal{S}$ have the same values for $(v, \varepsilon, 0, \tau^{\pm})$ and $(v, \varepsilon_*, 0, \tau_*^{\pm})$. Moreover, for the other terms we have $D(r^{N-n} v^n)(I) = D(r^{N-n} v_*^n)(I)$ for $n = 1, \ldots, N-1$ and

$$F_0(v(0), \varepsilon) + \sigma_{N-1}\, f_N^{\infty} D(v^N)(I) = F_0(v_*(0), \varepsilon_*) + \sigma_{N-1}\, f_N^{\infty} D(v_*^N)(I)$$

whence the equality

$$\begin{aligned} F_0(v_*(0), \varepsilon_*) + \sigma_{N-1} \sum_{1 \le n \le N} f_n^{\infty} D(r^{N-n} v_*^n)(I) &= \\ = F_0(v(0), \varepsilon) + \sigma_{N-1} \sum_{1 \le n \le N} f_n^{\infty} D(r^{N-n} v^n)(I) \end{aligned}$$

follows. Hence, from the definition of $\mathcal{F}_\mu$ and from (29) and (30) we compute

$$\begin{aligned} 0 \le \mathcal{F}_\mu(v_*, \varepsilon_*, 0, \tau_*^{\pm}) - \mathcal{F}_\mu(v, \varepsilon, 0, \tau^{\pm}) = \overline{F}_{\text{vol}}(v_*) - \overline{F}_{\text{vol}}(v) &= \\ = \frac{\sigma_{N-1}}{N} \left\{ \left[w(J_*) - w^{\infty} J_* \right] - \left[w(J) - w^{\infty} J \right] \right\} \end{aligned}$$

which gives a contradiction to the minimality of $(v, \varepsilon, 0, \tau^{\pm})$ because the function $t \mapsto w(t) - w^{\infty} t$ is strictly decreasing and $J < J_*$. This proves that $v(0) = \varepsilon$ when $R = 0$ and $\tau^- = \tau^+$.

Next, we assume that $\tau^- < \tau^+$ and we notice that it must be $\tau^- = \varepsilon$ otherwise, if it were $\varepsilon < \tau^-$, we could redefine τ^- to be equal to ε and this would make the term $\mathcal{S}$ strictly smaller and the term F_{hole} strictly larger whereas all other terms in $\mathcal{F}_\mu$ would remain unaffected as they do not depend on τ^-. Also, it must be $\tau^+ = v(0)$ otherwise, choosing $v_0 \in [\tau^+, v(0)]$, we could repeat the very same argument of the previous case $\tau^- = \tau^+$ to get a contradiction. Therefore, we have $\varepsilon = \tau^- < \tau^+ = v(0)$ and the generalized graph $(v, \varepsilon_*, R, \tau_*^{\pm}) \in \Gamma(\lambda)$ where $\varepsilon_* = \tau_*^{\pm} = v(0)$ is also a generalized minimizer of $\mathcal{F}_\mu$ because $\mathcal{F}_\mu$ takes the same values at $(v, \varepsilon_*, R, \tau_*^{\pm})$ and $(v, \varepsilon, R, \tau^{\pm})$. This completes the proof of (a).

(b) First, we prove that $v(0) = \varepsilon = 0$. By definition, it is $\varepsilon = 0$ and, if it were $v(0) > 0$, we could choose $v_0 \in [0, v(0)]$ and repeat the very same argument of the first part of the proof of (a) in the interval $(0, R)$ to get a contradiction to the minimality of $(v, 0, R, \tau^{\pm})$. As to the equality $\tau^{\pm} = v(R^{\pm})$, if $\tau^- = \tau^+$, then Theorem 4.2 yields the continuity of v at $r = R$ whence the equality $v(R) = v(R^{\pm}) = \tau^{\pm}$ follows. If $\tau^- < \tau^+$ instead, by the very same argument of the corresponding case of (a) we obtain that $v(R^{\pm}) = \tau^{\pm}$ and this establishes (b)

(c) In order to prove that the function v associated with a minimizing element $(v, \varepsilon, R, \tau^{\pm})$ satisfies the Dirichlet boundary condition $v(1) = \lambda$, regardless of the value of R and the validity of the equality $\tau^- = \tau^+$ or not, we assume by contradiction that $v(1) < \lambda$ and, recalling that $R \in [0, R_0]$, we perform outer variations of the function v in the interval $(R, 1)$. For the sake of definiteness, we assume $R \in (0, R_0]$ so that v is given by (54) in the interval $(R, 1)$ and has constant Jacobian $Jv(r) = J_+$ in the same interval. The same argument works in the case $R = 0$. Therefore, let

$$v_\delta(r) = v(r) + \delta\varphi(r), \qquad r \in I \quad (\delta \in \mathbb{R}),$$

where $\varphi \in C_c^\infty(\mathbb{R})$ is a nonnegative, smooth function such that $\operatorname{spt}(\varphi) \subset (R, +\infty)$ and $\varphi(1) = 1$. Since v' is positive and remains away from zero in the interval $(R, 1)$, the outer variations v_δ are positive, strictly increasing and such that $v_\delta(1) \le \lambda$ for all sufficiently small $|\delta| > 0$ and we assume throughout the rest of the proof that δ is such that this holds, so that $(v_\delta, \varepsilon, R, \tau^{\pm})$ is an admissible element in $\Gamma(\lambda)$ too. Then, we compute the limit as $\delta \to 0$ of the (rescaled) differential quotient $\Delta(\delta)$ defined by (48) which vanishes because of the minimality of $(v, \varepsilon, R, \tau^{\pm})$. Since the generalized graph $(v_\delta, \varepsilon, R, \tau^{\pm})$ and the minimizer $(v, \varepsilon, R, \tau^{\pm})$ share the same values of ε, R and $\tau^{\pm}$, the area term $\mathcal{S}$ and the terms F_0 and F_{hole} have the same values for $(v_\delta, \varepsilon, R, \tau^{\pm})$ and $(v, \varepsilon, R, \tau^{\pm})$ Moreover, we have

$$\sigma_{N-1}\, f_n^\infty D(r^{N-n} v_\delta^n)(I) + F_1(v_\delta(1)) = \sigma_{N-1}\, f_n^\infty D(r^{N-n} v^n)(I) + F_1(v(1))$$

for $n = 1, \dots, N$ so that from (29) and (30) we see that only the volume term $\overline{F}_{\text{vol}}$ on the interval $(R, 1)$ is involved in the computation of $\Delta(\delta)$, i.e.

$$\begin{aligned}\Delta(\delta) &= \frac{\overline{F}_{\text{vol}}(v_\delta) - \overline{F}_{\text{vol}}(v)}{\sigma_{N-1}\delta} = \\ &= \frac{1}{\delta}\int_R^1 r^{N-1}\left\{\left[w(Jv_\delta(r)) - w^\infty Jv_\delta(r)\right] - \left[w(J_+) - w^\infty J_+\right]\right\} dr\end{aligned}$$

and hence, recalling that v_δ is smooth in $(R, 1)$, we easily check that we can pass to the limit as $\delta \to 0$ within the integral to find

$$\begin{aligned}\lim_{\delta\to 0} \Delta(\delta) &= \sigma_{N-1}\left[w'(J_+) - w^\infty\right]\int_R^1 (v^{N-1}\varphi)'\, dr = \\ &= \sigma_{N-1}\left[w'(J_+) - w^\infty\right][v(1)]^{N-1} = 0.\end{aligned}$$

Since $v(1) > 0$ and $w'(t) < w^\infty$ for every $t > 0$ due to the strict convexity of w, we have got a contradiction and this completes the proof of (c). □

4.3 Existence of Minimizers

At last, we can prove the existence of solutions of the minimization problem for generalized Varga materials in arbitrary dimension in the class $\mathcal{A}(\lambda)$. This theorem extends to arbitrary dimension $N \geq 3$ the result stated for $N = 3$ as Theorem 1.1.

Theorem 4.4 *Let F_μ be the energy functional defined by* (8) *where F and Σ are defined by* (16) *and by* (18) *and* (19) *respectively with $c_n \geq 0$ ($n = 1, \ldots, N-1$) and w satisfying hypotheses* (H1),…,(H5). *Then, for every $\lambda \geq 1$ and $\mu > 0$ the minimum problem*

$$\inf \left\{ F_\mu(\overline{v}) \mid \overline{v} \in \mathcal{A}(\lambda) \right\} \tag{55}$$

has a solution $v \in \mathcal{A}(\lambda)$. Moreover, the minimizer $v(r)$, $r \in I$, is of one of the following three types:

(a) *linear: $v(r) = \lambda r$;*

(b) *cavitation: $v(r) = \left(Jr^N + k\right)^{1/N}$ with $v(0) = k > 0$ and $J = \lambda^N - k > 0$;*

(c) *fracture:*

$$v(r) = \begin{cases} Kr & \text{if } 0 < r < R \\ \left(Jr^N + k\right)^{1/N} & \text{if } R < r < 1 \end{cases}$$

with $R \in (0, R_0]$, $K > 0$, $JR^N + k > (KR)^N$ and $J = \lambda^N - k > 0$.

Proof Let $(v, \varepsilon, R, \tau^\pm) \in \Gamma(\lambda)$ be a generalized minimizer of the relaxed energy functional $\mathcal{F}_\mu$ (Theorem 3.2). Then, the function v is positive, strictly increasing and satisfies the Dirichlet condition $v(1) = \lambda$ (Theorem 4.3).

If $R = 0$, we can assume that $v(0) = \varepsilon = \tau^\pm$ (Theorem 4.3) and the function v is given by (53) with $J > 0$ and $k = \varepsilon^N$ (Theorem 4.2). Therefore, $v \in \mathcal{A}(\lambda)$ and, depending on whether $\varepsilon = 0$ or $\varepsilon > 0$, the function v is linear as in (a) or cavitating as in (b). It is then easy to check that

$$\begin{aligned} \mathcal{F}_\mu(v, v(0), 0, \tau^\pm) &= \mathcal{F}(v, v(0), 0, \tau^\pm) + \mu\, \mathcal{S}(v, v(0), 0, \tau^\pm) = \\ &= F(v) + \mu\, \Sigma(v) = F_\mu(v) \end{aligned}$$

and, since

$$F_\mu(v) = \mathcal{F}_\mu(v, v(0), 0, \tau^\pm) = \min_{\Gamma(\lambda)} \mathcal{F}_\mu \leq \inf \left\{ F_\mu(\bar{v}) \mid \bar{v} \in \mathcal{A}(\lambda) \right\},$$

we conclude that v is a minimizer of (55).

If $R \in (0, R_0]$ instead, then $\varepsilon = v(0) = 0$ and $\tau^\pm = v(R^\pm)$ (Theorem 4.3). Therefore, v is linear as in (a) if $v(R^-) = v(R^+)$ and features a fracture at $r = R$ as in (c) if $v(R^-) < v(R^+)$. In both cases, we have $v \in \mathcal{A}(\lambda)$ and, recalling (25), (26) and (18) and (19), it is again easy to check that

$$\mathcal{F}_\mu(v, \varepsilon, R, \tau^\pm) = \overline{F}(v) - F_{\text{hole}}(R, \tau^\pm) \qquad \text{and} \qquad \mathcal{S}(v, \varepsilon, R, \tau^\pm) = \Sigma(v)$$

and this implies that v is a minimizer of (55) as before. □

5 Occurrence of Cavitation and Fracture

In this section we discuss the occurrence of cavitation and fracture (see (b) and (c) in the statement of Theorem 1.1 or 4.4) for generalized Varga materials, mainly in the model case (14).

5.1 The Normalization Condition

Let $v \colon I \to \mathbb{R}$ be a critical point of the energy functional F_μ such that $v(0) = 0$ and assume in addition that v is discontinuous at some point $R \in (0, 1)$ and that v is smooth in the interval $I_R = (0, R)$. We look for necessary conditions in order that v is a critical point of F_μ in absence of Dirichlet conditions.

Since v is smooth in I_R, inner variations lead to the equation $Jv(r) = J$ for $r \in I_R$ whence $v(r) = Kr$ for some $K > 0$ follows. We then consider the restriction of the energy functional F_μ to the interval I_R which is given (up to the dimensional factor σ_{N-1}) by

$$\sum_{n=1}^{N-1} f_n^\infty D(r^{N-n} v^n)(I_R) + \int_0^R r^{N-1} w(Jv(r))\, dr + \mu \left(R^2 + [v(R)]^2 \right)^{(N-1)/2},$$

where $v(R) = v(R^-)$ and we exploit outer variations around v with test functions $\varphi \in C_c^\infty(I)$ such that $\varphi(R) = 1$. Computing the derivative with respect to δ, we obtain the balance equation

$$\sum_{n=1}^{N-1} n f_n^\infty R^{N-n} [v(R)]^{n-1} + w'(J)[v(R)]^{N-1} + \\ + \mu\, (N-1) \left(R^2 + [v(R)]^2 \right)^{(N-3)/2} v(R) = 0.$$

If $v(r) = Kr$, we have $v(R) = KR$ and $J = K^N$ and hence, dividing by R^{N-1}, we get

$$\sum_{n=1}^{N-1} n f_n^\infty K^{n-1} + w'(K^N) K^{N-1} + \frac{\mu}{R} (N-1) \left(1 + K^2 \right)^{(N-3)/2} K = 0.$$

In case $R = 1$, letting $K = 1$, we obtain the equation

$$-w'(1) = \sum_{n=1}^{N-1} n f_n^\infty + C(N)\mu = 0 \qquad \text{where} \qquad C_N := (N-1) 2^{(N-3)/2}$$

which, recalling the definition of the coefficients f_n^∞ in (23), is the normalization condition (20).

In the physical dimension $N = 3$, letting $t = K^3$, we obtain the balance equation

$$- w'(t) = g_R(t) \qquad \text{where} \qquad g_R(t) = c_1 t^{-2/3} + 2c_2 t^{-1/3} + \frac{2\mu}{R} t^{-1/3}. \tag{56}$$

Since $g_R(t) > g_1(t)$ for every $t > 0$ and $0 < R < 1$, by the normalization condition $-w'(1) = g_1(1)$ we infer that the equation $-w'(t) = g_R(t)$ has a solution $t_R \in (0, 1)$ for every $0 < R < 1$ provided that $-t^{2/3} w'(t) \to +\infty$ as $t \to 0^+$. We notice that the latter condition is satisfied in the model case (13).

5.2 Balance Equation for Energy Minimizers

As we have seen, energy minimizers $v \in \mathcal{A}(\lambda)$ of the energy functional F_μ must be of one of the three types listed in Theorem 4.4.

In the case of a solution v featuring cavitation, i.e.

$$v(r) = \left(J r^N + s^N\right)^{1/N}, \qquad r \in I,$$

for suitable numbers $J, s > 0$ such that $J + s^N = \lambda^N$, we can perform outer variations of the energy functional F_μ around v with test functions $\varphi \in C^\infty(I)$ such that $\varphi(0) = 1$ and $\varphi(r) = 0$ for $r > 1/2$ as in Theorem 4.3 and we obtain the balance equation

$$w'(J) = \mu \frac{N-1}{v(0)} \tag{57}$$

which relates the constant Jacobian $Jv(r) = J$ of v with the radius of the cavity $v(0) = s > 0$.

Similarly, in the case of a solution v with fracture, i.e.

$$v(r) = \begin{cases} Kr & \text{if } r \in (0, R) \\ \left(J r^N + s^N\right)^{1/N} & \text{if } r \in (R, 1). \end{cases}$$

for some $R \in (0, R_0]$ and suitable numbers $K, J, s > 0$ such that

$$K R < \left(J R^N + s^N\right)^{1/N} < \left(J + s^N\right)^{1/N} = \lambda,$$

by performing outer variations of the energy functional F_μ around v with test functions $\varphi \in C_c^\infty(I)$ such that $\varphi(R) = 1$, we get the balance equation

$$\sum_{n=1}^{N-1} n f_n^{\infty} R^{N-n} \left([v(R^+)]^{n-1} - [v(R^+)]^{n-1} \right) +$$
$$+ \left\{ w'(J_-)[v(R^-)]^{N-1} - w'(J_+)[v(R^+)]^{N-1} \right\} =$$
$$= \mu(N-1) \sum_{\kappa=\pm} \left(R^2 + [v(R^\kappa)]^2 \right)^{(N-3)/2} v(R^\kappa),$$

where $J_- = K^N$, $J_+ = J$ and $v(R^-) = KR < \left(J R^N + s^N \right)^{1/N} = v(R^+)$.

5.3 Cavitation vs Linear Solution

Let $\lambda > 1$ be fixed and assume that $v \in \mathcal{A}(\lambda)$ is a minimizer of the energy functional F_μ featuring cavitation. Then, denoting the radius of the cavity by $v(0) = s \in (0, \lambda)$, the function $v = v_s$ is given by

$$v_s(r) := \left((\lambda^N - s^N) r^N + s^N \right)^{1/N}, \qquad r \in I, \tag{58}$$

where $J = \lambda^N - s^N > 0$ is the constant Jacobian of v_s and the balance equation (57) becomes

$$w'(\lambda^N - s^N) = \mu \frac{N-1}{s}. \tag{59}$$

By hypotheses (H2) and (H3) of the energy density w, it turns out that the function $s \in (0, \lambda) \mapsto w'(\lambda^N - s^N)$ is strictly decreasing and onto $(-\infty, w'(\lambda^N))$ and hence the balance equation (59) has no solutions $s \in (0, \lambda)$ if $w'(\lambda^N) \leq \mu(N-1)/\lambda$. In particular, cavitation may occur only if $w'(\lambda^N) > J_{\text{cr}}$ where J_{cr} is the unique critical point of w. If we assume in addition that w' is strictly concave in $(0, +\infty)$, as it happens for the model case (13), it turns out that there exists a threshold $\lambda_0 > 1$ such that the balance equation (59) has no solution for $\lambda < \lambda_0$, has a unique solution for $\lambda = \lambda_0$ and has two distinct solutions for $\lambda > \lambda_0$. Assuming $\lambda > \lambda_0$ and denoting the two solutions of (59) by $0 < s_-(\lambda) < s_+(\lambda)$, we easily compute

$$F_\mu(v_s) = \sigma_{N-1} \sum_{n=1}^{N-1} f_n^{\infty} \lambda^n + \frac{\sigma_{N-1}}{N} w(\lambda^N - s^N) +$$
$$+ \mu \sigma_{N-1} \left\{ s^{N-1} + \left(1 + \lambda^2 \right)^{(N-1)/2} \right\}$$

whence we get

$$\frac{\mathrm{d}}{\mathrm{d}s} F_\mu(v_s) = \sigma_{N-1} s^{N-1} \left\{ \mu \frac{N-1}{s} - w'(\lambda^N - s^N) \right\} > 0$$

for every $s \in (0, s_-(\lambda)) \cup (s_+(\lambda), \lambda)$ so that we obtain

$$F_\mu(v_{s_+(\lambda)}) < F_\mu(v_{s_-(\lambda)}).$$

Moreover, we can compare the energy of the linear function defined by $v_0(r) = \lambda r$, $r \in I$, with the energy of the critical point v_s corresponding to $s = \lambda/2^{1/N}$:

$$\begin{aligned} F_\mu(v_{\lambda/2^{1/N}}) - F_\mu(v_0) = \\ = \frac{\sigma_{n-1}}{N} w(\lambda^N/2) + \mu\,\sigma_{n-1} \left(\frac{\lambda}{2^{1/N}}\right)^{N-1} - \frac{\sigma_{n-1}}{N} w(\lambda^N) \to -\infty \end{aligned}$$

as $\lambda \to +\infty$. This shows that the critical point with cavity $v_{s_+(\lambda)}$ is energetically favorable if λ is sufficiently large and that a cavity radius greater than the positive number $2\mu/a$ occurs abruptly, when increasing the parameter $\lambda > 1$ of the boundary displacement condition.

5.4 Occurrence of Fractures

In this part we prove that, for suitable values of the parameters $a, b > 0$ in the model case (13) in the physical dimension $N = 3$, there exists a threshold $\overline{\lambda} > 0$ such that, for $\lambda > \overline{\lambda}$, the energy functional F_μ has a minimizer $v \in \mathcal{A}(\lambda)$ which features a fracture at a positive radius $r = R$.

To this aim, we consider the strain energy density of generalized Varga materials in dimension $N = 3$ defined by (14) and we choose, for example, $a = \mu > 0$ and $b = 3^7\mu$ so that $c_1 = c_2 = (b - a - 2\mu)/3 = (3^6 - 1)\mu$. Then, the balance equation (59) becomes

$$1 - \frac{3^7}{(\lambda^3 - s^3)^2} = \frac{2}{s} \tag{60}$$

and it is easy to check that for $\lambda = 3\sqrt[3]{4}$ the unique solution of (60) is given by $s = 3$. We set $\lambda_0 = 3\sqrt[3]{4}$ and $s_0 = 3$ so that the Jacobian of the corresponding critical point v_{s_0} of F_μ in (58) is $J_0 = \lambda_0^3 - s_0^3 = 3^4$.

For such values $\lambda_0 = 3\sqrt[3]{4}$ and $s_0 = 3$, we compare the energy of the corresponding linear function $v_0(r) = 3\sqrt[3]{4}r$ with the energy of the unique critical point of F_μ featuring cavitation and satisfying the boundary condition $v_{s_0}(1) = 3\sqrt[3]{4}$ which is given by

$$v_3(r) := \left(J_0 r^3 + s_0^3\right)^{1/3} = \left(3^4 r^3 + 3^3\right)^{1/3}, \qquad r \in I. \tag{61}$$

By elementary computations we get $F_\mu(v_0) < F_\mu(v_3)$, i.e. the linear function $v_0(r) = 3\sqrt[3]{4}r$ has lower energy than the energy of the unique critical point with cavity v_3 given by (61). Yet, the linear function v_0 fails to be a energy minimizer. In fact, taking e.g.

$$\overline{v}(r) = \begin{cases} 3^{7/6}\, r & \text{if } r \in (0, 1/2) \\ 3 \cdot 4^{1/3}\, r & \text{if } r \in (1/2, 1), \end{cases}$$

we get by easy computation

$$F(\overline{v}) - F(v_0) = \\ = \frac{3\pi}{2}\mu\,(3^6 - 1)\left(3^{1/6} + 3^{4/3} - 4^{1/3} - 3 \cdot 4^{2/3}\right) + \frac{9\pi}{2}\mu\left(2 \cdot 3^{1/2} - \frac{19}{4}\right)$$

and

$$\mu\,\Sigma(\overline{v}) - \mu\,\Sigma(v_0) = \pi\mu\left(2 + 3^{7/3} + 3^2 \cdot 4^{2/3}\right)$$

so that we definitely obtain $F_\mu(\overline{v}) < F_\mu(v_0)$.

As a consequence, it is conceivable that there exists a threshold $1 < \overline{\lambda} < \lambda_0$ such that for $\overline{\lambda} < \lambda < \lambda_0$ the minimal profile function $v_{\min}$ has a discontinuity point at some radius $R \in (0, 1)$ and hence it is given by

$$v(r) = \begin{cases} Kr & \text{if } r \in (0, R) \\ \left(J(r^3 - R^3) + [v(R^+)]^3\right)^{1/3} & \text{if } r \in (R, 1) \end{cases}, \tag{62}$$

where $K = t^3 < 1$ with t the smallest positive solution to equation (56). Moreover, the balance system connecting the values J and K can be computed by performing outer variations as in Sec. 5.2. Therefore, a fracture occurs at some radius R and the minimal solution preserves the same fractured configuration as in (62) for λ growing from $\overline{\lambda}$ with a slope K (and radius R) fixed independently of the parameter λ.

5.5 The Super-Linear Growth Case

Assume now that $w^\infty = +\infty$ in (5). In that case, we consider the class $\mathcal{W}(\lambda)$ of strictly increasing functions $v \in W^{1,1}(I)$ such that $v(0) \geq 0$ and $v(1) = \lambda$. For any sequence $\{v_h\} \subset \mathcal{W}(\lambda)$ satisfying

$$\sup_{h \geq 1} F_\mu(v_h) < \infty,$$

we can find a subsequence converging in $L^1(I)$ to a function v in the same class $\mathcal{W}(\lambda)$. Therefore, for each $\mu > 0$ the energy minimum

$$\min\left\{F_\mu(v) \mid v \in \mathcal{W}(\lambda)\right\}$$

is attained. Moreover, since our analysis of critical points continues to hold, this time we infer that the energy minimizer is continuous and hence it satisfies one of the first two alternatives of Theorem 4.4, where in case $v(0) > 0$ the constant Jacobian determinant J satisfies the balance equation (57).

5.6 Comparison with Previous Results

A similar cavitation phenomenon is observed in [8], where a radially symmetric model with an energy density containing a term with superlinear growth in the gradient is analyzed.

In case $\mu = 0$, i.e. in absence of surface terms, the dynamics of smooth critical points of generalized Varga materials in the physical dimension $N = 3$ is analyzed in [15].

In that case, if a critical point $v(r)$ is smooth on I and $v(0) > 0$, so that a cavitation occurs at the origin, the balance equation of outer variations implies that the limit at zero of the radial component T_{rr} of the Cauchy stress tensor is equal to zero. As a consequence, equation $w'(Jv(r)) = 0$ holds and hence $Jv(r) \equiv J_{\rm cr}$, the unique critical point of the energy density $w(t)$. This conclusion is consistent with (57) when $\mu = 0$. Therefore, critical points with a cavitation occur for every $\lambda > J_{\rm cr}^{1/3}$ and with a positive cavity radius $s_\lambda(0) = (\lambda^3 - J_{\rm cr})^{1/3}$ such that $s_\lambda(0) \to 0^+$ as $\lambda \to (J_{\rm cr}^{1/3})^+$. On the other hand, in the case $\mu > 0$ analyzed here, by taking e.g. the model case (13), cavitation occurs abruptly, with a cavity radius greater than a positive constant, namely $v(0) > 2a/\mu$, due to equation (57).

Therefore, the results in [15] are qualitatively different from those obtained here. Dynamics of materials as in our model are influenced by the presence of an energy term depending on the area $4\pi[v(0)]^2$ of the surface of the cavity or on the area $4\pi(2R^2 + [KR]^2 + [v(R^+)]^2)$ of the boundary of the graph $\mathcal{G}_u$ of the fractured radially symmetric deformation u with a profile function v as in (62).

6 Conclusions

We have given in any dimension an analytical motivation to the existence and regularity of minimizers of a wide class of energy functionals that in the physical dimension $N = 3$ corresponds to isotropic deformations of generalized Varga materials. When the volume term has a linear growth at infinity, due to the presence of an additional term accounting for surface tension, we exclude plasticity phenomena and presence of diffuse (Cantor–type) fractures, and we prove that minimizers may display cavitation or appearance of a radially symmetric fracture. In the physical dimension $N = 3$ and for suitable choices of the coefficients in the model energy density we show the existence of a threshold for the Dirichlet datum above which a fractured deformation is energetically favorable with respect to both the solution with a cavitation and the linear one.

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